Subharmonic Methods in Banach Algebra Theory

Abstract

In this Part III essay, a class of functions is introduced that includes harmonic functions on a domain in the complex plane. It is proved that the spectral radius – composed with an analytic function on a domain in $\mathbb{C}$ with values in a Banach algebra – belongs to that class of functions (Vesentini's theorem). Applications of Vesentini's theorem such as analytic variation of isolated spectral elements, automatic continuity of algebra homomorphisms and spectral addtivity are studied. These results are applied to the Banach algebra of all compact endomorphisms on a Hilbert space.

Introduction

The main purpose of this essay is to study the methods used when applying results originating from the theory of subharmonic functions to Banach algebra theory. The first chapter introduces subharmonic functions and studies their properties, the second one is mainly concerned with proving the upper semi-continuity of the spectral function and Vesentini's theorem, often referred to as "folklore results" of Banach algebra theory. The third chapter studies various applications of these theorems, whereas the fourth one is concerned with demonstrating the use of the obtained results (and those proven in the appendix) with concrete examples. In the appendix, some more results concerning the spectral function are studied and another example for subharmonic methods is presented.

Throughout the essay, a Banach algebra is understood to be non-trivial and over the complex field. This restriction permits the use of analytic methods which would otherwise be inefficient. The following notations are used: $A \subset_o B$ means A is an open subset of B, $A \subset_c B$ means A is a closed subset of B. Also, all analytic functions are defined on domains of the complex plane so that attention need not be paid to issues of connectedness when applying various maximum principles.

The topic is a truely beautiful area of mathematics and demonstrates, in my opinion, the way in which different branches of mathematics should be combined to lead to a deeper understanding. As an introduction, I would like to quote from B.S. Yadav's paper on the algebraization of toplogy [30]:

> *It is aphoristically said that mathematics consists mainly of three basic disciplines: algebra, analysis and geometry, and the rest is their applications. However, in todays mathematics, the interplay of those disciplines is so intertwined and they are blended into one another to such an extent that it has become almost impossible to draw a line of demarcation between them. Each intrudes very often on territory of the others to give rise to new disciplines and in turn gets greatly stimulated in its own growth. Applications of topology (which is essentially analysis) to algebra and geometry have changed their entire fabric beyond recognition.*

This essay was written under the supervision of Dr. G.R. Allan whom I herewith would like to thank. It is mainly based on the book "A Primer on Spectral Theory" by B. Aupetit and on the paper "Schurian Algebras and Spectral Additivity" by L.A. Harris & R.V. Kadison.[1]

The reader is assumed to be familiar with complex analysis, elementary algebra, Lebesgue integration theory, functional analysis and Banach algebra theory. As references covering these areas I mainly used the books by Alhfors [1], Rudin [26] and Fischer & Lieb [12], Cameron [7], Alt [4], Rudin [27] and Heuser [16], as well as Dales [10].

For convenience of the reader, all results using representation theory are collected in the appendix. All other chapters do not assume any prior knowledge about representations.

[1]Typeset by TEX.

Contents

I Subharmonic Functions

In this chapter we will briefly study the properties of subharmonic functions. For a more detailed approach the reader might want to consult Hayman and Kennedy [15], Ahlfors [1], Conway [8], or Rudin [26]. We will also investigate the relationship between subharmonic, analytic, and harmonic functions.

The notion of subharmonicity first arose in potential theory where it is used to solve the Dirichlet problem for the Laplace equation.[2] Ahlfors [1] describes subharmonic functions as a generalisation of convex functions in that they lie "below" all (suitable) harmonic functions (in 2D)[3] just as a convex function lies "below" all (suitable) linear functions (in 1D)[4].

1 Semi-continuous Functions

First of all, we define semi-continuous functions:

Definition 1.1 Let X be a metric space. Then $u : X \to \mathbb{R} \cup \{-\infty\}$ is said to be *upper semi-continuous* or *usc* iff the set $\{x \in X \mid u(x) < \alpha\} \subset_o X$ for all real numbers α. u is called *lower semi-continuous* or *lsc* iff $-u$ is upper semi-continuous. Following Růžička in [25], we define the *epigraph* of u, $\mathrm{epi}(u)$, by

$$\mathrm{epi}(u) = \{(x, \alpha) \in X \times \mathbb{R} \mid u(x) \leq \alpha\}.$$

Clearly, every continuous function from a metric space into $\mathbb{R}$ is both upper and lower semi-continuous. The converse also holds true which can be shown by means of the following lemma (which is straight-forward to prove):

Lemma 1.2 *Let X be a metric space, $u : X \to \mathbb{R} \cup \{-\infty\}$. Then the following are equivalent:*

(*i*) *u is upper semi-continuous*

(*ii*) $x_n \to_n x \text{ in } X \Rightarrow u(x) \geq \limsup_{n\to\infty} u(x_n)$

(*iii*) $\forall\, x \in X : u(x) \geq \limsup_{y \to x} u(y) := \inf_{\epsilon > 0} \sup_{d(x,y) < \epsilon} u(y)$

The equivalent holds for lower semi-continuous functions.

Next, we will see how the epigraph of a semi-continuous function behaves.

[2]For details see [1] (p. 237ff).

[3]see paragraph I.3

[4]Note that the harmonic functions in 1D are exactly the linear functions.

Lemma 1.3 *Let X be a metric space, $u : X \to \mathbb{R} \cup \{-\infty\}$. Then* $\operatorname{epi}(u) \subset_c X \times \mathbb{R}$ *iff u is lower semi-continuous and convex iff u is convex.*

Proof. Let $\operatorname{epi}(u)$ be closed. Let $\alpha \in \mathbb{R}$, $S_\alpha := \{x \in X \,|\, u(x) > \alpha\}$, $x_0 \in S_\alpha$, $x_n \to_n x_0$ in X. Then by 1.2, $\alpha < u(x_0) \leq \liminf_{n\to\infty} u(x_n)$ and consequently $x_n \in S_\alpha \, \forall n \geq n_0$. Hence S_α is open and u is lsc.
Now suppose that u is lsc. Let $(x_n, \alpha_n) \to_n (x, \alpha)$ in $X \times \mathbb{R}$, $(x_n, \alpha_n) \in \operatorname{epi}(u)$. Then $u(x) \leq \liminf_{n\to\infty} u(x_n) \leq \lim_{n\to\infty} \alpha_n = \alpha$. Thus $(x, \alpha) \in \operatorname{epi}(u)$ and $\operatorname{epi}(u)$ is closed. The rest of the proof is left to the reader. $\square$

Lemma 1.4 *Upper semi-continuous functions defined on the same metric space X have the following properties:*

(i) u_1,u_2 *usc* $\Rightarrow u_1 + u_2$ *and* $u_1 \vee u_2 := \max(u_1, u_2)$ *are usc*

(ii) u *usc*, $\lambda \geq 0 \Rightarrow \lambda u$ *is usc*

(iii) $(u_\lambda)_{\lambda\in\Lambda}$ *usc* $\Rightarrow \inf_{\lambda\in\Lambda} u_\lambda$ *is usc*

(iv) $(u_n)_{n\in\mathbb{N}}$ *usc and* $u_n \to_n u$ *locally uniformly on* $X \Rightarrow u$ *is usc*

(v) u *usc and* $\phi : \mathbb{R} \to \mathbb{R}$ *continuous and increasing* $\Rightarrow \phi \circ u$ *is usc*
(*where* $\phi(-\infty) := \lim_{x\to-\infty} \phi(x) \in \mathbb{R} \cup \{-\infty\}$)

(vi) u *usc* $\Rightarrow u$ *is a Borel function*

(vii) *If X is compact and u usc then u is bounded above on X and attains its upper bound.*

(viii) *If X is compact, u usc and μ a bounded, regular Borel measure on X then $\int_X u \, d\mu$ is well defined as an element of $\mathbb{R} \cup \{-\infty\}$.*

The equivalent holds for lower semi-continuous functions.

Proof. We will exemplarily prove (iv), (vi) and (vii).

(iv) Let $x \in X$, $K \subset X$ compact s.t. $u_n \rightrightarrows u$ on K. Let $\epsilon > 0$, $\delta > 0$ s.t. the open disc $B(x; \epsilon) \subset K$. $\Rightarrow \exists n_0 \in \mathbb{N} : \sup_{y\in K} |u_n(y) - u(y)| < \delta \; \forall n \geq n_0$. Thus for all $n \geq n_0$ and all $y \in B(x; \epsilon)$:

$$\begin{aligned} u(x) - u(y) &= (u(x) - u_n(x)) + (u_n(x) - u_n(y)) + (u_n(y) - u(y)) \\ &> -\delta + (u_n(x) - u_n(y)) - \delta \\ \Rightarrow \sup_{d(x,y)<\epsilon} u_n(y) + u(x) &\geq -2\delta + \sup_{d(x,y)<\epsilon} u(y) + u_n(x) \\ \Rightarrow \limsup_{y\to x} u_n(y) + u(x) &\geq -2\delta + \limsup_{y\to x} u(y) + u_n(x) \\ &\geq -2\delta + \limsup_{y\to x} u(y) + \limsup_{y\to x} u_n(y) \\ \Rightarrow u(x) &\geq -2\delta + \limsup_{y\to x} u(y) \end{aligned}$$

Since δ was arbitrary, u is upper semi-continuous.

(*vi*) The sets of the form $[-\infty, \alpha)$ and (α, ∞) ($\alpha \in \mathbb{R}$) form a basis of the topology on $\mathbb{R} \cup \{-\infty\}$. Their preimages under u are $A_\alpha := \{x \in X \mid u(x) < \alpha\}$ and $B_\alpha := \{x \in X \mid \alpha < u(x)\}$ respectively. Since u is usc, A_α is open. Also, $B_\alpha = (\bigcap_{n=1}^{\infty} A_{\alpha+\frac{1}{n}})^c$ and hence is Borel measurable. Therefore, u is a Borel function.

(*vii*) For $\alpha \in \mathbb{R}$ let $A_\alpha := \{x \in X \mid u(x) < \alpha\}$, then A_α is open and $X = \bigcup_{\alpha \in \mathbb{R}} A_\alpha$. Whence since X is compact it is covered by a finite union of A_{α_i}. Since the A_{α_i} are ordered by inclusion, $X = A_{\alpha_{i_0}}$, so u is bounded above by α_{i_0}. Now either $u \equiv -\infty$ and thus attains its supremum or $\alpha_{i_0} \in \mathbb{R}$. In this case, let $\alpha_n \nearrow \alpha_{i_0}$ and observe that for each n there exist $x_n \in X \setminus A_{\alpha_n}$. But X is compact so wlog $x_n \to_n x \in X$. Thus, $\alpha_{i_0} \geq u(x) \geq \limsup_{n\to\infty} u(x_n) \geq \alpha_{i_0}$ which concludes the proof. $\square$

Examples

1. Let $F \subset X$ be open/closed. Then its characteristic function χ_F is lower/upper semi-continuous.

2. Let $u : X \to \mathbb{R}$ be bounded. Then its *modulus of continuity*[5]

$$\omega_u(x) := \inf_{\epsilon>0}(\sup\{|u(x_1)-u(x_2)| \mid \operatorname{dist}(x_1, x) < \epsilon \text{ and } \operatorname{dist}(x_2, x) < \epsilon\}), \ (x \in X)$$

is upper semi-continuous.

3. Let $u : X \to \mathbb{R} \cup \{-\infty\}$ be bounded above, $u^*(x) := \limsup_{y\to x} u(y)$ its *upper regularisation*. Then u^* is upper semi-continuous.

The link to Banach algebra theory comes from upper semi-continuity properties of the spectrum and its radius which will be studied in chapter II.

Sketches:

[5]The modulus of continuity is a tool from approximation theory, see [23].

2 Subharmonic Functions

Let D be a domain in $\mathbb{C}$. Some of the results in this section can be reformulated for superharmonic functions, but this would take us a bridge to far.

Definition 2.1 $u : D \to \mathbb{R} \cup \{-\infty\}$ is said to satisfy the *mean inequality* iff $\forall z_0 \in D \, \forall r > 0$ s.t. $\overline{B(z_0; r)} \subset D$:

$$u(z_0) \leq \frac{1}{2\pi} \int_0^{2\pi} u(z_0 + re^{i\theta}) \, d\theta$$

(in particular, the integral must be defined). $u : D \to \mathbb{R} \cup \{-\infty\}$ is called *subharmonic* or *sh* iff it is upper semi-continuous and satisfies the mean inequality. It is called *superharmonic*, if $-u$ is subharmonic.

Remarks The integral in the mean inequality is well defined for any upper semi-continuous function defined on D by lemma 1.4. The name *subharmonic* will become clear in lemma 3.6. The name *mean inequality* arises from the fact that the integral on the right-hand side is the mean of u on the circle $C(z_0; r)$.

Standard examples arise from the fact that the absolute value of an analytic function on D and its logarithm are subharmonic:

Proposition 2.2 *Let $f : D \to \mathbb{C}$ be analytic. Then $|f|$ and $\log |f|$ are sh.*

Proof. Since f is continuous, $|f|$ is upper semi-continuous. log is continuous and increasing hence by lemma 1.4 $\log |f|$ also is upper semi-continuous. Cauchy's integral formula now shows that $|f|$ is subharmonic. It can be shown[6] by means of Jensen's formula[7] that $\log |f|$ satisfies the mean inequality. $\square$

Lemma 2.3 *Subharmonic functions defined on the same domain $D \subset \mathbb{C}$ have the following properties:*

(*i*) u_1, u_2 *sh* $\Rightarrow u_1 + u_2$ *and* $u_1 \vee u_2$ *are sh*

(*ii*) u *sh*, $\lambda \geq 0 \Rightarrow \lambda u$ *is sh*

(*iii*) $(u_\lambda)_{\lambda \in \Lambda}$ *sh*, $u := \sup u_\lambda$ *usc* $\Rightarrow u$ *is sh*

(*iv*) $(u_n)_{n \in \mathbb{N}}$ *sh and* $u_n \downarrow_n u$ *pointwise on* $D \Rightarrow u$ *is sh*

(*v*) $(u_n)_{n \in \mathbb{N}}$ *sh and* $u_n \to_n u$ *locally uniformly on* $D \Rightarrow u$ *is sh*

[6]see [3] (p. 37)
[7]see [26] (p. 368)

(*vi*) *u sh and $\phi : \mathbb{R} \to \mathbb{R}$ convex and increasing $\Rightarrow \phi \circ u$ is sh (where again $\phi(-\infty) := \lim_{x\to-\infty} \phi(x) \in \mathbb{R} \cup \{-\infty\}$)*

Special case of (vi): ($\phi = \exp$)
Let $u : D \to \mathbb{R} \cup \{-\infty\}$ with $\log u$ subharmonic. Then u is subharmonic.

Proof. We will only prove (*iv*) and (*vi*), the others being straight-forward.

(iv) By lemma 1.4, u is usc. Also, $u(z_0) \leq u_n(z_0) \leq \frac{1}{2\pi}\int_0^{2\pi} u_n(z_0 + re^{i\theta})\, d\theta$. Hence by Lebesgue's theorem on monotone convergence, u satisfies the mean inequality since u_1 is bounded above on $[0, 2\pi]$ by lemma 1.4.

(vi) ϕ is convex and increasing $\Rightarrow$ ϕ is continuous $\Rightarrow$ $\phi \circ u$ is usc and Borel measurable by lemma 1.4. Jensen's convexity inequality[8] ensures the mean inequality: if μ is a Borel probability measure on a compact Hausdorff space Ω, $u \in L^1(\mu)$ and ϕ convex on an open interval including $\mathrm{im}(f \setminus \{-\infty\})$, then $\phi(\int_\Omega u\, d\mu) \leq \int_\Omega (\phi \circ u\, d\mu)$ (applied to $\Omega = [0, 2\pi]$). □

Lemma 2.4 *Let $u : D \to \mathbb{R} \cup \{-\infty\}$. Then*

(*i*) *If u is locally integrable on D and satisfies the mean inequality, and if its upper regularisation u^* satisfies $u^*(z) < \infty$ for all $z \in D$, then u^* is subharmonic on D.*

(*ii*) *If u is a Borel function, bounded above, and satisfying the mean inequality, then its upper regularisation u^* is subharmonic and bounded above.*

Proof.

(i) By example 1 on p. 3, u^* is usc on D. u satisfies the mean inequality, hence $u(z) \leq \frac{1}{2\pi}\int_0^{2\pi} u(z + re^{i\theta})\, d\theta \leq \frac{1}{2\pi}\int_0^{2\pi} u^*(z + re^{i\theta})\, d\theta$ whenever r is small enough. By taking the limes superior we get the result using Fatou's lemma and the fact that $\overline{B(z_n; r)} \subset D$ for all sufficiently large n.

(ii) Since u is bounded above, u^* is well defined. Let M be a bound for u, then $u^*(z) = \limsup_{y\to z} u(y) \leq \limsup_{y\to z} M = M$ whence u^* is bounded above. Finally, using (*i*), u^* is subharmonic. □

Again, the link to Banach algebra theory comes from a subharmonicity property of the spectral radius (see chapter II).

[8]see [26] (p. 74)

3 Harmonic Functions

In this paragraph, we will study the relationship between harmonic and subharmonic functions. To this end, let D be a domain in $\mathbb{C}$.

Definition 3.1 $u : D \to \mathbb{R}$ is *harmonic* iff $u \in C^2(D)$ and $\nabla^2 u = 0$ on D. Harmonic functions are also called *potential functions*.

Theorem 3.2
Let $u : D \to \mathbb{R}$ be harmonic, $z_0 \in D$. Then $\frac{1}{2\pi}\int_0^{2\pi} u(z_0 + re^{i\theta})\,d\theta =$ const. independent of $r > 0$ as long as $\overline{B(z_o; r)} \subset D$.

