On a Contraction free Sequent Calculus for the Modal Logic S4

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Theorem proving in the modal logic S4 is notoriously difficult, because in conventional sequent style calculi for this logic lengths of deductions are not bounded in terms of the length of their endsequent. This means that the usual depth first search strategy for backwards construction of deductions of given sequents may give rise to infinite search paths and is not guaranteed to terminate. Thus using such a search strategy prevents us not only from obtaining a decision procedure for the logic in question, but even from arriving at a complete proof procedure. There are two well known approaches for overcoming this problem: both approaches rely on the fact that all formulas which occur in a deduction are subformulas of the endsequent and that out of these formulas one may only form finitely many "essentially different" sequents. Thus although there is no bound on the length of *all* deductions of a given sequent, we know that for any given sequent there is a number such that if the sequent is deducible at all, then it has a deduction of length smaller than this number. Hence by only considering deductions of appropriately bounded length one may obtain a decision procedure. But due to the fact that using such a procedure one is forced to construct many redundant inferences – one will, for instance, have to consecutively apply the same inference many times – this approach is considered rather inefficient. Instead the usual technique for deciding provability of formulas in S4 is based on loop checking: If "essentially the same" sequent occurs twice on a branch of a constructed deduction, then there is a shorter deduction with the same endsequent which does not show this redundancy, and one may backtrack. Although more efficient in terms of run time than the previous approach this loop checking method requires quite involved implementation techniques. Now in the context of intuitionistic propositional logic recently a third approach has been found, which is based on so called contraction free sequent calculi, i.e. calculi for which there is a certain measure such that for all rules of the calculi all premisses have smaller measure then the conclusion (cf. Dyckhoff, Hudelmaier.) Thus in a contraction free calculus *all* deductions of a given sequent are bounded in length by some function of the length of their endsequent and a decision procedure is obtained by simple depth first backwards application of the rules. In this paper we show that there is a contraction free sequent for S4, too. This calculus is more complicated than the corresponding calculi for intuitionistic propositional logic, but still it gives rise to a simpler decision procedure for S4 than conventional methods.

§1. *Introduction*

We consider a language of sequents, i.e. pairs of multisets of formulas built up from propositional variables, the Boolean connectives $\neg$ and $\vee$ and the modal connective $\Box$.

We start from a calculus LM_0 for the modal logic S4 consisting of axioms of the form $M,a \Rightarrow a,N$, where a is a propositional variable, the well known Boolean rules

$$\mathrm{E}\neg\ \frac{M \Rightarrow N, v}{M, \neg v \Rightarrow N} \qquad\qquad \mathrm{I}\neg\ \frac{M, v \Rightarrow N}{M \Rightarrow N, \neg v}$$

$$\mathrm{E}\vee\ \frac{M, u \Rightarrow N \qquad M, v \Rightarrow N}{M, u \vee v \Rightarrow N} \qquad\qquad \mathrm{I}\vee\ \frac{M \Rightarrow N, u, v}{M \Rightarrow N, u \vee v}$$

and the two modal rules

$$\mathrm{E}\Box\ \frac{M, \Box v, v \Rightarrow N}{M, \Box v \Rightarrow N} \qquad\qquad \mathrm{I}\Box\ \frac{M^0 \Rightarrow v}{M \Rightarrow N, \Box v}$$

where M^0 results from M by omitting all formulas not of the form $\Box v$.

Obviously the so called weakening rule is an admissible rule of this calculus: If a sequent $M \Rightarrow N$ is deducible in LM_0 by a deduction of length n, then both the sequent $M \Rightarrow N, v$ and the sequent $M, v \Rightarrow N$ are deducible by deductions of length $\leq n$. Moreover it is immediately clear that the Boolean rules of LM_0 and the rule E□ are invertible: If a conclusion of one of these rules is deducible in LM_0 by a deduction of length n, then all its premisses are deducible by a deduction of length $\leq n$. Therefore the following holds:

LEMMA 1:
a) Every LM_0-deduction of a sequent $M,v,v \Rightarrow N$ may be transformed into a deduction of the sequent $M,v \Rightarrow N$ of smaller or equal length.
b) Every LM_0-deduction of a sequent $M \Rightarrow v,v,N$ may be transformed into a deduction of the sequent $M \Rightarrow v,N$ of smaller or equal length.

PROOF:
a) is true of axioms, and if $M,v,v \Rightarrow N$ is the conclusion of an inference I different from I□ with principal formula different from v, then the induction hypothesis applies to the premisses and by an application of I to the transformed premisses the sequent $M,v \Rightarrow N$ may be obtained. If $M,v,v \Rightarrow N$ is the conclusion of an I□-inference, then either both occurrences of v are contained in M^0, in which case the induction hypothesis applies to it and $M,v \Rightarrow N$ may be obtained as before, or neither occurrence of v is in M^0 and one of them may be introduced by weakening. If $M,v,v \Rightarrow N$ is the conclusion of a Boolean rule B with principal formula v, then the premisses may be transformed according to the inversion principle for the Boolean rules and the resulting sequents may be transformed by the induction hypothesis; finally to these transformed sequents the rule B may be applied again, thereby obtaining a deduction of $M,v \Rightarrow N$. If $M,v,v \Rightarrow N$ is the conclusion of an E□-inference with principal formula v, then v is of the form $\Box u$ and the premiss reads $M,\Box u,\Box u, u \Rightarrow N$. The induction hypothesis applied to this premiss yields the sequent $M, \Box u, u \Rightarrow N$; and the rule E□ applied to this gives the required deduction of $M,v \Rightarrow N$.
b) is also true of axioms, and if $M \Rightarrow v,v,N$ is the conclusion of an inference I different from I□, then the deduction of $M \Rightarrow v,N$ is constructed as above. But if $M \Rightarrow v,v,N$ is the conclusion of an I□-inference, then from its premiss we directly obtain $M \Rightarrow v,N$ by a different application of I□ with the same principal formula, vic. by introducing only one occurrence of v into the conclusion. QED

