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On photon spheres and 2+1 dimensional General Relativity

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Contents

Abstract

In a recent paper C. Cederbaum and G. Galloway established a uniqueness result of photon spheres in four dimensional static vacuum asymptotically flat spacetimes by adapting Bunting and Masood-ul Alam's proof of static black hole uniqueness. In this work, we[1] present all concepts necessary to understand this proof as well as give a of the proof itself. Before that, we will introduce *photon surfaces* and *photon spheres* together with various properties and characteristics. Moreover, we propose two different approaches to derive the photon sphere in the Schwarzschild solution.

Furthermore, we try to duplicate the techniques used by C. Cederbaum and G. Galloway for a three dimensional spacetime. In order to do so, we discuss major differences between three and four dimensional Relativity in general as well as in-depth analysis of the specific $(2+1)$-dimensional *Pseudo-Schwarzschild solution*, a three dimensional spacetime introduced e.g. in [Foertsch et al., 2003] in the context of photon surfaces. We point out why the notion of asymptotical flatness as well as mass turn out to be completely different in $(2+1)$-dimensional General Relativity and finish with a comparison of the Schwarzschild and the Pseudo-Schwarzschild solution.

[1]The author would like to thank his supervisor JProf. Dr. Carla Cederbaum for all her support and patience. Additional thanks are directed to Axel Fehrenbach and the participants as well as the lecturers and organizers of the summer school 'Between Geometry and Relativity', ESI Vienna, 2017, for interesting discussions and sharing their knowledge.
This work is typeset in LaTeX with extra packages by Dr. Christopher Nerz. All graphics are made with `tikz`.

1 Mathematical setup

As this chapter's headline indicates, we introduce the mathematical concepts needed in the following chapters. The reasons why the basics are presented, are on the one hand to make it possible for readers with less knowledge of General Relativity to comprehend this work and on the other hand to clarify the notations and conventions used later. This is important as they differ quite a lot in differential geometry based publications. Throughout this chapter several results are taken directly from Prof. Dr. S. Teufel's lecture 'Classical Mechanics', winter semester 2015/2016, University of Tübingen, and JProf. Dr. C. Cederbaum's lecture 'Mathematical Relativity', winter semester 2016/2017, University of Tübingen.

1.1 Manifolds and tensors

General Relativity is written in the language of semi-Riemannian geometry. The main concept in this theory is a space (*manifold*) equipped with a map (*tensor*) which tells us in a specific way how space is curved. To define those structures, we first take a look at a basic space.

Definition 1.1.1 (Topological space)
A *topological space* is a pair $(M, \mathcal{O})$ consisting of a set M and a set $\mathcal{O}$ of subsets of M (called *open sets*) satisfying the following axioms:
(i) $\emptyset$ and $M \in \mathcal{O}$,
(ii) Any (finite or infinite) union of members of $\mathcal{O}$ still belongs to $\mathcal{O}$,
(iii) The intersection of any finite number of members of $\mathcal{O}$ still belongs to $\mathcal{O}$.
A topological space $(M, \mathcal{O})$ is called *Hausdorff*, if for all $x, y \in M$, $x \neq y$, there are open sets $U, V \in \mathcal{O}$, where $x \in U$, $y \in V$ and $U \cap V = \emptyset$.

Definition 1.1.2 (Charts)
Let M be a topological space. A *chart on M* is a tuple (V, φ) consisting of an open set $V \subset M$

and a homeomorphism $\varphi : V \to \varphi(V) \subset \mathbb{R}^n$ on an open subset $\varphi(V)$.
Two charts (V_1, φ_1) and (V_2, φ) are called *compatible*, if $V_1 \cap V_2 = \emptyset$ or the sensible restricted function $\varphi_1 \circ \varphi_2^{-1}$ is a diffeomorphism between open domains of $\mathbb{R}^n$.
The functions $\varphi_1 \circ \varphi_2^{-1}$ and $\varphi_2 \circ \varphi_1^{-1}$ are called *overlap* or *transition charts*.

Definition 1.1.3 (Atlas)
An *atlas* $\mathcal{A}$ is an indexed family of pairwise compatible charts $\mathcal{A} = \bigcup_{j \in J} (V_j, \varphi_j)$, into $\mathbb{R}^n$ with the property $M = \bigcup_{j \in J} V_j$.
Two atlases $\mathcal{A}_1, \mathcal{A}_2$ are called *compatible* or *equivalent* if and only if for all charts $(V_1, \varphi_1) \in \mathcal{A}_1$ and $(V_2, \varphi_2) \in \mathcal{A}_2$, $\varphi_1 \circ \varphi_2^{-1}$ is a diffeomorphism.

Corollary 1.1.4 The equivalence of two atlases is an equivalence relation.

With the notion of a Hausdorff space and an atlas, we are able to introduce the main structure we are going to use in this work.

Definition 1.1.5 (Differentiable manifold)
A topological Hausdorff space M together with an equivalence class of atlases of M is called *differentiable manifold*. Its dimension is defined as the dimension of the image space of the charts.
The union of all compatible atlases on M is called *differentiable structure*.

Remark 1.1.6 A differentiable manifold $(M, \mathcal{A})$ is called C^k-manifold with a C^k-atlas, if all transition charts are of class C^k. A manifold is called *smooth* if it has a C^∞-atlas.

As we will see later, for a lot of concepts we do not need to consider the whole manifold all the time. We are interested in different parts with specific properties.

Definition 1.1.7 (Submanifold, hypersurface)
A subset N of an n-dimensional differentiable manifold $(M, \mathcal{A})$ is called *k-dimensional submanifold*, if for all $p \in M$ there is a chart $(V, \varphi) \in \mathcal{A}$, $p \in V$, such that

$$\varphi(N \cap V) = \left(\mathbb{R}^k \times \{0\}\right) \cap \varphi(V).$$

If $k = \dim(N) = n - 1$ we call N an embedded *hypersurface*.

In order to understand *spacetime*, we need to generalize the term of a metric. Up to now, there is no metrical structure on a differentiable manifold. Distances or angles between points are not defined. To do so, we will look at so called *tangent spaces* and *tensor fields*.

Definition 1.1.8 (Tangent space, tangent vector)
Let $(M, \mathcal{A})$ be a differentiable manifold. A *tangent vector* v on M at a point $p \in M$ is an equivalence class $[c]_p$ of curves $c : I \to M$, $I = (-\varepsilon, \varepsilon) \subset \mathbb{R}$, $c(0) = p$, $c \in C_p$ where c_1 and c_2 are called equivalent if for every chart $(V, \varphi) \in \mathcal{A}$ with $p \in V$:

$$\frac{\mathrm{d}}{\mathrm{d}t}(\varphi \circ c_1)(t)\bigg|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t}(\varphi \circ c_2)(t)\bigg|_{t=0}.$$

The set T_pM of tangent vectors on M in p is called *tangent sapce*.

Corollary 1.1.9 Let $(M, \mathcal{A})$ be a differentiable manifold. For every chart $(V, \varphi) \in \mathcal{A}$ and $p \in V$ the function

$$T\varphi : T_pM \to \mathbb{R}^n, \qquad [c]_p \mapsto \frac{\mathrm{d}}{\mathrm{d}t}(\varphi \circ c)(t)\bigg|_{t=0}$$

is a bijection. The of $T\varphi$ on T_pM induced vector space structure is independent of φ which makes T_pM a real vector space in a natural way. The dimension of T_pM equals the dimension of M.

Definition 1.1.10 (Tangent bundle)
The *tangent bundle* TM of a differentiable manifold M is the disjoint union of the tangent spaces

$$TM := \bigcup_{p \in M} (\{p\} \times T_pM).$$

Definition 1.1.11 (Vector field)
A *smooth vector field* X on a differentiable manifold M is a linear map $X : C^\infty(M) \to C^\infty(M)$ such that

$$X(fg) = fX(g) + X(f)g \quad \text{for all } f, g \in C^\infty(M).$$

In other words, a vector field is a *section* of the tangent bundle TM. The collection of all smooth vector fields on a smooth manifold M is denoted by $\Gamma(TM)$.

Definition 1.1.12 (Tensor)
Let M be a differentiable manifold and $p \in M$. A multilinear map

$$t : \underbrace{T_p^*M \times \cdots \times T_p^*M}_{\text{r copies}} \times \underbrace{T_pM \times \cdots \times T_pM}_{\text{s copies}} \to \mathbb{R}$$

is called *tensor of type (r,s)*. The dual space $(T_pM)^* = T_p^*M$ of T_pM is called *cotangent space*. We define $T_s^r(T_pM)$ as the space of all *(r,s) tensors* at $p \in M$.

Definition 1.1.13 (Metric tensor)
Let M be a differentiable manifold and $p \in M$. A tensor $g \in T_2^0(T_pM)$ is called
(i) *symmetric* if $g(u,v) = g(v,u)$ for all $u, v \in T_1^0(T_pM) = T_pM$,
(ii) *positive definite* if $g(v,v) > 0$ for all $v \in T_pM, v \neq 0$,
(iii) *non-degenerate* if $g(u,v) = 0$ for all $u \in T_pM$ implies $v = 0$.

Definition 1.1.14 (Signature)
Let M be a differentiable manifold and $p \in M$. The *signature* of a symmetric tensor $g \in T_2^0(T_pM)$ is a triple $(\mu, \nu, r) \in (\mathbb{N} \cup \{0\})^3$ containing the number of positive, negative and zero eigenvalues of the associated matrix representation.

Remark 1.1.15 For any non-degenerated tensor g the signature simplifies to (μ, ν), since $r = 0$.

Definition 1.1.16 (Tensor bundle, tensor field)
The *bundle of (r,s) tensors* on a differentiable manifold M is

$$T_s^rM := \bigcup_{p \in M} (\{p\} \times T_s^r(T_pM)).$$

A *tensor field* $\mathcal{T}_s^rM$ on M is a smooth section of some tensor bundle T_s^rM.

1.2 Spacetime, connection and the Christoffel symbols

With all the basics of the last section, we are able to define the main structure we are working with, a *spacetime*. Essentially, the idea is to define a system where time and space dimensions are distinguishable. This is made possible by using the signature of a metric tensor.

Definition 1.2.1 (Spacetime)
Let M be a smooth manifold and $g \in \mathcal{T}_2^0 M$ a smooth, symmetric, non-degenerate tensor field. We call

(i) g a *semi-Riemannian metric*,

(ii) (M, g) a *semi-Riemannian manifold*.

Let (μ, ν) be the signature of g. We call

(iii) g a *Riemannian metric* if $\nu = 0$,

(iv) g a *Lorentzian metric* if $\nu = 1$ and $\dim(M) \geq 2$.

The tuple (M, g), containing a smooth manifold M together with a Lorentzian metric g, is called *Lorentzian manifold* or *spacetime*.

Remark 1.2.2 (i) Since g in Definition 1.2.1 is smooth and non-degenerate, the values of μ and ν stay the same for all $p \in M$.

(ii) Another quite common sign-convention in literature is interchanging μ and ν.

(iii) Whenever possible, we will generalize any upcoming definition on semi-Riemannian manifolds. Most of the time, however, we only make use of the Lorentzian manifold and Riemannian manifold case.

In order to stop any notational confusion, we will name a general Lorentzian manifold $(\mathfrak{L}, \mathfrak{g})$ in all upcoming chapters. Whenever it seems necessary, we will indicate the dimension of a manifold or a metric by upper indices, e.g. a four dimensional spacetime $(\mathfrak{L}^4, {}^4\mathfrak{g})$.
One of the most famous examples of a spacetime is the *Minkowski space*.

Example 1.2.3 The pair $(\mathbb{R}^{n,1}, \eta)$, where $\mathbb{R}^{n,1} := \mathbb{R}^{n+1}$ and $\eta := -(\,\mathrm{d}x^0)^2 + \sum_{i=1}^{n} (\,\mathrm{d}x^i)^2$ is a spacetime called $(n+1)$*-dimensional Minkowski space*.

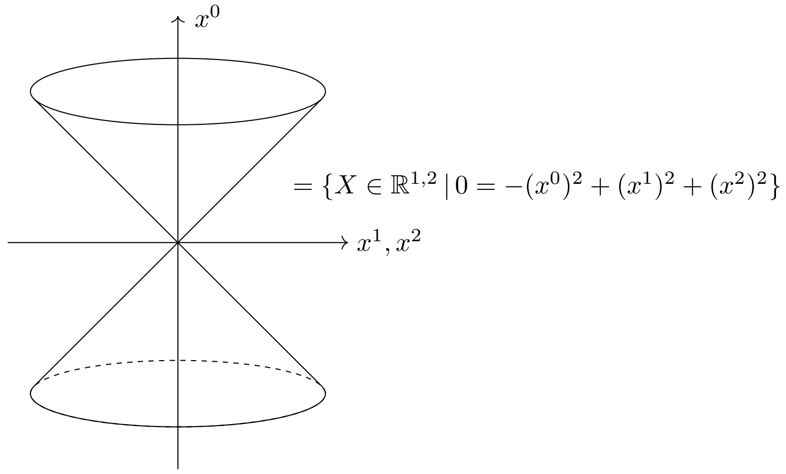


Figure 1.1: The light cone in the $(1+2)$-dimensional Minkowski space.