Proof. see [1] (p. 164, Theorem 22). $\square$

Similar to the case of analytic functions we have

Theorem 3.3 (Maximum Principle for Harmonic Functions)
Let $u : D \to \mathbb{R}$ be harmonic and $u(z) \leq u(z_0)$ on an open neighbourhood of z_0, then u is constant.

Proof. see [1] (p. 164, Theorem 23). $\square$

Lemma 3.4 *If u is harmonic on D it is also sub- and superharmonic on D.*

Proof. We prove this for an open disc $\Delta \subset D$:
$u : \Delta \to \mathbb{R}$ is continuous so it is upper semi-continuous. One can show that u is the real part of some analytic function f on Δ hence by Cauchy's theorem

$$u(z_0) = Re(f(z_0)) = Re(\frac{1}{2\pi}\int_0^{2\pi} f(z_0 + re^{i\theta})\,d\theta) = \frac{1}{2\pi}\int_0^{2\pi} u(z_0 + re^{i\theta})\,d\theta$$

for all r small enough. This shows the statement for Δ. But if $\overline{B(z_0; r)} \subset D$ then there exists an open disc $\overline{B(z_0; r)} \subset \Delta \subset D$ and hence the statement follows for D. Since $-u$ also is harmonic, it follows that u is superharmonic. $\square$

Remark The notion of subharmonicity truely generalises harmonicity. Consider for example $u(z) := |z|$. Then u is harmonic iff $0 \notin D$ but subharmonic on all of $\mathbb{C}$ by proposition 2.2.

Remark A continuous $u : D \to \mathbb{R}$ is said to have the *mean-value property* iff for each $z \in D$ there is a sequence $(r_n)_{n\in\mathbb{N}}$ converging to zero s.t.

$$u(z) = \frac{1}{2\pi} \int_0^{2\pi} u(z + r_n e^{i\theta})\, d\theta$$

for all n. Rudin proves in [26] (p. 284ff) – using the Poisson formula[9] – that a continuous function is harmonic on D iff it satisfies the mean-value property.

Now, we can show the converse of lemma 3.4:

Theorem 3.5
If $u : D \to \mathbb{R}$ is sub- and superharmonic it is harmonic.

Proof. Clearly u is continuous. It satisfies the mean-value property since $u(z) \le \frac{1}{2\pi}\int_0^{2\pi} u(z + r_n e^{i\theta})\, d\theta$ and $-u(z) \le \frac{1}{2\pi}\int_0^{2\pi}(-u(z + r_n e^{i\theta}))\, d\theta$ for all $z \in D$ and $n \in \mathbb{N}$ and all sequences $(r_n)_{n\in\mathbb{N}}$ that converge to zero. Hence, by the above remark, u is harmonic. $\square$

Theorem 3.6
Let $u : D \to \mathbb{R} \cup \{-\infty\}$ be subharmonic, $K \subset D$ compact, $h \in C(D) \cap C^2(\operatorname{int} K)$ harmonic and satisfying $u(z) \le h(z)$ on ∂K. Then $u(z) \le h(z)$ on all of K.

Proof. Let $v := u - h$. v is subharmonic by theorem 3.4. Suppose there exists $z \in \operatorname{int} K$ s.t. $v(z) > 0$. By lemma 1.4, continuity of h, and compactness of K, v attains its maximum m on K. Hence

$$E := \{z \in K \,|\, v(z) = m\} = \{z \in K \,|\, v(z) < m\}^c$$

is non-empty and compact. Let $z_0 \in \partial E$ and $r > 0$ s.t. $\overline{B(z_0; r)} \subset \operatorname{int} K$ but a non-trivial subarc of $C(z_0; r)$ lies in E^c. Then $v(z_0) = m > \frac{1}{2\pi}\int_0^{2\pi} v(z_0 + re^{i\theta})\, d\theta$ which contradicts the subharmonicity of v. $\square$

Remarks

1. Conway proves[10] that a usc function defined on a domain in $\mathbb{C}$ is sh iff for any compact $K \subset D$, $h \in C(D) \cap C^2(\operatorname{int} K)$ harmonic and satisfying $u(z) \le h(z)$ on ∂K, $u(z) \le h(z)$ on all of K.

2. Theorem 3.6 can also be shown for any open $U \subset D$ instead of $\operatorname{int} K$.[11]

[9] This formula stems from the study of the Dirichlet problem for the Laplace equation and can be found in Ahlforses book [1] (p. 237ff).

[10] in [8] (p. 221f)

[11] see [1] (p. 237ff)

3. It is well-known[12] that the Dirichlet problem for the Laplace equation has a solution for all suitably smooth prescribed boundary values/domains. Thus, theorem 3.6 can be reinterpreted to state that for every subharmonic function $u : D \to \mathbb{R} \cup \{-\infty\}$ and every (smooth) compact $K \subset D$ we can find a harmonic function $h : \mathrm{int}K \to \mathbb{R}$ s.t. $u(z) \leq h(z)$ on $\mathrm{int}K$ (because ∂K is compact and hence we can take the boundary values to be constantly equal to the supremum of u on K by lemma 1.4). This explains the name "subharmonic".

Theorem 3.7
If $u \in C^2(D)$, then it is subharmonic iff $\nabla^2 u \geq 0$ on D.

Proof. Suppose first that $\nabla^2 u \geq 0$ in D. For $z_0 \in D$ and r sufficiently small define $N(r) := \frac{1}{2\pi} \int_0^{2\pi} u(z_0 + re^{i\theta})\, d\theta$. N is differentiable since $u \in C^2(D)$. Let $\mu(r) := rN'(r)$. Then evidently $\mu(0) = 0$ and μ is increasing since $\nabla^2 u \geq 0$. Thus $N'(r) \geq 0$ and consequently $N(r) \geq \lim_{r' \to 0} N(r') = u(z_0)$. Since u is continuous, is must be subharmonic on D.
Now suppose conversely that $\nabla^2 u(z_0) < 0$ at some point $z_0 \in D$. Since $\nabla^2 u$ is continuous, there is an open neighbourhood of z_0 with that property. Hence μ decreases and is negative for small r. Thus $N(r) < N(0)$ for sufficiently small r, i.e. u is not subharmonic, a contradiction. $\square$

4 Useful Properties

We will now try to establish properties of subharmonic functions similar to those studied in paragraph I.3 and to those known for analytic functions. Again, many of the results could be restated for superharmonic functions. Let D be a domain in $\mathbb{C}$.

Definition 4.1 Let u be subharmonic on D, $z \in D$ and $r > 0$ s.t. $\overline{B(z;r)} \subset D$. We introduce the following notation:

$$N(z, r, u) := \frac{1}{2\pi} \int_0^{2\pi} u(z + re^{i\theta})\, d\theta \text{ and}$$

$$M(z, r, u) := \max_{0 \leq \theta \leq 2\pi} u(z + re^{i\theta}),$$

i.e. $N(z, r, u)$ is the mean, $M(z, r, u)$ is the maximum of u on the circle $C(z; r)$.

Our first analogy is the maximum principle:

[12] and cited in [1] (p. 243)

Theorem 4.2 (Maximum Principle for Subharmonic Functions) *Let $U : D \to \mathbb{R} \cup \{-\infty\}$ be subharmonic. Suppose there exists $z_0 \in D$ s.t. $u(z) \leq u(z_0)$ for all $z \in D$. Then u is constant.*

Proof. The set $E := \{z \in D \,|\, u(z) < u(z_0)\}$ is open since u is usc. Also, if $z_1 \in E^c$ and r small enough:

$$\begin{aligned} u(z_0) = u(z_1) &\leq \frac{1}{2\pi} \int_0^{2\pi} u(z_1 + re^{i\theta})\, d\theta \\ &\leq \frac{1}{2\pi} \int_0^{2\pi} u(z_0)\, d\theta \end{aligned}$$

Hence $\int_0^{2\pi} u(z_1 + re^{i\theta}) - u(z_0)\, d\theta = 0$. Since the integrand is non-negative there must be a set N_r of measure zero s.t. $u(z_0) = u(z_1 + re^{i\theta})$ for all θ outside N_r (r small enough). But if $u(z_1 + re^{i\theta}) < u(z_0)$ for some r (small enough) and some $\theta \in N_r$, then $z_1 + re^{i\theta} \in E$ hence there exists open neighbourhood of $z_1 + re^{i\theta}$ in E, a contradiction. In consequence, $B(z_1; r) \subset E^c$ which shows that E is also closed. Since D is connected and E^c non-empty, u must be constant. □

Remark In contrast to the case of harmonic functions, the conclusion would not follow if z_0 were only a *local* maximum, consider e.g. the function $u(z) = |z| \vee \frac{1}{2}$ on $D = B(0; 1)$.

Inspired by theorem 3.2 we formulate

Corollary 4.3 *Let u be subharmonic on D, $z_0 \in D$. Then*

$$u(z_0) = \limsup_{z \to z_0, z \neq z_0} u(z) = \lim_{r \searrow 0} N(z_0, r, u) = \lim_{r \searrow 0} M(z_0, r, u)$$

Proof. Clearly $u(z_0) \geq \limsup_{z \to z_0} u(z)$. But if $u(z_0) > \limsup_{z \to z_0} u(z)$ then there is $r > 0$ s.t. on $B(z_0; r) \setminus \{z_0\} : u(z) < u(z_0)$ hence by the maximum principle $u(z) = u(z_0)$ on $B(z_0; r)$, a contradiction.
We deduce that $\lim_{r \searrow 0} M(z_0, r, u) = u(z_0)$ and hence the conclusion follows from

$$u(z_0) \leq \frac{1}{2\pi} \int_0^{2\pi} u(z_0 + re^{i\theta})\, d\theta \leq M(z_0, r, u)$$

for all r small enough. □

Corollary 4.4 *Let D be a bounded domain in $\mathbb{C}$ and u subharmonic on D. Suppose there exists a bound M s.t. $\limsup_{z \to x, z \in D} u(x) \leq M$ for all $x \in \partial D$. Then u is constant or $u(z) < M$ on D. If D is unbounded, we have the same result if additionally $\limsup_{z \to \infty, z \in D} u(z) \leq M$.*

Proof. Let M be the smallest such constant.
Suppose D is bounded. Define $u(z) := M$ on ∂D. Then u is upper semi-continuous and thus attains its maximum on $\overline{D}$ at $z_0 \in \overline{D}$. If $z_0 \in D$ then the maximum principle tells us that u is constant, if $z_o \in \partial D$, $M = \sup_{z\in D} u(z)$ since $\limsup_{z\to x, z\in D} u(x) \leq M$ for all $x \in \partial D$. Whence we have the same game again: either u is constant or it does not attain the bound M, i.e. $u(z) < M \,\forall\, z \in D$. The unbounded case can be handled on the Riemannian sphere. $\square$

Without giving a proof[13] we state the following theorem which relies on some properties of mollifiers[14] and the Lebesgue integral:

Theorem 4.5 (Radó's Theorem)
Let $u : D \to \mathbb{R}$ be subharmonic. Then there exists an increasing sequence of open and D-relatively compact sets $D_n \subset D$ s.t. $\bigcup_{n=1}^{\infty} D_n = D$, as well as a decreasing sequence of subharmonic functions $u_n : D_n \to \mathbb{R}$ with $u_n \in C^\infty(D_n)$ and $u(z) = \lim_{n\to\infty} u_n(z)$ on D.

Theorem 4.6 (Hadamard's Three Circle Theorem)
Let $u : D \to \mathbb{R}$ be subharmonic and $z_0 \in D$. Then $r \mapsto M(z_0, r, u)$ is an increasing, convex function of $\log r$ (as long as $\overline{B(z_o; r)} \subset D$).

Proof. Set $M(r) := M(z_0, r, u)$. By corollary 4.3 we have

$$\lim_{r\to 0} M(r) = u(0) \leq M(r) \text{ for all } r > 0 \text{ with } \overline{B(z_0; r)} \subset D.$$

By theorem 4.2 we conclude $M(r) = \max\{u(z) \,|\, |z| \leq r\}$, i.e., M is increasing. Let $0 < r_1 < r < r_2$ and $\overline{B(z_0; r_2)} \subset D$. Define $M_j := M(r_j)$ $(j = 1, 2)$ and

$$h(z) := \frac{\log|z| - \log r_1}{\log r_2 - \log r_1} M_2 + \frac{\log r_2 - \log|z|}{\log r_2 - \log r_1} M_1.$$

If $|z| = r_j$, $h(z) = M_j$ and since the coefficients add up to one, h is a $\log r$-dependent convex combination of M_1, M_2. $-h$ is harmonic if $|z| \neq 0$ and hence sh by lemma 3.4. Apply theorem 4.2 to the annulus $\{z \in \mathbb{C} \,|\, r_1 \leq |z| \leq r_2\}$ and note that on its boundary $u - h \leq 0$. Hence $M(r) \leq h(r)$ in the annulus. Now set $t := \log r$, $t_j := \log r_j$ and define $\alpha := \frac{t_2 - t}{t_2 - t_1}$. Then $h(r) = \alpha M_1 + (1-\alpha) M_2$, $t = \alpha t_1 + (1-\alpha) t_2$, hence

$$\begin{aligned} M(r) &= M(\exp(\alpha t_1 + (1-\alpha)t_2)) \\ &\leq h(\exp(\alpha t_1 + (1-\alpha)t_2)) \\ &= \alpha M_1 + (1-\alpha) M_2 \end{aligned}$$

i.e. M is convex as a function of $\log r$. $\square$

[13] A proof can be found in [8] (p. 226f).
[14] For some properties of mollifiers see [25].

Motivated by Liouville's theorem for analytic functions we claim

Theorem 4.7 (Liouville's Theorem for Subharmonic Functions)
If u is subharmonic on the whole complex plane and $\liminf_{r\to\infty} \frac{M(0,r,u)}{\log r} = 0$, then u is constant.

Proof. If u is not identically zero, $M(0, e^{t_0}, u) > -\infty$ for some $t_0 \in \mathbb{R}$. Since $t \mapsto M(0, e^t, u) > -\infty$ is convex and increasing by Hadamard's three circle theorem 4.6, we have

$$\begin{aligned} 0 &\leq \frac{M(0,e^t,u) - M(0,e^{t_0},u)}{t - t_0} \\ &\leq \lim_{t\to\infty} \frac{M(0,e^t,u) - M(0,e^{t_0},u)}{t - t_0} \\ &= \lim_{t\to\infty} \frac{M(0,e^t,u)}{t} \\ &= \lim_{r\to\infty} \frac{M(0,r,u)}{\log r} \\ &= 0 \end{aligned}$$

So we have $M(0, e^t, u) = M(0, e^{t_0}, u)$ for all $t \geq t_0$. Hence by monotony of $t \mapsto M(0, e^t, u) > -\infty$ and theorem 4.2 u must be constant on $\mathbb{C}$. □

Corollary 4.8 *If u is subharmonic and bounded above on $\mathbb{C}$ it must be constant.*

Proof. cf. definition of $M(0, r, u)$. □

Proposition 2.2 can be generalised, too:

Theorem 4.9
If $D \subset_o \mathbb{C}$, $f : D \to \mathbb{R}$ is analytic, $f(D) \subset U \subset_o \mathbb{C}$, $u : U \to \mathbb{R}$ subharmonic, then $u \circ f$ is subharmonic on D.

Proof. We only prove this in the case $u \in C^2(D)$. A general proof can be found in [15] (p. 53f) or, for the reader familiar with the theory of distributions, in [8] (p. 220ff). By the rules of calculus and the real and imaginary parts of f are harmonic we have $\nabla^2(u \circ f) = (\nabla^2 u) \circ f \cdot [(\operatorname{Re} f_x)^2 + (\operatorname{Im} f_y)^2] \geq 0$ by theorem 3.7 since u is sh. This implies that $u \circ f$ is sh. □

II Vesentini's Theorem

In the preceding chapter, we studied subharmonic functions defined on domains in the complex plane. Our only motivation – apart from the beauty of the subject, that is – was the promise that we would soon be able to formulate an upper semi-continuity property of the spectrum as well as a subharmonicity property of the spectral radius. This promise will be delivered on in this chapter at the heart of which lies *Vesentini's theorem.*

This means we can give a (partial) answer to the main question in spectral theory[15]: "What can be said about the *spectral function* $x \mapsto \operatorname{Sp} x$ as x varies in a Banach algebra A?"

More details can be found in Aupetit [5]. Applications of Vesentini's theorem will be studied in the next chapter. Let A be a Banach algebra with identity throughout this chapter, D a domain in $\mathbb{C}$.

1 Semi-continuous Multifunctions

So what can we say about the spectral function? First of all, it is *multivalued.* This makes it somewhat harder to formulate any kind of continuity or even analyticity properties it might have. To overcome this hurdle we need to define

Definition 1.1 Let X be a metric space. Let $K(X) \subset P(X)$ be the collection of non-empty, compact subsets of X, $O(X) \subset P(X)$ the collection of all non-empty, open subsets of X. Define the *Hausdorff distance* Δ on $K(X)$ by

$$\Delta(K_1, K_2) := \max\{\sup_{z \in K_2} \operatorname{dist}(z, K_1), \sup_{z \in K_1} \operatorname{dist}(z, K_2)\}.$$

For $r > 0$, $K \in K(X)$ define $K + r := \{z \in X \mid \operatorname{dist}(K, z) \leq r\}$.

Lemma 1.2 *Δ is a metric on $K(X)$ and $K_1 \subset K_2 + \Delta(K_1, K_2)$ for any two non-empty compact K_1, K_2.*

Proof. The proof is left to the reader. □

Sketches:

[15] according to Aupetit, [5] (p. 48)

Now, we are able to formulate a notion of continuity for multi-valued functions with range in $K(X)$, X a metric space: A multi-function $F : Y \to K(X)$ on a metric space Y can be attributed any property of mappings between metric spaces if we equip $K(X)$ with the Hausdorff distance. Since the spectrum of each element in a Banach algebra is compact and non-empty, we can immediately ask questions about its continuity, uniform continuity or Lipschitz continuity, to name but a few.

