Advanced Time Series

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Overview of today's exercise

- Programming rules in GAUSS
- Writing the log-likelihood function AR(1) - Theory
- Using CML in GAUSS
- Estimating the parameters of an AR(1) process
- Writing the log-likelihood function MA(1) - Theory
- Estimating the parameters of a MA(1) process

Programming rules in GAUSS I

- ALWAYS start simple!!! Start with the simple calculations! Next enrich your program step by step!

- Start in a procedure ALWAYS from the inside and then go outside!

- CHECK frequently the results of your programming! (check them in the output window.)

- ALWAYS start with small n!

Programming rules in GAUSS II

- Comment your program!!!
- Use useful and sensible names for your variables and programs!
- Create your own program collection!

Likelihood function AR(1) for 1st and 2nd observation

$$Y_t = c + \phi Y_{t-1} + \varepsilon_t$$

$$\varepsilon_t = iidN(0, \sigma^2)$$

density of first observation

$$Y_1 \sim N(c/(1-\phi), \sigma^2/(1-\phi^2))$$

$$f_{Y_1}(y_1; c, \phi, \sigma^2) = \frac{1}{\sqrt{2\pi}\sqrt{\sigma^2/(1-\phi^2)}} \exp\left[\frac{-\{y_1 - [c/(1-\phi)]\}^2}{2\sigma^2/(1-\phi^2)}\right]$$

density of second observation

$$Y_2 = c + \phi Y_1 + \varepsilon_2$$

$$(Y_2|Y_1 = y_1) \sim N((c + \phi y_1), \sigma^2)$$

$$f_{Y_2|Y_1}(y_2|y_1;\boldsymbol{\theta}) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[\frac{-(y_2 - c - \phi y_1)^2}{2\sigma^2}\right]$$

$$\varepsilon_2 = y_2 - c - \phi y_1$$

$$f_{Y_2|Y_1}(y_2|y_1;\boldsymbol{\theta}) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[\frac{-(\varepsilon_2)^2}{2\sigma^2}\right]$$

joint density of first and second observation

$$f_{Y_2,Y_1}(y_2, y_1;\boldsymbol{\theta}) = f_{Y_1}(y_1;\boldsymbol{\theta}) \cdot f_{Y_2|Y_1}(y_2|y_1;\boldsymbol{\theta})$$

Writing the joint likelihood function AR(1)

$$
\begin{aligned}
f_{Y_T,Y_{T-1},\ldots,Y_1}(y_T, y_{T-1}, \ldots, y_1; \boldsymbol{\theta}) &= f_{Y_1}(y_1;\boldsymbol{\theta}) \cdot \prod_{t=2}^{T} f_{Y_t|Y_{t-1}}(y_t|y_{t-1};\boldsymbol{\theta}) \\
f_{Y_t|Y_{t-1}}(y_t|y_{t-1};\boldsymbol{\theta}) &= \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[\frac{-(y_t - c - \phi y_{t-1})^2}{2\sigma^2}\right] \\
f_{Y_t|Y_{t-1}}(y_t|y_{t-1};\boldsymbol{\theta}) &= \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[\frac{-(\varepsilon_t)^2}{2\sigma^2}\right]
\end{aligned}
$$

Taking logs yields

$$
\mathscr{L}(\boldsymbol{\theta}) = \log f_{Y_1}(y_1;\boldsymbol{\theta}) + \sum_{t=2}^{T} \log f_{Y_t|Y_{t-1}}(y_t|y_{t-1};\boldsymbol{\theta})
$$

Writing the log-likelihood function AR(1)

$$
\begin{aligned}
\mathscr{L}(\boldsymbol{\theta}) &= \log f_{Y_1}(y_1;\boldsymbol{\theta}) \rightarrow \text{deterministic} \\
&+ \log f_{Y_2|Y_1}(y_2|y_1;\boldsymbol{\theta}) \rightarrow \log\left(\frac{1}{\sqrt{2\pi\sigma^2}}\exp\left[\frac{-(\varepsilon_2)^2}{2\sigma^2}\right]\right) \\
&\dots \\
&+ \log f_{Y_T|Y_{T-1}}(y_T|y_{T-1};\boldsymbol{\theta}) \rightarrow \log\left(\frac{1}{\sqrt{2\pi\sigma^2}}\exp\left[\frac{-(\varepsilon_T)^2}{2\sigma^2}\right]\right)
\end{aligned}
$$