At this point, we classify different objects in a spacetime based on their behaviour with the metric.

Definition 1.2.4 (Spacelike, timelike, lightlike, light cone)
Let (M, g) be a semi-Riemannian manifold, $p \in M$, $X \in T_pM$, $Y \in \Gamma(TM)$ and $\gamma : I \to M$ a regularly parametrized (i.e. $\dot{\gamma}(s) := \frac{\mathrm{d}}{\mathrm{d}s}\gamma(s) \neq 0$) curve on M, $I \subseteq \mathbb{R}$ interval. Then we use the following expressions:

	X	Y	γ
spacelike	$g_p(X, X) > 0$	$g(Y, Y) > 0$	$g_\gamma(\dot{\gamma}, \dot{\gamma})\big\|_s > 0$ $\forall s \in I$
timelike	$g_p(X, X) < 0$	$g(Y, Y) < 0$	$g_\gamma(\dot{\gamma}, \dot{\gamma})\big\|_s < 0$ $\forall s \in I$
lightlike, null	$g_p(X, X) = 0$	$g(Y, Y) = 0$	$g_\gamma(\dot{\gamma}, \dot{\gamma})\big\|_s = 0$ $\forall s \in I$
causal	light- or timelike	light- or timelike	light- or timelike

The set $C := \{X \in T_pM \mid g_p(X,X) = 0\}$ is called *light cone*.

The next thing we want to introduce is a map called *connection*. As we are interested in different curvatures of a manifold, one may first think of curvature of curves in that manifold. In Euclidean space, we can make sense of curvature as a second derivative. However, in the general case, the difference quotient to obtain a second derivative contains a sum of vectors in different tangent spaces. The connection, as the name indicates, makes sense of such a sum by 'connecting' different spaces. We will see how this map helps to define *geodesics*, that is, generalized 'straight lines'.

Definition 1.2.5 (Connection)
Let (M, g) be a semi-Riemannian manifold. A smooth map

$$\nabla : \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM), \quad (X, Y) \mapsto \nabla_X Y,$$

is called *connection*, if

(i) $\nabla_{\cdot} Y$ is $C^\infty(M)$ linear for all $Y \in \Gamma(TM)$,
(ii) $\nabla_X \cdot$ is $\mathbb{R}$ linear for all $X \in \Gamma(TM)$,
(iii) ∇ satisfies the product rule $\nabla_X (fY) = X(f)Y + f\nabla_X Y$ for all $X, Y \in \Gamma(TM),\ f \in C^\infty(M)$.

Definition 1.2.6 (Torsion-free, metric connection)
Let (M, g) be a semi-Riemannian manifold and ∇ a connection on (M, g). Then ∇ is called

(i) *torsion-free*, if $\nabla_X Y - \nabla_Y X = [X, Y] := X(Y) - Y(X)$ for all $X, Y \in \Gamma(TM)$,
(ii) *metric*, if $(\nabla_X g)(Y, Z) = 0$ for all $X, Y, Z \in \Gamma(TM)$.

As analysis involving a connection on a manifold would change by changing the connection itself, the above defined specifications allow the Fundamental Theorem of Riemannian Geometry.

Theorem 1.2.7 (Fundamental Theorem of Riemannian Geometry)
Let (M, g) be a Riemannian or semi-Riemannian manifold. Then there is a unique torsion-free, metric connection ∇ on M, called the *Levi-Civitá connection*. It is characterized by the

Koszul formula

$$\begin{aligned}2g(\nabla_V W, X) =& V(g(W,X)) + W(g(X,V)) - X(g(V,W)) \\ & - g(V,[X,W]) + g(W,[X,V]) + g(X,[V,W]), \quad \text{for all } V,W,X \in \Gamma(TM).\end{aligned}$$

Proof. The proof uses the Koszul formula for both, uniqueness and existence, see e.g. [O'Neill, 1983]. □

To see how the Levi-Civita connection appears in components we look at the so called *Christoffel symbols*, a measurement of curvature in different directions. As we will see later, most of the objects we are interested in can be expressed using Christoffel symbols.

Definition 1.2.8 (Christoffel symbols)
Let $x^0, ..., x^n$ be a local coordinate system on some neighborhood $U \subset M$ of a semi-Riemannian manifold (M, g). The *Christoffel symbols* are given by real-valued functions ${\Gamma_{ij}}^k : U \to \mathbb{R},\ 0 \leq i,j,k \leq n$, such that

$$\nabla_{\partial_i}\partial_j = \sum_{k=0}^{n} {\Gamma_{ij}}^k \partial_k = {\Gamma_{ij}}^k \partial_k, \quad \partial_i := \frac{\partial}{\partial x^i}.$$

The last step uses the Einstein summation convention. In a coordinate chart, the Christoffel symbols are given by

$$(1.1) \qquad {\Gamma_{ij}}^k = \frac{1}{2} g^{ks}(\partial_j g_{si} + \partial_i g_{sj} - \partial_s g_{ij}).$$

Remark 1.2.9 (i) Since the Levi-Civita connection ∇ is no tensor (not linear over C^∞ in the second argument), the Christoffel symbols don't behave like tensors when changing the coordinates.

(ii) The Christoffel symbols are symmetric in the lower indices, since the Levi-Civita connection is torsion-free and $[\partial_i, \partial_j] = 0$.

Future calculations will make use of the Einstein summation convention without explicitly mentioning it.
The last thing in this section is a generalization of the *Laplace operator*. We will need it as it appears in a transformation formula for the scalar curvature later on (Section 1.7).

Definition 1.2.10 (Laplace operator)
The *Laplace operator* with respect to g acting on a smooth function f is given by the divergence of the gradient:

$$
\begin{aligned}
{}^{g}\Delta f := \operatorname{div}(\nabla f) := \nabla_{\partial_i}\nabla^i f &= \frac{1}{\sqrt{|\det(g)|}}\,\partial_j\left(g^{jk}\sqrt{|\det(g)|}\partial_k f\right) \\
&= g^{jk}\,\partial_j\partial_k f + \partial_j g^{jk}\,\partial_k f + \frac{1}{2}g^{jk}g^{il}\,\partial_j g_{il}\,\partial_k f \\
&= g^{jk}\,\partial_j\partial_k f - g^{jk}\,\Gamma_{jk}{}^{l}\,\partial_l f.
\end{aligned}
$$

Here, ∇f is the *gradient* of f, that is, the unique vector field on M such that $g(\nabla f, X) = \mathrm{d}f(X),\ X \in \Gamma(TM)$.

1.3 Geodesics

Geodesics, the Riemannian generalizations of straight lines, are modelling the trajectory of freely falling particles and photons in a given spacetime in General Relativity. They are defined as curves not exposed to any 'acceleration' than curvature itself. Technically, this is easier to work with than assuming a length minimizing curve between two nearby points. *Lightlike geodesics* (sometimes called *null geodesics*) are important to define *photon surfaces*, a concept analysed more deeply in Chapter 2.
Throughout this section, (M, g) denotes a semi-Riemannian manifold and $\gamma : I \to M$, $I \subseteq \mathbb{R}$ interval, denotes a regularly parametrized curve.

Definition 1.3.1 (Vector field along a curve)
A smooth map $X : I \to TM$ such that $X(s) \in T_{\gamma(s)}M$ for every $s \in I$ is called a *vector field along* γ. We let $\Gamma(\gamma)$ denote the space of all vector fields along γ.

Example 1.3.2 (i) $\dot\gamma : I \to TM$ is a vecor field along γ, called the *velocity field of* γ. (Here the overdot denotes differentiation with respect to an affine parameter along the curve).

(ii) For all $X \in \Gamma(TM)$: $X\big|_{\gamma} : I \to TM$ is a vecor field along γ.

Definition 1.3.3 (Geodesic)
A regular parametrized curve γ is a *geodesic* if its *acceleration field* $\ddot{\gamma} := \nabla_{\dot{\gamma}}\dot{\gamma}$ vanishes, i.e. $\nabla_{\dot{\gamma}}\dot{\gamma} \equiv 0$.

Example 1.3.4 (i) The images of geodesics in the Minkowski space $(\mathbb{R}^{n,1}, \eta)$ are straight lines.
(ii) On a sphere $(\mathbb{S}^n, \mathrm{d}\Omega^2)$, the images of geodesics are the great circles, i.e. the intersection of the sphere and a plane that passes through the center point of the sphere.

To verify talking about *lightlike, timelike* and *spacelike geodesics* we prove the following proposition:

Proposition 1.3.5 Geodesics don't change their causal character (see 1.2.4).

Proof. Let $\gamma : I \to M$ denote a geodesic, i.e. $\nabla_{\dot{\gamma}}\dot{\gamma} \equiv 0$. The product rule gives

$$\frac{\mathrm{d}}{\mathrm{d}s} g(\dot{\gamma}(s), \dot{\gamma}(s)) = 2\, g\left(\underbrace{\nabla_{\dot{\gamma}}\dot{\gamma}(s)}_{=0}, \dot{\gamma}(s)\right) = 0.$$

This means, fixing $s_0 \in I$, $g(\dot{\gamma}(s_0), \dot{\gamma}(s_0)) = g(\dot{\gamma}(s), \dot{\gamma}(s))$ for all $s \in I$. □

Theorem 1.3.6 (Existence and uniqueness of geodesics)
For all $p \in M$, $X \in T_pM$, $s_0 \in \mathbb{R}$, there is a open interval $I \subseteq \mathbb{R}$, $s_0 \in I$, such that a unique geodesic $\gamma : I \to M$ exists with $\gamma(s_0) = p$ and $\dot{\gamma}(s_0) = X$.

Proof. The proof heavily relies on the Picard–Lindelöf Theorem, see e.g. [Lee, 2006]. □

Remark 1.3.7 Theorem 1.3.6 can be expanded such that there exists a unique maximal geodesic, i.e. a geodesic that cannot be continued on a larger interval.

Definition 1.3.8 (Geodesically complete)
A semi-Riemannian manifold (M, g) is called *geodesically complete*, if every maximal geodesic is definend on whole $\mathbb{R}$.

Example 1.3.9 (i) The Minkowski space $(\mathbb{R}^{1,n}, \eta)$ and the sphere $(\mathbb{S}^n,\ \mathrm{d}\Omega^2)$ are geodesically complete.

(ii) The Minkowski space without the origin $(\mathbb{R}^{1,n} \setminus \{0\}, \eta)$ is not geodesically complete. This spacetime has a 'simulated singularity'.

1.4 Curvature and Einstein equations

The Einstein equations in the general theory of relativity describe a connection between the distribution of matter and the curvature of a spacetime. John Archibald Wheeler wrote: "Spacetime tells matter how to move; matter tells spacetime how to curve."[1] We introduce a global notion of curvature, the *Riemann curvature tensor*. A motivation why this tensor measures curvature is found e.g. in [Lee, 2006].

Definition 1.4.1 (Riemann curvature tensor)
Let (M, g) be a semi-Riemannian manifold and $X, Y, Z \in \Gamma(TM)$. Then

$$XYZ \coloneqq \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

is called *Riemann curvature tensor*. In coordinates, we have

$$\begin{aligned} {}_{ijk}^{\ \ \ l} &:= ((\partial_i, \partial_j)\partial_k)^l \\ &= \partial_i {\Gamma_{jk}}^l - \partial_j {\Gamma_{ik}}^l + {\Gamma_{is}}^l {\Gamma_{jk}}^s - {\Gamma_{js}}^l {\Gamma_{ik}}^s. \end{aligned}$$

The two other main measurements of curvature are the trace of the Riemann curvature tensor, the *Ricci tensor*, and the *scalar curvature* as the trace of the Ricci tensor:

Definition 1.4.2 (Ricci tensor, scalar curvature)
Let (M, g) be a semi-Riemannian manifold, $X, Y \in \Gamma(TM)$ and $(E_i)_1^n$ an orthonormal frame. Then

$$\mathrm{Ric}\,(X, Y) \coloneqq ((E_i, X)Y)^i, \qquad R \coloneqq (E_i, E_j)g(E_i, E_j)$$

[1] Geons, Black Holes, and Quantum Foam: A Life in Physics (2000), p. 235

are called the *Ricci curvature tensor* and the *scalar curvature*. In coordinates, we have

$$(1.2) \qquad {}_{ij} ={}^{k}_{ikj}= \partial_k \Gamma_{ij}{}^{k} - \partial_j \Gamma_{ik}{}^{k} + \Gamma_{ij}{}^{s}\Gamma_{ks}{}^{k} - \Gamma_{ik}{}^{s}\Gamma_{js}{}^{k}, \qquad R = g^{ij}_{ij}.$$

Remark 1.4.3 (i) The Ricci tensor is a symmetric $(0, 2)$ tensor field.

(ii) One dimensional spacetimes are always flat, i.e. $\equiv\equiv R \equiv 0$.

