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WIRTSCHAFTS- UND
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Chair of Statistics, Econometrics and Empirical Economics

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S414
Advanced Mathematical Methods
Exercises

WS 2020/21

Vector Calculus

Exercise 1 **Derivation rules**

Derive the following expressions:

a) $\frac{d}{dx}\left(-12x^3+5x^4+1\right)$

b) $\frac{d}{dx}\left(2x^7-6^x\right)$

c) $\frac{d}{dx}\left(2x^7\cdot 6^x\right)$

d) $\frac{d}{dx}\left(\sqrt[3]{x}\cdot\ln(x)\right)$

e) $\frac{d}{dx}\left[\frac{x^4+3x^2}{x-1}\right]$

f) $\frac{d}{dv}\left[\frac{-e^v\cdot\sin(v)}{2v^2+1}\right]$

g) $\frac{d}{dw}\left(w\cdot\exp(2w^8+4)\right)$

h) $\frac{d}{dx}\left(\sqrt{2x^4+\sin(x)}\right)$

i) $\frac{d}{du}\left(\ln(1+u^4)\right)$

j) $\frac{d}{dx}\left(x^2\cdot\sin(2x)\right)$

k) $\frac{d}{dx}\left[\left(x^{1/2}-\frac{1}{x^2}\right)^3\right]$

l) $\frac{d}{dx}\left(x\cdot\ln(x)\cdot\ln(x^2)\right)$

Exercise 2 **General partial derivatives**

Find the following derivatives:

a) $\frac{\partial}{\partial x}\ln(x^2+2y)$

b) $\frac{\partial^2}{\partial x^2}\ln(x^2+2y)$

c) $\frac{\partial^3}{\partial y\partial x^2}\ln(x^2+2y)$

d) $\frac{\partial^3}{\partial x^2\partial y}\exp(2x+y^2)$

e) $\frac{\partial^n}{\partial x^n}\exp(2x+y^2)$

Exercise 3 **Total partial derivative**

Determine for the function

$$z=e^{x^2}+y^2\,e^{xy}$$

with $x=2t+3s$ and $y=t^2s^3$ the the total partial derivative $\left(\frac{\partial z}{\partial s}\right)$.

Exercise 4 **Open Sets**

Determine which of the following sets are open:

a) $U = \{(x, y)|2 < x^2 + y^2 < 3\} \subseteq \mathbb{R}^2$

b) $U = \{(x, y)|x + y = 2\} \subseteq \mathbb{R}^2$

c) $U = \{(x, y, z)|xyz > 0\} \subseteq \mathbb{R}^3$

Exercise 5 **Real-valued and vector-valued function**

Find the domain and the range of the function:

$$f(x, y) = \frac{x - 3y^2}{x^2 - y}$$

Exercise 6 **Real-valued and vector-valued function**

Find the domain of the function $\boldsymbol{F} : U \subseteq \mathbb{R}^2 \to \mathbb{R}^3$ defined by

$$\boldsymbol{F}(x, y) = \left(\frac{y}{x^2 + y^2}, \frac{x}{x^2 + y^2}, \sqrt{xy} \right)$$

Exercise 7 **Gradient**

Determine the gradient of $f : \mathbb{R}^2 \to \mathbb{R}$ which is given by $(x, y) \mapsto e^{xy} + \sin xy$:

Exercise 8 **Derivative of a function**

Compute the derivative of the function $\boldsymbol{F}$ at the point $\boldsymbol{a}$:

a) $\boldsymbol{F}(x, y) = (y, x, 11), \boldsymbol{a} = (0, 0)$

b) $\boldsymbol{F}(x, y, z) = (ln(x^2 + y^2 + z^2), 2xy + z), \boldsymbol{a} = (1, 1, 0)$

c) $f(x, y, z) = ||x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}||^2, \boldsymbol{a} = (a_1, a_2, a_3)$

Exercise 9 **Taylor rule**

Derive the fourth order Taylor expansion of the function $f(x) = cos(x)$ around a suitable expansion point x_0.

Solution Exercise 1:

a) $-36x^2 + 20x^3$

b) $14x^6 - \ln(6) \cdot 6^x$

c) $14x^6 \cdot 6^x + 2x^7 \ln(6)6^x$

d) $\frac{1}{3} \frac{\ln(x)+3}{\sqrt[3]{x^2}}$

e) $\frac{3x^4-4x^3+3x^2-6x}{(x-1)^2}$

f) $e^v \frac{-(\sin(v)+\cos(v))(2v^2+1)+\sin(v)\cdot 4v}{(2v^2+1)^2}$

g) $\exp(2w^8 + 4)(16w^8 + 1)$

h) $\frac{1}{2} \cdot \frac{1}{\sqrt{2x^4+\sin(x)}} \cdot (8x^3 + \cos(x))$

i) $\frac{4u^3}{1+u^4}$

j) $2x(\sin(2x) + x\cos(2x))$

k) $3 \cdot (x^{\frac{1}{2}} - \frac{1}{x^2})^2 \cdot (\frac{1}{2}x^{-\frac{1}{2}} + 2 \cdot \frac{1}{x^3})$

l) $(\ln(x) + 1)\ln(x^2) + 2\ln(x)$

Solution Exercise 2:

a) $\frac{2x}{x^2+2y}$

b) $\frac{-2x^2+4y}{(x^2+2y)^2}$

c) $\frac{12x^2-8y}{(x^2+2y)^3}$

d) $8y\exp(2x + y^2)$

e) $2^n \exp(2x + y^2)$

Solution Exercise 3:

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} = 3(\exp(x^2)2x + y^3\exp(xy)) + 3t^2s^2(2y\exp(xy) + xy^2\exp(xy))$$

Solution Exercise 4:

a) U is open in $\mathbb{R}^2$

b) U is not open in $\mathbb{R}^2$

c) U is open in $\mathbb{R}^3$

Solution Exercise 5:
$f : \mathbb{R}^2 \setminus \{(x, y)|y = x^2\} \to \mathbb{R}$

Solution Exercise 6:
$f : \big\{(x, y) \in \mathbb{R}^2 : \{x \geq 0 \cap y \geq 0\} \cup \{x \leq 0 \cap y \leq 0\} \setminus \{x = 0 \cap y = 0\}\big\} \to \mathbb{R}^3$

Solution Exercise 7:

$$\nabla f(x, y) = (e^{xy} + \cos xy)(y\boldsymbol{i} + x\boldsymbol{j})$$

Solution Exercise 8:

a) $\begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{pmatrix}$

b) $\begin{pmatrix} 1 & 1 & 0 \\ 2 & 2 & 1 \end{pmatrix}$

c) $\begin{pmatrix} 2a_1 & 2a_2 & 2a_3 \end{pmatrix}$

Solution Exercise 9:
$x_0 = 0$
$T(x) = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4$