Lemma 1.3 *If $E \subset A$ is a cone, i.e. $\alpha E \subset E$ for all $\alpha > 0$, then $\operatorname{Sp} : A \to K(\mathbb{C})$ is uniformly continuous on E iff it is Lipschitz continuous on E.*

Proof. "$\Leftarrow$" is clear.
"$\Rightarrow$" Let $\epsilon > 0$ and consider the restriction of Sp to E. Then there exists $\delta > 0$ s.t. $\|x - y\| \leq \delta$ implies $\Delta(\operatorname{Sp} x, \operatorname{Sp} y) \leq \epsilon$. Let $L := \frac{\epsilon}{\delta}$. If $\|x - y\| = \delta$ then $\Delta(\operatorname{Sp} x, \operatorname{Sp} y) \leq \epsilon = L\delta = L\|x - y\|$, if for $\alpha > 0$ $\|x - y\| = \alpha\delta$ then $\|\frac{x}{\alpha} - \frac{y}{\alpha}\| = \delta$ whence $\Delta(\operatorname{Sp} \frac{x}{\alpha}, \operatorname{Sp} \frac{y}{\alpha}) \leq L\|\frac{x}{\alpha} - \frac{y}{\alpha}\|$. Since for all $\alpha > 0$ we have $\operatorname{Sp}(\alpha x) = \alpha \operatorname{Sp} x$ and thus $\Delta(\alpha \operatorname{Sp} x, \alpha \operatorname{Sp} y) = \alpha\Delta(\operatorname{Sp} x, \operatorname{Sp} y)$ we conclude that Sp is Lipschitz continuous with Lipschitz constant L. $\square$

Only seldom will we be in this lucky situation though; usually the spectral function fails to even be continuous. But it is always *upper semi-continuous*:

Theorem 1.4 (Semi-continuity Property of the Spectral Function) *The spectral function on a Banach algebra A is upper semi-continuous in the sense that $\forall x \in A \,\forall U \subset_o \mathbb{C}$ with $\operatorname{Sp} x \subsetneq U \;\exists \delta > 0 : \|x - y\| < \delta \Rightarrow \operatorname{Sp} y \subsetneq U$.*

Proof. Suppose not. Then there exists $x \in A$, $U \subset_o \mathbb{C}$ with $\operatorname{Sp} x \subset U$, and $\forall \delta > 0 \,\exists y_\delta \in A : \|x - y_\delta\| < \delta$ but $\operatorname{Sp} y_\delta \not\subseteq U$. Choose $\delta = \frac{1}{n}$, then $y_{\frac{1}{n}} \to_n x$. Let $\lambda_n \in \operatorname{Sp} y_{\frac{1}{n}} \cap U^c$. $\lambda_n \in \operatorname{Sp} y_{\frac{1}{n}} \Rightarrow |\lambda_n| \leq \|y_{\frac{1}{n}}\| \Rightarrow (\lambda_n)$ is bounded. Hence by the Bolzano-Weierstrass theorem it converges to a limit λ (wlog). U is open, $\lambda \in U^c \Rightarrow \lambda \notin \operatorname{Sp} x$, so $\lambda \mathbf{1} - x$ is invertible and hence $\lambda_n \mathbf{1} - y_{\frac{1}{n}} \in G(A) \,\forall n \geq n_0$ since $G(A)$ is open; a contradiction. $\square$

Remark This is called usc since "being strictly included in an open set U" can be regarded as "being bounded above". A very general definition of upper semi-continuity could read:

Definition 1.5 Let X be a topological space, Y, Z arbitrary spaces s.t. $Y \cup Z$ is partially strictly ordered. Let $f : X \to Y$. f is called upper semi-continuous whenever $\forall z \in Z : \{y \in X \mid f(y) < z\} \subset_o X$.

Examples

1. $\mathrm{Sp} : A \to K(\mathbb{C})$, A a Banach algebra, $Z = O(\mathbb{C})$ and $Z \cup K(\mathbb{C})$ ordered by strict inclusion (by theorem 1.4).
2. If $u : D \to \mathbb{R} \cup \{-\infty\}$ with D a domain in $\mathbb{C}$, then u is usc in the classical sense iff it is usc in the sense of the above definition with $Z = \mathbb{R}$ and $\mathbb{R} \cup \{-\infty\}$ strictly ordered by $<$ as usual.

Upper semi-continuous multi-functions have nice properties:

Lemma 1.6 *Let X be a topological space, $f : \mathbb{C} \to \mathbb{R}$ continuous, $H : X \to K(\mathbb{C})$ upper semi-continuous w.r.t. inclusion and $O(\mathbb{C})$. Then $f \circ H : X \to K(\mathbb{R})$ is upper semi-continuous w.r.t. inclusion and $O(\mathbb{R})$.*

Proof. Let $U \in O(\mathbb{R})$. We have to show that

$$M := \{x \in X \mid f(H(x)) \subsetneqq U\} \subset_o X.$$

To this end, let $x \in M$ and observe that $H(x) \subset f^{-1}(f(H(x))) \subsetneqq f^{-1}(U)$. Since f is continuous $f^{-1}(U) \subset_o \mathbb{C}$ and since H is upper semi-continuous

$$N := \{y \in X \mid H(y) \subsetneqq f^{-1}(U)\} \subset_o X.$$

Thus, $M \subset N \subset_o X$. On the other hand, if $y \in N$ we have $H(y) \subsetneqq f^{-1}(U)$ so also $f(H(y)) \subsetneqq U$ i.e. $N = M$. $\square$

Lemma 1.7 *Let X be a topological space, $H : X \to K(\mathbb{R})$ upper semi-continuous w.r.t. inclusion and $O(\mathbb{R})$. Then $S : X \to \mathbb{R} : x \mapsto \sup\{H(x)\}$ is upper semi-continuous in the classical sense.*

Proof. The proof is left to the reader. $\square$

2 Points of Continuity or Discontinuity of the Spectral Function

First of all, we need the following lemma concerning the spectral radius $\rho(x)$:

Lemma 2.1 *Let $x \in A$, $\alpha \notin \mathrm{Sp}\, x$. Then*

$$\mathrm{dist}(\alpha, \mathrm{Sp}\, x) = \frac{1}{\rho((\alpha\,\mathbf{1} - x)^{-1})}$$

Proof. Let $\mathrm{Sp}\, x \subset \Omega \subset_o \mathbb{C}$ with $\alpha \notin \Omega$. Then $f(\lambda) := \frac{1}{\alpha - \lambda}$ is analytic on Ω. By the holomorphic functional calculus we have

$$\rho((\alpha\,\mathbf{1} - x)^{-1}) = \sup\{\frac{1}{|\alpha - \lambda|} \mid \lambda \in \mathrm{Sp}\, x\} = \frac{1}{\inf\{|\alpha - \lambda| \mid \lambda \in \mathrm{Sp}\, x\}} = \frac{1}{\mathrm{dist}(\alpha, \mathrm{Sp}\, x)}.$$

$\square$

Although in general one cannot prove more than semi-continuity, there are cases where the spectral function behaves nicely, for example:

Theorem 2.2
Let $x, y \in A$ commute. Then $\operatorname{Sp} y \subset \operatorname{Sp} x + \rho(x-y)$ and consequently

$$\Delta(\operatorname{Sp} x, \operatorname{Sp} y) \leq \rho(x-y) \leq \|x-y\|.$$

In particular, if A is commutative, Sp *is uniformly continuous on A.*

Proof. Let $xy = yx$. Suppose the inclusion were false, i.e. exists $\alpha \in \operatorname{Sp} y$ s.t. $\operatorname{dist}(\alpha, \operatorname{Sp} x) > \rho(x-y)$. This implies $\rho((\alpha\,\mathbf{1} - x)^{-1})\rho(x-y) < 1$ by lemma 2.1. Thus $\rho((\alpha\,\mathbf{1} - x)^{-1}(x-y)) < 1$ since $(\alpha\,\mathbf{1} - x)^{-1}$ and $x - y$ commute. But

$$\alpha\,\mathbf{1} - y = (\alpha\,\mathbf{1} - x) + (x - y) = (\alpha\,\mathbf{1} - x)[\mathbf{1} + (\alpha\,\mathbf{1} - x)^{-1}(x-y)],$$

a contradiction. We then swap the roles of x and y to get the inclusion the other way round, this gives us the inequality. □

In the finite dimensional case the situation is also quite nice:

Proposition 2.3 *Let $A = M_n(\mathbb{C})$ be the Banach algebra of all complex $n \times n$-matrices normed as linear mappings on $\mathbb{C}^n$. Then the spectral radius is continuous on A, but not uniformly continuous.*

Proof. The continuity of ρ can be proved by the implicit function theorem or by Newburgh's corollary 2.7. ρ is not even uniformly continuous if $n = 2$: Suppose it were. Define

$$a_k := \begin{pmatrix} k^2 & 1 \\ k^2(k-k^2) & k-k^2 \end{pmatrix},\ b_k := \begin{pmatrix} k^2 & 1 \\ k^2(k-k^2) & k-k^2-\frac{1}{k} \end{pmatrix}$$

Then $\operatorname{Sp}(a_k) = \{0, k\}$ and $\operatorname{Sp}(b_k) = \{\frac{k}{2} - \frac{1}{2k} \pm \sqrt{\frac{1}{4}k^2 - \frac{1}{2} + \frac{1}{4k^2} + k}\}$. $M_2(\mathbb{C})$ is a cone, hence by lemma 1.3 it is Lipschitz continuous, i.e. $\exists L > 0:$

$$\Delta(\operatorname{Sp} a_k, \operatorname{Sp} b_k) \leq L\|a_k - b_k\| = \frac{L}{k}.$$

But $\Delta(\operatorname{Sp} a_k, \operatorname{Sp} b_k) \geq \frac{1}{2}$ if k is big enough, a contradiction. □

Remarks

- This can be generalised to $\mathfrak{LC}(X)$, the Banach algebra of all compact endomorphisms of a Banach space X (with adjoint identity), see corollary 2.7.
- It can also be shown that the spectral function is uniformly continuous on the subset of $\mathfrak{L}(H)$ – the Banach algebra of all linear endomorphisms of a Hilbert space H – consisting of all self-adjoint operators[16], and that the spectral function is continuous at all normal operators, see [9].

[16]see [5] (Theorem 6.2.1)

Next, we will study an example where the spectral function behaves badly:

Example 2.4 (Kakutani) The spectral function is not continuous on $\mathfrak{L}(H)$, where H is any infinite dimensional Hilbert space.

Proof. Let $V \subset H$ be an infinite dimesional, closed, separable subspace. Let $(e_n)_{n\in\mathbb{N}}$ be an orthonormal basis of V. Observe that $\forall\, n \in \mathbb{N}$ there exist unique $k, l \in \mathbb{N}$ s.t. $n = 2^k(2l+1)$; define $a_n := e^{-k} \geq 0$. Let $T : H \to H$ be zero on $V^\perp$ and the weighted shift with weight $(a_n)_{n\in\mathbb{N}}$ on V. T is linear and continuous by the theorem on projections in Hilbert space. For each $k \in \mathbb{N}$ define $T_k \in \mathfrak{L}(H)$ by zero on $V^\perp$ and by

$$T_k e_n := \begin{cases} 0 & n = 2^k(2l+1) \text{ for some } l \\ a_n e_{n+1} & \text{otherwise} \end{cases}$$

on V. Then T_k is nilpotent with order at most 2^{k+1} (easy to check), and it follows that $\operatorname{Sp} T_k = \{0\}$. Also

$$(T - T_k)e_n = \begin{cases} e^{-k} e_{n+1} & \text{if } n \equiv 2^k \mod 2^{k+1} \\ 0 & \text{otherwise.} \end{cases}$$

$\Rightarrow$ $\|T - T_k\| \leq e^{-k}$ $\forall k \in \mathbb{N}$ $\Rightarrow$ $T_k \rightrightarrows_k T$ on H, i.e. $T_k \to_k T$ in $\mathfrak{L}(H)$. But Aupetit [5] (p. 49, Example) shows that $\operatorname{Sp} T \neq \{0\}$. Thus, the spectral function is discontinuous at T. □

Now for positive results:

Theorem 2.5 (Kuratowski)
Let A be a Banach algebra, then $C := \{a \in A \,|\, \operatorname{Sp}$ is continuous at $a\}$ is a dense G_δ-set, i.e. a countable intersection of open sets.

Proof. The proof is quite technical and can be found in [5] (p. 50f). We just give a sketch:

- The algebra $B = C_{\mathbb{R}}(\mathbb{C})$ of continuous real-valued functions on $\mathbb{C}$ is separable. Let $(f_n)_{n\in\mathbb{N}}$ be a dense sequence in B.
- Use theorem 1.4 and lemmas 1.6, 1.7 to show that $\hat{f}_n(x) := \sup\{f_n(\operatorname{Sp} x)\}$ is usc in the classical sense ($x \in A$, $n \in \mathbb{N}$).
- Show that Sp is continuous at $a \in A$ iff $\hat{f}_n$ is continuous at a for all $n \in \mathbb{N}$. (using the density of $(f_n)_{n\in\mathbb{N}}$ and Urysohn's lemma)
- Convince yourself that the sets C_n of points of continuity of $\hat{f}_n$ are dense G_δ-sets in A.

- Use Baire's Category theorem to prove that the set $C = \cap_{n=1}^{\infty} C_n$ is a dense G_δ-set in A. $\square$

Theorem 2.6 (Newburgh)
Let A be a Banach algebra, $x \in A$, $U \cap V = \emptyset$, $U, V \subset_o \mathbb{C}$, $\operatorname{Sp} x \subset U \cup V$ and $\operatorname{Sp} x \cap U \neq \emptyset$. Then there exists $r > 0$ s.t. $\|x - y\| < r$ implies $\operatorname{Sp} y \cap U \neq \emptyset$.

Proof. Wlog $V \neq \emptyset$, otherwise the conclusion follows from the upper semi-continuity of Sp (theorem 1.4). Now suppose the theorem were false, i.e. for all $r > 0\, \exists y_r \in A$ with $\|x - y\| < r$ but $\operatorname{Sp} y_r \cap U = \emptyset$. Set $r = \frac{1}{n}$ and observe that $y_{\frac{1}{n}} \longrightarrow_n x$ in A. Since U, V are disjoint the function

$$f : U \cup V \to \{0, 1\} : x \mapsto \begin{cases} 0 & x \in V \\ 1 & x \in U \end{cases}$$

is analytic. By holomorphic functional calculus we have $f(y_{\frac{1}{n}}) = 0$ for all $n \geq n_0$ and thus by continuity of f $f(x) = 0$. Thus $\{0\} = \operatorname{Sp} f(x) = f(\operatorname{Sp} x) \ni 1$ since $\operatorname{Sp} x \cap U \neq \emptyset$, a contradiction. $\square$

Remarks

1. This means that the spectrum does not "retract" from components under small variations of λ.

2. A topological space is called *totally disconnected* iff all its components are singletons. For example, every discrete space is totally disconnected. But not every tot. disconnected space is discrete, consider e.g. $\{\frac{1}{n} \mid n \in \mathbb{N}\} \cup \{0\}$.

Corollary 2.7 (Newburgh) *If the spectrum of $a \in A$ is totally disconnected, then* Sp *is continuous at a. This applies in particular to all elements with finite or countable spectra.*

Proof. Let $\epsilon > 0$. $\operatorname{Sp} a$ is totally disconnected and compact and hence included in the union U of a finite number of disjoint open balls $B(z_i; \epsilon_i), i = 1, ..., n$ with $\epsilon_i < \epsilon$ that all meet $\operatorname{Sp} a$. By upper semi-continuity (theorem 1.4) there exists $r_0 > 0$ s.t. $\|x - a\| < r_0$ implies $\operatorname{Sp} x \subset U$. Now apply Newburgh's theorem subsequently to $B(z_i; \epsilon_i), i = 1, ..., n$ and their respective complements in U to get $r_i > 0$ s.t. $\|x - a\| < r_i$ implies $\operatorname{dist}(\lambda, \operatorname{Sp} x) < \epsilon_i$ for all $\lambda \in \operatorname{Sp} a$. Hence $\|x - a\| < \min\{r_i \mid i = 0, ..., n\}$ implies $\Delta(\operatorname{Sp} a, \operatorname{Sp} x) < \epsilon$. $\square$

Remark Recall the remark on p. 15 which says that the spectral function is continuous on $\mathfrak{LC}(X)$. This is now an easy consequence of Newburgh's Corollary since the spectrum of any compact endomorphism on a Banach space is at most countable by the theorem of Riesz and Schauder[17]. The equivalent follows for $M_n(\mathbb{C})$ as stated in proposition 2.3.

[17] see, e.g. [4] (p. 363ff)

3 A Maximum Principle for the Spectral Function

In the 1960's, A. Brown and R.G. Douglas considered the problem of formulating a maximum principle for the multi-function $\mathrm{Sp} \circ f$, where f is analytic.[18] This question was solved by E. Vesentini in 1968-70 with the help of subharmonic functions. In this section we will prove his theorem and combine it with our previous results to get the desired maximum principle.