(iii) If $\dim(M) \leq 3$, the Riemann tensor is fully determined by the Ricci tensor and the scalar curvature, i.e. in local coordinates

$$g_{ls}{}^{s}_{ijk} ={}_{lk} g_{ij} +{}_{ij} g_{lk} -{}_{lj} g_{ik} -{}_{ik} g_{lj} - \frac{1}{2}R(g_{lk}g_{ij} - g_{lj}g_{ik}).$$

Now we are able to present the *Einstein equations* (sometimes *Einstein-Field Equations* or *Einstein-Hilbert equations*), a system of non-linear partial differential equations.

Definition 1.4.4 (Einstein equations)
A spacetime $(\mathfrak{L}, \mathfrak{g})$ satisfies the *Einstein equations in vacuum*, if

$$(1.3) \qquad -\frac{1}{2}R\mathfrak{g} = 0$$

on $\mathfrak{L}$. Let be a symmetric, smooth $(0, 2)$ tensor field on M. Then is called *mathematical stress-energy-momentum tensor* and the spacetime $(\mathfrak{L}, \mathfrak{g})$ satisfies the *Einstein equations to the matter model T*, if

$$(1.4) \qquad -\frac{1}{2}R\mathfrak{g} = \frac{8\pi G}{c^4}$$

on $\mathfrak{L}$. Here G denotes the gravitational constant and c denotes the speed of light.

Example 1.4.5 (i) If T fully vanishes on M, i.e. $\equiv 0$, we have a vacuum solution.

(ii) Electromagnetic stress-energy tensor: ${}^{\mu\nu} = \frac{1}{\mu_0}(F^{\mu\alpha}g_{\alpha\beta}F^{\nu\beta} - \frac{1}{4}g^{\mu\nu}F_{\delta\gamma}F^{\delta\gamma})$, where $F_{\mu\nu}$ is the electromagnetic field tensor.

(iii) Perfect fluid in thermodynamic equilibrium: $T^{\mu\nu} = \left(\rho + \frac{p}{c^2}\right) u^\mu u^\nu + pg^{\mu\nu}$, where ρ is the mass–energy density (kilograms per cubic meter), p is the hydrostatic pressure (pascals) and u^μ is the fluid's four velocity.

The stress-energy-momentum tensor is the part where physical aspects are encoded. One can set different requirements to this tensor to, in a way, measure how physically accurate a specific spacetime might be. These are the so called *energy conditions*.

Definition 1.4.6 (Energy conditions)
The stress-energy-momentum tensor T fulfils the *weak energy condition* if for every timelike vector field $X \in \Gamma(TM)$ it holds that

$$_{ij}X^i X^j \geq 0.$$

It fulfils the *dominant energy condition* if in addition to the weak energy condition, for every future-pointing causal vector field $Y \in \Gamma(TM)$ the vector field $-{}^{i}_{j}Y^j$ is a future-pointing causal vector field.

The strong energy condition is a standard assumption to ensuring in particular non-negative scalar curvature on all time slices of a static spacetime.
The Einstein equations can be written in a nice way for the special case of a vacuum or a *static* spacetime.

Definition 1.4.7 (Static)
A n-dimensional spacetime $(\mathfrak{L}, \mathfrak{g})$ is called *static* if there is a smooth $(n-1)$-dimensional Riemannian manifold (M, g) such that

$$\mathfrak{L} = \mathbb{R} \times M, \qquad \mathfrak{g} = -N^2 \, \mathrm{d}t^2 + g,$$

where $N : M \to \mathbb{R}^+$ denotes a smooth *lapse* (i.e. time independent) function. The Riemannian manifold $M = \{t = 0\} \cong \{t = \text{const.}\}$ is called *canonical time slice*.

Proposition 1.4.8 The Einstein equations in vacuum (1.3) simplify to $= 0$. If the spacetime is additionally static (Definition 1.4.7) with lapse function N, we obtain the *static vacuum equations*

$$\begin{aligned} N &= \nabla^2 N \\ \Delta N &= 0, \end{aligned}$$

on a time slice. Here, Δ denotes the Laplacian with respect to the induced metric g on the time slice and ∇^2 the covariant Hessian. Those equations force every time slice to be scalar flat (taking trace of the first equation and substituting in the second one).

1.5 Asymptotic flatness and mass

As much as all the solutions of the Einstein equations might be mathematically interesting, for interpretation in physics we need some more assumptions. The first thing that comes to mind is modelling isolated systems, that is, one central object of mass surrounded by vacuum. It seems sensible that the space far away from the object, what might be a star or a black hole or something similar, would flatten out. By flat, we mean the metric approaches, in some sense, the Minkowskiean respectively the Euclidean (flat) metric, depending on whether we are talking about the whole spacetime or some time slice of it. In this section, we just discuss dimensions $n \geq 3$ as the two dimensional case is somewhat special and will be analysed in more depth in Chapter 3.

Definition 1.5.1 (Asymptotically flat)
A n-dimensional Riemannian manifold (M^n, g) is *asymptotically flat (with one end)* if there exists a compact set $C \subset M^n$, a constant $R > 0$, and a diffeomorphism $\Phi : M^n \backslash C \to \mathbb{R}^n \backslash \overline{B}_R$ such that, in the x-coordinate chart defined by Φ,

$$g = g_{ij}(x)\, \mathrm{d}x^i\, \mathrm{d}x^j, \tag{1.5}$$

$$g_{ij} = \delta_{ij} + \mathcal{O}_k\Big(r^{-p}\Big), \tag{1.6}$$

as $r \to \infty$. Here, $r : M^n \setminus C \to \mathbb{R}_+,\ \ x \mapsto |\Phi(x)|$, is the Euclidean distance function and we assume $p > (n-2)/2$ and $k \geq 2$. Furthermore, we require the scalar curvature R to satisfy certain additional fall-off conditions, namely

$$|R| = \mathcal{O}\Big(r^{-q}\Big), \tag{1.7}$$

for some $q > n$.

There are various inequivalent definitions of asymptotic flatness in literature, but the underlying principles are similar. The quoted definition can be found, e.g., in [Cabrera Pacheco, 2016]

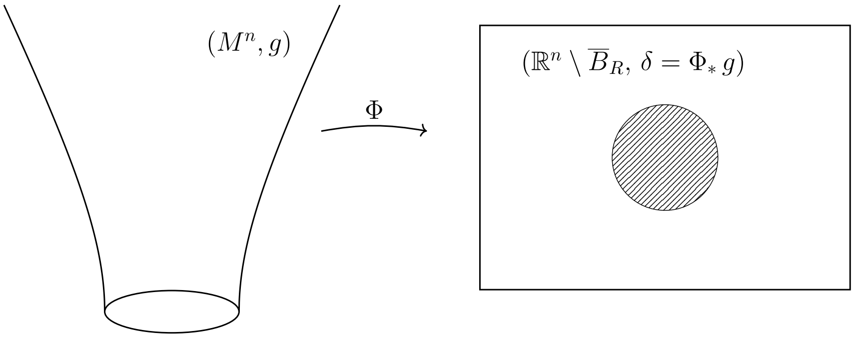


Figure 1.2: The map Φ between a Riemannian manifold (e.g. a time slice of a static spacetime) and the flat space without a ball.

or [Schoen, 1989]. Besides the modelling of isolated systems, asymptotic flatness is the standard requirement to define *ADM-mass*, a characteristic first considered in [Arnowitt et al., 1961].

Definition 1.5.2 (ADM-mass)
The *ADM-mass* of a n-dimensional, asymptotically flat Riemannian manifold (M^n, g) is defined by

$$m_{ADM} := \frac{1}{2(n-1)\omega_{n-1}} \lim_{r\to\infty} \int_{\mathbb{S}_r} \sum_{i,j} \left(\partial_i g_{ij} - \partial_j g_{ii}\right) \nu^j \mathrm{d}\xi(r), \tag{1.8}$$

where ω_{n-1} is the surface area of the Euclidean unit $(n-1)$-sphere, $\mathbb{S}_r$ is the coordinate sphere of radius r, ν its Euclidean outward unit normal and $\mathrm{d}\xi(r)$ is the Euclidean area element on $\mathbb{S}_r$.

It is not immediately obvious that the limit in (1.8) always exists. Its existence, in fact, is due to the fall-off condition of the scalar curvature (1.7). We have the following Corollary (see e.g. [Schoen, 1989]):

Corollary 1.5.3 The ADM-mass is well-defined, that is, the limit exists.

Proof. Using the asymptotic assumptions and the formulas of scalar curvature (1.2) and

Christoffel symbols (1.1), we obtain the following expression:

$$R = \sum_{i,j} (\partial_i \partial_j g_{ij} - \partial_j \partial_j g_{ii}) + E(x),$$

where the error term $E(x)$ has the asymptotic $\mathcal{O}(r^{-(2p+2)})$, $2p+2 > n$. Thus, once $r > r_0$ for some radius r_0, we have a bound of the form

$$|E(x)| \leq C\, r^{-(2p+2)}, \qquad C \in \mathbb{R}_+.$$

The Divergence Theorem implies

$$\int_{\mathbb{S}_r} \sum_{i,j} (\partial_i g_{ij} - \partial_j g_{ii})\, \nu^j \mathrm{d}\xi(r) = \int_{B_r} \sum_{i,j} \partial_j (\partial_i g_{ij} - \partial_j g_{ii})\, \mathrm{d}V(r),$$

where B_r is the coordinate ball of radius r and $\mathrm{d}V(r)$ its Euclidean volume element. The integrand on the right hand side only differs from the scalar curvature by the error $E(x)$. We know

$$\int_{\overline{B}_{r_0}} E(x) < \infty$$

by compactness of $\overline{B}_{r_0}$ and continuity of R. For $r > r_0$, we estimate

$$\begin{aligned} \left| \int_{M^n \setminus \overline{B}_{r_0}} E(x) \right| &\leq C \int_{r_0}^{\infty} \int_{\mathbb{S}_r} r^{-(2p+2)} \mathrm{d}\xi(r)\, \mathrm{d}r \\ &\leq C A_n \int_{r_0}^{\infty} r^{n-1-(2p+2)}\, \mathrm{d}r \\ &\leq C A_n \int_{r_0}^{\infty} r^{-(1+\varepsilon)}\, \mathrm{d}r \\ &< \infty \end{aligned}$$

for some $\varepsilon > 0$, where A_n denotes the surface area of the unit sphere. So the error term is integrable and, for the very same asymptotic reason (Equation (1.7)), the scalar curvature is.

Thus,

$$\begin{aligned}\lim_{r\to\infty}\int_{\mathbb{S}_r}\sum_{i,j}\left(\partial_i g_{ij}-\partial_j g_{ii}\right)\nu^j \mathrm{d}\xi(r) &= \lim_{r\to\infty}\int_{B_r}\sum_{i,j}\partial_j\left(\partial_i g_{ij}-\partial_j g_{ii}\right)\mathrm{d}V(r)\\ &= \lim_{r\to\infty}\int_{B_r}\left(R-E(x)\right)\mathrm{d}V(r)\\ &< \infty.\end{aligned}$$

□

Sometimes, we want to be more restrictive than the general notion of asymptotic flatness. To be precise, we want the lowest order term in the asymptotic assumption (1.6) to have a specific form.

Definition 1.5.4 (Asymptotic Schwarzschildean)
Assume the settings of Definition 1.5.1. We call (M^n, g) *asymptotically Schwarzschildean* by replacing Equation (1.6) with

$$g_{ij} = \left(1+\frac{m}{2r^n}\right)^{4/(n-2)}\delta_{ij} + \mathcal{O}_k\Big(r^{-(p+1)}\Big), \qquad m \in \mathbb{R}.$$

The name and the specific factor occurring in Definition 1.5.4 are going to be explained in Section 1.8.

A keystone result regarding the ADM-mass is the *Positive Mass Theorem*. It ensures the ADM-mass to be positive given certain prerequisites. For simplicity we only consider the three dimensional case, see e.g. [Schoen and Yau, 1994]:

Theorem 1.5.5 (Positive Mass Theorem)
Let (M^3, g) be a complete, asymptotically flat Riemannian manifold with ADM-mass m_{ADM}. If the scalar curvature R is non-negative, then

$$m_{ADM} \geq 0,$$

with equality if and only if $(M^3, g) \cong (\mathbb{R}^3, \delta)$, where δ denotes the standard flat metric.

Remark 1.5.6 The same result can be achieved with weaker regularity, see e.g. [Bartnik, 1986] or [Lee and LeFloch, 2015].

1.6 Submanifolds

We already learned about time slices of a static spacetime as hypersurfaces of a Lorentzian manifold. Similar to Definition 1.2.4, we classify different kinds of submanifolds with the help of the metric.

Definition 1.6.1 (Hypersurfaces)
Let (M^{n+1}, g) be a semi-Riemannian manifold and $\Sigma^n \hookrightarrow M^{n+1}$ an embedded hypersurface. We call Σ^n a

(i) *spacelike hypersurface*, if $T\Sigma^n$ consits only spacelike vectors,
(ii) *timelike hypersurface*, if the induced metric is Lorentzian,
(iii) *lightlike hypersurface*, if the induced metric is degenerate.