CML procedure

- Numerical optimization of a function using an algorithm
- Input: function to be minimized and starting values for parameters, and data
- Output: vector of parameters and function value at minimum

CML procedure-CALL

$$\{ \text{x,f,g,cov,retcode} \} = \text{CML(dataset,vars,\&fct,start)}$$

INPUT

dataset - name of data matrix

Important Note: the rows(dataset) needs to be equal to the number of likelihood contributions

DATA goes into the &fct as a global variable

vars - character vector of labels selected for analysis

take vars = 0;

fct - the name of a procedure that returns the log-likelihood, e.g. &malikeliproc

start - a Kx1 vector of start values

CML procedure-CALL

$$\{ \text{x,f,g,cov,retcode} \} = \text{CML(dataset,vars,\&fct,start)}$$

OUTPUT

x - Kx1 vector, estimated parameters

f - scalar, function at minimum (mean log-likelihood)

g - Kx1 vector, gradient evaluated at x

cov - KxK matrix, covariance matrix of the parameters

retcode - scalar, return code

CML procedure-GLOBALS

Example:

```
_cml_Algorithm=1;
_cml_LineSearch=1;
_cml_DirTol = 1e-5;
_cml_CovPar_=1;
```

CML Global variables I

CML global: `_cml_DirTol`=0.000000001;

`_cml_DirTol` = scalar is a convergence tolerance for gradient of estimated coefficients.

Default = 1e-5.

When this criterion has been satisfied CML will exit the iterations.

Important!!

Some applications demand a small value in order to prevent convergence on a local minimum!!!!! (local vs. global optima)

CML Global variables II

CML global:`_cml_Algorithm`= scalar indicator for optimization method

`_cml_Algorithm`

= 1, BFGS (Broyden, Fletcher, Goldfarb, Shanno)

= 2, DFP (Davidon, Fletcher, Powell)

= 3, NEWTON (Newton-Raphson)

= 4, BHHH

CML Global variables III

`_cml_LineSearch`;

= 1 One

= 2, STEPBT (default)

= 3, HALF (step-halving)

= 4, BRENT

= 5, BHHHSTEP

MA(1) Process

$$\begin{aligned} Y_t &= c + \theta\varepsilon_{t-1} + \varepsilon_t \\ \varepsilon_t &= iidN(0, \sigma^2) \\ \boldsymbol{\theta} &= (c, \theta, \sigma^2)' \end{aligned}$$

Conditional density of the t^{th} observation

$$
\begin{aligned}
Y_t|_{\varepsilon_{t-1}} &\sim N((c+\theta\varepsilon_{t-1}),\sigma^2) \\
f_{Y_t|\varepsilon_{t-1}}(y_t|\varepsilon_{t-1};\boldsymbol{\theta}) &= \frac{1}{\sqrt{2\pi\sigma^2}}\exp\left[\frac{-(y_t-c-\theta\varepsilon_{t-1})^2}{2\sigma^2}\right] \\
\varepsilon_t &= y_t-c-\theta\varepsilon_{t-1} \\
f_{Y_t|\varepsilon_{t-1}}(y_t|\varepsilon_{t-1};\boldsymbol{\theta}) &= \frac{1}{\sqrt{2\pi\sigma^2}}\exp\left[\frac{-\varepsilon_t^2}{2\sigma^2}\right]
\end{aligned}
$$

Writing the sample likelihood function MA(1)

$$f_{Y_T,Y_{T-1},..,Y_1|\varepsilon_0=0}(y_T, y_{T-1}, .., y_1|\varepsilon_0 = 0; \boldsymbol{\theta}) =$$

$$f_{Y_1|\varepsilon_0=0}(y_1|\varepsilon_0 = 0; \boldsymbol{\theta}) \prod_{t=2}^{T} f_{Y_t|Y_{t-1},Y_{t-2},..,Y_1,\varepsilon_0=0}(y_t|y_{t-1}, y_{t-2}, .., y_1, \varepsilon_0 = 0; \boldsymbol{\theta})$$

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Conditional log likelihood function MA(1)

$$\mathscr{L}(\boldsymbol{\theta}) =$$

$$\log f_{Y_T, Y_{T-1}, \ldots. Y_1 | \varepsilon_0 = 0}(y_T, y_{T-1}, \ldots, y_1 | \varepsilon_0 = 0; \boldsymbol{\theta}) =$$

$$-T \log(\sqrt{2\pi\sigma^2}) - \sum_{t=1}^{T} \frac{\varepsilon_t^2}{2\sigma^2}$$

