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Chair of Statistics, Econometrics and Empirical Economics

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# S414
# Advanced Mathematical Methods

Exercises

WS 2023/24

# Parameter Estimation

## Exercise 1 **Properties of an estimator**

**(a)** Illustrate graphically a biased and an unbiased estimator for the parameter $\theta$.

**(b)** Illustrate graphically a consistent estimator for the parameter $\theta$.

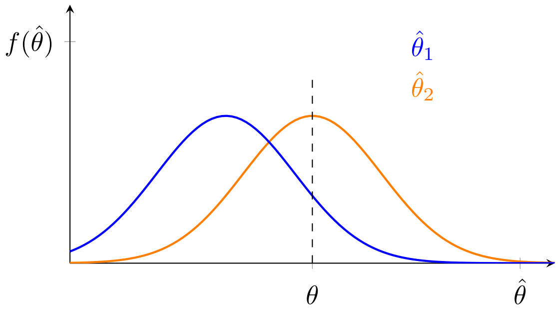
**(c)** Assume there are two alternative estimators $\hat{\theta}_1$ and $\hat{\theta}_2$ for the parameter $\theta$. Both estimators are unbiased, however, $\hat{\theta}_1$ is more efficient than $\hat{\theta}_2$. Illustrate this in a graph.

**(d)** Decompose the mean squared error (MSE) of an estimator into its components bias and variance.

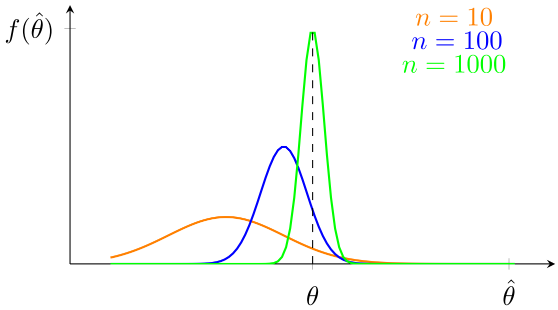
**(e)** Using a graph, explain the possible bias variance trade-off of an estimator (draw two estimators).
Comprehension question: Why does an estimator have a variance? Why is the estimator a random variable?

**Solution:**

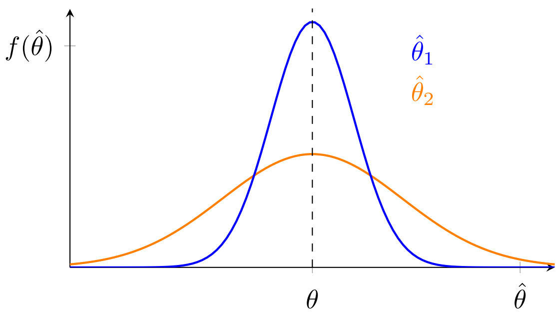
**(a)**




**(b)**



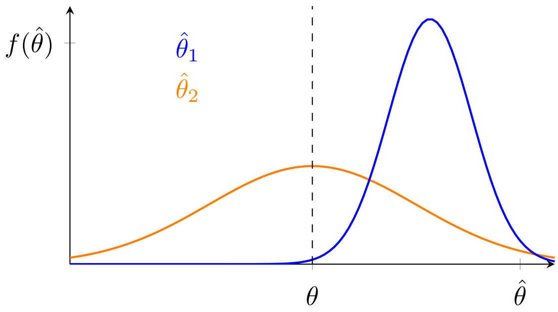

**(c)**




**(d)**

$$
\begin{aligned}
MSE &= \mathbb{E}[(\hat{\theta}-\theta)^2] \\
&= \mathbb{E}\left\{[(\hat{\theta}-\mathbb{E}(\hat{\theta})) + (\mathbb{E}(\hat{\theta})-\theta)]^2\right\} \\
&= \mathbb{E}\left\{(\hat{\theta}-\mathbb{E}(\hat{\theta}))^2 + 2(\hat{\theta}-\mathbb{E}(\hat{\theta}))(\mathbb{E}(\hat{\theta})-\theta) + (\mathbb{E}(\hat{\theta})-\theta)^2\right\} \\
&= \mathbb{E}\left[(\hat{\theta}-\mathbb{E}(\hat{\theta}))^2\right] + 2\,\mathbb{E}\left[(\hat{\theta}-\mathbb{E}(\hat{\theta}))(\mathbb{E}(\hat{\theta})-\theta)\right] + \mathbb{E}\left[(\mathbb{E}(\hat{\theta})-\theta)^2\right] \\
&= \mathbb{E}\left[(\hat{\theta}-\mathbb{E}(\hat{\theta}))^2\right] + 2\,\mathbb{E}\left[\hat{\theta}\,\mathbb{E}(\hat{\theta}) - \hat{\theta}\theta - \mathbb{E}(\hat{\theta})^2 + E(\hat{\theta})\theta\right] + (\mathbb{E}(\hat{\theta})-\theta)^2 \\
&= \mathbb{E}\left[(\hat{\theta}-\mathbb{E}(\hat{\theta}))^2\right] + 2\Big[\mathbb{E}(\hat{\theta})^2 - \mathbb{E}(\hat{\theta})\theta - \mathbb{E}(\hat{\theta})^2 + \mathbb{E}(\hat{\theta})\theta\Big] + (\mathbb{E}(\hat{\theta})-\theta)^2 \\
&= \mathbb{E}\left[(\hat{\theta}-\mathbb{E}(\hat{\theta}))^2\right] + \left[\mathbb{E}(\hat{\theta})-\theta\right]^2 \\
&= Var(\hat{\theta}) + Bias(\hat{\theta})^2
\end{aligned}
$$

**(e)**




Estimators are measurable functions of random variables and thus, random variables themselves.