Example 1.6.2 (i) In a Riemannian manifold, every hypersurface is spacelike.
(ii) In the Minkowski space $(\mathbb{R}^{1,n}, \eta)$, the hyperboloid $\{X \in \mathbb{R}^{1,n} \,|\, \eta(X,X) = -1\}$ is spacelike.

Besides the *intrinsic* curvature parameters of a hypersurface itself, we can introduce new interesting objects to measure the curvature of a hypersurface lying in the semi-Riemannian manifold, that is, 'how crooked the hypersurface looks from the outside'. Those objects are often referred as *extrinsic curvature*.

Definition 1.6.3 (Second fundamentel form, mean curvature)
Let (M^{n+1}, g) be a semi-Riemannian manifold and $\Sigma^n \hookrightarrow M^{n+1}$ a spacelike or timelike hypersurface with normal vector ν. Then the *second fundamental form* is defined as

$$\begin{aligned} h : T\Sigma^n \times T\Sigma^n &\to \mathbb{R} \\ (X, Y) &\mapsto h(X,Y) := g\left({}^{n+1}\nabla_X \nu, Y\right). \end{aligned}$$

Here ${}^{n+1}\nabla$ denotes the Levi-Civita connection on (M^{n+1}, g). The n-dimensional trace of the second fundamental form h is called *mean curvature* $H := \mathrm{tr}(h)$.

For the rest of this section (M^{n+1}, g) is a semi-Riemannian manifold, $\Sigma^n \hookrightarrow M^{n+1}$ a spacelike or timelike hypersurface with normal vector ν and h its second fundamental form.

Proposition 1.6.4 (i) The second fundamental form h is a symmetric $(0,2)$-tensor field.
(ii) It holds that $h(X,Y) = -g({}^{n+1}\nabla_X Y, \nu)$, for all $X, Y \in \Gamma(T\Sigma^n)$.

Proof. (i) Straightforward calculation.
(ii) $0 = X(g(Y,\nu)) = g({}^{n+1}\nabla_X Y, \nu) + \underbrace{g({}^{n+1}\nabla_X \nu, Y)}_{=h(X,Y)}$.

□

An important correlation between the $(n+1)$-dimensional and n-dimensional Levi-Civita connection is given by the Gauß formula.

Proposition 1.6.5 (Gauß formula)
For all $X, Y \in \Gamma(T\Sigma^n)$ and $\sigma = g(\nu,\nu)$ it holds that

$$^{n+1}\nabla_X Y =^n \nabla_X Y - \sigma h(X,Y)\nu.$$

Proof. The proof involves decomposing ${}^{n+1}\nabla_X Y$ in its tangent and normal parts as well as properties of the Levi-Civita connection, see e.g. [Lee, 2006]. □

Another important concept is a *black hole horizon*. It is defined as by using extrinsic curvature assumptions.

Definition 1.6.6 (Static black hole horizon) A static spacetime $(\mathfrak{L}^{n+1} = \mathbb{R} \times M^n, \mathfrak{g})$ possesses a *static black hole horizon* if there is a hypersurface $S^{n-1} \hookrightarrow M^n$, where $S^{n-1} \cong \mathbb{S}^{n-1}$, and

(i) the mean curvature H of S vanishes,
(ii) the lapse function N of $(\mathfrak{L}^{n+1}, \mathfrak{g})$ fulfils $N|_{S^{n-1}} \equiv 0$.

Typically, black hole horizons occur as the inner boundary of a time slice. We will see two examples in later sections.

1.7 Conformal geometry

A way to classify Riemannian manifolds is to consider *conformal equivalence classes* of a Riemannian metric. This is valuable whenever working with conformal invariants (e.g. the the Weyl tensor in dimensions higher than three). We need a conformal transformation to prove a uniqueness result in Chapter 2.

Definition 1.7.1 (Conform equivalent)
Two Riemannian manifolds (M^n, g), $(\widetilde{M}^n, \tilde{g})$, $n \geq 2$, are said to be *conformally equivalent with respect to the diffeomorphism* $\Psi : M^n \to \widetilde{M}^n$, if

$$\Psi^* \, \tilde{g} = \rho \, g$$

for a smooth, positive function $\rho : M^n \to \mathbb{R}^+$ (called *conformal factor*). Here, $\Psi^* \tilde{g}$ denotes the pullback of $\tilde{g}$ along Ψ. If $\widetilde{M} = M$ and Ψ is the identity, we say that $\tilde{g}$ is a conformal deformation of g. If $\tilde{g}$ is the flat metric, we call g *conformally flat*.

Remark 1.7.2 The motivation for calling deformations of the form $\tilde{g} = \rho \, g$ conformal is their property to preserve *angles*. To clarify, let $X, Y \in T_pM^n$ for some $p \in M^n$. Then the *angle* between X and Y with respect to the metric g can be computed by (see e.g. [Lee, 2006], p. 23)

$$^{g}\!\cos(\angle(X,Y)) = \frac{g_p(X,Y)}{\|X\|_{g_p} \|Y\|_{g_p}},$$

where $\|\cdot\|_{g_p} := \sqrt{g_p(\cdot,\cdot)}$ evaluates the *length* of a vector. We observe

$$\|X\|_{\tilde{g}_p} = \sqrt{\tilde{g}_p(X,X)} = \sqrt{\rho \, g_p(X,X)} = \sqrt{\rho}\sqrt{g_p(X,X)} = \sqrt{\rho}\, \|X\|_{g_p}.$$

Therefore,

$$\begin{aligned} ^{\tilde{g}}\!\cos(\angle(X,Y)) &= \frac{\tilde{g}_p(X,Y)}{\|X\|_{\tilde{g}_p} \|Y\|_{\tilde{g}_p}} \\ &= \frac{\rho \, g_p(X,Y)}{\rho \, \|X\|_{g_p} \|Y\|_{g_p}} \\ &= \frac{g_p(X,Y)}{\|X\|_{g_p} \|Y\|_{g_p}} = {}^{g}\!\cos(\angle(X,Y)). \end{aligned}$$

The basic question that arises considering conformal changes is the behaviour of curvature. In particular, the transformation of scalar curvature is of special interest, as its values are important assumptions in certain theorems. The most general transformation is given by (see e.g. [Cabrera Pacheco, 2016])

$$\tilde{R} = \rho^{-1} R - (n-1)\rho^{-2}\, {}^{g}\Delta\rho - \frac{1}{4}(n-1)(n-6)\rho^{-3}\,|\nabla\rho|^2.$$

We are most interested in special cases for the conformal factor, that is, where terms involving the gradient cancel out.

(i) $n = 2$: Let $\rho = e^{2u}$, where $u : M^2 \to \mathbb{R}$ is smooth. Then

$$\tilde{R} = e^{-2u}(R - 2\, {}^{g}\Delta u).$$

(ii) $n > 2$: Let $\rho = u^{\frac{4}{n-2}}$, where $u : M^n \to \mathbb{R}^+$ is smooth. Then

$$\tilde{R} = u^{-\frac{n+2}{n-2}}\left(Ru - \frac{4(n-1)}{n-2}\, {}^{g}\Delta u\right).$$

1.8 The Schwarzschild solution

The Schwarzschild solution, named after Karl Schwarzschild (1873 – 1916), was the first non-trivial solution to the Einstein equations. It was derived by solving the Einstein equations assuming spherical symmetry, vacuum and staticity. Spherical symmetry and vacuum already ensure the metric to be static (this result is known as Birkhoff's theorem (see e.g. [Wald, 1984], p. 125)). In this section, we collect properties of the Schwarzschild solution which we need later. Those can be found in standard literature, e.g. [Wald, 1984].

The Schwarzschild solution is a sensible model of a spacetime with one central, static and spherically symmetric source of gravitation, for example a star or a black hole of a certain mass $m > 0$. Thus, moving away from the centre, the space becomes flat and the metric converges to the Minkowski metric.

The exterior Schwarzschild solution $(\mathfrak{L}^4 := \mathbb{R} \times (2m, \infty) \times \mathbb{S}^2, \mathfrak{g})$ of mass $m > 0$ is given by

$$\mathfrak{g} := \mathrm{d}s^2 = -N^2(r)\,\mathrm{d}t^2 + N^{-2}(r)\,\mathrm{d}r^2 + r^2\,\mathrm{d}\Omega^2, \qquad N := \left(1 - \frac{2m}{r}\right)^{1/2},$$

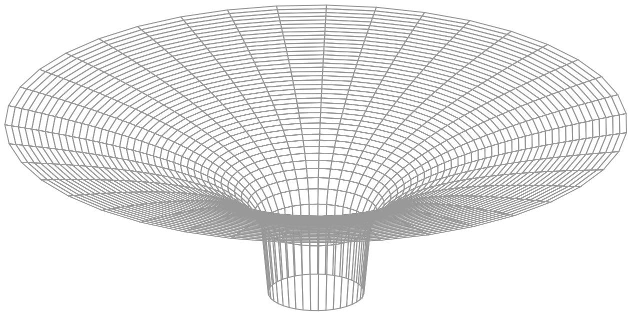


Figure 1.3: A plot of Flamm's paraboloid which models the Schwarzschild exterior for some $\{t \equiv \text{const.}\}$ and $\theta \equiv \frac{\pi}{2}$.

with $\mathrm{d}\Omega^2 := \mathrm{d}\theta^2 + \sin^2(\theta)\,\mathrm{d}\varphi^2$ denoting the canonical metric on $\mathbb{S}^2$.

As a vacuum solution, the Schwarzschild spacetime is Ricci flat and, therefore, scalar flat. As a static spacetime, all time slices are scalar flat as well. The canonical time slice $M = (2m, \infty) \times \mathbb{S}^2 = \{t = 0\}$ is conformally flat, which can be seen easily in *isotropic coordinates*

$$-N^2(r)\,\mathrm{d}t^2 + N^{-2}(r)\,\mathrm{d}r^2 + r^2\,\mathrm{d}\Omega^2 = -\left(\frac{1 - m/2s}{1 + m/2s}\right)^2 \mathrm{d}t^2 + \left(1 + \frac{m}{2s}\right)^4 \left(\mathrm{d}s^2 + s^2\,\mathrm{d}\Omega^2\right).$$

In fact, it is the only conformally flat, maximally extended solution of the static vacuum equations. In isotropic coordinates, the spatial Schwarzschild solution is asymptotically Schwarzschildean which implies it is asymptotically flat.
A straightforward computation shows $m_{ADM} = m$. If $m > 0$, the spacetime possesses a black hole horizon at $r = 2m$.

2 Photon surfaces and spheres

In this chapter we will give the definitions of photon surfaces and photon spheres. We prove some main characterizations and establish different properties. Furthermore, we will present two different ways of deriving the photon sphere in the Schwarzschild solution. This chapter finishes with a recent uniqueness result about photon spheres in static vacuum asymptotically flat spacetimes.

2.1 Basic definitions and properties

Throughout this section, $(\mathfrak{L}^{n+1}, \mathfrak{g})$ denotes an $(n+1)$-dimensional static spacetime with lapse function N.
As the name *photon surface* indicates, the formal definition should have something to do with the trajectory of photons:

Definition 2.1.1 (Photon surface, [Cederbaum, 2014]) A *photon surface* is a timelike embedded hypersurface $\Sigma^n \hookrightarrow \mathfrak{L}^{n+1}$ that is "totally null geodesic", i.e. any lightlike geodesic tangent to Σ^n remains tangent to Σ^n as long as it exists.

A *photon sphere* is a more special photon surface. The photons need to be 'trapped' in a specific way.

Definition 2.1.2 (Photon sphere, [Cederbaum, 2014]) A photon surface P^n is called *photon sphere* if the lapse function N of the spacetime is constant on each connected component of P^n.

In order to prove one main characterization of photon surfaces we start with the following proposition.

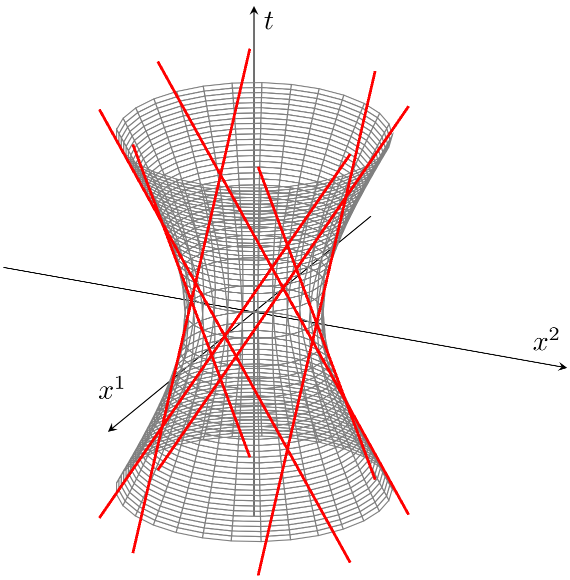


Figure 2.1: A plot of a hyperboloid $(x^1)^2 + (x^2)^2 - t^2 = \text{const.}$ with lightlike geodesics in it. In the $2+1$-dimensional Minkowski space, geodesics are just straight lines.