This shows that the calculus LM_0 is equivalent to the more frequently encountered S4-calculi for sequents made up of pairs of *sets* of formulas (cf. FITTING), and in particular the so called cut rule is also a derived rule of LM_0: If two sequents $M \Rightarrow c$ and $M,c \Rightarrow v$ are deducible by LM_0, then so is the sequent $M \Rightarrow v$.

§2. *Reduction to clausal form*
In order to determine deducibility of sequents by our calculus LM_0 we may restrict the language to so called *clausal sequents*: There is a simple procedure which associates to every sequent s of the full language a clausal sequent $C(s)$, such that s is derivable by LM_0 if and only if $C(s)$ is derivable by LM_0.

DEFINITION: (Cf. MINTS)
a) A modal literal is either a propositional variable, a negated propositional variable, a formula of the form $\Box a$, where a is a propositional variable or a formula of the form $\neg\Box a$, where a is a propositional variable.
b) A modal clause is a formula of the form $l_0\vee(l_1\vee\ldots)\ldots)$ or $\Box(l_0\vee(l_1\vee\ldots)\ldots))$, where the l_i are modal literals.
c) A clausal sequent is a sequent of the form $c_1, \ldots, c_m \Rightarrow a_1, \ldots, a_n$, where the c_i are modal clauses and the a_i are propositional variables.
For simplifying notation we let the expression $[v_0, \ldots, v_n]$ ($n\geq 0$) denote the formula $v_0\vee(v_1\vee(\ldots\vee v_n)\ldots)$. Thus a formula $[v_0, \ldots, v_n]$ or $\Box[v_0, \ldots, v_n]$ is a modal clause iff all the v_i are modal literals.

Then the following holds:

LEMMA 2: *The sequents in the first column of the following table are deducible by LM_0 if and only if the corresponding sequents of the second column are deducible:*

$M \Rightarrow v, N$	$M, \neg v \Rightarrow N$
$M, [A, \neg\neg v, B] \Rightarrow N$	$M, [A, v, B] \Rightarrow N$
$M, [A, \neg(u \vee v), B] \Rightarrow N$	$M, [A, \neg p, B], [p, \neg u], [p, \neg v] \Rightarrow N$
$M, [A, u \vee v, B] \Rightarrow N$	$M, [A, u, v, B] \Rightarrow N$
$M, [A, \neg\Box\neg v, B] \Rightarrow N$	$M, [A, \neg\Box p, B], \Box[p, v] \Rightarrow N$
$M, [A, \neg\Box(u \vee v), B] \Rightarrow N$	$M, [A, \neg\Box p, B], \Box[p, \neg u], \Box[p, \neg v] \Rightarrow N$
$M, [A, \neg\Box\Box v, B] \Rightarrow N$	$M, [A, \neg\Box v, B] \Rightarrow N$
$M, [A, \Box\neg v, B] \Rightarrow N$	$M, [A, \Box p, B], [\neg p, \neg v] \Rightarrow N$
$M, [A, \Box(u \vee v), B] \Rightarrow N$	$M, [A, \Box p, B], \Box[\neg p, u, v] \Rightarrow N$
$M, [A, \Box\Box v, B] \Rightarrow N$	$M, [A, \Box v, B] \Rightarrow N$
$M, \Box[A, \neg\neg v, B] \Rightarrow N$	$M, \Box[A, v, B] \Rightarrow N$
$M, \Box[A, \neg(u \vee v), B] \Rightarrow N$	$M, \Box[A, \neg p, B], \Box[p, \neg u], \Box[p, \neg v] \Rightarrow N$
$M, \Box[A, u \vee v, B] \Rightarrow N$	$M, \Box[A, u, v, B] \Rightarrow N$
$M, \Box[A, \neg\Box\neg v, B] \Rightarrow N$	$M, \Box[A, \neg\Box p, B], \Box[p, v] \Rightarrow N$
$M, \Box[A, \neg\Box(u \vee v), B] \Rightarrow N$	$M, \Box[A, \neg\Box p, B], \Box[p, \neg u], \Box[p, \neg v] \Rightarrow N$
$M, \Box[A, \neg\Box\Box v, B] \Rightarrow N$	$M, \Box[A, \neg\Box v, B] \Rightarrow N$
$M, \Box[A, \Box\neg v, B] \Rightarrow N$	$M, \Box[A, \Box p, B], \Box[\neg p, \neg v] \Rightarrow N$
$M, \Box[A, \Box(u \vee v), B] \Rightarrow N$	$M, \Box[A, \Box p, B], \Box[\neg p, u, v] \Rightarrow N$
$M, \Box[A, \Box\Box v, B] \Rightarrow N$	$M, \Box[A, \Box v, B] \Rightarrow N$

(Here the formulas p on the right hand sides are propositional variables which do not occur on the corresponding left hand sides.)