## EXERCISE 2 **Method of Moments and Maximum Likelihood**

**(a)** Assume that in the population the random variable $X$ follows a Poisson distribution with parameter $\lambda$. Propose a method of moment (MM) estimator for the parameter $\lambda$. What do you need for the MM estimator? Are there any further possible estimators?

**(b)** Assume that in the population the random variable $X$ follows an Exponential distribution with parameter $\lambda$. Propose a MM estimator for the parameter $\lambda$. Justify your choice.

**(c)** Assumptions as in (a): Derive the maximum likelihood (ML) estimator for the parameter $\lambda$ of the Poisson distribution.

**(d)** Assumptions as in (b): Derive the ML estimator for the parameter $\lambda$ of the Exponential distribution.

**Solution:**

**(a)** $X \sim Po(\lambda)$, $\mathbb{E}[X] = \lambda$, $Var[X] = \lambda$
Idea: Set theoretical moments equal to empirical moments to obtain the MM estimator
$\hat{\lambda}_1 = \frac{1}{n}\sum_{i=1}^n x_i$
$\hat{\lambda}_2 = \frac{1}{n}\sum_{i=1}^n x_i^2 - \left(\frac{1}{n}\sum_{i=1}^n x_i\right)^2$

**(b)** $X \sim Ex(\lambda)$ , theoretical moments: $\mathbb{E}[X] = \frac{1}{\lambda}$, $Var[X] = \frac{1}{\lambda^2}$
Solve for $\lambda$:
$\lambda = \frac{1}{\mathbb{E}(X)}$
$\lambda = \frac{1}{\sqrt{Var(X)}} = \frac{1}{\sqrt{\mathbb{E}(X^2) - \mathbb{E}(X)^2}}$
Replace theoretical by empirical moments:
$\hat{\lambda}_1 = \frac{1}{\frac{1}{n}\sum_{i=1}^n x_i}$
$\hat{\lambda}_2 = \frac{1}{\sqrt{\frac{1}{n}\sum_{i=1}^n x_i^2 - (\frac{1}{n}\sum_{i=1}^n x_i)^2}}$

**(c)**

$$
\begin{aligned}
\mathcal{L}(\tilde{\lambda}) = \mathcal{L}(x_1, ..., x_n; \tilde{\lambda}) &= \prod_{i=1}^n \frac{\tilde{\lambda}^{x_i}}{x_i!} e^{-\tilde{\lambda}} \\
ln(\mathcal{L}(x_1, ..., x_n; \tilde{\lambda})) &= ln\left[\prod_{i=1}^n \frac{\tilde{\lambda}^{x_i}}{x_i!} e^{-\tilde{\lambda}}\right] \\
&= \sum_{i=1}^n ln\left[\frac{\tilde{\lambda}^{x_i}}{x_i!} e^{-\tilde{\lambda}}\right] \\
&= \sum_{i=1}^n \left[ln(\tilde{\lambda}^{x_i}) - ln(x_i!) - \tilde{\lambda}\right] \\
&= \sum_{i=1}^n \left[x_i \cdot ln(\tilde{\lambda}) - ln(x_i!) - \tilde{\lambda}\right] \\
FOC: & \\
\frac{\partial\, ln\mathcal{L}(\tilde{\lambda})}{\partial \tilde{\lambda}} &= \sum_{i=1}^n \left[x_i \frac{1}{\tilde{\lambda}} - 1\right] \stackrel{!}{=} 0 \\
\hat{\lambda} &= \frac{1}{n}\sum_{i=1}^n x_i
\end{aligned}
$$

**(d)**

$$
\begin{aligned}
\mathcal{L}(x_1, ..., x_n; \tilde{\lambda}) &= \prod_{i=1}^{n} \tilde{\lambda} e^{-\tilde{\lambda} x_i} \\
ln(\mathcal{L}(x_1, ..., x_n; \tilde{\lambda})) &= \sum_{i=1}^{n} \Big[ ln\big(\tilde{\lambda} e^{-\tilde{\lambda} x_i}\big) \Big] \\
&= n \cdot ln(\tilde{\lambda}) - \tilde{\lambda} \sum_{i=1}^{n} x_i \\
FOC: & \\
\frac{\partial \, ln\mathcal{L}(\tilde{\lambda})}{\partial \tilde{\lambda}} &= \frac{n}{\tilde{\lambda}} - \sum_{i=1}^{n} x_i \overset{!}{=} 0 \\
\hat{\lambda} &= \frac{1}{\frac{1}{n} \sum_{i=1}^{n} x_i}
\end{aligned}
$$