How to get ε_t? - Recursion

$$
\begin{aligned}
\varepsilon_1 &= y_1 - c \quad \text{with} \quad \varepsilon_0 = 0 \\
\varepsilon_2 &= y_2 - c - \theta\varepsilon_1 \\
&\dots \\
\varepsilon_T &= y_T - c - \theta\varepsilon_{T-1}
\end{aligned}
$$

Writing the conditional log-likelihood function MA(1)

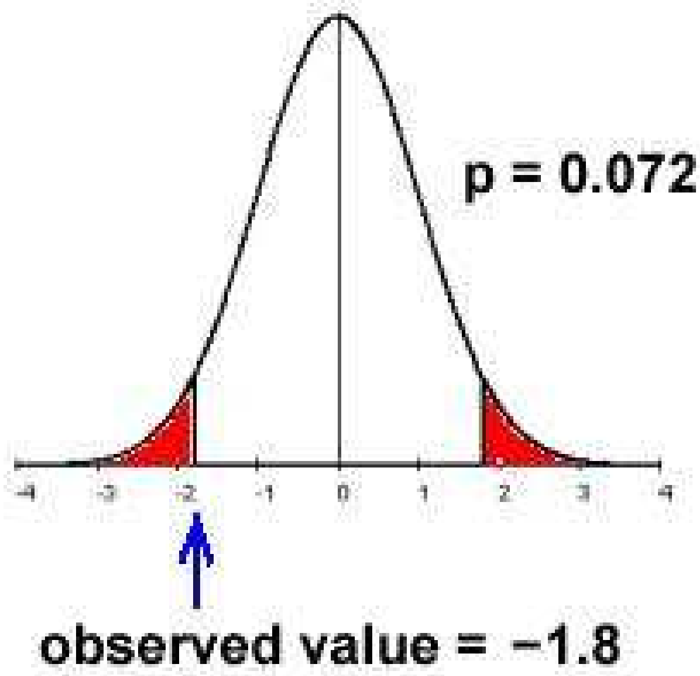
$$\begin{aligned}
\mathscr{L}(\boldsymbol{\theta}) \quad &= \quad f_{Y_1|\varepsilon_0=0}(\cdot;\boldsymbol{\theta}) \rightarrow \log\left(\frac{1}{\sqrt{2\pi\sigma^2}}\exp\left[\frac{-(\varepsilon_1)^2}{2\sigma^2}\right]\right) \\
&+ \quad \log f_{Y_2|Y_1,\varepsilon_0=0}(\cdot;\boldsymbol{\theta}) \rightarrow \log\left(\frac{1}{\sqrt{2\pi\sigma^2}}\exp\left[\frac{-(\varepsilon_2)^2}{2\sigma^2}\right]\right) \\
&\quad\ \dots \\
&+ \quad \log f_{Y_T|Y_{T-1},\dots,Y_1,\varepsilon_0=0}(\cdot;\boldsymbol{\theta}) \rightarrow \log\left(\frac{1}{\sqrt{2\pi\sigma^2}}\exp\left[\frac{-(\varepsilon_T)^2}{2\sigma^2}\right]\right)
\end{aligned}$$

Writing the conditional log-likelihood function MA(1)

$$\begin{aligned}
\mathscr{L}(\boldsymbol{\theta}) &= f_{Y_1|\varepsilon_0=0}(\cdot;\boldsymbol{\theta}) \rightarrow -\log\left(\sqrt{2\pi\sigma^2}\right) - \left[\frac{\varepsilon_1^2}{2\sigma^2}\right] \\
&+ \log f_{Y_2|Y_1,\varepsilon_0=0}(\cdot;\boldsymbol{\theta}) \rightarrow -\log\left(\sqrt{2\pi\sigma^2}\right) - \left[\frac{\varepsilon_2^2}{2\sigma^2}\right] \\
&\quad \ldots \\
&+ \log f_{Y_T|Y_{T-1},\ldots,Y_1,\varepsilon_0=0}(\cdot;\boldsymbol{\theta}) \rightarrow -\log\left(\sqrt{2\pi\sigma^2}\right) - \left[\frac{\varepsilon_T^2}{2\sigma^2}\right]
\end{aligned}$$

Reminder: P-value

Two Sided Test: Example



Reminder: P-value

```
se = sqrt(diag(cov));
t_test = thx./se;
p_val = 2*cdfnc(t_test);
```

General:

One-sided `p_val = cdfnc(t_test);`

Two-sided `p_val = 2*cdfnc(t_test);`