Advanced Mathematical Methods

WS 2023/24

5 Parameter Estimation

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WIRTSCHAFTS- UND SOZIALWISSENSCHAFTLICHE FAKULTÄT

Online References

Applied Econometrics Lecture (by Prof. Grammig, available on TIMMS)

- Lecture 37: Law of Large Numbers
- Lecture 38: Central Limit Theorem

Implications of a random sample

$\{X_1, X_2, ..., X_n\}$ is a random sample if all draws of the random variable are **independently** and **identically distributed (iid)**.

Implications:

$$F_{X_1,X_2,...,X_n}(x_1, x_2, ..., x_n) \overset{\text{independent}}{=} F_{X_1}(x_1) \cdot F_{X_2}(x_2) \cdot ... \cdot F_{X_n}(x_n)$$
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Asymptotic results ($n \to \infty$)

For a random sample $\{X_1, X_2, \ldots, X_n\}$ with finite $\mathbb{E}(X_i)$ and $Var(X_i)$ and an appropriately large n, following concepts apply:

1. Law of Large Numbers (LLN)
2. Central Limit Theorem (CLT)

Law of Large Numbers

Law of Large Numbers

$$\lim_{n\to\infty} P\left[\left|\frac{1}{n}\sum_{i=1}^{n} X_i - \mathbb{E}(X)\right| > \varepsilon\right] = 0 \quad \text{for any } \epsilon > 0.$$

When the LLN holds, moments of the distribution of a population can be consistently estimated by moments of a random sample. I.e., we can write:

- $\text{plim} \frac{1}{n}\sum_{i=1}^{n} X_i = \mathbb{E}(X) = \mu$
- $\frac{1}{n}\sum_{i=1}^{n} X_i \xrightarrow{p} \mathbb{E}(X) = \mu$

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LLN and convergence in probability

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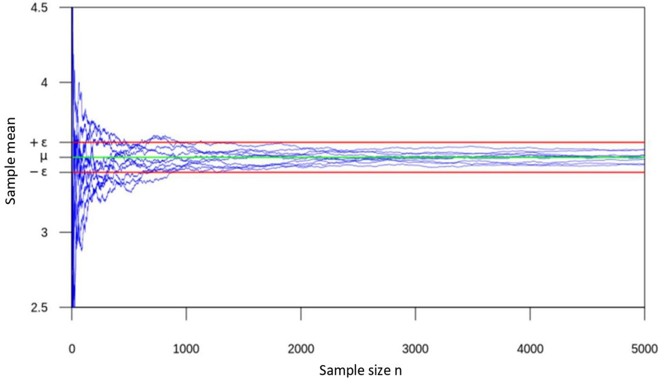
Illustration of convergence in probability using LLN:



$\Rightarrow$ A sequence $\{X_i\}$ converges in probability to a constant μ

Law of Large Numbers

Central Limit Theorem

Properties of the sample average

Central Limit Theorem

Properties of the sample average

Central Limit Theorem

Central Limit Theorem

$$\sqrt{n}\left[\frac{1}{n}\sum_{i=1}^{n} X_i - \mathbb{E}(X)\right] \overset{a}{\sim} \mathcal{N}(0, Var(X))$$

E.g. if $\{X_i\}$ is **iid** with $\mathbb{E}(X_i) = \mu$ and $Var(X_i) = \sigma^2$ and $\bar{z}_n = \frac{1}{n}\sum_{i=1}^{n} X_i \xrightarrow{p} \mu$ then,

$$\sqrt{n}\frac{(\bar{X}_n - \mu)}{\sigma} \xrightarrow{d} z \sim \mathcal{N}(0, 1)$$

$$\text{or} \quad \bar{X}_n - \mu \overset{a}{\sim} \mathcal{N}(0, \frac{\sigma^2}{n})$$

$$\text{or} \quad \bar{X}_n \overset{a}{\sim} \mathcal{N}(\mu, \frac{\sigma^2}{n})$$