Lemma 3.1 *Let X be a complex Banach space, D a domain in $\mathbb{C}$, $f : D \to X$ analytic. Then $\|f(\cdot)\|$ and $\log \|f(\cdot)\|$ are subharmonic on D.*

Proof. By lemma I.2.3 we only need to prove the second claim. $\log \|f(\cdot)\|$ is continuous as a function into $\mathbb{R} \cup \{-\infty\}$ and hence upper semi-continuous. Let $B \subset X^*$ be the closed unit ball of X^*. Then, since log is increasing, we have (for all $r > 0$ sufficiently small):

$$\begin{aligned} \log \|f(\lambda)\| &= \sup_{\chi \in B} (\log |\chi \circ f(\lambda)|) \\ &\leq N(\lambda, r, \sup_{\chi \in B} (\log |\chi(f(\cdot))|)) \\ &= N(\lambda, r, \log \|f(\cdot)\|) \end{aligned}$$

since $\chi \circ f$ is analytic and by lemmas I.2.2 and I.2.3. $\square$

Theorem 3.2 (Vesentini)
Let D be a domain in $\mathbb{C}$, A a complex Banach algebra with identity, $f : D \to A$ analytic. Then $\rho \circ f$ and $\log \circ \rho \circ f$ are subharmonic on D.

Proof. Again, the first follows from the second by lemma I.2.3. Define

$$\phi_n(\lambda) := \log \|f(\lambda)^{2^n}\|^{\frac{1}{2^n}} = \frac{1}{2^n} \log \|f(\lambda)^{2^n}\| \ (n \in \mathbb{N},\ \lambda \in D,)$$

inspired by the Beurling-Gelfand formula $\rho(x) = \lim_{n\to\infty} \|x^n\|^{\frac{1}{n}}$ $(x \in A)$. Since f^{2^n} is analytic the above lemma tells us that ϕ_n is subharmonic for all n. We also know that $\phi_n \searrow \log(\rho \circ f)$ pointwise as $n \to \infty$ since

$$\begin{aligned} \phi_{n+1}(\lambda) &= \frac{1}{2^{n+1}} \log \|f(\lambda)^{2^{n+1}}\| \\ &\leq \frac{1}{2^{n+1}} \log(\|f(\lambda)^{2^n}\| \|f(\lambda)^{2^n}\|) \\ &= \frac{1}{2^n} \log \|f(\lambda)^{2^n}\| = \phi_n(\lambda) \end{aligned}$$

Thus, by lemma I.2.3, $\log \circ \rho \circ f$ is subharmonic. $\square$

[18] says Aupetit in [5] (p. 52)

Vesentini's theorem has many important applications some of which will be presented in chapter III. The interested reader can find more applications in Aupetit [5] (chapters III §4 and VI). Before we start with the applications though we need to convince ourselves that Vesentini's theorem really provides a maximum principle for the multi-function $\operatorname{Sp} \circ f$, where f is analytic.

Lemma 3.3 *Let $f : D \to A$ be analytic, $\alpha \notin \operatorname{Sp} f(\lambda)\, \forall \lambda \in D$. Then the functions $\lambda \mapsto \frac{1}{\operatorname{dist}(\alpha, \operatorname{Sp} f(\lambda))}$ and $\lambda \mapsto -\log \operatorname{dist}(\alpha, \operatorname{Sp} f(\lambda))$ are subharmonic on D.*

Proof. By lemma 2.1 $\operatorname{dist}(\alpha, \operatorname{Sp} f(\lambda)) = \frac{1}{\rho((\alpha\mathbf{1} - f(\lambda))^{-1})}$. Now apply both cases of Vesentini's theorem to $\lambda \mapsto (\alpha\mathbf{1} - f(\lambda))^{-1}$ which is analytic. $\square$

Remark If the complement of $A \subset \mathbb{C}$ has exactly one unbounded component U, we write $\hat{A} := U^c$. The spectrum of a Banach algebra element has this property, see [2].

Theorem 3.4 (Spectral Maximum Theorem)
Let $f : D \to A$ be analytic and suppose $\exists \lambda_0 \in D$ s.t. $\operatorname{Sp} f(\lambda) \subset \operatorname{Sp} f(\lambda_0)$ for all $\lambda \in D$. Then $\partial \operatorname{Sp} f(\lambda_0) \subset \partial \operatorname{Sp} f(\lambda)$ and $\operatorname{Sp} f(\lambda_0)\hat{} = \operatorname{Sp} f(\lambda)\hat{}$ for all $\lambda \in D$. In particular if $\operatorname{Sp} f(\lambda_0)$ has no interior points, or if $\operatorname{Sp} f(\lambda)$ does not seperate the plane for all $\lambda \in D$, then $\operatorname{Sp} \circ f$ is constant.

Proof. Suppose that $z_0 \in \partial \operatorname{Sp} f(\lambda_0)$ and $z_0 \notin \partial \operatorname{Sp} f(\lambda_1)$ for some $\lambda_1 \in D$. Of course z_0 ca not be an interior point of $\operatorname{Sp} f(\lambda_1)$ because in that case it would also be interior to $\operatorname{Sp} f(\lambda_0)$. Hence there must exist $r > 0$ with $\overline{B(z_0; r)} \cap \operatorname{Sp} f(\lambda_1) = \emptyset$. Since z_0 lies on the boundary of $\operatorname{Sp} f(\lambda_0)$ there exists z_1 exterior to $\operatorname{Sp} f(\lambda_0)$ with $|z_1 - z_0| < \frac{r}{3}$. Then by geometric considerations $\operatorname{dist}(z_1, \operatorname{Sp} f(\lambda_0)) < \frac{r}{3}$ and $\operatorname{dist}(z_1, \operatorname{Sp} f(\lambda_1)) > \frac{2r}{3}$. But by hypothesis $\operatorname{Sp} f(\lambda) \subset \operatorname{Sp} f(\lambda_0)$ and in consequence $\operatorname{dist}(z_1, \operatorname{Sp} f(\lambda)) \geq \operatorname{dist}(z_1, \operatorname{Sp} f(\lambda_0))$ for all $\lambda \in D$ since z_1 lies in the exterior of the latter. So by lemma 3.3 and the maximum principle for subharmonic functions I.4.2 we get $\operatorname{dist}(z_1, \operatorname{Sp} f(\lambda)) = \operatorname{dist}(z_1, \operatorname{Sp} f(\lambda_0))$ for all $\lambda \in D$ which leads to a contradiction at λ_1.

Now let $U(\lambda)$ be the unbounded component of $\mathbb{C} \setminus \operatorname{Sp} f(\lambda)$. Then clearly $U(\lambda_0) \subset U(\lambda)$. Suppose they were not equal. Then there is $z \in U(\lambda) \cap \operatorname{Sp} f(\lambda_0)\hat{}$. Let z be connected to infinity by an arc Γ included in $U(\lambda)$. Since $\operatorname{Sp} f(\lambda_0)$ is compact and $\Gamma \cap \operatorname{Sp} f(\lambda_0) \neq \emptyset$ there must be $z_0 \in \partial \operatorname{Sp} f(\lambda_0) \cap \Gamma$ by the intermediate value theorem. But by the above, $\partial \operatorname{Sp} f(\lambda_0) \subset \partial \operatorname{Sp} f(\lambda)$, a contradiction to $z_0 \in U(\lambda)$.

In particular, if $\operatorname{Sp} f(\lambda_0)$ has no interior points, then so has not $\operatorname{Sp} f(\lambda)$ for any $\lambda \in D$. Thus $\partial \operatorname{Sp} f(\lambda) = \operatorname{Sp} f(\lambda) \subset \operatorname{Sp} f(\lambda_0) = \partial \operatorname{Sp} f(\lambda_0) \subset \partial \operatorname{Sp} f(\lambda)$ by the above and the spectral function must be constant.

If the set $\mathrm{Sp}\, f(\lambda)$ does not separate the complex plane for any $\lambda \in D$, i.e., their respective complements are connected, then we can complete the proof by noting that $\mathrm{Sp}\, f(\lambda) = \mathrm{Sp}\, f(\lambda)\hat{} = \mathrm{Sp}\, f(\lambda_0)\hat{} = \mathrm{Sp}\, f(\lambda_0)$. $\square$

As in the single-valued case, we can derive

Theorem 3.5 (Liouville's Spectral Theorem)
If $f : \mathbb{C} \to A$ is entire, $C \subset \mathbb{C}$ bounded and $\mathrm{Sp}\, f(\lambda) \subset C\, \forall \lambda \in \mathbb{C}$, then $\mathrm{Sp}\, f(\lambda)\hat{}$ is constant.

Proof. The proof relies heavily on Liouville's theorem for subharmonic functions and can be found in [5]. $\square$

Remark Note that we only claimed that $\mathrm{Sp}\, f(\lambda)\hat{}$ be constant, no claim has been made as to the constancy of $\mathrm{Sp}\, f(\lambda)$. This has the simple reason that we can produce counter-examples that satisfy all the assumptions we made but do not have constant spectrum, even in the endomorphism algebras on Hilbert spaces as common as $l^2(\mathbb{Z})$ - the space of all square-summable double infinite sequences - see [5].

4 Spectral Diameters

In this paragraph, we will encounter the spectral diameter - the subharmonicity of which will prove to be a useful tool for using subharmonic methods in Banach algebra theory. It is a consequence of Vesentini's theorem 3.2.

Definition 4.1 Let X be a metric space, $n \in \mathbb{N}$, $K \in K(X)$. The *n-th diameter* of K is defined by

$$\delta_n(K) := \max\Bigg(\prod_{1\le i<j\le n+1} |\lambda_i - \lambda_j|\Bigg)^{\frac{2}{n(n+1)}}.$$

the maximum being taken over all possible values $\lambda_1, \dots \lambda_{n+1} \in K$, $\delta(K)$ being a shorthand for $\delta_1(K)$. If A, as always, is a Banach algebra with identity, $x \in A$, then the n-th diameter of its spectrum is called the *n-th spectral diameter* of x. (notation: $\delta_n(x) := \delta_n(\mathrm{Sp}\, x)$)

Lemma 4.2 $\delta_n(x) \le \delta(x)\, \forall x \in A$, $\forall n \in \mathbb{N}$.

Proof. $|\{(i,j) \,|\, 1 \le i < j \le n+1\}| = \sum_{k=1}^{n} k = \frac{n(n+1)}{2}$ for all $n \in \mathbb{N}$. Thus

$$\begin{aligned}\delta_n(x) &= \max\Bigg(\prod_{1\le i<j\le n+1} |\lambda_i - \lambda_j|\Bigg)^{\frac{2}{n(n+1)}} \\ &\le \max\Bigg(\prod_{1\le i<j\le n+1} \delta(x)\Bigg)^{\frac{2}{n(n+1)}} = \max \delta(x) \text{ by the above}\end{aligned}$$ $\square$

Lemma 4.3 *Let $f : D \to A$ be analytic. Define*

$$u(\lambda) := \max\{\operatorname{Re} u \,|\, u \in \operatorname{Sp} f(\lambda)\}$$
$$v(\lambda) := \min\{\operatorname{Re} v \,|\, v \in \operatorname{Sp} f(\lambda)\}$$

Then $u(\lambda) = \log \rho(e^{f(\lambda)})$ and $v(\lambda) = -\log \rho(e^{-f(\lambda)})$ and $u, -v$ are subharmonic.

Proof. $v(\lambda) = -\max\{\operatorname{Re} v \,|\, v \in \operatorname{Sp}(-f(\lambda))\}$ hence it is enough to show that u fulfils the claims. Set $g(z) := e^z$; then we deduce by holomorphic functional calculus since log is increasing:

$$\begin{aligned}\log \rho(e^x) &= \log \max\{|e^u| \,|\, u \in \operatorname{Sp} x\} \\ &= \max\{\log |e^u| \,|\, u \in \operatorname{Sp} x\} \\ &= \max\{\log e^{\operatorname{Re} u} \,|\, u \in \operatorname{Sp} x\} \\ &= \max\{\operatorname{Re} u \,|\, u \in \operatorname{Sp} x\}\end{aligned}$$

for all $x \in A$. Now apply Vesentini's theorem. □

Theorem 4.4 (Diametric Vesentini)
If $f : D \to A$ is analytic, then $\delta_n \circ f$ is subharmonic for all $n \in \mathbb{N}$.

Proof. We only prove this for $n = 1$ the other cases being much more complicated[19]. Let $x \in A$, $|\alpha| = 1$. By lemma 4.3, the length of the projection of $\operatorname{Sp} x$ on $\{t\overline{\alpha} \,|\, t \in \mathbb{R}\}$ is given by $\log \rho(e^{\alpha x}) + \log \rho(e^{-\alpha x})$. This implies $\delta(x) = \max\{\log \rho(e^{\alpha x}) + \log \rho(e^{-\alpha x}) \,|\, \alpha \in \mathbb{C}, |\alpha| = 1\}$. By lemma 4.3, $\lambda \mapsto \log \rho(e^{\alpha f(\lambda)}) + \log \rho(e^{-\alpha f(\lambda)})$ is subharmonic and whence satisfies the mean inequality. Thus, by lemma I.2.3, $\delta \circ f$ satisfies the mean inequality. From theorem 1.4 and lemma 1.7 it follows that $\delta \circ f$ is subharmonic. □

Sketches:

[19] says Aupetit in [5] (p. 62). A proof can be found in this very book, theorems 7.1.3 and 7.1.13.

III Subharmonicity and Banach Algebras

Now that Vesentini's theorem II.3.2 has become familiar to us, we will address several of its applications to Banach algebra theory. As mentioned before, the interested reader is referred to Aupetit's book [5] (chapters III §4 and VI) for a more complete survey.

The applications we will discuss in this essay are spectral variation and automatic continuity (in this chapter), characterisation of commutative Banach algebras and spectral additivity (in the appendix), as well as several special results for matrix or operator algebras (in chapter IV).

As always, let A be a Banach algebra with identity and D a domain in $\mathbb{C}$, $f : D \to A$ analytic.

1 Spectral Variation

In the preceding chapter we adressed the question of variation of the spectral function and the spectral radius if composed with an analytic function. We came to the conclusions that

(1) The spectral function Sp is upper semi-continuous. (II.1.4)

(2) Sp does not "retract from components" (Newburgh, II.2.6)

(3) $\rho \circ f$ and $\delta \circ f$ are sh. (Vesentini, II.3.2 and diametric Vesentini, II.4.4)

(4) $\mathrm{Sp} \circ f$ satisfies a maximum principle. (spectral maximum principle, II.3.4)

(5) f entire and $\mathrm{Sp} \circ f$ "bounded" $\Rightarrow$ f constant. (spectral Liouville, II.3.5)

In this paragraph, we want to study the variation of the spectrum in more depth and see e.g. that isolated spectral values vary analytically.

Our first result concerns spectral elements in the periphery:

Definition 1.1 Let A be a Banach algebra, $x \in A$, $\lambda \in \mathrm{Sp}\, x$ with $|\lambda| = \rho(x)$. Then λ is called an element of the *peripherical spectrum* $\mathrm{Sp}_p(x)$ of x.

Sketches:

Theorem 1.2 (Peripherical Maximum Principle)
Suppose there exists $\lambda_0 \in D$ s.t. $\rho(f(\lambda)) \leq \rho(f(\lambda_0))$ for all $\lambda \in D$. Then the peripherical spectrum of f, $\operatorname{Sp}_p(f(\cdot))$, is constant.

Proof. $\rho \circ f$ must be constant by the maximum principle for sh functions I.4.2. If it is constantly zero, we are done since spectra are non-empty. So let us consider the case $\rho(f(\lambda)) =: c > 0$. Suppose there were $\lambda_1, \lambda_2 \in D$ and $z \in D$ with $z \in \operatorname{Sp}(f(\lambda_1))$, $z \notin \operatorname{Sp}(f(\lambda_2))$ and $|z| = c$. Since for fixed $a > 0$ the function $g : D \to A : \lambda \mapsto f(\lambda) + az\,\mathbf{1}$ is analytic, Vesentini's theorem II.3.2 tells us that $\rho \circ g$ is subharmonic. Thus the inclusion

$$\operatorname{Sp} g(\lambda) \subset \overline{B(az; c)} \subset \overline{B(0; (a+1)z)}$$

is valid for all $\lambda \in D$ (the second inclusion follows from a geometrical argument). This implies $\rho(g(\lambda_2)) < (a+1)c$ and $\rho(g(\lambda)) \leq (a+1)c = \rho(g(\lambda_1))$ for all $\lambda \in D$, contradicting the maximum priniple. □

Corollary 1.3 *Suppose* $\operatorname{Sp} f(\lambda) \subset \mathbb{R}$ *for all* $\lambda \in D$. *Then* $\operatorname{Sp} \circ f$ *is constant.*

Proof. Let $\lambda_0 \in D$ be fixed and $E := \{\lambda \in D \mid \operatorname{Sp} f(\lambda) = \operatorname{Sp} f(\lambda_0)\}$. We will show that E is open and closed and are then done by connectedness of D. To this end let $\lambda_1 \in E$. Replacing f by $\lambda \mapsto \alpha f(\lambda) + \beta\,\mathbf{1}$ with appropriate complex numbers α and β we can assume that $\operatorname{Sp} f(\lambda_1) \subset (0, 2\pi)$ and thus by upper semi-continuity of the spectral function (theorem II.1.4) there is a neighbourhood N_1 of λ_1 with $\operatorname{Sp} f(\lambda) \subset (0, 2\pi)$ for all $\lambda \in N_1$. By the holomorphic functional calculus we know that $g : D \to A : \lambda \mapsto e^{if(\lambda)}$ is analytic and has its spectrum contained in the circle $C(0; 1)$. Thus $\operatorname{Sp} g(\lambda) = \operatorname{Sp}_p g(\lambda)$ and $\rho(g(\lambda)) = 1 = \rho(g(\lambda_1))$ for all $\lambda \in D$, hence by the above theorem the spectrum of g must be constant. But, again by holomorphic functional calculus,

$$e^{i \operatorname{Sp} f(\lambda)} = \operatorname{Sp} g(\lambda) = \operatorname{Sp} g(\lambda_1) = e^{i \operatorname{Sp} f(\lambda_1)}.$$

Since $x \mapsto e^{ix}$ is injective on $(0, 2\pi)$ we conlude $N_1 \subset E$ and hence E must be open. Leading the same argument with $\lambda_2 \in E^c$ we conclude that E is also closed. Actually, we could reformulate this proof to show that $\operatorname{Sp} \circ f$ is locally constant and hence constant. □

Remark In the case $A = \mathbb{C}$ this leads to the following theorem which is well-known in complex analysis: If $f : D \to A$ is analytic and has its range included in $\mathbb{R}$, then f is constant.

