ON THE COMLEXITY OF COMMUTATIVE AND NONCOMMUTATIVE LINEAR LOGIC

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We consider a fragment NC of intuitionistic non commutative linear logic based upon the connectives \ (left-to right-implication) and $\wedge$ (additive conjunction), i.e. with axioms $a \Rightarrow a$ and rules

$$\text{E}\backslash \ \frac{M \Rightarrow a \qquad L,b,N \Rightarrow r}{L,M,a\backslash b,N \Rightarrow r} \qquad\qquad \text{I}\backslash \ \frac{a,M \Rightarrow b}{M \Rightarrow a\backslash b}$$

$$\text{E}\wedge \ \frac{M,a,N \Rightarrow r}{M,a\wedge b,N \Rightarrow r} \quad \frac{M,b,N \Rightarrow r}{M,a\wedge b,N \Rightarrow r} \qquad\qquad \text{I}\wedge \ \frac{M \Rightarrow a \quad M \Rightarrow b}{M \Rightarrow a\wedge b}$$

We show:

Theorem: The set of sequents provable in NC is PSPACE-complete.

We consider the sequents of intuitionistic propositional logic called CD-sequents in [1]. Provabilty of such sequents is PSPACE-complete. Therefore it suffices to show that we can reduce provability of CD-sequents to NC-provability. For this purpose we translate any formula $(a\to b)\to c$ on the left hand side of a given CD-sequent to $(((a\wedge\mathbf{1})\backslash b)\backslash(c\wedge\mathbf{1}))\wedge\mathbf{1}$, any other implication $a\to b$ on the left hand side of such a sequent to $(a\backslash(b\wedge\mathbf{1}))\wedge\mathbf{1}$, and any propositional variable a to $a\wedge\mathbf{1}$, where $\mathbf{1}$ is an abbreviation for $a\backslash a$. The right hand side of our sequents remain unchanged. Then we impose a fixed order on the formulas of the left hand side which is compatible with the order associated with our CD-sequent, i.e. such that a formula $a\backslash b$ does not occur to the right of a formula $c\backslash d$ if d is greater according to this order than b. Moreover we write all the formulas $a\wedge\mathbf{1}$, where a is a propositional variable to the left of all formulas coming from complex formulas of our CD-sequent. In this way we obtain from every CD-sequent s its translation $\tau(s)$. If $\tau(s)$ is provable in NC, then obviously s is provable in intuitionistic propositional logic, so it remains to show the converse, i.e. to show that the usual structural rules are admissible for NC and for translations of CD-sequents in the sense that if s' results from s by exchange or contraction or weakening and $\tau(s)$ is provable in NC, then so is $\tau(s')$. Here exchange is trivial, because due to the presupposed ordering $\tau(s)$ and $\tau(s')$ are the same sequent. Also weakening is trivial, because all the formulas on the left hand side of $\tau(s')$ are of the form $v\wedge\mathbf{1}$, and by the rules of NC and the admissibility of the cut rule we may add such formulas at will. thus it remains to prove that contraction is admissible.

For this purpose we use three lemmas:

Lemma 1: The following calculus NC1 is complete for our sequents $\tau(s)$: NC1 has axioms $M\wedge\mathbf{1},a\wedge\mathbf{1},N\wedge\mathbf{1} \Rightarrow a$ and rules:

$$\mathrm{E}\backslash\ \frac{M \Rightarrow a \qquad L,b\wedge\mathbf{1},N \Rightarrow r}{L,M,(a\backslash(b\wedge\mathbf{1}))\wedge\mathbf{1},N \Rightarrow r}$$

$$\mathrm{E}\backslash\wedge\ \frac{M \Rightarrow a \qquad M \Rightarrow b \quad L,c\wedge\mathbf{1},N \Rightarrow r}{L,M,((a\wedge b)\backslash(c\wedge\mathbf{1}))\wedge\mathbf{1},N \Rightarrow r}$$

$$\mathrm{E}\backslash\backslash\ \frac{a\wedge\mathbf{1},M \Rightarrow b \qquad L,c\wedge\mathbf{1},N \Rightarrow r}{L,M,(((a\wedge\mathbf{1})\backslash b)\backslash(c\wedge\mathbf{1}))\wedge\mathbf{1},N \Rightarrow r}$$

This is obvious by the invertibility of the rules I$\wedge$ and I\.
Now we know

Lemma 2: There is a transformation converting any NC1-deduction of a given sequent into another deduction of the same sequent such that in the new deduction every rightmost premiss of one of the rules is an axiom.

This is proved by shifting applications od the rules which do not obey this restriction upwards acoording to well known techniques. (cf. [1]).

This lemma shows that the following calculus NC2 is complete: NC2 has the same axioms as NC1 and the rules

$$\mathrm{E}\backslash\ \frac{M \Rightarrow a}{L,M,(a\backslash(b\wedge\mathbf{1}))\wedge\mathbf{1},N \Rightarrow b}$$

$$\mathrm{E}\backslash\wedge\ \frac{M \Rightarrow a \qquad M \Rightarrow b}{L,M,((a\wedge b)\backslash(c\wedge\mathbf{1}))\wedge\mathbf{1},N \Rightarrow c}$$

$$\mathrm{E}\backslash\backslash\ \frac{a\wedge\mathbf{1},M \Rightarrow b}{L,M,(((a\wedge\mathbf{1})\backslash b)\backslash(c\wedge\mathbf{1}))\wedge\mathbf{1},N \Rightarrow c}$$

Now in an NC2-deduction of a sequent $\tau(s)$ the right hand sides of premisses are strictly smaller according to the order associated with the CD-sequent s than the right hand side of the conclusion. Therefore a straightforward induction on the lengths of deductions shows

Lemma 3: If $M,\tau(a\rightarrow b),N \Rightarrow c$ is provable by NC2, where b is greater than c, then $M,N \Rightarrow c$ is provable, too.

Now suppose a sequent $M,v\wedge\mathbf{1},v\wedge\mathbf{1},N \Rightarrow r$ is provable in NC2: If it is an axiom, then so is $M, v\wedge\mathbf{1},N \Rightarrow r$. Otherwise if $v\wedge\mathbf{1}$ is not the principal formula of the last inference, then both formulas $v\wedge\mathbf{1}$ are present in all premisses and by the induction hypothesis one of them may be dropped. If it is the principal formula of this final inference, then v is some $\tau(a\rightarrow b)$ and r is b and the premisses have smaller right hand sides than b. Therefore by the preceding lemma the formula $v \wedge \mathbf{1}$ may be dropped from the premiss and applying the inference with principal formula $v \wedge \mathbf{1}$ again we arrive at a deduction of $M, v \wedge \mathbf{1}, N \Rightarrow r$.

Corollary: The same construction a fortiori works for commutative linear logic, yielding a proof of the PSPACE-completeness of the $\wedge,\rightarrow$-fragment of this logic.

Reference:

[1] Hudelmaier, J.: On a hierarchy of fragments of intuitionistic propositional logic, manuscript, Univ. Tübingen, 1994