PROOF:
This lemma is well known: For instance from a deduction of $M,\Box[A,\neg\Box(u\vee v),B] \Rightarrow N$ we obtain the required deduction of $M,\Box[A,\neg\Box p,B],\Box[p,\neg u]\ \Box[p,\neg v] \Rightarrow N$ by a cut with the LM_0-deducible sequent $\Box[A,\neg\Box p,B]$, $\Box[p,\neg u],\Box[p,\neg v] \Rightarrow \Box[A,\neg\Box(u\vee v),B]$, whereas given a deduction of $M,\Box[A,\neg\Box p,B],\Box[p,\neg u]$, $\Box[p,\neg v] \Rightarrow N$ we obtain the required deduction of $M,\Box[A,\neg\Box(u\vee v),B] \Rightarrow N$ by changing all occurences of p to $u\vee v$ and cutting out the two LM_0-deducible formulas $\Box[u\vee v,\neg u]$ and $\Box[u\vee v,\neg v]$ from the resulting endsequent. QED

From this lemma we easily obtain a procedure for reducing a given sequent of arbitrary form to a clausal sequent. This procedure consists of two steps:
a)Applying the property expressed by the first row of the above table to the formulas v_i on the right hand side of a given sequent $a_1, \dots ,a_m,\Box d_1, \dots ,\Box d_n \Rightarrow v_1, \dots ,v_q, p_1, \dots ,p_r$ (where the a_i are not of the form $\Box b$, the v_i are not propositional variables and the p_i are propositional variables), we arrive at a new sequent which has only propositional variables on its right hand side and which has at most twice as many connectives as the given sequent.
b) Applying the remaining rows to the resulting sequent $[a_1], \dots ,[a_m],\Box[d_1], \dots , \Box[d_n],[\neg v_1], \dots ,[\neg v_q] \Rightarrow p_1, \dots ,p_r$ it is easily seen, that at each step the number of connectives inside one of the square brackets of this sequent, which do not belong to modal literals, decreases. This number is bounded by the number of all connectives of the sequent which is itself bounded by twice the number of connectives of the original sequent. Since at each step the number of connectives of a sequent is at most increased by 1, this shows the required

LEMMA 3: *There is a procedure which converts any sequent s into a clausal sequent C(s) such that the number of connectives of C(s) is at most four times the number of connectives of s and s is deducible by LM_0 iff C(s) is deducible.* QED

Now unfortunately in an LM_0-deduction of a clausal sequent there will usually occur sequents which aren't clausal, vic. sequents occurring as premisses of applications of E$\neg$ and having a formula $\Box a$ on their right hand side. But the following lemma shows that we may combine such applications of E$\neg$ with an immediately preceding application of I$\Box$ into a single new rule, thus preserving clausal forms for all sequents of a deduction:

Lemma 4: *There is a transformation sending every LM_0-deduction of a given sequent into a deduction of the same sequent such that in the new deduction every premiss of an application of* E$\neg$ *with principal formula* ,$\neg\Box v$ *is the conclusion of an application of* I,*with principal formula* $\Box v$*.*

Proof:
We consider a maximal application I of E$\neg$ with principal formula $\neg\Box v$ and premiss s for which $\Box v$ is not the principal formula of the inference leading to s and we use recursion on the maximal number of successive sequents preceding s in which $\Box v$ occurs on the right hand side to turn it into an E$\neg$-inference having the required property: If this number is 1, then either s is an axiom or $\Box v$ must have disappeared because of some application of I$\Box$ with principal formula different from $\Box v$ and in both cases the E$\neg$-inference I may be dropped. If this number is greater than 1 and the inference leading to s is either a Boolean inference with principal formula not of the form $\neg\Box w$ or an E$\Box$-inference, then this inference may be shifted down past the E$\neg$-inference, thereby lowering the recursion parameter. But if the inference leading to s is another E$\neg$-inference J with principal formula $\neg\Box w$, then its premiss is the conclusion of an I$\Box$-inference with principal formula $\Box w$ and again the inference I may be dropped. Using this technique we will eventually turn all E$\neg$-inferences of a given deduction into inferences of the form required by this lemma. QED

Using this lemma we may now replace any pair of inferences consisting of an application of I$\Box$ with principal formula $\Box v$ and an application of E$\neg$ with principal formula $\neg\Box v$ immediately following it by a new inference leading directly from the premiss of the I$\Box$-inference to the conclusion of the E$\neg$-inference. This gives us the new calculus LM_1 which has the axioms and the rules E$\vee$ and E$\Box$ of LM_0 and the following divided E$\neg$-rule:

$$\mathrm{E}\neg\mathrm{s}\ \frac{M \Rightarrow a, N}{M, \neg a \Rightarrow N} \qquad (a \text{ a propositional variable}) \qquad\qquad \mathrm{E}\neg\mathrm{d}\ \frac{M^0 \Rightarrow a}{M, \neg\Box a \Rightarrow N}$$

Now in an LM_1-deduction of a clausal sequent there occur only clausal sequents, and the rules just mentioned are the only rules of LM_0 which are applicable to clausal sequents. Therefore since LM_1 shall only be used for clausal sequents, it does not need any further rules; in particular it does not need any I-rules for the following lemma to hold:

Lemma 5: *The calculi LM_0 and LM_1 are equivalent as regards deducibility of clausal sequents.*

QED

Therefore the rules E$\neg$s and E$\vee$ are invertible rules of this calculus, too, and we may safely extend the E$\vee$-rule to disjunctions of arbitrary length, showing the following:

Lemma 6: *The following calculus LM_2 is equivalent to LM_1: LM_2 has the axioms and the rule* E$\neg$d *of LM_1 and it has he rules* E$\vee$ *and* E$\Box$ *in the form*

$$\mathrm{E}\vee\ \frac{\begin{array}{c} M, a_1 \Rightarrow N \ \ldots\ M, a_p \Rightarrow N \\ M \Rightarrow b_1, N \ \ldots\ M \Rightarrow b_q, N \\ M, \Box c_1 \Rightarrow N \ \ldots\ M, \Box c_r \Rightarrow N \\ M, \neg\Box d_1 \Rightarrow N \ \ldots\ M, \neg\Box d_s \Rightarrow N \end{array}}{M, v \Rightarrow N} \qquad \mathrm{E}\Box\ \frac{\begin{array}{c} M, \Box v, a_1 \Rightarrow N \ \ldots\ M, \Box v, a_p \Rightarrow N \\ M\Box v \Rightarrow b_1, N \ \ldots\ M\Box v \Rightarrow b_q, N \\ M, \Box v, \Box c_1 \Rightarrow N \ \ldots\ M, \Box v, \Box c_r \Rightarrow N \\ M, \Box v, \neg\Box d_1 \Rightarrow N \ \ldots\ M, \Box v, \neg\Box d_s \Rightarrow N \end{array}}{M, \Box v \Rightarrow N}$$

where v is the formula $[a_1, \ldots, a_p, \neg b_1, \ldots, \neg b_q, \Box c_1, \ldots, \Box c_r, \neg\Box d_1, \ldots, \neg\Box d_s]$, the a_i, b_j, c_k and d_l are propositional variables, $p+q+r+s > 0$ and v is meant to represent any permutation of its subformulas. QED

§3. *The calculus*
Here we note, that the rule E¬s of LM_1 is a special case of the rule E∨ and that due to the invertibility of the E¬s-rule the formulas b_i in the rules E∨ and E□ may be put to the right hand side of the premisses in the second row, whereas the formulas $\neg\Box d_i$ have to remain on the left hand side, because the rule E¬d is not directly invertible. Still we show in the next lemma that any LM_2-deduction may be transformed in such a way that these formulas can immediately be put to the right hand sind, too. For this purpose we call the premisses in the first and second rows of the E∨- and E□-rules α-premisses, those in the third row β-premisses and those in the fourth row γ-premisses.

Lemma 7: *There is a transformation sending any LM_2-deduction of a given sequent into another deduction of the same sequent, such that in the new deduction every γ-premiss is the conclusion of an application of* E¬d.

Proof:
In order to construct such a deduction from a given arbitrary deduction we consider a maximal γ-premiss $s = M,\Box v,\neg\Box d \Rightarrow N$ of an inference I of our deduction, which isn't the conclusion of an application of E¬d (here $v =[a_1, \ldots, a_p, \neg b_1, \ldots, \neg b_q, \Box c_1, \ldots, \Box c_r, \neg\Box d_1, \ldots, \neg\Box d_s, \neg\Box d]$), and we use recursion on the maximal number of successive sequents preceding s in which $\neg\Box d$ occurs: If it is 0, then either s is an axiom or it is the conclusion of an application of E¬d with principal formula different from $\neg\Box d$. In both cases the inference whose premiss s is may be dropped. If this number is greater than 0, and the inference J leading to s is an application of E¬s, then we may shift J down past I in the usual way and lower the recursion parameter. But if J is by an application of a multipremiss rule, e.g. E□, then we proceed as follows:
Let w be $[e_1, \ldots, e_t, \neg f_1, \ldots, \neg f_u, \Box g_1, \ldots, \Box g_v, \neg\Box h_1, \ldots, \neg\Box h_w]$, let M be $L,\Box w$ and let $L,\Box w,\Box v \Rightarrow N$ be the conclusion of I, let $L,\Box w,\Box v,a_i \Rightarrow N$ and $L,\Box w,\Box v \Rightarrow b_j,N$ be its α-premisses, $L,\Box w,\Box v,\Box c_k \Rightarrow N$ its β-premisses and $L,\Box w,\Box v,\neg\Box d_l \Rightarrow N$ and s its γ-premisses: then the premisses of J are of the form $L,\Box w,\Box v, \neg\Box d,e_{i'} \Rightarrow N$ resp. $L,\Box w,\Box v, \neg\Box d \Rightarrow f_{j'},N$ resp. $L,\Box w,\Box v, \neg\Box d,\Box g_{k'} \Rightarrow N$ resp. $L,\Box w,\Box v, \neg\Box d,\neg\Box h_{l'} \Rightarrow N$. Now since I was a maximal inference not obeying the lemma, the sequents $L,\Box w,\Box v, \neg\Box d,\neg\Box h_{l'} \Rightarrow N$ are immediately preceded by sequents $L^0,\Box w,\Box v \Rightarrow b_{l'}$. Thus we may shift the J-inference below the I-inference:
From the I-premisses different from s we obtain by weakening sequents $L,\Box w,\Box v,a_i,e_{i'} \Rightarrow N$, sequents $L, \Box w,\Box v,e_{i'} \Rightarrow, b_j,N$, sequents $L,\Box w, \Box v,\Box c_k,e_{i'} \Rightarrow N$ and sequents $L,\Box w,\Box v,\neg\Box d_l,e_{i'} \Rightarrow N$ and we combine these with the J-premiss $L,\Box w,\Box v, \neg\Box d,e_{i'} \Rightarrow N$ to form the premisses of an E□-inference leading to $L,\Box w,\Box v,e_{i'} \Rightarrow N$. Similar E□-inferences provide us with the sequents $L,\Box w,\Box v \Rightarrow f_{j'},N$ and $L,\Box w,\Box v, \Box g_{k'} \Rightarrow N$, and combining these with the sequents $L,\Box w,\Box v,\neg\Box h_{l'} \Rightarrow N$ which result from $L^0,\Box w,\Box v \Rightarrow b_{l'}$ we obtain all the premisses of an E□-inference leading to $L,\Box w,\Box v \Rightarrow N$, i.e. the conclusion of I. Now in the new deduction the recursion parameter has decreased and at the same time − although the total number of E□-inferences has increased − the sum of the recursion parameters of inferences on the single branches of our deduction has not increased. Therefore we may successively eliminate all the E□-inferences in our deduction which violate the condition expressed in this lemma.
QED