Liouville's spectral theorem II.3.5 can be strengthened in the following way:

Theorem 1.4
Let $f : \mathbb{C} \to A$ be analytic. Then either $\widehat{\operatorname{Sp} f(\lambda)}$ *is constant or* $\bigcup_{\lambda \in \mathbb{C}} \widehat{\operatorname{Sp} f(\lambda)}$ *is dense in* $\mathbb{C}$.

Proof. A proof involving polynomially convex subsets of $\mathbb{C}$ can be found in [5] (p. 57). $\square$

Remarks One can define the notion of *capacity* of a subset of the complex plane and prove that the latter case in the above theorem implies that $\mathbb{C}\setminus\bigcup_{\lambda\in\mathbb{C}}\operatorname{Sp} f(\lambda)\hat{}$ is a G_δ-set having zero capacity. This would be a bridge to far though for this essay. The interested reader can find the definition of capacity in [5] (appendix, p. 177ff) and a proof for the above statement in chapter VII of the same work.

We will now embark on showing that, in some sense, the composite of the spectral function with an analytic function is also analytic.

Lemma 1.5 *Let* $0 \le r \le s$, $0 < \theta_2 - \theta_1 < 2\pi$, $f : D \to A$ *analytic. Let* $\Omega = \{z \in \mathbb{C} \,|\, |z| > s,\ \theta_1 < \arg z < \theta_2\}$. *Suppose that* $\operatorname{Sp} f(\lambda) \subset D := \Omega \cup B(0; r)$ *for all* $\lambda \in D$. *Define*

$$u(\lambda) := \max\{\arg u \,|\, u \in \operatorname{Sp} f(\lambda) \cap \Omega\}$$
$$v(\lambda) := \min\{\arg v \,|\, v \in \operatorname{Sp} f(\lambda) \cap \Omega\}$$

Then $u, -v$ *are subharmonic.*

Proof. Again, the proof for v is similar to the one for u. Wlog $\Omega \cap \operatorname{Sp} f(\lambda) \neq \emptyset$ for all λ. On Ω we consider the branch of the logarithm $\log z = \log |z| + i \arg z$ and define

$$h(z) = \begin{cases} -i\log z & \text{on } \Omega \\ \alpha & \text{on } B(0; r) \end{cases}$$

where $\alpha < \theta_1$ is fixed. Then h is analytic on D and by holomorphic functional calculus we have

$$\operatorname{Sp} h(f(\lambda)) = h(\operatorname{Sp} f(\lambda)) \subset \{-i \log z \ \,|\, z \in \operatorname{Sp} f(\lambda) \cap \Omega\} \cup \{\alpha\}.$$

So $u(\lambda) = \max\{\operatorname{Re} z \,|\, z \in \operatorname{Sp} h(f(\lambda))\}$. Apply now lemma II.3.3 to $h \circ f$. $\square$

Theorem 1.6
Let $f : D \to A$ *be analytic and suppose that* $\operatorname{Sp} f(\lambda) = \{0, \alpha(\lambda)\}$ *for all* $\lambda \in D$, *where* $\alpha : D \to \mathbb{C}$. *Then* α *is analytic on the whole of* D.

Proof. We already know that α must be continuous (Newburgh's corollary, II.2.7). Let $D' := \{\lambda \in D \,|\, \alpha(\lambda) \neq 0\}$. Hence D' is open. If it is empty we are done, so suppose it is not. By Radó's extension theorem (see [26] (p. 315)) it is enough to show that α is analytic on D'. Let $\lambda_0 \in D$. By continuity of α there exists $\delta > 0$, $0 \le r \le s$, and $0 < \theta_2 - \theta_1 < 2\pi$ s.t. $|\lambda - \lambda_0| < \delta$ implies $\alpha(\lambda) \in \Omega \cup B(0; r)$, where Ω is defined as in the lemma. By definition of u and v we conclude that $u = v$ and thus "both" must be harmonic on $B(\lambda_0; \delta)$ by the above lemma and theorem I.3.5. Thus, there must be an analytic function $k : B(\lambda_0;\delta) \to \mathbb{C}$ satisfying $u = \operatorname{Im} k$. But then, taking $g : B(\lambda_0;\delta) \to A : \lambda \mapsto e^{-k(\lambda)} f(\lambda)$, and observing that $\operatorname{Sp} g(\lambda) \subset \mathbb{R}$ for all $\lambda \in B(\lambda_0;\delta)$ we can use corollary 1.3 to show that $\operatorname{Sp} g$ is constant on its domain of definition. Using the equality

$$\operatorname{Sp} g(\lambda) = e^{-k(\lambda)} \operatorname{Sp} f(\lambda) = e^{-k(\lambda)}\{0, \alpha(\lambda)\} = \{0, e^{-k(\lambda)}\alpha(\lambda)\} = \{0, C\}$$

we conclude that $\alpha(\lambda) = Ce^{k(\lambda)}$ on $B(\lambda_0;\delta)$ which shows that α is analytic at λ_0. Since λ_0 was arbitrary, α must be analytic on D. □

Similarly, one can prove:

Corollary 1.7 *Let $f : D \to A$ be analytic,* $\operatorname{Sp} f(\lambda) = \{\alpha(\lambda)\}$ *for all $\lambda \in D$ where $\alpha : D \to \mathbb{C}$. Then α is analytic.*

Theorem 1.8
Let $f : D \to A$ be analytic and suppose that $\operatorname{Sp} f(\lambda)$ *lies on the same vertical segment for all $\lambda \in D$. Then there exists an analytic $h : D \to \mathbb{C}$ and $K \in K(\mathbb{R})$ s.t.* $\operatorname{Sp} f(\lambda) = h(\lambda) + iK$.

Proof. Using the notation from lemma II.4.3 we have $u = v$ on D and thus, by lemma I.3.5, u is harmonic. Let $h(\lambda)$ be the element of $\operatorname{Sp} f(\lambda)$ having smallest imaginery part – this works since spectra are compact. Fix $\lambda_0 \in D$ and let $\delta > 0$ be s.t. $\overline{B(\lambda_0;\delta)} \subset D$. Since u is harmonic there is an analytic function k on $B(\lambda_0;\delta)$ satisfying $\operatorname{Re} k = u$. Let $g_1 : B(\lambda_0;\delta) \to A : \lambda \mapsto -i(f(\lambda) - k(\lambda)\,\mathbf{1})$, so $\operatorname{Sp} g_1(\lambda) \subset \mathbb{R}$. Now we can use corollary 1.3 which tells us that $\operatorname{Sp} g_1(\cdot)$ is constant. By definition of h it follows that h is analytic at λ_0, so, since λ_0 was arbitrary, h must be analytic. Using the same argument with $g_2 : B(\lambda_0;\delta) \to A : \lambda \mapsto -i(f(\lambda) - h(\lambda)\,\mathbf{1})$ we get the result. □

Motivated by the example of $\mathfrak{LC}(X)$ which will be explained in paragraph IV.2, we formulate:

Theorem 1.9 (Holomorphic Variation of Isolated Spectral Values)
Let $f : D \to A$ be analytic and suppose there exist $\lambda_0 \in D$, $\alpha_0 \in$ $\operatorname{Sp} f(\lambda_0)$ *and positive r, δ s.t. $|\lambda - \lambda_0| < \delta$ implies $\lambda \in D$ and* $\operatorname{Sp} f(\lambda) \cap B(\alpha_0; r) = \{\alpha(\lambda)\}$ *where $\alpha : B(\lambda_0;\delta) \to \mathbb{C}$. Then α is analytic on a neighbourhood of λ_0.*

Proof. Assume first that $\operatorname{Sp} f(\lambda_0) = \{\alpha_0\}$. By upper semi-continuity of the spectral function (theorem II.1.4) there must be $\delta \geq \delta' > 0$ s.t. $\operatorname{Sp} f(\lambda) = \{\alpha(\lambda)\}$ for all $\lambda \in B(\alpha_0;\delta')$. By corollary 1.7 α is analytic on $B(\alpha_0;\delta')$.

Now assume that $\{\alpha_0\} \subsetneqq \operatorname{Sp} f(\lambda_0)$. Again since the spectral function is usc we have $\operatorname{Sp} f(\lambda) \cap \partial B(\alpha_0; r) = \emptyset$ for all λ sufficiently close to λ_0. In consequence,

$\mathrm{Sp}\, f(\lambda_0) \cap \{z \in \mathbb{C} \,|\, |z - \alpha_0| > r\} \neq \emptyset$. By Newburgh's theorem II.2.6 we can conclude wlog that $\mathrm{Sp}\, f(\lambda) = \{\alpha(\lambda)\} \cup S$ where $\alpha(\lambda) \in B(\alpha_0; r)$ and $S \neq \emptyset$, $S \subset \{z \in \mathbb{C} \,|\, |z - \alpha_0| > r\}$. Let $h : \mathbb{C} \setminus C(\alpha_0; r) \to \mathbb{C}$, h equal to the identity inside the disc of radius r and equal to zero outside. Then h is analytic and by the holomorphic functional calculus $h \circ f$ is well defined and $\mathrm{Sp}\, h(f(\lambda)) = \{0, \alpha(\lambda)\}$ for all λ close enough to λ_0. Hence the result is obtained by theorem 1.6. $\square$

Remarks

- It is not true in general that if $\mathrm{Sp}\, f(\lambda_0)$ contains an isolated singleton so must $\mathrm{Sp}\, f(\lambda)$ for λ nearby (see first example on p. 35). Newburgh's theorem (II.2.6) only ensures that there will be a small component of $\mathrm{Sp}\, f(\lambda)$ close to α_0 if λ and λ_0 are sufficiently close. The holomorphic variation theorem only states that *if* there is an isolated spectral value close to α_0 for all λ close enough to λ_0, *then* it varies analytically.

- The holomorphic variation theorem can be generalised, see chapter VII in [5].

2 Automatic Continuity

It is a well-known fact from classical Banach algebra theory that all complete algebra norms on a commutative, *semi-simple* Banach algebra are equivalent. In the 1950's, I. Kaplansky conjectured that the same result is true for non-commutative semi-simple algebras. This problem was solved in 1967 by B.E. Johnson. His proof is based on representation theory and will thus not be treated in this essay. It can be found in [3] (p. 34f). We will give two proofs using subharmonic methods. These proofs are due to B. Aupetit and T.J. Ransford (1989) respectively.

First of all, we give all the necessary definitions to formulate the theory:

Definition 2.1 Let A be an algebra with identity over $\mathbb{C}$. Then the *Jacobson radical* of A, $\mathrm{Rad}(A)$, is defined by

$$\mathrm{Rad}(A) := \bigcap \{I \subset A \,|\, I \text{ is a maximal left ideal in } A\}$$

A is called *semi-simple* iff $\mathrm{Rad}(A) = \{0\}$. $a \in A$ is called *quasi-nilpotent* iff $\rho(a) = 0$.

Remark It can be shown[20] that

$$\mathrm{Rad}(A) = \bigcap \{I \subset A \,|\, I \text{ is a maximal right ideal in } A\}$$

[20] see [5] (p. 34)

and that $A/\operatorname{Rad}(A)$ is semi-simple for any Banach algebra A with identity[21]. The Jacobson radical is included in the set of all quasi-nilpotent elements of A[22] and equals this set if A is commutative[23].

Definition 2.2 Let X, Y be topological vector spaces, $T : X \to Y$ linear. The *seperating space* $\Sigma := \Sigma(T)$ of T is defined by

$$\Sigma(T) := \{y \in Y \mid \exists\, x_n \to_n 0 \text{ in } X \text{ with } Tx_n \to_n y \text{ in } Y\}.$$

Before we prove Johnson's theorem which will directly lead to the verification of his conjecture, we need to study the properties of the seperating space.

Lemma 2.3 *Let X, Y be Banach spaces, $T : X \to Y$ linear. The seperating space of T has the following properties:*

(*i*) *$\Sigma(T) \subset_c Y$ is a closed subspace*

(*ii*) *If X, Y are Banach algebras and T is an algebra homomorphism whose image $T(X)$ is dense in Y, then $\Sigma(T) \subset_c Y$ is a closed 2-sided ideal.*

(*iii*) *T is continuous if and only if $\Sigma(T) = \{0\}$*

Proof. The algebraic properties are straight-forward to prove. We will just consider the topological claims.

(*i*) The closedness of Σ follows from a diagonal sequence argument.

(*iii*) T is continuous iff it is continuous at 0. The claim then follows from the closed graph theorem. □

Lemma 2.4 *Let A, B be Banach algebras, $T : A \to B$ an algebra-with-identity homomorphism. Then $\rho_B(Ta) \leq \rho_A(a)$ for all $a \in A$. If $S : A \to B$ is an arbitrary mapping satisfying $\rho_B(Sa) \leq \rho_A(a)$ for all $a \in A$, we call S* spectrally decreasing.

Proof. Let $a \in A$, $\lambda \notin \operatorname{Sp}_A(a)$ and x_λ the inverse of $\lambda\mathbf{1} - a$. Then Tx_λ is an inverse of $\lambda\mathbf{1} - Ta$ and thus $\lambda \notin \operatorname{Sp}_B(Ta)$. The conclusion is immediate. □

Theorem 2.5 (Johnson)
Let A, B be Banach algebras with identity, B semi-simple, $T : A \to B$ a surjective algebra-with-identity homomorphism. Then T is continuous.

The first proof we present is due to B. Aupetit.

[21] see [5] (p. 35)
[22] see [5] (p. 36)
[23] see [26] (p. 443)

Proof. We will show that $\Sigma(T) = \{0\}$ and are then done by lemma 2.3. By the same lemma, we know that Σ is a closed 2-sided ideal in B. If we were able to show that Σ is quasi-nilpotent, we could argue that Σ must be included in the Jacobson radical $\mathrm{Rad}(B)$ of B[24] and whence since B is semi-simple $\Sigma = \{0\}$. This is what we are now proving:

Let $z \in \Sigma$, $x_n \in A$, $x_n \to_n 0$ in A, $Tx_n \to_n z$ in B. Since T is surjective there is $x \in A$ with $Tx = z$. Then $\lambda x_n + x \to_n x$ in A and $\lambda T x_n + Tx \to_n (\lambda + 1)z$ in B for all $\lambda \in \mathbb{C}$. Define $\phi_n(\lambda) := \rho_B(\lambda T x_n + Tx)$ and $\psi_n(\lambda) := \rho_A(\lambda x_n + x)$ for all $n \in \mathbb{N}$, $\lambda \in \mathbb{C}$. By the above lemma $\rho_B(Ta) \leq \rho_A(a)$ for all $a \in A$ and hence $\phi_n(\lambda) \leq \psi_n(\lambda)$ for all $\lambda \in \mathbb{C}$. Consequently,

$$\phi(\lambda) := \limsup_{n\to\infty} \phi_n(\lambda) \leq \limsup_{n\to\infty} \psi_n(\lambda) \leq \rho_A(x)$$

since ρ_A is upper semi-continuous. By Vesentini's theorem II.3.2 ϕ_n is subharmonic on $\mathbb{C}$, hence by lemma I.1.4 $\phi : \mathbb{C} \to \mathbb{R} \cup \{-\infty\}$ is a Borel function. Recall that $\phi(\lambda) \leq \rho_A(x)$ for all complex λ, i.e., ϕ is bounded above. We will subsequently show that ϕ satisfies the mean inequality and conclude that its upper regularisation ϕ^* is subharmonic and bounded above by lemma I.2.4. By the maximum principle for subharmonic functions I.4.2, ϕ^* must then be constant. So $\rho_B(z) = \phi(0) \leq \phi^*(0) = \phi^*(\lambda)$ for all complex λ. On the other hand,

$$\phi(\lambda) = \limsup_{n\to\infty} \phi_n(\lambda) \leq \rho_B((\lambda+1)z) = |\lambda + 1|\rho_B(z)$$

by upper semi-continuity of ρ_B. Thus, we have

$$0 \leq \phi^*(\lambda) \leq \limsup_{\mu\to\lambda} |\mu + 1|\rho_B(z) = |\lambda + 1|\rho_B(z)$$

and whence $\phi^*(\lambda) = \phi^*(-1) = 0$ for all $\lambda \in \mathbb{C}$. Since $\rho_B(z) = \phi(0) \leq \phi^*(0) = 0$ we have shown that $z = Tx$ is quasi-nilpotent.

It remains to be shown that ϕ satisfies the mean inequality. Let $\lambda_0 \in \mathbb{C}$, $r > 0$. Since ϕ_n is subharmonic we can conclude by Fatou's lemma that

$$\begin{aligned}\phi(\lambda_0) &= \limsup_{n\to\infty} \phi_n(\lambda_0) \\ &\leq \limsup_{n\to\infty} N(\lambda_0, r, \phi_n) \\ &\leq N(\lambda_0, r, \phi),\end{aligned}$$

i.e., ϕ satisfies the mean inequality. □

[24] see [3] (p. 33, Corollary 4.1.6)

Corollary 2.6 (Johnson's Uniqueness of Norm Theorem) *Let A be a unital Banach algebra w.r.t. $\|\cdot\|$; $|\cdot|$ an arbitrary Banach algebra norm on A. Suppose further that A is semi-simple. Then both norms are equivalent on A.*

Proof. Take A to be normed with $|\cdot|$, $B := A$ to be normed with $\|\cdot\|$. Then B is unital and semi-simple and $T : A \to B : a \mapsto a$ is a surjective algebra homomorphism. Hence by Johnson's theorem 2.5 T must be continuous. Considering T as a linear, continuous, and surjective mapping between two Banach spaces and applying the open mapping theorem we secure the continuity of its inverse. Thus T is a Banach algebra isomorphism and the conclusion follows. $\square$

Remarks

- It is remarkable that the purely algebraic notion of semi-simplicity has such severe topological consequences.
- Remembering that there always is an equivalent unital norm on a Banach algebra, we can reinterpret our result to state that any two Banach algebra norms on a semisimple algebra must be equivalent.
- Note that we can always apply this reformulated theorem to $A/\operatorname{Rad}(A)$ since this algebra is semi-simple.