Parameter estimation

Underlying principle:

1. Assumption about the distribution of a random variable in the population: $\mathbb{E}(X) = \mu < \infty$, $Var(X) = \sigma^2 < \infty$
2. Draw of a random sample
3. Use estimation functions to estimate the unknown parameters of the distribution: μ, σ^2

Estimation functions (short: estimators) are measurable functions of random variables $\Rightarrow \widehat{\theta}_n = \widehat{\theta}_n(X_1, X_2, ..., X_n)$

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Quality of estimators

Finite sample properties of estimators:

- Bias
- Variance
- Efficiency (MSE)

Asymptotic concepts ($n \to \infty$):

- Consistency
- Asymptotic normality

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Quality of estimators

Bias of $\widehat{\theta}_n$:

- If $\mathbb{E}(\widehat{\theta}_n) = \theta \Rightarrow$ unbiased estimator
- If $\mathbb{E}(\widehat{\theta}_n) \neq \theta \Rightarrow$ biased estimator

Quality of estimators

Example

Consider a random variable X that follows a known distribution with $\mathbb{E}(X) = \mu$ and $Var(X) = \sigma^2$.

We want to estimate the parameter $\mu = \mathbb{E}(X)$ and can draw a random sample $\{X_1, \ldots, X_n\}$.

We have four different candidate estimation functions (estimators) for the parameter μ:

$(i) \quad \widehat{\theta}_n^{(a)} = X_3$

$(ii) \quad \widehat{\theta}_n^{(b)} = \frac{X_1 + X_n - 1}{2}$

$(iii) \quad \widehat{\theta}_n^{(c)} = \frac{1}{n} \sum_{i=1}^{n} X_i$

$(iv) \quad \widehat{\theta}_n^{(d)} = \frac{1}{n-1} \sum_{i=1}^{n} X_i$

Quality of estimators

Example

Quality of estimators

Variance of $\widehat{\theta}_n$:

Preferable: small variance of the estimation function $\Rightarrow$ small $Var[\widehat{\theta}_n(X_1, X_2, ..., X_n)]$

Quality of estimators

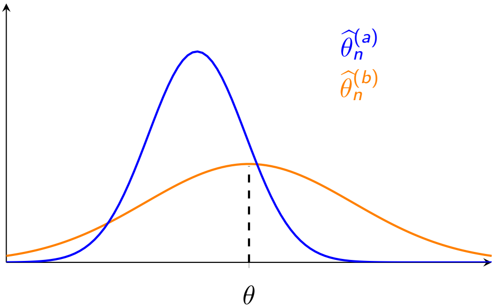
Example

Quality of estimators

Quality of estimators

Mean Squared Error (MSE) of $\widehat{\theta}_n$:

- Preferable: an unbiased estimator with a small variance
  $\Rightarrow$ Efficiency
- $\text{MSE}(\widehat{\theta}_n) \equiv \mathbb{E}[(\widehat{\theta}_n - \theta)^2] = Var(\widehat{\theta}_n) + Bias(\widehat{\theta}_n)^2$
- Trade-off: variance vs. bias of an estimator



Quality of estimators

Mean Squared Error (MSE)

Quality of estimators

Consistency

Quality of estimators

Consistency of $\widehat{\theta}_n$:

- The bigger the sample size gets, the smaller the sampling error $|\widehat{\theta}_n - \theta|$ should become
- Formally $\widehat{\theta}_n$ is consistent if

$$\lim_{n\to\infty} P(|\widehat{\theta}_n - \theta| > \varepsilon) = 0$$

- Short-hand notation: $\widehat{\theta}_n \xrightarrow{p} \theta$ or $\operatorname*{plim}_{n\to\infty} \widehat{\theta}_n = \theta$

Quality of estimators

Example

Quality of estimators

Convergence in distribution and asymptotic normality of $\widehat{\theta}_n$

Convergence in distribution: $z_n \xrightarrow{d} z$

If the c.d.f. of z_n converges to the c.d.f. of z at each point of continuity, then z_n converges in distribution to z.