Proposition 2.1.3 Let Σ^n be an embedded timelike or spacelike hypersurface in $(\mathfrak{L}^{n+1}, \mathfrak{g})$. Then the second fundamental form h of Σ^n vanishes if and only if any geodesic initially tangent to Σ^n remains tangent to Σ^n as long as it exists.

Proof. „⇒“ Assume the second fundamental form vanishes, i.e. $h \equiv 0$. The Gauß formula (Proposition 1.6.5) tells us

$$^{n+1}\nabla_X Y \;=\; {}^n\nabla_X Y - \sigma \underbrace{h(X,Y)}_{=0} \nu \;=\; {}^n\nabla_X Y, \quad \text{for all } X, Y \in \Gamma(T\Sigma^n).$$

So for every geodesic in Σ^n, $\gamma : (-\varepsilon, \varepsilon) \to \Sigma^n$, $\dot\gamma(0) = v \in T_p\Sigma^n$, it holds that

$$0 \;=\; {}^n\nabla_{\dot\gamma}\dot\gamma \;=\; {}^{n+1}\nabla_{\dot\gamma}\dot\gamma.$$

This means, that every geodesic in Σ^n is also a geodesic $\mathfrak{L}^{n+1}$. Since geodesics are unique (Theorem 1.3.6), every geodesic in $\mathfrak{L}^{n+1}$ initially tangent to Σ^n remains tangent to it as long as it exists.

„⇐“ Assume every geodesic initially tangent to Σ^n remains tangent to it as long as it exists. Let $\gamma : (-\varepsilon, \varepsilon) \to \mathfrak{L}^{n+1}$, $\dot\gamma(0) = v \in T_p\Sigma^n$ be a geodesic satisfying the condition above. Thus $\dot\gamma \in \Gamma(T\Sigma^n)$. As long as $\gamma(s) \in \Sigma^n$, uniqueness (Theorem 1.3.6) states γ is also a geodesic of Σ^n, i.e.

$$0 = {}^{n}\nabla_{\dot\gamma}\dot\gamma = {}^{n+1}\nabla_{\dot\gamma}\dot\gamma.$$

Again, substituting this in the Gauß formula (Proposition 1.6.5), we obtain

$$\begin{aligned} {}^{n+1}\nabla_{\dot\gamma}\dot\gamma &= {}^{n}\nabla_{\dot\gamma}\dot\gamma - \sigma h(\dot\gamma, \dot\gamma)\nu \\ &= {}^{n+1}\nabla_{\dot\gamma}\dot\gamma - \sigma h(\dot\gamma, \dot\gamma)\nu. \end{aligned}$$

This implies $\sigma h(\dot\gamma, \dot\gamma)\nu = 0$. Since Σ^n is timelike or spacelike, we know $\sigma = \mathfrak{g}(\nu, \nu) \neq 0$ and $\nu \neq 0$. In particular, this means $h_p(v, v) = 0$ for all $v \in T_p\Sigma^n$. By polarization, we have

$$h_p(v, w) = \frac{1}{4}\left(\underbrace{h_p(v + w, v + w)}_{=0} - \underbrace{h_p(v - w, v - w)}_{=0}\right) = 0 \quad \forall v, w \in T_p\Sigma^n,$$

thus $h \equiv 0$. □

Now the main characterization of photon surfaces:

Proposition 2.1.4 ([Claudel et al., 2001]) Let (Σ^n, σ) be an embedded timelike hypersurface in $(\mathfrak{L}^{n+1}, \mathfrak{g})$. Then Σ^n is a photon surface if and only if Σ^n is totally umbilic.

Proof. „$\Leftarrow$“ Assume Σ^n is totally umbilic, i.e. there exists an $\lambda \in \mathbb{R}$ such that $h_p(v, w) = \lambda\sigma_p(v, w)$ for all $p \in \Sigma^n$ and $v, w \in T_p\Sigma^n$. By Definition 1.2.4, for any lightlike geodesic γ in Σ^n it holds that $\sigma(\dot\gamma, \dot\gamma) = 0$. This gives

$$h(\dot\gamma, \dot\gamma) = \lambda\underbrace{\sigma(\dot\gamma, \dot\gamma)}_{=0} = 0.$$

Proposition 2.1.3, „$\Rightarrow$“, leads to the claim.

„$\Rightarrow$“ Assume Σ^n is a photon surface. Proposition 2.1.3, „$\Leftarrow$“, gives

$$h_p(v, v) = 0 \quad \text{for all } p \in \Sigma^n \text{ and lightlike } v \in T_p\Sigma^n. \tag{2.1}$$

Let $\{e_1, \dots, e_n\}$ be an orthonormal basis of $T_p\Sigma^n$ such that $\sigma_p(e_1, e_1) = -1$ and $\sigma_p(e_i, e_i) = 1$

for $i = 2, \dots, n$. Then vectors of the form $v_i^\pm := e_1 \pm e_i$ are lightlike:

$$
\begin{aligned}
\sigma_p\left(v_i^\pm, v_i^\pm\right) = \sigma_p(e_1 \pm e_i, e_1 \pm e_i) &= \sigma_p(e_1, e_1 \pm e_i) \pm \sigma_p(e_i, e_1 \pm e_i) \\
&= \left[\sigma_p(e_1, e_1) \pm \underbrace{\sigma_p(e_1, e_i)}_{=0}\right] \pm \left[\underbrace{\sigma_p(e_i, e_1)}_{=0} \pm \sigma_p(e_i, e_i)\right] \\
&= \underbrace{\sigma_p(e_1, e_1)}_{=-1} + \underbrace{\sigma_p(e_i, e_i)}_{=1} \\
&= 0.
\end{aligned}
$$

Combining this result with equation (2.1)

$$
0 = h_p\left(v_i^+, v_i^+\right) = h_p(e_1 + e_i, e_1 + e_i) = h_p(e_1, e_1) + 2h_p(e_1, e_i) + h_p(e_i, e_i) \tag{2.2}
$$

$$
0 = h_p\left(v_i^-, v_i^-\right) = h_p(e_1 - e_i, e_1 - e_i) = h_p(e_1, e_1) - 2h_p(e_1, e_i) + h_p(e_i, e_i). \tag{2.3}
$$

Adding and subtracting both the equations (2.2) and (2.3) leads to

$$
-h_p(e_1, e_1) = h_p(e_i, e_i) \tag{2.4}
$$

$$
h_p(e_1, e_i) = 0. \tag{2.5}
$$

We define $\lambda := -h_p(e_1, e_1) = h_p(e_i, e_i)$ and $v_{ij} := e_i + e_j + \sqrt{2}e_1$, $i, j = 2, \dots, n$. These vectors are also lightlike:

$$
\sigma_p(v_{ij}, v_{ij}) = \underbrace{\sigma_p(e_i, e_i)}_{=1} + \underbrace{\sigma_p(e_j, e_j)}_{=1} + 2\underbrace{\sigma_p(e_1, e_1)}_{=-1} = 0.
$$

Finally, using (2.4), (2.5) and (2.1)

$$
\begin{aligned}
0 = h_p(v_{ij}, v_{ij}) &= h_p(e_i, e_i) + h_p(e_j, e_j) + 2h_p(e_i, e_j) + 2h_p(e_1, e_1) \\
&\qquad + 2\sqrt{2}h_p(e_i, e_1) + 2\sqrt{2}h_p(e_j, e_1) \\
&= \lambda + \lambda - 2\lambda + 2h_p(e_i, e_j) + 0 + 0 \\
&= 2h_p(e_i, e_j)
\end{aligned}
$$

implies $h_p(e_i, e_j) = 0$. Now, we know $h_p \equiv \lambda\sigma_p$ on a basis of $T_p\Sigma^n$, hence, by linearity,

$h_p \equiv \lambda \sigma_p$ on $T_p \Sigma^n$ for all $p \in \Sigma^n$ and Σ^n is totally umbilic. □

2.2 Photon sphere in Schwarzschild

The photon sphere in the Schwarzschild solution is well known in literature. We present to different approaches to deriving it. One is based on physics and one more mathematically oriented.

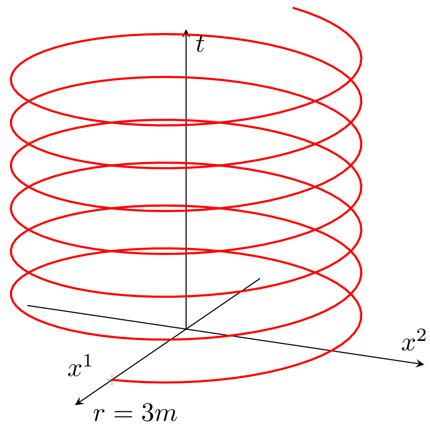


Figure 2.2: A plot of the trajectory of a photon going forward in time at fixed radial component $r = 3m$.

2.2.1 A physical approach

We want to calculate circular orbits of photons in the Schwarzschild solution using classical mechanics. The fact that this is working confirms that classical mechanics and General Relativity are connected. To fully understand this calculation we need to know about *conservation laws* and the principle of *equilibrium* (i.e. equilibrium happens at the critical points of the potential energy). Further explanations can be found in standard literature [Goldstein, 2011] or [Kibble and Berkshire, 2004]. This derivation follows the lecture on General Relativity of Leonard Susskind at Stanford University, Fall 2012[1].

[1]This lecture can be found on is website *http://theoreticalminimum.com/courses/general-relativity/2012/fall*, last visited 10.2017

To derive the equations of motion, one may start with the *action* (dimensions: [energy]·[time], in geometric unit system [mass]·[time]) of the particle, which is the product of mass and the proper time of the orbit between two points:

$$\begin{aligned}
\text{Action} &= m \int \mathrm{d}s \\
&= m \int \sqrt{-N^2(r)\,\mathrm{d}t^2 + N^{-2}(r)\,\mathrm{d}r^2 + r^2\,\mathrm{d}\Omega^2} \\
&= m \int \sqrt{-N^2(r)\,\mathrm{d}t^2 + N^{-2}(r)\,\mathrm{d}r^2 + r^2\,\mathrm{d}\theta^2} \\
&= m \int \sqrt{-N^2(r) + N^{-2}(r)\frac{\mathrm{d}r^2}{\mathrm{d}t^2} + r^2\frac{\mathrm{d}\theta^2}{\mathrm{d}t^2}}\,\mathrm{d}t \\
&= m \int \sqrt{-N^2(r) + N^{-2}(r)\dot{r}^2 + r^2\dot{\theta}^2}\,\mathrm{d}t.
\end{aligned}$$

We used the definition of the Schwarzschild metric and assumed the trajectory of the particle to be a circle. Hence, without loss of generality, we can assume the circle to be parametrized by a single angle θ, thus $\mathrm{d}\Omega^2$ simplifies to $\mathrm{d}\theta^2$.

In addition, we know the the integrand of the action is the *Lagrangian* $\mathcal{L}$

$$\text{Action} = m \int \mathrm{d}s = \int \mathcal{L}\,\mathrm{d}t,$$

which implies

$$\mathcal{L} = m\sqrt{-N^2(r) + N^{-2}(r)\dot{r}^2 + r^2\dot{\theta}^2}.$$

Given the Lagrangian, we can now calculate the *angular momentum* L, which is a conserved quantity

$$L = \frac{\partial \mathcal{L}}{\partial \dot{\theta}} = \frac{m r^2 \dot{\theta}}{\sqrt{-N^2(r) + N^{-2}(r)\dot{r}^2 + r^2\dot{\theta}^2}} =: m\Lambda. \tag{2.6}$$

The other conserved quantity we need is *energy* (conserved since the Lagrangian is time independent). The general expression for energy is the *Hamiltonian*

$$\mathcal{H} = \frac{\partial \mathcal{L}}{\partial \dot{\theta}}\dot{\theta} + \frac{\partial \mathcal{L}}{\partial \dot{r}}\dot{r} - \mathcal{L} = \frac{N^2(r)m}{\sqrt{-N^2(r) + N^{-2}(r)\dot{r}^2 + r^2\dot{\theta}^2}}. \tag{2.7}$$

Substituting equation (2.6) in (2.7) and using the fact that $\dot{r} = 0$ for cricular orbits, we obtain

$$\mathcal{H} = \frac{mN(r)\sqrt{\Lambda^2 - r^2}}{r}. \tag{2.8}$$

Since photons have no mass, we study the limit of the Hamiltonian as m approaches zero, leaving the conserved quantities energy and angular momentum constant. We observe

$$\begin{aligned} &\text{const.} = L = m\Lambda &&\Longrightarrow \quad \Lambda \to \infty \text{ as } m \to 0 \\ \Longrightarrow \quad &\Lambda^2 \gg r^2 \text{ as } m \to 0 &&\Longrightarrow \quad \sqrt{\Lambda^2 - r^2} \approx \Lambda \text{ as } m \to 0. \end{aligned} \tag{2.9}$$

Combining (2.8) and (2.9):

$$\begin{aligned} \lim_{m\to 0} \mathcal{H} &= \lim_{m\to 0} \frac{mN(r)\sqrt{\Lambda^2 - r^2}}{r} \\ &= \lim_{m\to 0} \frac{mN(r)\Lambda}{r} \\ &= L\frac{N(r)}{r} \\ &= L\frac{\sqrt{1 - \frac{2M}{r}}}{r} \end{aligned}$$

The result is a function of energy only dependent of r. Its first derivative tells us the critical points

$$\frac{\partial}{\partial r}\mathcal{H}(r) = L\frac{3M - r}{\sqrt{1 - \frac{2M}{r}}r^3} = 0 \quad \Longleftrightarrow \quad r = 3M.$$

Since $\partial_r^2\mathcal{H}(3M) < 0$, the energy has a maximum at $r = 3M$, which corresponds to an unstable equilibrium position. This means, the only chance of a photon having a circular orbit is in the hypersurface $\{r = 3M\}$.