We will now study the proof of T.J. Ransford. He actually proved a slightly more general form of Johnson's theorem which does not assume that T is an algebra homomorphism but only that it be linear and spectrally decreasing. By lemma 2.4 any algebra-with-identity homomorphism is spectrally decreasing and thus Ransford's version of Johnson's theorem is stronger than the one presented in theorem 2.5. To keep things simple, we will only prove Ransford's version in the case when T is an algebra-with-identity-homomorphism.

In the following, we will prove two lemmata that will lead to the announced result. The first one is a step in the proof of Dini's theorem[25], the second one is due to Ransford. Ransford's lemma is a special case of the Hadamard's three circle theorem I.4.6 although no explicit use of subharmonicity occurs.

Lemma 2.7 (Dini) *Let X be a compact metric space, $(f_n)_{n\in\mathbb{N}}$ a decreasing sequence of non-negative, continuous, $\mathbb{R}$-valued functions on X, $f(x) = \lim_{n\to\infty} f_n(x)$ for all $x \in X$. Then*

$$\sup_{x\in X} f_n(x) \to_n \sup_{x\in X} f(x).$$

[25] Thanks to Dr. Aupetit for bringing the proof of both the theorem and the lemma of Dini to my attention. Dini's theorem can be found in [10] (p. 789).

Proof. By lemma I.1.4 f is usc, and since $f_n \geq f$ it is bounded above. Also, it is clear that

$$\infty > s_{n+1} := \sup_{x \in X} f_{n+1}(x) \geq \sup_{x \in X} f_n(x) \geq \sup_{x \in X} f(x) =: s \geq 0$$

for all $n \in \mathbb{N}$ since X is compact. $(s_n)_{n\in\mathbb{N}}$ is a decreasing sequence that is bounded below and is hence convergent to a limit s'. Clearly $s' \geq s$. To show the converse, let $\epsilon > 0$ and fix $x \in X$. Since $f_n(x) \to_n f(x)$ there is $n_x \in \mathbb{N}$ s.t. for all $n \geq n_x$ we have $f_n(x) \leq f(x) + \epsilon \leq s + \epsilon$. Since f_n is continuous, there is a neighbourhood $B(x; r_x)$ s.t. $f_n(y) \leq s + \epsilon$ for all $y \in B(x; r_x)$. The open balls $B(x; r_x)$ cover X. Hence since X is compact it is covered by a finite union of them. Take n_0 to be the maximum of the finitely many numbers n_x and note that $f_n(x) \leq s + \epsilon$ for all $x \in X$ and $n \geq n_0$. Hence $s' \leq s$ and the conclusion follows. $\square$

Remark Note that, unlike Dini's theorem itself, there is no hypothesis about the continuity of f; but the conditions do necessarily imply that f is upper semi-continuous.

Lemma 2.8 (Ransford) *Let A be a Banach algebra, $p \in A[z]$ – the set of all polynomials in a complex variable z with coefficients in A –, $R > 1$. Then*

$$\rho(p(1))^2 \leq \sup_{|z|=R} \rho(p(z)) \sup_{|z|=\frac{1}{R}} \rho(p(z)).$$

Proof. Let $q \in A[z]$. By the Hahn-Banach theorem there is $\chi \in A^*$ with $\|\chi\| = 1$ and $\chi(q(1)) = \|q(1)\|$. Set $F := \chi \circ q$, then $F \in \mathbb{C}[z]$. Applying the maximum modulus theorem to the analytic function $z \mapsto F(z)F(\frac{1}{z})$ on the annulus $\{z \in \mathbb{C} \mid \frac{1}{R} \leq |z| \leq R\}$ we get:

$$\|q(1)\|^2 = |F(1)|^2 \leq \sup_{|z|=R} |F(z)| \sup_{|z|=\frac{1}{R}} |F(z)| \leq \sup_{|z|=R} \|q(z)\| \sup_{|z|=\frac{1}{R}} \|q(z)\|.$$

Choose $q = p^{2^n}$ for $n \in \mathbb{N}$ to get

$$\|p(z)^{2^n}\|^2 \leq \sup_{|z|=R} \|p(z)^{2^n}\| \sup_{|z|=\frac{1}{R}} \|p(z)^{2^n}\|.$$

Take the 2^n-th root of this and let n tend to infinity. By the Beurling-Gelfand formula the sequence $(\|p(z)^{2^n}\|^{2^{-n}})_{n\in\mathbb{N}}$ converges pointwise to $\rho(p(z))$. It also satisfies all the other assumptions that are needed to apply Dini's lemma and hence the proof is complete. $\square$

Now we will study Ransford's proof:

Theorem 2.9 (Johnson)
Let A, B be Banach algebras, B semi-simple. Let $T : A \to B$ be a surjective algebra-with-identity homomorphism. Then T is continuous.

Proof. Again, we will show that $\Sigma(T)$ is quasi-nilpotent. Let $b \in \Sigma(T)$ and choose $(a_n) \subset A$ with $a_n \to_n 0$ in A and $Ta_n \to_n b$ in B. Since T is surjective, we can choose a preimage a of b. For all $n \in \mathbb{N}$, $z \in \mathbb{C}$ let the polynomial $p_n \in A[z]$ be defined by $p_n(z) := zTa_n + (Ta - Ta_n)$; then $p_n(1) = Ta = b$ and hence $\rho_B(p_n(1)) = \rho_B(Ta) = \rho_B(b)$. Also

$$\rho_B(p_n(z)) \leq \|p_n(z)\| \leq |z|\|Ta_n\| + \|Ta - Ta_n\|.$$

T decreases the spectral radius, hence

$$\rho_B(p_n(z)) \leq \rho_A(za_n + (a - a_n)) \leq |z|\|a_n\| + \|a - a_n\|.$$

By Ransford's lemma 2.8,

$$\begin{aligned}\rho_B(b)^2 &\leq \sup_{|z|=R} \rho_B(p_n(z)) \sup_{|z|=\frac{1}{R}} \rho_B(p_n(z)) \\ &\leq (R\|a_n\| + \|a - a_n\|)(\frac{1}{R}\|Ta_n\| + \|Ta - Ta_n\|)\end{aligned}$$

Letting n tend to infinity we get $\rho_B(b)^2 \leq \|a\|(\frac{1}{R}\|b\|)$, letting R tend to infinity this implies $\rho_B(b) = 0$. The rest follows as in Aupetit's proof (since we only consider homomorphisms and not spectrally decreasing operators). $\square$

Remarks

- There is an unsolved problem in the theory of automatic continuity: suppose that A, B are Banach algebras, B semi-simple, $T : A \to B$ a homomorphism with range dense in B. Is T necessarily continuous?
- It can be shown that this is equivalent to the following question[26]: suppose that A, B are Banach algebras and let $T : A \to B$ be a homomorphism. Then $\Sigma(T)$ is quasi-nilpotent. Note that in this equivalent formulation, B is *not* assumed to be semi-simple.
- The corresponding problem in Ransford's formulation has been disproven, see Dales' monograph [10] (p. 601). By this, it is meant that there are Banach algebras A, B, B semi-simple and $T : A \to B$ linear and spectrally decreasing s.t. $\overline{T(A)} = B$, yet T is discontinuous.
- One can generalise the notion of a spectrally decreasing linear operator to the notion of a spectrally bounded one, i.e., a linear operator satisfying $\rho_B(Ta) \leq M\rho_A(a)$ with a universal constant M. This seems to be closely related to the theory of automatic continuity and is presented in M. Mathieu and G.J. Schick's paper [20].

[26] see [10] (p. 601)

IV Applications

In the last two chapters, we have seen many theorems on Banach algebras. This chapter is intended to illustrate the applicability of these theorems (and those proven in the appendix) in other branches of mathematics. As our time and space are very limited, we will only consider subalgebras of the operator algebra on a Hilbert space. The writer is aware that this is a very limiting choice that deprives the reader of many beautiful insights. Sources for stimulating further reading include the books by Heuser [16], Zeidler [31, 32], and Douglas [11].

1 Models of Banach Algebra Axioms

There are many models of the Banach algebra axioms that are frequently used in mathematics and theoretical physics. Also, some models are used in engineering and computer science. Without providing further details we will now list a few of them, some of which we have already met. Let K be a compact Hausdorff topological space.

(1) $C(K)$, the algebra of all continuous complex-valued functions on K, normed by the uniform norm

(2) $C_A(K)$, the algebra of all continuous A-valued functions on K, normed by the uniform norm, where A is a Banach algebra

(3) If X is any complex Banach space, then $\mathfrak{L}(X)$, the algebra of all continuous linear operators on X, w.r.t. the operator norm is a Banach algebra. A closed subalgebra of this is the algebra $\mathfrak{LC}(X)$ of all compact linear operators on X.

(4) If G is a locally compact Hausdorff commutative topological group[27] and μ its Haar-measure, then $L^1(\mu)$ is a Banach algebra w.r.t. convolution, i.e.

$$(f * h)(x) := \int f(x-t)g(t)d\mu(t).$$

Examples are $l^1 = L^1(\mathbb{Z})$, $L^1(\Omega)$ where $\Omega \subset_o \mathbb{R}^n$ etc.

(5) The algebra l^1 can be identified with the *Wiener algebra* W of all continuous functions f on $[0, 2\pi]$ with Fourier coefficients $(a_n)_{n\in\mathbb{Z}}$ satisfying

$$\|f\| := \sum_{n\in\mathbb{Z}} |a_n| < \infty.$$

[27] For a definition of topological groups, their properties and the construction of the Haar measure, the reader is referred to Rudin's book [28].

Also, closed subalgebras, product, quotient, and matrix algebras of Banach algebras are Banach algebras with the usual norms. Important examples of these include:

- *uniform algebras*, i.e. norm-closed subalgebras of $C(K)$ containing the constant functions and seperating the points of K. If $K \subset \mathbb{C}^n$, then $P(K) \subset R(K) \subset O(K) \subset A(K)$, the respective closures of polynomials, rational functions without poles in K, functions analytic on some open set containing K, and functions analytic on the interior of K, are uniform algebras; there are examples for $n \geq 2$ for which all the inclusions are strict.
- *Calkin algebras* $\mathfrak{L}(X)/\mathfrak{LC}(X)$, where X is a complex Banach space
- $M_n(\mathbb{C})$ normed as operator algebra on $\mathbb{C}^n$

Convolution algebras are extremely important in theoretical physics, engineering, computer science, and in the theory of Wavelets[28], especially those associated with the Fourier transform.

2 Compact Operators on a Hilbert Space

In this paragraph, we will mainly consider the Banach algebra $A := \mathfrak{LC}(H) \oplus \mathbb{C}\,\mathrm{id}$ on the Hilbert space H. If H is finite dimensional A contains all linear operators (or, equivalently, matrices) on H since these are always compact. Let D be a domain in $\mathbb{C}$ and $L : D \to A$ be analytic.

The algebra A has many nice properties which are consequences of our work in the previous chapters:

Theorem 2.1 (Compact Operators)
Let $A := \mathfrak{LC}(H) \oplus \mathbb{C}\,\mathrm{id}$ *on the Hilbert space* H, $L : D \to A$ *analytic on a domain* $D \subset \mathbb{C}$. *The following statements hold true:*

(1) *The spectral function is continuous on* A. *(Newburgh's corollary, II.2.7)*

(2) *It is not uniformly continuous since it is not even so if* H *is finite dimensional. (proposition II.2.3)*

(3) $\rho \circ L$ *and* $\delta \circ L$ *are subharmonic on* A. *(Vesentini, II.3.2 and diametric Vesentini, II.4.4)*

(4) $\widehat{\operatorname{Sp} L(\lambda)} = \operatorname{Sp} L(\lambda)$ *and* $\operatorname{Sp} L(\lambda)$ *has no interior points for all* $\lambda \in D$

(5) *If there exists* $\lambda_0 \in D$ *with* $\operatorname{Sp} L(\lambda) \subset \operatorname{Sp} L(\lambda_0)$ *for all* $\lambda \in D$ *then* $\operatorname{Sp} L(\cdot)$ *is constant on* D. *(spectral maximum theorem, II.3.4)*

[28] see [19]

(6) *Suppose* L *is entire and there exists* $C \subset \mathbb{C}$ *s.t.* $\operatorname{Sp} L(\lambda) \subset C$ *for all complex* λ, *then* $\operatorname{Sp} L(\cdot)$ *is constant. (spectral Liouville, II.3.5)*

(7) *Suppose there exists* $\lambda_0 \in D$ *s.t.* $\rho(L(\lambda)) \leq \rho(L(\lambda_0))$ *for all* $\lambda \in D$, *then the peripherical spectrum* $\operatorname{Sp}_p(L(\cdot))$ *is constant. (peripherical maximum principle, III.1.2)*

(8) *If* L *is entire, then either* $\operatorname{Sp} L(\cdot)$ *is constant or* $\bigcup_{\lambda \in \mathbb{C}} \operatorname{Sp} L(\lambda)$ *is dense in* $\mathbb{C}$. *(theorem III.1.4)*

The spectrum of any compact operator on a Banach space is at most countable by the theorem of Riesz and Schauder[29]. Its only possible accumulation point is zero. An *eigenvalue* of a linear operator $T \in \mathfrak{L}(H)$ is a complex number α for which there exists a so-called *eigenvector* $x \in H \setminus \{0\}$ satisfying $Tx = \alpha x$. If $T \in A$ is of the form $T = K + \mu \mathbf{1}$ and $\alpha \neq \mu$ is an eigenvalue of T, then its *multiplicity* $n \in \mathbb{N}$ is defined by the dimension of the space spanned by all eigenvectors corresponding to the eigenvalue α. These considerations enable us to strengthen the theorem on holomorphic variation of isolated spectral values III.1.9 for the algebra A.

Theorem 2.2 (Holomorphic Variation of Spectral Values)
Let H *be a Hilbert space,* D *a domain in* $\mathbb{C}$, A *as before,* $L : D \to A$ *analytic. Then, for each* $\lambda_0 \in D$ *and* $\alpha_0 \in \operatorname{Sp} L(\lambda_0) \setminus \{0\}$ *having mutiplicity one, there exist* $r, \delta > 0$ *s.t.* $|\lambda - \lambda_0| < \delta$ *implies* $\lambda \in D$ *and* $\operatorname{Sp} L(\lambda) \cap B(\alpha_0; r) = \{\alpha(\lambda)\}$ *where* $\alpha : B(\lambda_0; \delta) \to \mathbb{C}$ *is analytic. If* H *is finite dimensional,* $\alpha_0 \in \operatorname{Sp} L(\lambda_0)$ *may take the value* 0 *if it is an eigenvalue of multiplicity one.*

Proof. Suppose first that there is $\lambda_0 \in D$, $\alpha_0 \in \operatorname{Sp} L(\lambda_0) \setminus \{0\}$ s.t. $\forall r, \delta > 0$ there is $\lambda_{r,\delta} \in \mathbb{C}$ with $|\lambda_0 - \lambda_{r,\delta}| < \delta$ and $\lambda_{r,\delta} \notin D$ or $\operatorname{Sp} L(\lambda_{r,\delta}) \cap B(\alpha_0; r)$ is not a singleton. Since D is open we can choose δ small enough s.t. $|\lambda_0 - \lambda_{r,\delta}| < \delta$ implies $\lambda_{r,\delta} \in D$. Hence we only need to consider the case when $\operatorname{Sp} L(\lambda_{r,\delta}) \cap B(\alpha_0; r)$ is not a singleton.

Case 1: $\operatorname{Sp} L(\lambda_{r,\delta}) \cap B(\alpha_0; r) = \emptyset$:
Let $U := B(\alpha_0; r), V := \mathbb{C} \setminus \overline{B(\alpha_0; r)}$, $x := L(\lambda_0)$. Then the assumptions of Newburgh's theorem II.2.6 are fulfilled if r is suitably small. Hence there exists $\epsilon > 0$ s.t. $\|x - y\| < \epsilon$, $y \in A$ implies $\operatorname{Sp} y \cap U \neq \emptyset$. Since L is continuous there exists $\delta > 0$ s.t. $|\lambda - \lambda_0| < \delta$ implies $\|L(\lambda) - L(\lambda_0)\| < \epsilon$ and hence $\operatorname{Sp} L(\lambda) \cap U \neq \emptyset$, a contradiction.

Case 2: $\operatorname{Sp} L(\lambda_{r,\delta}) \cap B(\alpha_0; r)$ contains at least two elements:
Gohberg and Kreĭn show in [13] (Chapter II) that this is impossible[30].

[29] see, e.g. [4] (p. 363ff)
[30] says Aupetit in [5] (p. 59)

So we have shown that for all $\lambda_0 \in D$ and $\alpha_0 \in \operatorname{Sp} L(\lambda_0) \setminus \{0\}$ having multiplicity one, there exist $r, \delta > 0$ s.t. $|\lambda - \lambda_0| < \delta$ implies $\lambda \in D$ and $\operatorname{Sp} L(\lambda) \cap B(\alpha_0; r) = \{\alpha(\lambda)\}$ where $\alpha : B(\lambda_0; r) \to \mathbb{C}$. It remains to show that α is analytic. The theorem on holomorphic variation of isolated spectral values III.1.9 tells us that α is analytic on a neighbourhood of λ_0. We can then choose our δ even smaller so that $B(\lambda_0; \delta)$ is included in this neighbourhood.