Asymptotic normality of $\widehat{\theta}_n$:

The distribution of an estimator $\widehat{\theta}_n$ converges to the distribution of another random variable z, which is normally distributed.
Generally:

$z_n \xrightarrow{d} z \sim \mathcal{N}(0, 1)$

Applied to the Central Limit Theorem:

$\sqrt{n}(\widehat{\theta}_n - \theta) \xrightarrow{d} z \sim \mathcal{N}(0, Var(\widehat{\theta}_n))$

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Estimation techniques

1. Method of Moments
2. Maximum Likelihood Method
3. Ordinary Least Squares (OLS)

Method of Moments

Idea:

- Use the known relationship between parameters and theoretical moments
- Replace theoretical by empirical moments
- Method of Moment estimators are consistent

Method of Moments

Example

Method of Moments

Example

Method of Moments

Exponential distribution $f_X(x;\lambda) = \lambda e^{-\lambda x}$:

- theoretical moments:
  1. $\mathbb{E}(X) = \frac{1}{\lambda} \Longrightarrow \lambda = \frac{1}{\mathbb{E}(X)}$
  2. $Var(X) = \frac{1}{\lambda^2} \Longrightarrow \lambda = \frac{1}{\sqrt{Var(X)}} = \frac{1}{\sqrt{\mathbb{E}(X^2) - \mathbb{E}(X)^2}}$
- replace by empirical moments:
  1. $\widehat{\lambda}_1 = \frac{1}{\frac{1}{n}\sum_{i=1}^n X_i}$
  2. $\widehat{\lambda}_2 = \frac{1}{\sqrt{\frac{1}{n}\sum_{i=1}^n X_i^2 - (\frac{1}{n}\sum_{i=1}^n X_i)^2}}$

Maximum Likelihood Method

Idea:

- Choose $\widehat{\theta}_n$ such that the likelihood function $\mathcal{L}(\tilde{\theta})$ is maximized
- The likelihood function $\mathcal{L}(\tilde{\theta})$ is the joint density function of $X_1, ..., X_n$ with parameters $\tilde{\theta}$

$$\mathcal{L}(\tilde{\theta}) = f_{X_1, X_2, ..., X_n}(x_1, x_2, ...x_n; \tilde{\theta})$$

Intuition:

1. We decide on the distribution of $X \sim D(\theta)$ in the population
2. Draw a random sample of X
3. Keep the sample fixed and choose the parameter such that $\mathcal{L}(\tilde{\theta})$ is maximized $\Rightarrow$ "Maximize the probability to observe the realization of our sample"

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Recipe:

1. Set up the likelihood function and exploit the **iid** sample implications
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ML - 1. Set up the likelihood function

Use the **iid** property of a random sample:

$$\mathcal{L}(\tilde{\boldsymbol{\theta}}) = f_{X_1,X_2,\ldots,X_n}(x_1, x_2, \ldots x_n; \tilde{\boldsymbol{\theta}})$$

$$\overset{\text{independent}}{=} f_{X_1}(x_1; \tilde{\boldsymbol{\theta}}) \cdot f_{X_2}(x_2; \tilde{\boldsymbol{\theta}}) \cdot \ldots \cdot f_{X_n}(x_n; \tilde{\boldsymbol{\theta}})$$

$$\overset{\text{identical}}{=} f_X(x_1; \tilde{\boldsymbol{\theta}}) \cdot f_X(x_2; \tilde{\boldsymbol{\theta}}) \cdot \ldots \cdot f_X(x_n; \tilde{\boldsymbol{\theta}})$$

$$= \prod_{i=1}^{n} f_X(x_i; \tilde{\boldsymbol{\theta}})$$

ML - 2. Log-Transformation

Logarithmic rules

1. $ln(a \cdot b) \Leftrightarrow ln(a) + ln(b)$
2. $ln(\frac{a}{b}) \Leftrightarrow ln(a) - ln(b)$
3. $ln(a)^b \Leftrightarrow \ b \cdot ln(a)$
4. $ln(e^x) \Leftrightarrow e^{ln(x)} \Leftrightarrow x$