2.2.2 A mathematical approach

In this section we will present a rigorous proof that the submanifold $P^3 = \{r = 3M\}$ is a photon sphere in the Schwarzschild spacetime in the sense of Definition 2.1.2.

Theorem 2.2.1 Let $(\mathfrak{L}^{3+1}, \mathfrak{g})$ be the $(3+1)$-dimensional Schwarzschild solution. Then the

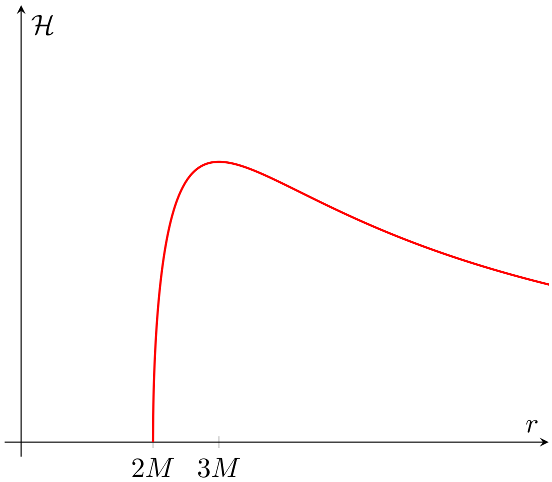


Figure 2.3: A plot of $\mathcal{H}(r)$ in the limit $m \to 0$ for $L = m\Lambda =$ const.. The potential energy is zero at $r = 2M$ and attains its maximum at $r = 3M$. As r grows $\mathcal{H}(r)$ falls like $1/r$, i.e. $\mathcal{H}(r)$ asymptotically approaches 0.

set $P^3 = \{r = 3M\}$ is a photon sphere in $\mathfrak{L}^{3+1}$.

Proof. At first we show that $P^3 = \{r = 3M\} \hookrightarrow \mathfrak{L}^{3+1}$ is a smooth, timelike hypersurface. Consider the smooth projection to the radial coordinate $\Phi_r : \mathfrak{L}^{3+1} \to (2M, \infty)$, $(t, r, \varphi, \theta) \mapsto r$. Since $r = 3M$ is a regular value of Φ_r and $P^3 = \{r = 3M\} = \Phi_r^{-1}(3M)$, P^3 is a smooth hypersurface by the regular level set Theorem. Furthermore, the tangent space of P^3 is given by

$$T_p P^3 = \{v \in T_p \mathfrak{L}^{3+1} v | \, \mathrm{d}r(v) = 0\}.$$

This allows a simple calculation of the induced metric p, namely

$$\begin{aligned} p = \mathfrak{g}\big|_{P^3} &= \left[-\left(1 - \frac{2M}{r}\right) \mathrm{d}t^2 + \frac{1}{1 - \frac{2M}{r}} \mathrm{d}r^2 + r^2 \, \mathrm{d}\Omega^2 \right]_{P^3} \\ &= \left[-\left(1 - \frac{2M}{r}\right) \mathrm{d}t^2 + r^2 \, \mathrm{d}\Omega^2 \right]_{P^3} \\ &= -\frac{1}{3} \mathrm{d}t^2 + 9M^2 \, \mathrm{d}\Omega^2. \end{aligned}$$

So the induced metric p is Lorentzian, which, by definition, means P^3 is timelike.

Next we want to rely on Proposition 2.1.4 and show that P^3 is totally umbilic. First we observe $\nu := \frac{1}{\sqrt{3}}\partial_r \in \Gamma(T\mathfrak{L}^4)$ is a unit normal vector field of P^3, since $\nu \perp P^3$ and

$$\begin{aligned}\mathfrak{g}(\nu,\nu) &= \mathfrak{g}\left(\frac{1}{\sqrt{3}}\partial_r, \frac{1}{\sqrt{3}}\partial_r\right)\\ &= \frac{1}{3}N^{-2}(r),\end{aligned}$$

so for any $p \in P^3 = \{r = 3M\}$ it holds that

$$\mathfrak{g}_p(\nu,\nu) = \frac{1}{3}N^{-2}(3M) = \frac{1}{3}\frac{1}{1-\frac{2M}{3M}} = 1.$$

Now, we can calculate the second fundamental form h of P^3. To do so, we will make use of the Koszul formula (Theorem 1.2.7) and $[\partial_i, \partial_j] = 0$ for $i, j \in \{t, \theta, \varphi\}$:

$$\begin{aligned}h(\partial_i,\partial_j) &= \mathfrak{g}\left({}^{n+1}\nabla_{\partial_i}\nu, \partial_j\right)\\ &= \frac{1}{\sqrt{3}}\,\mathfrak{g}\left({}^{n+1}\nabla_{\partial_i}\partial_r, \partial_j\right)\\ &= \frac{1}{\sqrt{3}}\frac{1}{2}\left(\partial_i(\mathfrak{g}(\partial_r,\partial_j)) + \partial_r(\mathfrak{g}(\partial_j,\partial_i)) - \partial_j(\mathfrak{g}(\partial_i,\partial_r))\right.\\ &\qquad \left.-\mathfrak{g}(\partial_r,[\partial_i,\partial_j]) - \mathfrak{g}(\partial_j,[\partial_r,\partial_i]) + \mathfrak{g}(\partial_i,[\partial_j,\partial_r])\right)\\ &= \frac{1}{2\sqrt{3}}(\partial_i(\mathfrak{g}(\partial_r,\partial_j)) + \partial_r(\mathfrak{g}(\partial_j,\partial_i)) - \partial_j(\mathfrak{g}(\partial_i,\partial_r)))\\ &= \frac{1}{2\sqrt{3}}\partial_r(\mathfrak{g}(\partial_j,\partial_i)),\end{aligned}$$

since additionally we know $\mathfrak{g}(\partial_i, \partial_j) = 0$ for $i \neq j$ ($\mathfrak{g}$ is diagonal). This allows an easy computation of the components:

$$\begin{aligned}h(\partial_t,\partial_\varphi) &= h(\partial_t,\partial_\theta) = h(\partial_\varphi,\partial_\theta) = 0\\ h(\partial_t,\partial_t) &= \frac{1}{2\sqrt{3}}\,\partial_r(g(\partial_t,\partial_t))\Big|_{r=3M} &&= -\frac{1}{2\sqrt{3}}\,\partial_r N^2(r)\Big|_{r=3M} &&= \frac{1}{3\sqrt{3}M}p(\partial_t,\partial_t)\\ h(\partial_\varphi,\partial_\varphi) &= \frac{1}{2\sqrt{3}}\,\partial_r(g(\partial_\varphi,\partial_\varphi))\Big|_{r=3M} &&= \frac{1}{2\sqrt{3}}\,\partial_r\left(r^2\sin^2(\theta)\right)\Big|_{r=3M} &&= \frac{1}{3\sqrt{3}M}p(\partial_\varphi,\partial_\varphi)\\ h(\partial_\theta,\partial_\theta) &= \frac{1}{2\sqrt{3}}\,\partial_r(g(\partial_\theta,\partial_\theta))\Big|_{r=3M} &&= \frac{1}{2\sqrt{3}}\,\partial_r r^2\Big|_{r=3M} &&= \frac{1}{3\sqrt{3}M}p(\partial_\theta,\partial_\theta)\end{aligned}$$

In conclusion, defining $\lambda := \frac{1}{3\sqrt{3}M}$, we obtain $h \equiv \lambda p$. This means P^3 is totally umbilic and therefore, by Proposition 2.1.4, is a photon surface. Since the lapse function $N(r)$ is only dependent of r, it is constant on $P^3 = \{r = 3M\}$, which makes P^3 a photon sphere. $\square$

2.3 Uniqueness of photon spheres

We cite the following proposition directly from the paper [Cederbaum, 2014]. This is the reference tool to prove the main theorem of this chapter, the uniqueness result of photon spheres in four dimensional static vacuum asymptotically flat spacetimes.

Proposition 2.3.1 ([Cederbaum, 2014]) Let $(\mathfrak{L}^4, \mathfrak{g})$ be a static, vacuum, asymptotically flat spacetime with lapse function N and let $(P^3, p) \hookrightarrow (\mathfrak{L}^4, \mathfrak{g})$ be a photon sphere arising as the inner boundary of $\mathfrak{L}^4$. Let $\mathfrak{H} : P^3 \to \mathbb{R}$ denote the mean curvature of $(P^3, p) \hookrightarrow (\mathfrak{L}^4, \mathfrak{g})$ and write

$$\left(P^3, p\right) = \left(\mathbb{R} \times \Sigma^2, -N^2 dt^2 + \sigma\right) = \bigcup_{i=1}^{I} \left(\mathbb{R} \times \Sigma_i^2, -N_i^2 dt^2 + \sigma_i\right), \tag{2.10}$$

where each $P_i^3 = \mathbb{R} \times \Sigma_i^2$ is a connected component of P^3. Then the mean curvature $\mathfrak{H}_i := \mathfrak{H}|_{P_i^3}$ is constant on each connected component $P_i^3 = \mathbb{R} \times \Sigma_i^2$ and the embedding $(\Sigma^2, \sigma) \hookrightarrow (\Sigma^2, g)$ is totally umbilic with constant mean curvature $H_i = \frac{2}{3}\mathfrak{H}_i$ on the component Σ_i^2. The scalar curvature of the component (Σ_i^2, σ_i), ${}^{\sigma_i}R$, is a non-negative constant, namely

$${}^{\sigma_i}R = \frac{3}{2} H_i^2. \tag{2.11}$$

Moreover, the normal derivative of the lapse function N in direction of the outward unit normal ν to Σ^2, $\nu(N)$, is also constant on every component (Σ_i^2, σ_i), $\nu(N)_i := \nu(N)|_{\Sigma_i^2}$. For each $i \in \{1, \ldots, I\}$, either $H_i = 0$ and Σ_i^2 is a totally geodesic flat torus or Σ_i^2 is an intrinsically and extrinsically round CMC sphere for which the above constants are related via

$$N_i H_i = 2\nu(N)_i, \tag{2.12}$$

$$(r_i H_i)^2 = \frac{4}{3}, \tag{2.13}$$

where

$$r_i := \sqrt{\frac{|\Sigma_i^2|_{\sigma_i}}{4\pi}} \tag{2.14}$$

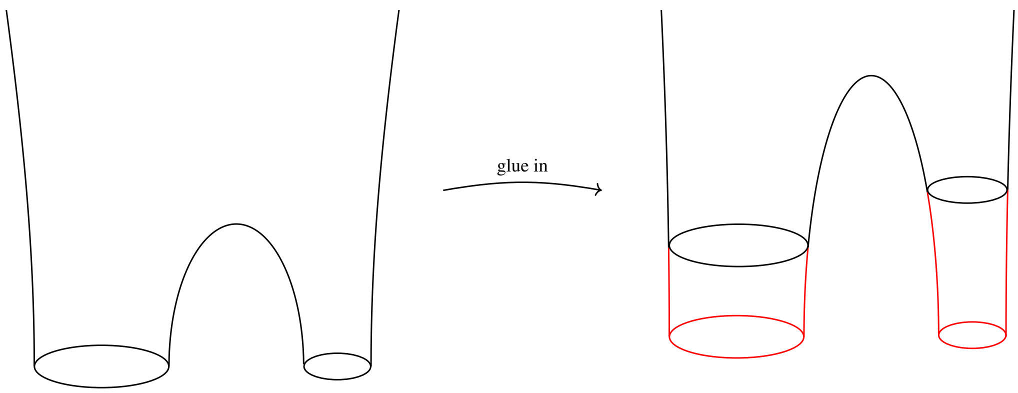
denotes the *area radius* of Σ_i^2.