Now as before we may combine any E∨- resp. E□-inference with the E¬d inferences leading to its γ-premisses into a single new inference and we obtain the calculus LM_3 consisting of the usual axioms and the two rules

$$\mathrm{E}\vee\frac{\begin{array}{c}M,a_1 \Rightarrow N\ \ldots\ M,a_p \Rightarrow N\\ M \Rightarrow b_1,N\ \ldots\ M \Rightarrow b_q,N\\ M,\Box c_1 \Rightarrow N\ \ldots\ M,\Box c_r \Rightarrow N\\ M^0 \Rightarrow d_1\ \ldots\ M^0 \Rightarrow d_s\end{array}}{M,v\ \Rightarrow\ N}\qquad \mathrm{E}\Box\frac{\begin{array}{c}M,\Box v,a_1 \Rightarrow N\ \ldots\ M,\Box v,a_p \Rightarrow N\\ M\Box v \Rightarrow b_1,N\ \ldots\ M\Box v \Rightarrow b_q,N\\ M,\Box v,\Box c_1 \Rightarrow N\ \ldots\ M,\Box v,\Box c_r \Rightarrow N\\ M^0,\Box v \Rightarrow d_1\ \ldots\ M^0,\Box v \Rightarrow d_s\end{array}}{M,\Box v\ \Rightarrow\ N}$$

which are to be read as before, except that in case $p+q+s=0$, the number r has to be >1. Now the E$\vee$-rule also comprises the E$\neg$d-rule of LM_2 thus we have shown:

LEMMA 8: *The calculi LM_2 and LM_3 are equivalent .* QED

Now the rule E$\vee$ isn't invertible any more, but it still holds that if its conclusion is deducible, then so are its α- and β-premisses. This shows:

LEMMA 9: *Every LM_3-deduction of a sequent $M,v,v \Rightarrow N$ may be transformed into a deduction of the sequent $M,v \Rightarrow N$ of smaller or equal length.*

PROOF:
This is true of axioms, and if $M,v,v \Rightarrow N$ is the conclusion of an inference I with principal formula different from v, then the premisses of this inference contain either both occurrences of v or none and to the former ones the induction hypothesis applies. Thus by an application of I to these transformed premisses and to those which contain no occurrence of v the sequent $M,\ v \Rightarrow N$ may be obtained. If $M,v,v \Rightarrow N$ is the conclusion of an E$\vee$-inference with principal formula v, then the above mentioned restricted inversion principle for E$\vee$ may be applied to the α- and β-premisses and then these premisses may be transformed according to the induction hypothesis, whereas the γ-premisses do not contain v. Thus by applying E$\vee$ to the transformed α- and β-premisses and to the original γ-premisses we arrive at the required deduction of $M,v \Rightarrow N$. Finally if our sequent is the conclusion of an E$\Box$-inference with principal formula v, then the induction hypothesis directly applies to all premisses and the required sequent may be derived by a similar application of E$\Box$ to the transformed premisses.
QED

Now while for the rule E$\vee$ all premisses have smaller length than the conclusion, this is not the case for E$\Box$. Therefore we need the following

LEMMA 10:
a) Every LM_3-deduction of a sequent $M,\Box[A,u] \Rightarrow N$ or a sequent $M,\Box[A,\Box u] \Rightarrow N$ may be transformed into a deduction of the sequent $M,\ \Box u \Rightarrow N$ of smaller or equal length.
b) A sequent $M,\Box[A,\Box u],\Box u \Rightarrow N$ is deducible by LM_3 iff the sequent $M,\Box u \Rightarrow N$ is deducible.

PROOF:
a) This is true for axioms, and it is trivially preserved under inferences with principal formulas different from $\Box[A,u]$ resp. $\Box[A,\Box u]$. For an E$\Box$-inference with principal formula $\Box[A,u]$ resp. $\Box[A,\Box u]$ one premiss contains u resp. $\Box u$ and to this premiss the induction hypothesis applies and yields a sequent with two occurrences of u resp. $\Box u$. Thus by an application of the preceding lemma we obtain deductions of the required sequents.
b) If $M,\Box[A,\Box u],\Box u \Rightarrow N$ is deducible, then a) and Lemma 9 show that $M,\Box u \Rightarrow N$ is deducible, too. If this latter sequent is dedcucible, then weakening shows that $M,\Box[A,\Box u],\Box u \Rightarrow N$ is deducible. QED

From this follows:

Lemma 11: *The calculus LM_4 consisting of the usual axioms, the rule* E$\vee$ *and the two* E$\Box$*-rules*

$$\text{E}\Box\text{s}\,\frac{\begin{array}{c} M,\Box v, a_1 \Rightarrow N \;\ldots\; M,\Box v, a_p \Rightarrow N \\ M,\Box v \Rightarrow b_1, N \;\ldots\; M,\Box v \Rightarrow b_q, N \\ M,\Box c_1 \Rightarrow N \;\ldots\; M,\Box c_r \Rightarrow N \end{array}}{M,\Box v \;\Rightarrow\; N} \qquad \text{E}\Box\text{d}\,\frac{\begin{array}{c} M,\Box\bar{v}, a_1 \Rightarrow N \;\ldots\; M,\Box\bar{v}, a_p \Rightarrow N \\ M\Box\bar{v} \Rightarrow b_1, N \;\ldots\; M\Box\bar{v} \Rightarrow b_q, N \\ M,\Box v,\Box c_1 \Rightarrow N \;\ldots\; M,\Box v,\Box c_r \Rightarrow N \\ M^0,\Box v \Rightarrow d_1 \;\ldots\; M^0,\Box v \Rightarrow d_s \end{array}}{M,\Box v \;\Rightarrow\; N}$$

where v is the formula $[a_1, \ldots ,a_p,\neg b_1, \ldots ,\neg b_q,\Box c_1, \ldots ,\Box c_r,\neg\Box d_1, \ldots ,\neg\Box d_s]$, $s = 0$ for E$\Box$s*, $s > 0$ for* E$\Box$d *and $\bar{v}$is the formula $[a_1, \ldots ,a_p,\neg b_1, \ldots ,\neg b_q,\Box c_1, \ldots ,\Box c_r]$, deduces all the sequents which LM_3 deduces. (Henceforth we shall call formulas v with $s = 0$ shallow formulas and those with $s > 0$ deep formulas.)*

Proof:
All we have to show, is that the E$\Box$-rule of LM_3 is admissible for this new calculus: So given all the premisses of an application of E$\Box$ Lemma 10 allows us to transform all its β-premisses into a form suitable for premisses of our new E$\Box$-rules. Moreover if the principal formula v of the given application of E$\Box$ is deep, then the same lemma allows us to transform this formula into $\bar{v}$ in all α-premisses. Thus we may deduce the conclusion of any application of E$\Box$ from its premisses by applying either the rule E$\Box$s or E$\Box$d. QED

This shows one half of the

Lemma 12: *The calculi LM_3 and LM_4 are equivalent.*

Proof:
To proof the other direction we have to show that both E$\Box$-rules of LM_4 are admissible for LM_3: For E$\Box$s this follows directly from Lemma 10. For E$\Box$d we rely on the equivalence of LM_3 and LM_0: Suppose we are given all the premisses of an application of E$\Box$d, then from the γ-premisses $M^0,\Box v \Rightarrow d_l$, we may deduce in LM_0 the sequents $M,\Box v,\neg\Box d_l \Rightarrow N$, and from the β-premisses we may as before deduce the sequents $M,\Box v, \Box c_k \Rightarrow N$. Thus by suitable applications of the rules E$\neg$ and E$\vee$ from all premisses we may deduce the sequent

$$M, \ldots \vee \neg(\neg\Box\bar{v} \vee \neg a_i) \vee \ldots \vee \neg(\neg\Box\bar{v} \vee \neg b_j) \vee \ldots \vee \neg(\neg\Box v \vee \neg\Box c_k) \vee \ldots$$
$$\ldots \vee \neg(\neg\Box v \vee \neg\neg\Box d_l) \vee \ldots \;\Rightarrow\; N.$$

But we may also deduce

$$M,\Box v \;\Rightarrow\; \ldots \vee \neg(\neg\Box\bar{v} \vee \neg a_i) \vee \ldots \vee \neg(\neg\Box\bar{v} \vee \neg b_j) \vee \ldots$$
$$\ldots \vee \neg(\neg\Box v \vee \neg\Box c_k) \vee \ldots \vee \neg(\neg\Box v \vee \neg\neg\Box d_l) \vee \ldots, N.$$

Thus by an application of the cut rule we may deduce the required conclusion $M,\Box v \Rightarrow N$ of E$\Box$d. QED

Now for LM_4 the β-premisses of both E$\Box$-rules and the α-premisses of E$\Box$d are shorter than the conclusion, whereas for the γ-premisses of both rules and the α-premisses of E$\Box$s this does not hold. We deal with the latter premisses first:

LEMMA 13: *There is a transformation which converts every LM_4-deduction of a given sequent into another LM_4-deduction of the same sequent in which no α-premiss of an application of* E□s *is the conclusion of an application of* E□d.