$$\mathcal{L}(\tilde{\boldsymbol{\theta}}) = \prod_{i=1}^{n} f_X(x_i; \tilde{\boldsymbol{\theta}}) \qquad | \, ln$$

$$ln\mathcal{L}(\tilde{\boldsymbol{\theta}}) = \sum_{i=1}^{n} ln \, f_X(x_i; \tilde{\boldsymbol{\theta}})$$

ML - 3. Log-likelihood maximization

Maximize the log-likelihood function:

$$\widehat{\boldsymbol{\theta}}_{ML} = \underset{\tilde{\boldsymbol{\theta}}}{\operatorname{argmax}}\, ln\mathcal{L}(\tilde{\boldsymbol{\theta}}) = \sum_{i=1}^{n} ln\, f_X(x_i; \tilde{\boldsymbol{\theta}})$$

Determine $\widehat{\boldsymbol{\theta}}_{ML}$ via the F.O.C. $\frac{\partial ln\mathcal{L}(\tilde{\boldsymbol{\theta}})}{\partial \tilde{\boldsymbol{\theta}}} \overset{!}{=} 0$:

$$\sum_{i=1}^{n} \frac{\partial ln\, f_X(x_i; \widehat{\boldsymbol{\theta}})}{\partial \tilde{\theta}_1} \overset{!}{=} 0$$

$$\vdots$$

$$\sum_{i=1}^{n} \frac{\partial ln\, f_X(x_i; \widehat{\boldsymbol{\theta}})}{\partial \tilde{\theta}_K} \overset{!}{=} 0$$

Maximum Likelihood Method

Example

Maximum Likelihood Method

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Properties of the ML estimator

If the likelihood function is correctly specified, the ML estimator has following properties:

- Consistency
- Asymptotic efficiency (for large n this estimator has the smallest MSE)
- Asymptotic normality

Conditional Maximum Likelihood (CML)

Procedure:

1. Specification of the conditional distribution $Y|X = x$
2. Specification of conditional moments (functions of X)
3. Insert conditional moments into the likelihood function
4. Maximize the (log-) likelihood function w.r.t. the unknown parameter

The marginal, joint and conditional distribution

Relationship of the marginal, joint, and conditional distribution:

$$f_{X_1|X_2} = \frac{f_{X_1,X_2}}{f_{X_2}}$$

$$\Leftrightarrow \quad f_{X_1,X_2} = f_{X_1|X_2} \cdot f_{X_2}$$

$$\Leftrightarrow \quad f_{X_2} = \frac{f_{X_1,X_2}}{f_{X_1|X_2}}$$

The marginal, joint and conditional distribution

Example: Exploiting this relationship allows us to write the joint density $f_{X_1,\ldots,X_5}$ as product of four conditional and one marginal density:

$$f_{X_1,X_2} = f_{X_2|X_1} \cdot f_{X_1} \tag{1}$$

$$\underline{f_{X_1,X_2,X_3}} = \underline{f_{X_3|X_1,X_2} \cdot f_{X_1,X_2}} \tag{2}$$

$$f_{X_1,X_2,X_3,X_4} = f_{X_4|X_1,X_2,X_3} \cdot \underline{f_{X_1,X_2,X_3}} \tag{3}$$

$$f_{X_1,X_2,X_3,X_4,X_5} = f_{X_5|X_1,X_2,X_3,X_4} \cdot f_{X_1,X_2,X_3,X_4} \tag{4}$$

Plugging in (1)-(4) $f_{X_1,\ldots,X_5}$ can be expressed as:

$$= f_{X_5|X_1,X_2,X_3,X_4} \cdot f_{X_4|X_1,X_2,X_3} \cdot f_{X_3|X_1,X_2} \cdot f_{X_2|X_1} \cdot f_{X_1}$$