Theorem 2.3.2 ([Cederbaum and Galloway, 2015]) Let $(\mathfrak{L}^4, \mathfrak{g})$ be a static, vacuum, asymptotically flat spacetime with lapse function N that possesses a (possibly disconnected) photon sphere $(P^3, p) \hookrightarrow (\mathfrak{L}^4, \mathfrak{g})$, arising as the inner boundary of $\mathfrak{L}^4$. Let m denote the ADM-mass of $(\mathfrak{L}^4, \mathfrak{g})$ and let $\mathfrak{H} : P^3 \to \mathbb{R}$ denote the mean curvature of $(P^3, p) \hookrightarrow (\mathfrak{L}^4, \mathfrak{g})$. Then $m = (\sqrt{3}\mathfrak{H})^{-1}$, with $\mathfrak{H} > 0$, and $(\mathfrak{L}^4, \mathfrak{g})$ is isometric to the region $\{r \geq 3m\}$ exterior to the photon sphere $\{r = 3m\}$ in the Schwarzschild spacetime of mass m. In particular, (P^3, p) is connected and a cylinder over a topological sphere.

Sketch of proof.
Cederbaum and Galloway adapted the argument of [Bunting and Masood, 1987] and their uniqueness result of static black holes. The idea is to construct a static vacuum black hole spacetime that extends $(\mathfrak{L}^4, \mathfrak{g})$ and then apply the result of Bunting-Masood. We skip various details of the proof and try to capture the important ideas.

As the spacetime is static, we are allowed to work in the time slice $(M^3 = \{t = 0\}, g)$, which is a Riemannian manifold. The proof is divided into four steps.

Step 1 Constructing a scalar flat, asymptotically flat manifold with minimal boundary.



We want to extend the time slice (M^3, g) by gluing in suitable Schwarzschild necks, that is, the cylindrical piece of a spatial Schwarzschild solution between its photon sphere and its black hole horizon. For each $i \in \{1, \dots, I\}$, Σ_i^2 denotes one photon sphere base of (M^3, g). The fitting Schwarzschild necks V_i^3 are uniquely determined by their mass. We choose

$$V_i^3 := [2\mu_i, 3\mu_i] \times \mathbb{S}^2, \qquad \mu_i := \frac{r_i}{3},$$

where $r_i = 2/\sqrt{3H_i}$ is the constant radius of the photon sphere base, H_i the constant mean curvature (see Proposition 2.3.1). We can now define a new metric $\tilde{g}$ by extending the old metric g on the glued in necks by

$$\tilde{g}\Big|_{V_i^3} := \frac{1}{1 - \frac{2\mu_i}{r}}\, \mathrm{d}r^2 + r^2\, \mathrm{d}\Omega^2, \quad i = 1, \dots, I.$$

Across the glueing surface, Σ_i^2 and the top of V_i^3 agree to the same induced metric since both are round spheres of the same radius. The second fundamental forms agree as well, since both photon spheres have to be totally umbilic with the same mean curvature (see Proposition 2.1.4).

We can extend the time slice back to a static black hole spactime

$$\widetilde{\mathfrak{L}}^4 := \mathbb{R} \times \widetilde{M}^3 := \mathbb{R} \times \left(M^3 \cup \left(\bigcup_{i=1}^{I} V_i^3 \right) \right), \qquad \tilde{\mathfrak{g}} := -\widetilde{N}^2\, \mathrm{d}t^2 + \tilde{g},$$

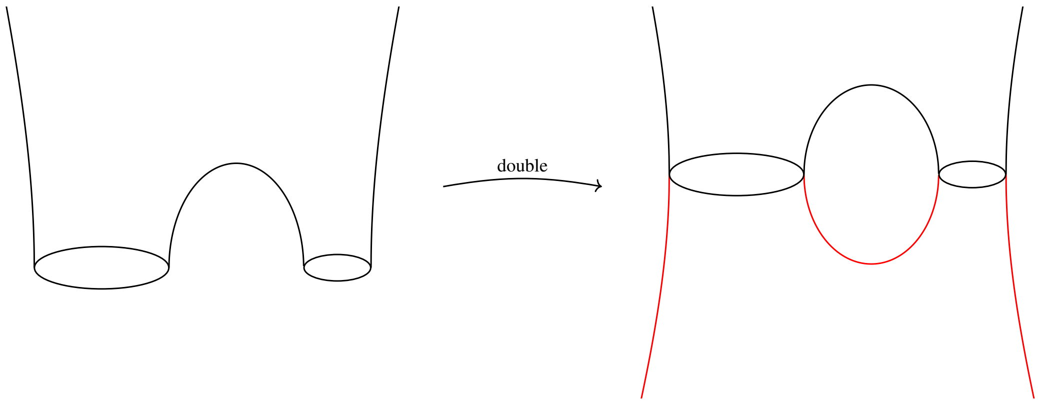
with new lapse function

$$\widetilde{N} := \begin{cases} N & \text{on } M^3 \\ \sqrt{3} N_i \sqrt{1 - \frac{2\mu_i}{r}} & \text{on } V_i^3,\ i = 1, \dots, I \end{cases}.$$

It remains to be shown that the constructed manifold $(\widetilde{M}^3, \tilde{g})$ together with the lapse are suitably regular across the glueing surfaces (which we are not going to present here).

Now, the uniqueness result of Bunting-Masood essentially applies if we can solve the regularity issues created by the glueing process.

Step 2 Doubling.

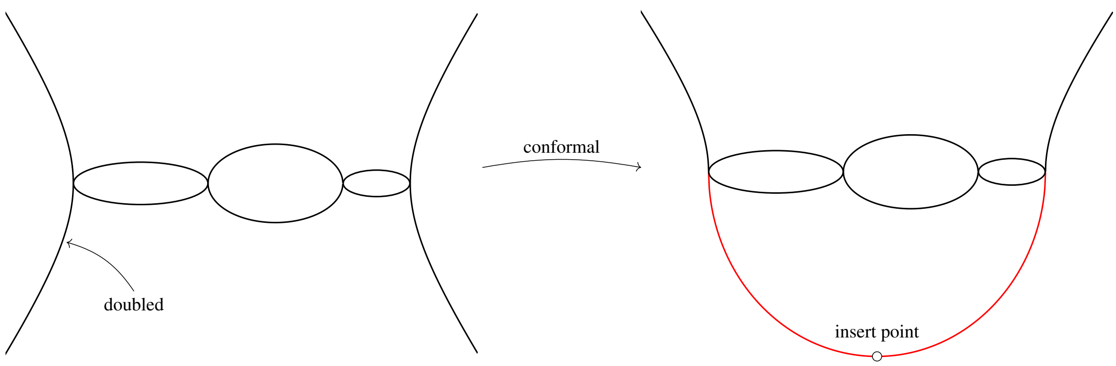


We rename $\widetilde{M}^3$ to $\widetilde{M}^+$ and generate a copy of $\widetilde{M}^+$ named $\widetilde{M}^-$. We glue the two copies to each other along their shared Schwarzschild black hole horizon (minimal boundary). We equip this new smooth manifold $\widetilde{M}^\pm := \widetilde{M}^+ \cup \widetilde{M}^-$ with the metric $\tilde{g}^\pm$ by setting

$$\tilde{g}^\pm = \begin{cases} \tilde{g} & \text{on } M^+ \\ \tilde{g} & \text{on } M^- \end{cases} \quad \text{and define a new lapse as} \quad \widetilde{N}^\pm = \begin{cases} \widetilde{N} & \text{on } M^+ \\ -\widetilde{N} & \text{on } M^- \end{cases}.$$

By construction, the manifold has two asymptotically flat ends of ADM-mass m. This time, the glueing surface is a horizon of two Schwarzschild black holes of the same mass. This means, the new lapse function, as we defined it, is indeed smooth across the shared horizon and thus is the metric.

Step 3 Conformal transformation to a scalar flat, geodesically complete manifold with vanishing ADM-mass.



We introduce the conformal transformation

$$\hat{g} := u^4\,\tilde{g}^{\pm}, \qquad u : \widetilde{M}^{\pm} \to \mathbb{R},\; p \mapsto \frac{1+\widetilde{N}^{\pm}(p)}{2} > 0.$$

By linearity, the conformal factor is $\tilde{g}^{\pm}$-harmonic as the original lapse function is g-harmonic because of the static vacuum equations (Proposition 1.4.8). Thus, we can easily compute the scalar curvature with the formula of Section 1.7

$$^{\hat{g}}R = u^{-5}\left(\underbrace{^{\tilde{g}}R}_{=0}\,u + 8\,\underbrace{^{\tilde{g}}\Delta u}_{=0}\right) = 0.$$

Further calculations show $(\widetilde{M}^{+}, \hat{g})$ is asymptotically flat with vanishing ADM-mass. Heuristically, assuming that (M^3, g) is asymptotically Schwarzschildean together with $N = 1 - \frac{m}{s} + \mathcal{O}_k(s^{-2})$, we find in $(\widetilde{M}^{-}, \hat{g})$

$$\begin{aligned}\hat{g}_{ij} &= u^4\,\widetilde{g}_{ij} = \left(\frac{1+\widetilde{N}^{\pm}}{2}\right)^4 \widetilde{g}_{ij} = \left(\frac{1-N}{2}\right)^4 g_{ij}\\ &= \left(\frac{m}{2s} + \mathcal{O}_k\big(s^{-2}\big)\right)^4 \left(\left(1+\frac{m}{2s}\right)^4 \delta_{ij} + \mathcal{O}_k\big(s^{-2}\big)\right) = \left(\frac{m}{2s}\right)^4 \delta_{ij} + \mathcal{O}_k\big(s^{-5}\big)\end{aligned}$$

as $s \to \infty$ in the original asymptotically flat coordinates of (M^3, g). This allows to perform an inversion in the sphere which justifies to glue in a point at $r = \infty$.

We end up with a geodesically complete, scalar flat Riemannian manifold $(\widehat{M}^3 := \widetilde{M}^3 \cup \{\infty\}, \hat{g}\,)$ with one asymptotically flat end of vanishing ADM-mass, that is smooth apart from some gluing surfaces and is sufficiently regular across them.

Step 4 Applying the Positive Mass Theorem.

The constructed manifold fulfils all assumptions of the weak regularity version the Positive Mass Theorem 1.5.5. As the ADM-mass is zero, the rigidity case yields that $(\widehat{M}^3, \hat{g})$ must be isometric to Euclidean space. This implies that the original manifold (M^3, g) must be conformally flat. However, as stated in Section 1.8, the only conformally flat, maximally extended solution of the static vacuum equations is the Schwarzschild solution. The algebraic relations of Proposition 2.3.1 give the claimed values.

□

3 Schwarzschild in (2+1) dimensions

3.1 Pseudo-Schwarzschild spacetime

In this section we will present an analysis of the restriction to the equatorial plane of the Schwarzschild solution. We call this new spacetime *Pseudo-Schwarzschild solution*. We will justify the name by showing this restriction still solves the Einstein equations (1.4) for a fitting stress-energy-momentum tensor. Furthermore, we discuss its singularities and energy conditions.

Definition 3.1.1 (Pseudo-Schwarzschild solution) The *Pseudo-Schwarzschild solution* $(\mathfrak{L}^3, \mathfrak{g})$ for positive parameter $m > 0$ is defined as

(3.1)

$$\mathfrak{L}^3 := \mathbb{R}\times(2m,\infty)\times\mathbb{S}, \quad \mathfrak{g} := -N^2\,\mathrm{d}t^2 + N^{-2}\,\mathrm{d}r^2 + r^2\,\mathrm{d}\varphi^2, \quad N = N(r) := \left(1 - \frac{2m}{r}\right)^{1/2}.$$

For negative parameter $m < 0$, the same metric still defines a spacetime on $\mathfrak{L}^3 := \mathbb{R} \times (0,\infty) \times \mathbb{S}$. If $m = 0$, this Pseudo-Schwarzschild solution degenerates to the Minkowski solution.

Remark 3.1.2 One of the most studied solution of the $(2+1)$-dimensional Einstein equations is the *BTZ black hole* (see e.g. [Banados et al., 1992] and [Birmingham et al., 2001]). The metric of this solution is given as

$$\mathfrak{g} := -N^2\,\mathrm{d}t^2 + N^{-2}\,\mathrm{d}r^2 + r^2(N^\varphi\,\mathrm{d}t + \mathrm{d}\varphi)^2, \quad N = N(r) := \left(-M + \frac{r^2}{l^2} + \frac{J^2}{4r^2}\right)^{1/2},$$
$$N^\varphi = N^\varphi(r) := -\frac{J}{2r^2},$$

for some constants M, J and $l \neq 0$. It is worth noticing that the Pseudo-Schwarzschild solution does not arise as a special case of the BTZ solution. In fact, as a solution of the vacuum equations with negative cosmological constant, the BTZ black hole has negative scalar curvature. However, as we prove in Proposition 3.1.3, the Pseudo-Schwarzschild solution is scalar flat.