PROOF:
Given an arbitrary LM_4-deduction w.l.o.g. we consider a maximal α-premiss $s = M,\Box v,a \Rightarrow N$ of an application of E□s, which is the conclusion of an application of E□d (here $v = [a,a_1,\ldots,a_p,\neg b_1,\ldots,\neg b_q,\Box c_1,\ldots, \Box c_r]$), and we use recursion on the maximal number of successive applications of E□d preceding s: This number cannot be 0, and thus we let $w = [e_1, \ldots ,e_t,\neg f_1, \ldots ,\neg f_u,\Box g_1, \ldots ,\Box g_v,\neg\Box h_1, \ldots ,\neg\Box h_w]$ be the principal formula of the application J of E□d leading to s, we let M be $L,\Box w$ and we let $L,\Box w,\Box v \Rightarrow N$ be the conclusion of I, the sequents s and $L,\Box w,\Box v,a_i \Rightarrow N$ and $L,\Box w,\Box v \Rightarrow b_j,N$ its α-premisses, $L,\Box w,\Box v,\Box c_k \Rightarrow N$ its β-premisses: then the premisses of J are of the form $L,\Box w,\Box v, a,e_{i'} \Rightarrow N$ resp. $L,\Box w,\Box v, a \Rightarrow f_{j'},N$ resp. $L,\Box w,\Box v,a, \Box g_{k'} \Rightarrow N$ resp. $L^0,\Box w,\Box v, \Rightarrow h_{l'}$ and the inference J is shifted down past I as follows:
We use the given sequent $L,\Box w,\Box v,a,e_{i'} \Rightarrow N$ and the sequents $L,\Box w,\Box v,a_i,e_{i'},\Rightarrow N$ and $L,\Box w,\Box v,e_{i'} \Rightarrow b_j,N$ and $L,\Box w,\Box v,\Box c_k,e_{i'} \Rightarrow N$ (where these latter sequents are obtained from the corresponding I-premisses by weakening) as premisses of an application of E□s leading to $L,\Box w,\Box v,e_{i'} \Rightarrow N$. Similarly we obtain deductions of the sequents $L,\Box w,\Box v \Rightarrow f_{j'},N$ resp. $L,\Box w,\Box v, \Box g_{k'} \Rightarrow N$ and from these and $L^0,\Box w,\Box v, \Rightarrow h_{l'}$ using an E□d-inference with principal formula $\Box w$ we arrive at a deduction of $L,\Box w,\Box v \Rightarrow N$ where the recursion parameter has decreased by 1. Now the number of E□s-inferences may have increased in this new deduction, but the maximal sum of recursion parameters on any branch of our deduction cannot have increased. Hence we may in this way eliminate all the α-premisses of E□s which are conclusions of applications of E□d. QED

This lemma is applied for proving completeness of the calculus LM_5 which in addition to the modal operator □ uses a new operator ○ to be substituted for □ in certain situations: LM_5 has besides the usual axioms the rules E∨, E□s and E□d in the following form:

$$\mathrm{E}\vee\ \frac{\begin{array}{c} M^2,a_1 \Rightarrow N \ \ldots\ M^2,a_p \Rightarrow N \\ M^2,\Rightarrow b_1,N \ \ldots\ M^2,\Rightarrow b_q,N \\ M^2,\Box c_1 \Rightarrow N \ \ldots\ M^2,\Box c_r \Rightarrow N \\ M^3 \Rightarrow d_1 \ \ldots\ M^3 \Rightarrow d_s \end{array}}{M,v \ \Rightarrow\ N}$$

$$\mathrm{E}\Box\mathrm{s}\ \frac{\begin{array}{c} M^1,\circ v,a_1 \Rightarrow N \ \ldots\ M^2,\circ v,a_p \Rightarrow N \\ M^1,\circ v \Rightarrow b_1,N \ \ldots\ M^1,\circ v \Rightarrow b_q,N \\ M^2,\Box c_1 \Rightarrow N \ \ldots\ M^2,\Box c_r \Rightarrow N \end{array}}{M,\Box v \ \Rightarrow\ N} \qquad \mathrm{E}\Box\mathrm{d}\ \frac{\begin{array}{c} M^2,\Box\bar{v},a_1 \Rightarrow N \ \ldots\ M^2,\Box\bar{v},a_p \Rightarrow N \\ M^2,\Box\bar{v} \Rightarrow b_1,N \ \ldots\ M^2,\Box\bar{v} \Rightarrow b_q,N \\ M^2,\Box c_1 \Rightarrow N \ \ldots\ M^2,\Box c_r \Rightarrow N \\ M^3,\Box v \Rightarrow d_1 \ \ldots\ M^3,\Box v \Rightarrow d_s \end{array}}{M,\Box v \ \Rightarrow\ N}$$

where M^1 results from M by replacing any formula $\Box v$ with deep v by $\circ v$, M^2 results from M by replacing any formula $\circ v$ by $\Box v$, and M^3 results from M by omitting all non modalized formulas, i.e. all formulas not of the form $\Box v$ or $\circ v$. There is no rule for introducing the operator ○, thus any application of one of the rules of LM_5 becomes a valid application of the corresponding LM_4-rule if we replace any ○ in all premisses and in the conclusion by a □. This shows that if $M,\circ v \Rightarrow N$ is deducible by LM_5, then $M,\Box v \Rightarrow N$ is deducible by LM_5 and by LM_4; and in general: if $M \Rightarrow N$ is deducible by LM_5, then $M^2 \Rightarrow N$ is deducible by LM_5 and by LM_4 – one half of the equivalence of LM_4 and LM_5. For the proof of the other direction we call a formula $\Box v$ distant in a deduction d of a sequent $M,\Box v \Rightarrow N$ iff below any conclusion of an inference with principal formula $\Box v$ there is a conclusion of an E∨-inference or a β-premiss. Then we show

Lemma 14: *Any LM_4-deduction d of a sequent $M \Rightarrow N$ may be transformed into an LM_5-deduction of the sequent $M^4 \Rightarrow N$, where M^4 results from M by replacing any number of formulas $\Box v$ distant in d by $\circ v$.*