Let H be finite dimensional. Let $\mu_0 \notin L(\lambda_0)$ and let $L'(\lambda) := L(\lambda) + \mu_0 \operatorname{id}$. Then $L' \in A$ and $0 \notin \operatorname{Sp} L'(\lambda_0)$. Hence we can apply the theorem to L' and subsequently substract μ_0 to get the desired result. □

Remark The assumptions that α_0 be a *non-zero* (in the infinite dimensional case) eigenvalue of $L(\lambda_0)$ having *multiplicity one* are necessary as the following examples demonstrate:

Example 2.3 Let H be a Hilbert space (at least of dimension two), V a two-dimensional subspace with basis $\{v, w\}$. Let $L : \mathbb{C} \to A$ be the linear operator defined by $L(\lambda)v := 2\lambda v$, $L(\lambda)w := (2 - 2\lambda)w$ and by zero on the orthogonal complement of V. Since $L(\lambda)$ has finite range for all $\lambda \in \mathbb{C}$, L is indeed a mapping with range in A. $\operatorname{Sp}(L(\lambda)) = \{2\lambda, 2 - 2\lambda, 0\}$ for all $\lambda \in \mathbb{C}$ (no zero if H is two-dimensional). Choose $\lambda_0 = \frac{1}{2}$ and $\alpha_0 = 1$. Clearly 1 is an isolated spectral value of $L(\frac{1}{2})$. But there is no neighbourhood N of $\frac{1}{2}$ and no $r > 0$ s.t. the set $\operatorname{Sp} L(\lambda) \cap B(1; r)$ is a singleton for all $\lambda \in N$ since $|2\lambda - 1| = |(2 - 2\lambda) - 1|$ for all complex λ.

Example 2.4 Let H be infinite dimensional. Let $L : \mathbb{C} \to A$ be constantly equal to $T \in \mathfrak{LC}(H)$. Then 0 is not an isolated spectral value of T and hence there cannot be any $\lambda_0 \in \mathbb{C}$, $r > 0$ (not to speak of a whole neighbourhood) s.t. $\operatorname{Sp} L(\lambda_0) \cap B(0; r)$ is a singleton.

The afore presented automatic continuity results also have immediate consequences for A:

Theorem 2.5 (Johnson)
If B is any Banach algebra and $L : B \to A(= \mathfrak{LC}(H) \oplus \mathbb{C}\operatorname{id})$ is a surjective algebra homomorphism, then L is continuous. In particular, there is no Banach algebra norm on A that is not equivalent to the operator norm.

Proof. One can show that A is semi-simple. Since it is unital w.r.t. the operator norm, too, Johnson's theorem III.2.5 and Johnson's uniqueness of norm theorem III.2.6 prove the result. □

These results, although valuable in their own right, might be very useful in the theory of integral equations. There, they could be used in the following way: Let

D be a domain in $\mathbb{C}$ and $L : D \to \mathfrak{LC}(H)$ an analytic family of integral operators – e.g. Fredholm or Volterra operators[31] – s.t. there is a parameter $\lambda_0 \in D$ for which we know that the equation $L(\lambda_0)x = \alpha x$ $(x \in H)$ has a one dimensional space of solutions for a certain non-zero eigenvalue α_0. We can then conclude by theorem 2.2 that there must be a neighbourhood of λ_0 in D on which α_0 is an *approximative* eigenvalue of multiplicity one, i.e., that there is an eigenvalue $\alpha(\lambda)$ of multiplicity one that depends on λ not only continuously but even analytically. It might also be possible to combine the results of this paragraph with the Fredholm alternative to give necessary and sufficient conditions for solvability of such integral equations, but this would be far beyond the scope of this essay.

It is shown in the appendix that every complex Banach algebra with identity is a *Schurian algebra*. Hence $A = \mathfrak{LC}(H) \oplus \mathbb{C}\,\mathrm{id}$ is such an algebra, and we can apply all theorems proven for Schurian algebras to A. Note that the centre of A consists of all scalar multiples of the identity.

Theorem 2.6 (Commutators and Spectral Additivity)

- *$a \in A$ satisfies $La \subset L$ for every maximal left ideal L iff it is a scalar multiple of the identity. (theorem A.3.1)*
- *For a fixed $a \in A$ $\mathrm{Comm}(a, b) + \mathbf{1}$ is invertible for all $b \in A$ iff a is a scalar multiple of the identity. (theorem A.3.2)*
- *There exist $a, b \in A$ s.t. $\mathrm{Comm}(a, b) + \mathbf{1}$ is not invertible. (corollary A.3.3)*
- *There exist $a, b \in A$ s.t. $\mathrm{Comm}(a, b)$ is not nilpotent. (corollary A.3.4)*
- *There even exist $a, b \in A$ s.t. $\mathrm{Comm}(a, b)$ is not quasi-nilpotent. (corollary A.3.5)*
- *$a \in A$ is spectrally additive iff it is a scalar multiple of the identity. (theorem A.4.6)*

Remark As all operators on a finite dimensional space are compact, all the above theorems also apply to matrix-algebras.

[31] see [16]

3 Self-adjoint Operators

To practice applying the results to operator algebras, we prove:

Lemma 3.1 *Let S denote the subset of $\mathfrak{L}(H)$ or $\mathfrak{LC}(H)$ containing all self-adjoint operators. Let $L : D \to S$ be analytic, where D is a domain in $\mathbb{C}$. Then $\operatorname{Sp} L(\cdot)$, $\rho \circ L$, and $\|L(\cdot)\|$ are constant.*

Proof. If T is self-adjoint, then $\operatorname{Sp} T \subset \mathbb{R}$ and $\|T\| = \rho(T)$. Thus, by corollary III.1.3, $\operatorname{Sp} \circ L$ is constant. This clearly implies the constancy of $\rho \circ L$ and hence of $\|L(\cdot)\|$. $\square$

One can even show that any analytic function with range in S must be constant:

Theorem 3.2 (Self-adjoint Operators)
Let S and L be as in the lemma. Then L is constant.

Proof. [32] Let $x \in H$, $f : D \to \mathbb{C} : \lambda \mapsto (L(\lambda)x; x)$ where $(\cdot;\cdot)$ denotes the scalar product in H. Then f is analytic on D since the scalar product is linear and continuous in its first component. Also, since $L(\lambda)$ is self-adjoint for each $\lambda \in D$, f has its range in $\mathbb{R}$ and is thus constant by the remark on p. 23. Since H is a complex Hilbert space, we can reconstruct $L(\lambda)$ using the identity $(L(\lambda)x; y) = \frac{(L(\lambda)(x+y); x+y) - (L(\lambda)x; x) - (L(\lambda)y; y)}{2}$ for all $\lambda \in D$ and all $x, y \in H$, and see that is must also be independent of λ. $\square$

[32] Thanks to Dr. Allan for helping me with this proof.

A Representations and Subharmonicity

This appendix assumes that the reader is familiar with representation theory for algebras (or, equivalently, with algebra modules). A concise introduction to the theory can be found in Dr. Allan's notes [3] (p. 31ff). We follow L.A. Harris and R.V. Kadison in their paper [14]. Let $\mathfrak{Lin}(X)$ be the algebra of all linear endomorphisms on a vector space X.

1 Schurian algebras

In this paragraph, we will introduce the class of *Schurian algebras* that includes unital Banach algebras over the complex field. In the following paragraphs we will then derive properties of Banach algebras stemming from the general ones proven for Schurian algebras.

We begin with the basic definitions:

Definition 1.1 Let A be a complex algebra with identity.

- A is said to be *Schurian* iff $La \subset L$ for a maximal left ideal L in A and some $a \in A$ implies that there is a $\lambda \in \mathbb{C}$ s.t. $\lambda\mathbf{1} - a \in L$.
- Let X be an A-module. A is said to act *transitively* on X iff all non-zero vectors in X are cyclic. This is equivalent to the statement that X is an irreducible A-module.
- A representation π of A on X is said to satisfy the *Schur condition* iff each linear endomorphism of X that commutes with $\pi(A)$ is effectively a multiplication by a scalar.

Remark The name "Schurian algebra" is motivated by the assumptions made in Schur's lemma.

Schurian algebras and transitive representations satisfying the Schur condition are in fact closely related:

Theorem 1.2
An algebra A over the complex field is Schurian iff each transitive representation of A on a complex vector space X satisfies the Schur condition.

Proof. Let A be an algebra represented transitively on a complex vector space X by π. Let x be a non-zero element in X and let $L_x := \{a \in A \,|\, \pi(a)x = 0\}$ be the annihilator of x. Clearly, L_x is a left ideal in A. Aupetit proves on p. 34 of [5] that L_x is in fact a maximal left ideal.

Assume that A is Schurian. Let T be a linear endomorphism on X that commutes with $\pi(A)$. Since π is transitive, there is an element $a_T \in A$ such that $\pi(a_T)x = Tx$. For each $b \in A$ we have $T\pi(b)x = \pi(b)Tx = \pi(b)\pi(a_T)x$. If $b \in L_x$, this implies $\pi(b)\pi(a_T)x = T\pi(b)x = 0$, i.e. $ba_T \in L_x$. It follows that $L_x a_T \subset L_x$. Since A is Schurian, there must be $\lambda \in \mathbb{C}$ such that $\lambda\mathbf{1} - a_T \in L_x$. Hence $\pi(a_T)x = \lambda x$ and for all $b \in A$ we have $T\pi(b)x = \pi(b)\lambda x = \lambda\pi(b)x$, in short, $\pi(b)x$ is an eigen-vector of T to the eigen-value λ. Since b was arbitrary and x is cyclic we conclude that $T = \lambda\,\mathrm{id}_X$ and whence that π satisfies the Schur condition.

Suppose next that each transitive representation of A satisfies the Schur condition. Let L be a maximal left ideal in A, $a \in A$ s.t. $La \subset L$. Let $X := A/L$ and let π be the left regular representation of A on X (i.e. left multiplication). We will show that π is transitive. Let $b + L \in X$ be non-zero, i.e. $b \in A \setminus L$. $Ab + L$ is a left ideal in A containing L properly. Thus, by maximality of L, $Ab + L = A$. Thus, $\pi(A)(b+L) = A(b+L) = Ab+L = A$, i.e., $b+L$ is cyclic. So π is transitive. Let $T : X \to X : b + L \mapsto ba + L$, then $T \in \mathfrak{Lin}(X)$ and T commutes with $\pi(A)$ since $\pi(a)T(b + L) = a(ba + L) = aba + L = T(ab + L) = T\pi(a)(b + L)$. Thus by the Schur condition, T must be multiplication by some complex scalar λ, i.e. $ba + L = \lambda b + L$ for all $b \in A$. Hence $\lambda\mathbf{1} + L = T(\mathbf{1} + L) = \mathbf{1}\,a + L = a + L$ which gives us $\lambda\mathbf{1} - a \in L$. It follows that A is Schurian. $\square$

Lemma 1.3 *$\tilde{A} := A/\operatorname{Rad}(A)$ is Schurian iff A is Schurian.*

Proof. The quotient mapping q carries the set of maximal left ideals in A into the corresponding one in $\tilde{A}$. If A is Schurian, $\tilde{L}$ a maximal left ideal in $\tilde{A}$ and $\tilde{a} \in \tilde{A}$ s.t. $\tilde{L}\tilde{a} \subset \tilde{L}$, then $L := q^{-1}(\tilde{L})$ and $a \in q^{-1}(\tilde{a})$ satisfy

$$La = q^{-1}(\tilde{L}\tilde{a}) \subset q^{-1}(\tilde{L}) = L$$

and thus, since A is Schurian, there exists $\lambda \in \mathbb{C}$ s.t. $\lambda\mathbf{1} - a \in L$. In consequence, $0 = q(\lambda\mathbf{1} - a) = \lambda(\mathbf{1} + \tilde{L}) - \tilde{a}$ and it follows that $\tilde{A}$ is Schurian. Suppose conversely that $\tilde{A}$ is Schurian, L is a maximal left ideal in A, $a \in A$ and $La \subset L$. Then $q(L)q(a) \subset q(L)$ and thus there exists $\lambda \in \mathbb{C}$ s.t. $q(\lambda\mathbf{1} - a) \in q(L)$. Since $\operatorname{Rad}(A) \subset L$ by def. of $\operatorname{Rad}(A)$, $\lambda\mathbf{1} - a \in L$ and it follows that A is Schurian. $\square$

2 Banach Algebras as Schurian Algebras

After having proven some basic facts about Schurian algebras, we will now show that finite dimensional algebras and complex Banach algebras with identity are indeed Schurian. Let A first be an n-dimensional complex algebra with identity. Suppose π is a representation of A on an m-dimensional complex vector space X. Wlog we can consider π as a representation of A in $M_m(\mathbb{C})$. Using the left-regular representation of A we can view A as a subalgebra of $M_n(\mathbb{C})$.

Lemma 2.1 *A representation π of the n-dim. algebra A on the m-dim. space X is transitive iff it satisfies the Schur condition.*

Proof. Let π be transitive. Then $m \leq n$. Let T be a linear endomorphism on X commuting with $\pi(A)$. Then the kernel and the range of T are invariant under $\pi(A)$. By transitivity, they are each either equal to $\{0\}$ or to X, i.e. T is either constantly zero or invertible. Since the determinant of $z\,\mathbf{1} - T$ is a complex polynomial in z it has a zero $\lambda \in \mathbb{C}$ by the fundamental theorem of algebra. As $\lambda\,\mathbf{1} - T$ commutes with $\pi(A)$ and is not invertible, we conclude $T = \lambda\,\mathbf{1}$., i.e., π satisfies the Schur condition.

Conversely, let π satisfy the Schur condition. Suppose $m > n$. Then there is a non-trivial vector $y \in X$ which is annihilated by all mappings $\pi(a)$ $(a \in A)$. Let $T : X \to X$ be defined by the identity map on the span of $\{y\}$ and by zero on $\pi(A)X$ (as well as zero in all other linearly independent directions). Then T commutes with $\pi(A)$ but is not equal to a constant multiple of the identity, a contradiction. □

The next corollary is a simple consequence of this lemma and theorem 1.2:

Corollary 2.2 *Every finite dimensional complex algebra is Schurian.*

Examples The following algebras are thus Schurian:

- $M_n(\mathbb{C})$
- the subalgebra of $M_n(\mathbb{C})$ containing all upper (lower) triangular matrices
- $\mathbb{C}^n$ with coordinate-wise multiplication

Remark Harris and Kadison prove in [14] (p. 5) that an arbitrary algebra A need not Schurian if it is not finite dimensional, using the example of $\mathbb{C}(x)$, the abstract algebra of all rational functions over $\mathbb{C}$. But if we are in the case that A is normed and complete (and has an identity), then A is Schurian, as we will see now:

Theorem 2.3
Every complex Banach algebra with identity is Schurian.

Proof. Let A be a complex Banach algebra with identity. Let X be a Banach space, $B := \mathfrak{L}(X)$ its algebra of continuous linear operators. Let π be a transitive representation of an algebra C on X. We will prove that if $T \in B$ commutes with each operator in $\pi(C)$, then T is effectively a multiplication by a scalar. To see this, let $\lambda \in \operatorname{Sp} T$. As in the finite dimensional case, the kernel and the range of $\lambda\mathbf{1} - T$ must be invariant under $\pi(C)$. Since π is transitive we are either in the case that $T = \lambda\mathbf{1}$ or its range is the whole of X. We thus only need to consider the case that $\lambda\mathbf{1} - T$ is surjective and its kernel must hence be trivial (note that this follows from a different argument than in the finite dimensional case). But this means that $\lambda\mathbf{1} - T$ is a continuous automorphism of X and hence invertible in B by the oppen mapping theorem.

Conversely, let L be a maximal left ideal in A and $a \in A$ s.t. $La \subset L$. We have to show that there exists $\lambda \in \mathbb{C}$ s.t. $\lambda\mathbf{1} - a \in L$. Let $X := A/L$, $T : X \to X : b + L \mapsto ba + L$, then $T \in \mathfrak{L}(X)$. Let $C := A$ and let π be the left regular representation of C on X. Since L is a maximal ideal, π must be transitive. Also, T commutes with $\pi(C)$. Hence, we have proven above that there exists $\lambda \in \mathbb{C}$ s.t. $T = \lambda\mathbf{1}$. In particular, $a + L = T(\mathbf{1} + L)$ and whence $a + L = \lambda\mathbf{1} + L$. This shows that A is Schurian. □

3 Characterisation of Commutative Schurian Algebras

The following theorems study the relationship between commutators, maximal left ideals, and the Jacobson radical:

Theorem 3.1
Let A be Schurian. Then $a \in A$ has the property $La \subset L$ for every maximal left ideal L iff the commutator *of a and b,* $\operatorname{Comm}(a, b) := ab - ba$, *lies in* $\operatorname{Rad}(A)$ *for all $b \in A$.*

Proof. Fix $a \in A$. Let $C_a : A \to A : b \mapsto \operatorname{Comm}(a, b)$. Suppose first that C_a has range in $\operatorname{Rad}(A)$. Let L be a maximal ideal in A and $b \in L$. Then $ab \in L$, $ab - ba \in \operatorname{Rad}(A) \subset L$, and hence $ba \in L$. Since $b \in L$ was arbitrary we conclude $La \subset L$.

Suppose conversely that every maximal left ideal $L \subset A$ satisfies $La \subset L$ and let $b \in A$. If L is an arbitrary maximal left ideal, define $L_b := \{z \in A \,|\, zb \in L\}$. Then L_b is a left ideal in A. We will now show that L_b is maximal iff $b \notin L$. Clearly, if $b \in L$, then $L_b = A$ and L_b is not a maximal left ideal. Let $b \notin L$. Then there is $z \notin L_b$ and thus $zb \notin L$. Since L is a maximal left ideal in A there is

$s \in A$ s.t. $szb - b \in L$. Thus $sz - \mathbf{1} \in L_b$. It follows that L_b is a maximal left ideal.