Conditional Maximum Likelihood (CML)

- A binary response model $\rightarrow$ outcome Y is either 0 or 1
- $Y|X = x \sim Be(p(x))$
- specify a model for the probability of succes $p(x)$. Some examples:
  (a) linear probability model

$$p(x) = E[Y|X = x] = \beta_0 + \beta_1 x$$

  (b) nonlinear probability model $\rightarrow$ probability for a certain event doesn't change linear in x - here: Probit model

$$p(x) = E[Y|X = x] = \underbrace{F(\beta_0 + \beta_1 x) = \int_{-\infty}^{\beta_0+\beta_1 x} \frac{1}{\sqrt{2\pi}} e^{\frac{z^2}{2}} dz}_{c.d.f.\ of\ a\ standard\ normal\ distribution}$$

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Conditional Maximum Likelihood (CML)

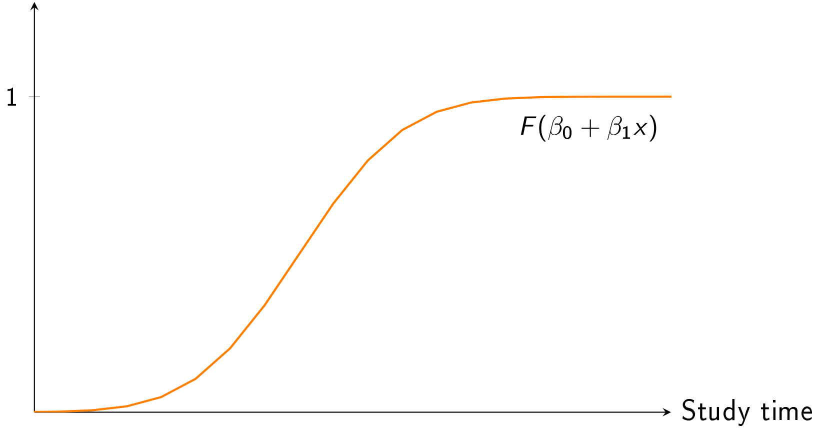
- Pass or fail in an exam given the amount of study time
- X: Study time
  Y = 1: Pass
  Y = 0: Fail
- Probability to pass the exam doesn't change linearly with increasing study time
- $P(Y = 1|X) = p(x) \rightarrow$ probability Pass

  $P(Y = 0|X) = 1 - P(Y = 1|X) = 1 - p(x) \rightarrow$ probability Fail

  $\Rightarrow$ Bernoulli distribution

Conditional Maximum Likelihood (CML)

Probability to pass the exam as a function of study time:



Conditional Maximum Likelihood (CML)

Estimating the parameters β_0 and β_1:

1. Set up conditional likelihood function using a random sample

$$\mathcal{L}(\tilde{\beta}_0, \tilde{\beta}_1) = \prod_{i=1}^{n} \underbrace{p(x_i)^{y_i}(1 - p(x_i))^{1-y_i}}_{\text{Bernoulli prob fct}}$$

2. Plug in expression for $p(x_i)$ and caluclate log-likelihood
3. Maximize the log-likelihood w.r.t. β_0 and β_1

$$\frac{\partial ln\mathcal{L}}{\partial \tilde{\beta}_0} \overset{!}{=} 0 \qquad \frac{\partial ln\mathcal{L}}{\partial \tilde{\beta}_1} \overset{!}{=} 0$$

4. Solve the F.O.C. for $\widehat{\beta}_0$ and $\widehat{\beta}_1$

Conditional Maximum Likelihood (CML)

Example

Conditional Maximum Likelihood (CML)

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Ordinary Least Squares (OLS)

Basic idea:

- Explain a variable Y (dependent variable) as linear combination of other variables $X_1, X_2, ..., X_K$ (explanatory variables/regressors) and the residual

$$Y = \tilde{\beta}_1 X_1 + \tilde{\beta}_2 X_2 + ... + \tilde{\beta}_K X_K + \tilde{u}$$