Proof:
Suppose we are given an LM_4-deduction of a sequent s. We may assume that it has the property expressed by Lemma 13. If the final inference of this deduction is an application of E$\vee$, then by the induction hypothesis the deductions of all premisses may be transformed into LM_5-deductions of the same sequents and these are the premisses of an E$\vee$-inference of LM_5-inference leading to the required sequent. If the final inference is an application of E$\Box$d, then by the induction hypothesis the α- and β-premisses are deducible by LM_5 and these are the corresponding premisses of an E$\Box$d-inference of LM_5 leading to the required sequent. But the γ-premisses necessary for this inference are obtained from the given γ-premisses by replacing sufficiently many distant formulas $\Box v$ by $\circ v$, because all the distant formulas of s are distant in all γ-premisses, too. Therefore the required sequent is deducible by LM_5. If the final inference I is an application of E$\Box$s with principal formula $\Box v$, where $v = [a_1, \ldots, a_p, \neg b_1, \ldots, \neg b_q, \Box c_1, \ldots, \Box c_r]$ and the formula $\Box v$ is not distant in an α-premiss P of I, then by the property of Lemma 13 there is a chain of α-premisses preceding P and such that one of these α-premisses is the conclusion of another inference with principal formula $\Box v$. In this case we may drop the inference I. Otherwise the formula $\Box v$ is distant in all α-premisses of I, and also all the distant formulas of our sequent $M, \Box v \Rightarrow N$ in its deduction d are distant formulas in all α-premisses, too. Now again by the property of Lemma 13 in the α-premisses at least all formulas $\Box w$ with w deep are distant. Thus by the induction hypothesis all sequents $M^4, \Box v, a_i \Rightarrow N$ resp. $M^4, \Box v \Rightarrow b_j, N$ are deducible, and they are the α-premisses of an LM_5-application of E$\Box$s leading to $M^4, \Box v \Rightarrow N$. Moreover as before the transformed deductions of the β-premisses again yield the β-premisses of a corresponding LM_5-inference leading to the required conclusion $M, \Box v \Rightarrow N$. Thus this latter sequent is deducible by LM_5. QED

The calculus LM_5 shows one single obstacle to contraction freeness, vic. the presence of the principal formula of an E$\Box$d-inference in the γ-premisses. This obstacle is removed by considering the calculus LM having the usual axioms, the rules E$\vee$ and E$\Box$s of LM_5 and the rule E$\Box$d in the form

$$\mathrm{E\Box d}\ \frac{\begin{array}{c} M^2, \Box\bar{v}, a_1 \Rightarrow N \ \ldots \ M^2, \Box\bar{v}, a_p \Rightarrow N \\ M^2, \Box\bar{v} \Rightarrow b_1, N \ \ldots \ M^2, \Box\bar{v} \Rightarrow b_q, N \\ M^2, \Box c_1 \Rightarrow N \ \ldots \ M^2, \Box c_r \Rightarrow N \\ M^3, \circ v \Rightarrow d_1 \ \ldots \ M^3, \circ v \Rightarrow d_s \end{array}}{M, \Box v \ \Rightarrow \ N}$$

In order to proof completeness of this calculus we call a formula $\Box v$ distant in a deduction d of a sequent $M, \Box v \Rightarrow N$ iff v is deep and below any conclusion of an inference with principal formula $\Box v$ there is a conclusion of an E$\vee$-inference or an α-premiss or a β-premiss. Then we reproof Lemma 14 as:

Lemma 15: *Any LM_5-deduction d of a sequent $M \Rightarrow N$ may be transformed into an LM-deduction of the sequent $M^4 \Rightarrow N$, where M^4 results from M by replacing any number of formulas $\Box v$ distant in d by $\circ v$.*

Proof:
If the last rule applied in a given LM_5-deduction of our sequent $s = M \Rightarrow N$ is E$\vee$, then all premisses are deducible by LM by the induction hypothesis, and moreover they are the LM-premisses of an inference leading to $M^4 \Rightarrow N$. Therefore this sequent is deducible by LM, too. If the last rule applied is E$\Box$s, then the induction hypothesis gives us LM-deductions of all β-premisses, and these are the required LM-premisses for an inference leading to M^4

$\Rightarrow N$. But in the α-premisses all deep formulas are marked by a $\circ$. Thus we may also use the α-premisses of our LM_5-inference as α-premisses of an LM-inference leading to $M^4 \Rightarrow N$. Finally if the last inference is by an application of E$\Box$d with principal formula $\Box v$ and there is a γ-premiss P of this inference in which $\Box v$ is not distant, then there is a chain of γ-premisses preceding P which contains another occurrence of P. In this case the last inference may be dropped. Otherwise the formula $\Box v$ is distant in the deductions of all γ-premisses, and furthermore all formulas distant in our given deduction of s are distant in all γ-premisses, too. Therefore by the induction hypothesis we obtain all the γ-premisses necessary for an LM-application of E$\Box$d leading to $M^4 \Rightarrow N$. But the α- and β-premisses for such an inference are obtained as before, hence in all cases we arrive at an LM-deduction of the required sequent $M^4 \Rightarrow N$. QED

The calculus LM now has the desired property that there is a measure μ, such that in every one of its rules the measure of the conclusion is greater than the measures of all premisses: namely we may take $\mu(s)$ for a sequent s to be (the total number of connectives of s times (the number of $\Box$'s plus the number of $\circ$'s)) minus the number of $\circ$'s, and moreover there holds the

THEOREM: *A sequent is valid in S4 if and only if it is deducible by LM.* QED

§4. *References.*

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