By assumption $L_b a \subset L_b$. By theorem 1.2 it follows, since A is Schurian, that there is an element $\lambda_b \in \mathbb{C}$ s.t. $\lambda_b \mathbf{1} - a \in L_b$, or, equivalently, $(\lambda_b \mathbf{1} - a)b \in L$. Thus $\lambda_b b - ab \in L$ for all $b \in A$. We will soon prove that we can actually choose $\lambda \in \mathbb{C}$ satisfying $\lambda b - ab \in L$ for all $b \in A$. In particular, $\lambda \mathbf{1} - a \in L$ and consequently $\lambda b - ba \in L$ for all $b \in A$. We can then conclude that $\mathrm{Comm}(a, b) \in L$. Since L was an arbitrary maximal left ideal, $\mathrm{Comm}(a, b) \in \mathrm{Rad}(A)$ for all $b \in A$.

It remains to be shown that we can choose $\lambda \in \mathbb{C}$ independent of $b \in A$. The choice of λ_b is unique if $b \notin L$; if $b \in L$, then every complex number will serve as λ_b. Therefore we only need to consider the case when $b \notin L$. Suppose first $\lambda z - b \in L$ for some $z \in A$, $\lambda \in \mathbb{C}$. Then $\lambda az - ab \in L$, $\lambda_b b - ab \in L$, and $\lambda_z z - az \in L$. Consequently, $\lambda \lambda_z z - \lambda_b b \in L$. But we have made the assumption that $\lambda_b \lambda z - \lambda_b b \in L$. Hence $\lambda(\lambda_z - \lambda_b)z \in L$ and thus ($\lambda \neq 0$, $z \neq 0$ since $b \notin L$) $\lambda_b = \lambda_z$.

Now let $b \notin L$, $z \notin L$, and $\lambda z - b \notin L$ for all complex numbers λ. Then $(\lambda_{b+z} \mathbf{1} - a)(b + z) \in L$, $\lambda_b b - ab \in L$, and $\lambda_z z - az \in L$. Thus

$$(\lambda_b - \lambda_{b+z})b + (\lambda_z - \lambda_{b+z})z \in L.$$

By the assumptions on b, z, and $b - \lambda z$ this implies $\lambda_b = \lambda_{z+b} = \lambda_z$. Since for $b, z \notin L$ either $\lambda z - b \in L$ for some complex λ or $\lambda z - b \notin L$ for all complex λ we have indeed shown that λ can be chosen independent of b which completes the proof. □

Theorem 3.2
Let A be Schurian, $a \in A$. Then a has the property that $\mathrm{Comm}(a, b) + \mathbf{1}$ is invertible for all $b \in A$ iff $C_a : A \to A : b \mapsto \mathrm{Comm}(a, b)$ has its range included in $\mathrm{Rad}(A)$. If A is semi-simple as well, then $\mathrm{Comm}(a, b) + \mathbf{1}$ is invertible for all $b \in A$ iff a lies in the centre *of A, i.e. in the set $\{z \in A \mid \mathrm{Comm}(a, z) = 0 \, \forall a \in A\}$.*

Proof. Fix $b \in A$. Suppose first that C_a has range in $\mathrm{Rad}(A)$. Then $\mathrm{Comm}(a, b)$ lies in every maximal left and every maximal right ideal of A and $\mathrm{Comm}(a, b) + \mathbf{1}$ lies in no such ideal. Hence $\mathrm{Comm}(a, b) + \mathbf{1}$ lies in no proper left or right ideal at all. From this we get that $A(\mathrm{Comm}(a, b) + \mathbf{1}) = A = (\mathrm{Comm}(a, b) + \mathbf{1})A$. In particular, $\mathrm{Comm}(a, b) + \mathbf{1}$ has left and right inverses which therefore must be equal, i.e., $\mathrm{Comm}(a, b) + \mathbf{1}$ is invertible.

Suppose conversely that $\mathrm{Comm}(a, b) + \mathbf{1}$ is invertible for all $b \in A$. If C_a does not have range in $\mathrm{Rad}(A)$, then by theorem 3.1 there is a maximal left ideal L in A s.t. $La \not\subseteq L$. Thus there is a $z \in L$ with $za \notin L$. Since L is a maximal left ideal, there is $s \in A$ s.t. $sza - \mathbf{1} \in L$. Now $asz \in L$ since L is a left ideal, and hence $\mathrm{Comm}(a, sz) + \mathbf{1} \in L$ and can therefore not be invertible. Choosing

$b = sz$, we get a contradiction.

In particular, if A is semi-simple, then $\mathrm{Comm}(a, b) + \mathbf{1}$ is invertible for all $b \in A$ iff $C_a \equiv 0$, i.e. $\mathrm{Comm}(a, b) = 0$ for all $b \in A$. It follows that a lies in the centre of A. □

The following corollary is immediate:

Corollary 3.3 *If A is semi-simple, Schurian and such that* $\mathrm{Comm}(a, b) + \mathbf{1}$ *is invertible for all $a, b \in A$, then A is commutative.*

Corollary 3.4 *If A is semi-simple and Schurian, then each commutator is nilpotent iff A is commutative.*

Proof. An easy consequence of the theorem on the Neumann series is the following: if n^k=0 for some $k \in \mathbb{N}$, then $\mathbf{1} - n + n^2 - \ldots (-1)^{k-1} n^{k-1}$ is inverse to $\mathbf{1} + n$. In particular, $\mathbf{1} + n$ is invertible for all nilpotent n. By corollary 3.3 we can deduce from the nilpotency of all commutators that A is commutative. Conversely, if A is commutative, then all commutators vanish. □

Remarks The semi-simplicity is an essential assumption in this last corollary. Consider for example the non-commutative algebra A of all upper triangular $n \times n$-matrices. We have shown in corollary 2.2 that this algebra is Schurian. It is easy to verify that it has the property that all commutators are nilpotent. Its radical consists of all upper triangular matrices with zero diagonal so that A is not semi-simple.

Applied to Banach algebras, corollary 3.3 gives the following characterisation of commutativity:

Corollary 3.5 *If A is a semi-simple complex Banach algebra with identity, then all commutators in A are quasi-nilpotent iff A is commutative.*

Proof. If $n \in A$ is quasi-nilpotent, then its Neumann series converges and thus $n + \mathbf{1}$ is invertible. Hence if al commutators are nilpotent corollary 3.3 tells us that A is commutative. The opposite direction follows as before. □

4 Spectral Additivity

A.A. Jafarian and A.R. Sourour show in their paper [18] that

Lemma 4.1 (Jafarian and Sourour) *If $L \in \mathfrak{L}(X)$, X a Banach space, then* $\mathrm{Sp}(T + L) \subset \mathrm{Sp}\, T$ *for every $T \in \mathfrak{L}(X)$ iff $L = 0$.*

L.A. Harris and R.V. Kadison call an element a of a complex Banach algebra A with identity *spectrally additive* iff a has this property. Then they show with the help of maximal commutative subalgebras that the centre of A is included in the set of all spectrally additive elements:

Lemma 4.2 *If $a, b \in A$ commute, then* $\mathrm{Sp}(a + b) \subset \mathrm{Sp}\, a + \mathrm{Sp}\, b$.

This has an immediate corollary:

Theorem 4.3
The centre of A is included in the set of spectrally additive elements in A.

M.S. Moslehian shows on his website [21] that there exist Banach algebras and elements therein that do not satisfy $\mathrm{Sp}(a + b) \subset \mathrm{Sp}(a) + \mathrm{Sp}(b)$. Hence it seems an interesting question to ask which elements of a Banach algebra are spectrally additive. His example is the following:

Let A be the matrix algebra $M_2(\mathbb{C})$, a a forward and b a backward shift. Then $\mathrm{Sp}\, a = \{0\}$, $\mathrm{Sp}\, b = \{0\}$ yet $\mathrm{Sp}(a + b) = \{-1, 1\}$.

We will now begin to characterise those elements of a Banach algebra A with identity that are spectraly additive:

Lemma 4.4 *Let A be a Banach algebra with identity. Then for each $a \in A$ we have $\mathrm{Sp}_A(a) = \mathrm{Sp}_{A/\mathrm{Rad}(A)}(q(a))$ where $q : A \to A/\mathrm{Rad}(A)$ is the quotient mapping.*

Proof. It suffices to prove that b is invertible in A iff $q(b)$ is invertible in $A/\mathrm{Rad}(A)$. If a is the inverse of b in A, then $q(a)$ is the inverse of $q(b)$ since q is multiplicative and takes the identity in A to the identity in $A/\mathrm{Rad}(A)$. Suppose conversely that $q(a)$ is the inverse of $q(b)$ in $A/\mathrm{Rad}(A)$. This implies $q(ba - \mathbf{1}) = 0 = q(ab - \mathbf{1})$ and thus $ba - \mathbf{1}$ and $ab - \mathbf{1}$ lie in the radical of A. By theorem 3.1.3 in [5] it follows that $ab(= ab - \mathbf{1} + \mathbf{1})$ and ba are invertible. We can conclude that b has both a left and a right inverse and that those must then be equal. $\square$

Corollary 4.5 *Let A and q be as in the lemma. $a \in A$ is spectrally additive in A iff $q(a)$ is spectrally additive in $A/\operatorname{Rad}(A)$.*

Proof. Let a be spectrally additive, $q(b) \in A/\operatorname{Rad}(A)$. Then

$$\operatorname{Sp}(q(b)+q(a)) = \operatorname{Sp}(q(b+a)) = \operatorname{Sp}(b+a) \subset \operatorname{Sp}(b)+\operatorname{Sp}(a) = \operatorname{Sp}(q(b))+\operatorname{Sp}(q(a)).$$

The other direction is similar to prove. □

Theorem 4.6
$a \in A$ is spectrally additive in A iff $\operatorname{Comm}(a,b) \in \operatorname{Rad}(A)$ for all $b \in A$. If A is semi-simple, then a is spectrally additive iff it lies in the centre of A.

Proof. Let $\operatorname{Comm}(a,b) \in \operatorname{Rad}(A)$ for all $b \in A$ and let q again be the quotient mapping. Then $q(a)$ lies in the centre of $A/\operatorname{Rad}(A)$. From 4.3 we know that $q(a)$ must hence be spectrally additive. Corollary 4.5 tells us that a is spectrally additive in A.

Let conversely a be spectrally additive. Then $\rho(a+b) \leq \rho(a) + \rho(b)$ for all $b \in A$. Fix $b \in A$ and define $f : \mathbb{C} \to A : \lambda \mapsto \exp(-\lambda b)a\exp(\lambda b)$, then f is entire. Moreover, $f(\lambda)$ is spectrally additive since it is the image of an automorphism of A and since we know that $\operatorname{Sp}(cd) \cup \{0\} = \operatorname{Sp}(dc) \cup \{0\}$ for all $c, d \in A$. Whence $\rho(f(\lambda) + c) \leq \rho(f(\lambda)) + \rho(c) = \rho(a) + \rho(c)$ for all complex λ and all $c \in A$. Let $g : \mathbb{C} \setminus \{0\} \to A : \lambda \mapsto \frac{f(\lambda)-f(0)}{\lambda}$. g is analytic on its domain of definition and $\lim_{\lambda\to 0} g(\lambda) = ab - ba$. Hence we can extend g to the whole complex plane by setting $g(0) := \operatorname{Comm}(a,b)$ and noting that g must be entire (as in Morera's theorem). Vesentini's theorem tells us that $\rho \circ g$ is subharmonic on $\mathbb{C}$. Note that for all $\lambda \neq 0$ we have

$$\rho(g(\lambda)) = \rho(\frac{f(\lambda) - f(0)}{\lambda}) \leq \frac{2}{|\lambda|}\rho(a),$$

and $\rho(g(0))$ is finite. By Liouville's theorem for subharmonic functions 4.7 we conclude that $\rho \circ g$ must be constant. ($\frac{M(0,r,\rho\circ g)}{\log r} \leq \frac{2\rho(a)}{r\log r}$ tends to zero as r tends to infinity.) We can also conclude that this constant must be zero since $\rho \circ g$ decays as $\frac{1}{r}$. Hence $\rho(\operatorname{Comm}(a,b)) = 0$, i.e. $\operatorname{Comm}(a,b)$ is quasi-nilpotent. It follows as before (Neumann series) that $\operatorname{Comm}(a,b)+\mathbf{1}$ is invertible for all $b \in A$. Since A is Schurian by theorem 2.3 we can apply theorem 3.2 and conclude that $\operatorname{Comm}(a,b) \in \operatorname{Rad}(A)$ for all $b \in A$.

In particular, if A is semi-simple, we have shown that $\operatorname{Comm}(a,b) = 0$ for all $b \in A$ if a is spectrally additive, i.e. a lies in the centre of A. The opposite direction follows from theorem 4.3. □

Remarks

- L.A. Harris and R.V. Kadison also give a second proof of this theorem that uses the Hahn-Banach theorem instead of subharmonic functions.

- If A is a complex Banach algebra with identity, then all elements of the form $c + r$ – where c lies in the centre of A and r in its Jacobson radical – are spectrally additive (remember that $\mathrm{Rad}(A)$ is a two-sided ideal).

- Not all spectrally additive elements need to be such a sum. For an example of this, we can take A to be the algebra of all upper triangular complex $n \times n$-matrices. In this case, the centre just consists of all multiples of the identity, and the radical, as seen before, of all strictly upper triangular matrices, i.e. those elements of A having zero diagonal. The elements considered above are those whose diagonal entries are all equal. But a diagonal matrix m with different entries still satisfies the property that $\mathrm{Comm}(m, a) \in \mathrm{Rad}(A)$ for all $a \in A$. (easy to check)

- The example $A = \mathfrak{LC}(H) \oplus \mathbb{C}\,\mathrm{id}$ is considered in chapter IV.

References

[1] L.V. Ahlfors, Complex Analysis, McGraw-Hill Book Company, 1966

[2] G.R. Allan, Introduction to Banach Spaces and Banach Algebras, Lecture Course at University of Cambridge, United Kingdom, 2003

[3] G.R. Allan, Banach Algebras, Lecture Course at University of Cambridge, United Kingdom, 2002

[4] H.W. Alt, Lineare Funktionalanalysis, Springer, 1990

[5] B. Aupetit, A Primer on Spectral Theory, Springer, 1990

[6] F.F. Bonsall & J. Duncan, Complete Normed Algebras, Ergebnisse der Mathematik und ihrer Grenzgebiete Band 80, Springer, 1973

[7] P.J. Cameron, Introduction to Algebra, Oxford Science Publications, Oxford University Press, 1998

[8] J.B. Conway, Functions of One Complex Variable II, Springer, 1995

[9] J.B. Conway & B.B. Morrel, Operators that are points of Spectral Continuity, Integral Equations and Operator Theory, Vol.2/2, Birkäuser Verlag (Switzerland), 1979

[10] H.G. Dales, Banach Algebras and Automatic Continuity, LMS Monographs 24, Clarendon Press, Oxford, 2000

[11] R.G. Douglas, Banach Algebra Techniques in Operator Theory, Springer, 1997

[12] W. Fischer & I. Lieb, Funktionentheorie, vieweg studium, 1988

[13] I.C. Gohberg & M.G. Kreĭn, Introduction à la théorie des opérateurs linéaires non auto-adjoints dans un espace hilbertien, Paris, 1971

[14] L.A. Harris & R.V. Kadison, Schurian Algebras and Spectral Additivity, J. Algebra 180, 175-186, 1996

[15] W.K. Hayman & P.B. Kennedy, Subharmonic Functions, Vol. 1, LMS Monographs 9, Academic Press, London, 1976

[16] H. Heuser, Funktionalanalysis, B.G. Teubner Stuttgart, 1991

[17] L. Hörmander, An Introduction to Complex Analysis in Several Variables, D.Van Nostrand Company, inc., 1966

[18] A.A. Jafarian & A.R. Sourour, Spectrum-Preserving Linear Maps, Journal of Functional Analysis 66, 255-261, 1986

[19] V.V. Kisil, Wavelets in Banach Algebras: Functional Calculus & Spectrum, http://amsta.leeds.ac.uk/∼kisilv/valencia.pdf

[20] M. Mathieu & G.J. Schick, First Results on Spectrally Bounded Operators, Studia Mathematica 152 (2), 2002

[21] M.S. Moslehian, www1.um.ac.ir/∼moslehian/cfa/BA.HTM

[22] M.A. Naimark, Normed Algebras, Wolters-Noordhoff Publishing Groningen, The Netherlands, 1972

[23] M.J.D. Powell, Approximation Theory and Methods, Cambridge University Press, 1981

[24] C.E. Rickart, Banach Algebras, D.Van Nostrand Company, inc., 1960

[25] M. Růžička, Funktionalanalysis II, Lecture Course at Albert-Ludwigs-Univeristät Freiburg, Germany, 2002

[26] W. Rudin, Reelle und Komplexe Analysis, Oldenbourg, 1999

[27] W. Rudin, Functional Analysis, The McGraw-Hill Companies, inc., 2000

[28] W. Rudin, Fourier Analysis on Groups, Interscience Publishers, 1962

[29] A.M. Sinclair, Automatic Continuity of Linear Operators, LMS Lecture Note Series 21, Cambridge University Press, 1976

[30] B.S. Yadaf, Algebraization of Topology, NAW 5/2 nr.2, 2001

[31] E. Zeidler, Applied Functional Analysis, Applications to Mathematical Physics, Applied Mathematical Sciences 108, Springer, 1995

[32] E. Zeidler, Applied Functional Analysis, Main Principles and Their Applications, Applied Mathematical Sciences 109, Springer, 1995

List of Symbols

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