- While Y and X are directly observable, $\tilde{u}$ is not $\Rightarrow$ $\tilde{u}$ depends on how we choose $\tilde{\beta}_1, \tilde{\beta}_2, ..., \tilde{\beta}_K$

$$\tilde{u}_i = \tilde{u}_i(\tilde{\beta}_1, \tilde{\beta}_2, ..., \tilde{\beta}_K) = y_i - \tilde{\beta}_1 X_{i1} - \tilde{\beta}_2 X_{i2} - ... - \tilde{\beta}_K X_{iK}$$

- How can $\tilde{\beta}_1, \tilde{\beta}_2, ..., \tilde{\beta}_K$ be chosen?

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Example for $k = 2$:

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Example for $k = 2$:

OLS - Choosing $\tilde{\beta}_1, \tilde{\beta}_2, ..., \tilde{\beta}_K$

Choosing $\tilde{\beta}_1, \tilde{\beta}_2, ..., \tilde{\beta}_K$ either by:

1. Minimization of the least squares function:

$$\widehat{\boldsymbol{\beta}} = \underset{\tilde{\beta}}{argmin} \sum_{i=1}^{n} (y_i - \tilde{\beta}' x_i)^2$$

2. Moment restrictions: $\widehat{u} = y_i - \widehat{\beta}_1 - \widehat{\beta}_2 x_{i2} - ... - \widehat{\beta}_K x_{iK}$

$$\frac{1}{n}\sum_{i=1}^{n} \widehat{u}_i x_{i1} \stackrel{!}{=} 0\,, \quad \frac{1}{n}\sum_{i=1}^{n} \widehat{u}_i x_{i2} \stackrel{!}{=} 0\,, \; \ldots\,, \frac{1}{n}\sum_{i=1}^{n} \widehat{u}_i x_{iK} \stackrel{!}{=} 0$$

$\Rightarrow$ Both approaches lead to the same result

$$\widehat{\beta}_{\boldsymbol{n}} = \left(\frac{1}{n}\sum_{i=1}^{n} \boldsymbol{x}_i' \boldsymbol{x}_i\right)^{-1} \left(\frac{1}{n}\sum_{i=1}^{n} \boldsymbol{x}_i' y_i\right) = (\boldsymbol{X}'\boldsymbol{X})^{-1}(\boldsymbol{X}'\boldsymbol{y})$$

OLS - Choosing $\tilde{\beta}_1, \tilde{\beta}_2, ..., \tilde{\beta}_K$

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$$\widehat{\boldsymbol{\beta}}_{\boldsymbol{n}} = \left(\frac{1}{n} \sum_{i=1}^{n} \boldsymbol{x_i' x_i} \right)^{-1} \left(\frac{1}{n} \sum_{i=1}^{n} \boldsymbol{x_i'} y_i \right) = (\boldsymbol{X'X})^{-1} (\boldsymbol{X'y})$$

OLS - classical assumptions

1. Linearity
   $Y = \beta_1 + \beta_2 X_2 + ... + \beta_K X_K$
2. Strict exogeneity
   $\mathbb{E}(u|X_1, X_2, ..., X_K) = 0$
3. Conditional homoscedasticity
   $Var(u|X_1, X_2, ..., X_K) = \sigma^2$
4. Distribution assumption
   $u|X_1, X_2, ..., X_K \sim \mathcal{N}(0, \sigma^2)$

OLS and ML

Example for $k = 2$:

OLS and ML

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OLS Example

We want to estimate the effect of more education on wage (β_2: return to schooling)

$$wage = \beta_1 + \beta_2 education + \beta_3 experience + \beta_4 experience^2 + u$$

Estimate $\widehat{\boldsymbol{\beta}} = (\boldsymbol{X'X})^{-1}(\boldsymbol{X'y})$

$\Rightarrow$ Under the above mentioned assumptions $\widehat{\beta}_2$ gives the estimated marginal effect (ceteris paribus) of schooling on wage