In order to work with the Pseudo-Schwarzschild metric, we need the Christoffel symbols. Recall formula (1.1):

$$\Gamma_{ij}{}^{k} = \frac{1}{2}\mathfrak{g}^{ks}(\partial_j \mathfrak{g}_{si} + \partial_i \mathfrak{g}_{sj} - \partial_s \mathfrak{g}_{ij}).$$

Since the metric in (3.1) is diagonal, the formula simplifies:

$$\begin{aligned}\Gamma_{ij}{}^{k} &= \frac{1}{2}\mathfrak{g}^{ks}(\partial_j \mathfrak{g}_{si} + \partial_i \mathfrak{g}_{sj} - \partial_s \mathfrak{g}_{ij})\\ &= \frac{1}{2}\mathfrak{g}^{kk}(\partial_j \mathfrak{g}_{ki} + \partial_i \mathfrak{g}_{kj} - \partial_k \mathfrak{g}_{ij}).\end{aligned}$$

This implies $\Gamma_{ij}{}^{k} \equiv 0$ for $i, j, k \in \{t, \varphi\}$, since the metrical components are independent of t and φ. The Christoffel symbols with three pairwise different indices also vanish due to the metric being diagonal. For $j \in \{t, \varphi\}$ and $i = k = r$, we have

$$\Gamma_{rj}{}^{r} = \frac{1}{2}\mathfrak{g}^{rr}(\partial_j \mathfrak{g}_{rr} + \partial_r \mathfrak{g}_{rj} - \partial_r \mathfrak{g}_{rj}) = 0.$$

For $i = j \neq k$ we know $\mathfrak{g}_{ki} \equiv \mathfrak{g}_{kj} \equiv 0$. This allows

$$\begin{aligned}\Gamma_{ij}{}^{k} &= \frac{1}{2}\mathfrak{g}^{kk}(\partial_j \mathfrak{g}_{ki} + \partial_i \mathfrak{g}_{kj} - \partial_k \mathfrak{g}_{ij})\\ &= -\frac{1}{2}\mathfrak{g}^{kk}(\partial_k \mathfrak{g}_{ii}),\end{aligned}$$

which implies $\Gamma_{rr}{}^{t} \equiv \Gamma_{rr}{}^{\varphi} \equiv 0$. For $k = r$ and $i = j \in \{t, \varphi\}$, we have

$$\begin{aligned}\Gamma_{ii}{}^{r} &= \frac{1}{2}\mathfrak{g}^{rr}(\partial_i \mathfrak{g}_{ri} + \partial_i \mathfrak{g}_{ri} - \partial_r \mathfrak{g}_{ii})\\ &= -\frac{1}{2}\mathfrak{g}^{rr}(\partial_r \mathfrak{g}_{ii}).\end{aligned}$$

So we obtain ${\Gamma_{tt}}^r$ and ${\Gamma_{\varphi\varphi}}^r$:

$$\begin{aligned}
{\Gamma_{tt}}^r &= -\frac{1}{2}\mathfrak{g}^{rr}(\partial_r \mathfrak{g}_{tt}) \\
&= -\frac{1}{2}\left(1-\frac{2m}{r}\right)\left(-\partial_r\left(1-\frac{2m}{r}\right)\right) \\
&= \frac{1}{2}\left(1-\frac{2m}{r}\right)\frac{2m}{r^2} \\
&= \frac{m}{r^2}N^2
\end{aligned}
\qquad
\begin{aligned}
{\Gamma_{\varphi\varphi}}^r &= -\frac{1}{2}\mathfrak{g}^{rr}(\partial_r \mathfrak{g}_{\varphi\varphi}) \\
&= -\frac{1}{2}\left(1-\frac{2m}{r}\right)\left(\partial_r r^2\right) \\
&= -\frac{1}{2}\left(1-\frac{2m}{r}\right)(2r) \\
&= -rN^2.
\end{aligned}$$

For $i = r$ and $k = j \in \{t, \varphi\}$, we have

$$\begin{aligned}
{\Gamma_{rj}}^j &= \frac{1}{2}\mathfrak{g}^{jj}(\partial_j \mathfrak{g}_{jr} + \partial_r \mathfrak{g}_{jj} - \partial_j \mathfrak{g}_{rj}) \\
&= \frac{1}{2}\mathfrak{g}^{jj}(\partial_r \mathfrak{g}_{jj}).
\end{aligned}$$

So we obtain ${\Gamma_{rt}}^t$ and ${\Gamma_{r\varphi}}^\varphi$:

$$\begin{aligned}
{\Gamma_{rt}}^t &= \frac{1}{2}\mathfrak{g}^{tt}(\partial_r \mathfrak{g}_{tt}) \\
&= -\frac{1}{2}\left(1-\frac{2m}{r}\right)^{-1}\left(-\partial_r\left(1-\frac{2m}{r}\right)\right) \\
&= \frac{1}{2}\left(1-\frac{2m}{r}\right)^{-1}\frac{2m}{r^2} \\
&= \frac{m}{r^2}N^{-2}
\end{aligned}
\qquad
\begin{aligned}
{\Gamma_{r\varphi}}^\varphi &= \frac{1}{2}\mathfrak{g}^{\varphi\varphi}(\partial_r \mathfrak{g}_{\varphi\varphi}) \\
&= \frac{1}{2}r^{-2}\left(\partial_r r^2\right) \\
&= \frac{1}{2}r^{-2}(2r) \\
&= \frac{1}{r}.
\end{aligned}$$

This leaves us with one remaining Christoffel symbol:

$$\begin{aligned}
{\Gamma_{rr}}^r &= \frac{1}{2}\mathfrak{g}^{rr}(\partial_r \mathfrak{g}_{rr} + \partial_r \mathfrak{g}_{rr} - \partial_r \mathfrak{g}_{rr}) \\
&= \frac{1}{2}\mathfrak{g}^{rr}(\partial_r \mathfrak{g}_{rr}) \\
&= \frac{1}{2}\left(1-\frac{2m}{r}\right)\left(\partial_r \frac{1}{1-\frac{2m}{r}}\right) \\
&= \frac{1}{2}\left(1-\frac{2m}{r}\right)\frac{-2m}{r^2\left(1-\frac{2m}{r}\right)^2} \\
&= \frac{-m}{r^2N^2}.
\end{aligned}$$

In conclusion, we have seven non-vanishing Christoffel symbols:

$$\Gamma_{\varphi\varphi}{}^{r} = -rN^2 = 2m - r \qquad \Gamma_{r\varphi}{}^{\varphi} = \Gamma_{\varphi r}{}^{\varphi} = \frac{1}{r}$$
$$\Gamma_{tt}{}^{r} = \frac{m}{r^2}N^2 = \frac{m}{r^3}(r-2m) \qquad \Gamma_{rt}{}^{t} = \Gamma_{tr}{}^{t} = \frac{m}{r^2}N^{-2} = \frac{m}{r(r-2m)}$$
$$\Gamma_{rr}{}^{r} = \frac{-m}{r^2N^2} = \frac{-m}{r(r-2m)}$$

Knowing all the Christoffel symbols, we can now prove the following result.

Proposition 3.1.3 The Pseudo-Schwarzschild solution $(\mathfrak{L}^3, \mathfrak{g})$ is scalar-flat, i.e. the Ricci tensor is trace free.

Proof. To prove this, we will calculate the components of the Ricci tensor. Recall formula (1.2):

$$_{ij} = \partial_k \Gamma_{ij}{}^{k} - \partial_j \Gamma_{ik}{}^{k} + \Gamma_{ij}{}^{s}\Gamma_{ks}{}^{k} - \Gamma_{ik}{}^{s}\Gamma_{js}{}^{k}.$$

Substituting the Christoffel symbols gives us this diagonal Ricci tensor:

$$\mathrm{Ric}_{tt} = \frac{m}{r^4}(2m-r) = \frac{-mN^2}{r^3} \qquad \mathrm{Ric}_{rr} = \frac{-m}{r^2(2m-r)} = \frac{m}{r^3N^2} \qquad \mathrm{Ric}_{\varphi\varphi} = \frac{-2m}{r} = N^2 - 1.$$

So the scalar curvature is given as

$$\begin{aligned} R = \mathfrak{g}^{ij}_{ij} &= \mathfrak{g}^{ii}_{ii} \\ &= -N^{-2}\frac{-mN^2}{r^3} + N^2\frac{m}{r^3N^2} + r^{-2}\big(N^2-1\big) \\ &= \frac{m}{r^3} + \frac{m}{r^3} + \frac{-2m}{r^3} \\ &= 0. \end{aligned}$$

□

Remark 3.1.4 As mentioned in the beginning of this Section, Proposition 3.1.3 justifies an own analysis of the Pseudo-Schwarzschild solution as it does not arise as a special case of the well known BTZ solution. A scalar flat spacetime cannot solve the vacuum Einstein equations with non-vanishing cosmological constant.

Proposition 3.1.5 The Pseudo-Schwarzschild solution satisfies the Einstein equations (1.4) to the diagonal matter model $:= \kappa$, where $\kappa := (8\pi G)^{-1}c^4$.

Proof. By Propsition 3.1.3, the Einstein equations (1.4) simplify to

$$-\frac{1}{2}R\mathfrak{g} == \frac{8\pi G}{c^4} \qquad \Longleftrightarrow \qquad = \frac{c^4}{8\pi G}.$$

□

The Pseudo-Schwarzschild metric defined in 3.1.1 has singularities at $r = 2m$ and at $r = 0$. We want to gain further information about this solution by introducing the same coordinate transformations as in [Wald, 1984] for the four-dimensional Schwarzschild solution. In these new coordinates, the metric will only have one singularity at $r = 0$. This singularity, also shown in [Wald, 1984], is not due to the coordinates but a geometrical property of the spacetime itself. The extension we want to make is called Kruskal Extension, named after Martin David Kruskal (1925 – 2006).

We introduce three coordinate changes. Because of the spherical symmetry of our metric, it is sufficient to study the two dimensional part

$$\tilde{\mathfrak{g}} = -\Big(1 - \frac{2m}{r}\Big)\,\mathrm{d}t^2 + \Big(1 - \frac{2m}{r}\Big)^{-1}\mathrm{d}r^2$$

to analyse the singularity at $r = 2m$. The first transformation is given as

$$\begin{aligned} u &:= t - \Big(r + 2m\ln\Big(\frac{r}{2m} - 1\Big)\Big), \\ v &:= t + \Big(r + 2m\ln\Big(\frac{r}{2m} - 1\Big)\Big). \end{aligned}$$

A simple calculation shows the Jacobian determinant is non-vanishing. The chain rule gives

$$\begin{aligned} \mathrm{d}u &= \mathrm{d}t - \Big(\mathrm{d}r + \frac{2m}{r - 2m}\,\mathrm{d}r\Big), \\ \mathrm{d}v &= \mathrm{d}t + \Big(\mathrm{d}r + \frac{2m}{r - 2m}\,\mathrm{d}r\Big). \end{aligned}$$

Therefore, the metric is given as

$$\mathfrak{g} = \tilde{\mathfrak{g}} + r^2\,\mathrm{d}\varphi^2 = -\Big(1 - \frac{2m}{r}\Big)\,\mathrm{d}u\,\mathrm{d}v + r^2\,\mathrm{d}\varphi^2, \qquad r = r(u,v).$$

For the second transformation we introduce the coordinates

$$\begin{aligned} U &:= -e^{-u/4m}, \\ V &:= e^{v/4m}. \end{aligned}$$

Again, the Jacobian determinant is non-vanishing and we have

$$\begin{aligned} \mathrm{d}U &= \frac{1}{4m} e^{-u/4m}\,\mathrm{d}u, \\ \mathrm{d}V &= \frac{1}{4m} e^{v/4m}\,\mathrm{d}v. \end{aligned}$$

This transforms our metric to

$$\mathfrak{g} = -\Big(1 - \frac{2m}{r}\Big)\,\mathrm{d}u\,\mathrm{d}v + r^2\,\mathrm{d}\varphi^2 = -\frac{32m^3}{r} e^{-r/2m}\,\mathrm{d}U\,\mathrm{d}V + r^2\,\mathrm{d}\varphi^2, \qquad r = r(U,V).$$

Finally, we introduce the *Kruskal coordinates*

$$\begin{aligned} T &:= \frac{1}{2}(V + U), \\ X &:= \frac{1}{2}(V - U). \end{aligned}$$

The Jacobian determinant of this transformation is also non-vanishing and the final metric is given as

$$\mathfrak{g} = -\frac{32m^3}{r} e^{-r/2m}\,\mathrm{d}U\,\mathrm{d}V + r^2\,\mathrm{d}\varphi^2 = \frac{32m^3}{r} e^{-r/2m}\Big(-\,\mathrm{d}T^2 + \,\mathrm{d}X^2\Big) + r^2\,\mathrm{d}\varphi^2,$$

where $r = r(T,X)$ is implicitly given by

$$X^2 - T^2 = \left(\frac{r}{2m} - 1\right) e^{r/2m}. \tag{3.2}$$

Equation (3.2) together with the condition $r > 0$ yields

$$X^2 - T^2 > -1,$$

which defines the allowed range of the coordinates T and X. In conclusion, the Pseudo-Schwarzschild metric in Kruskal coordinates is, as claimed, regular at $r = 2m$ and has one singularity at $r = 0$ left.

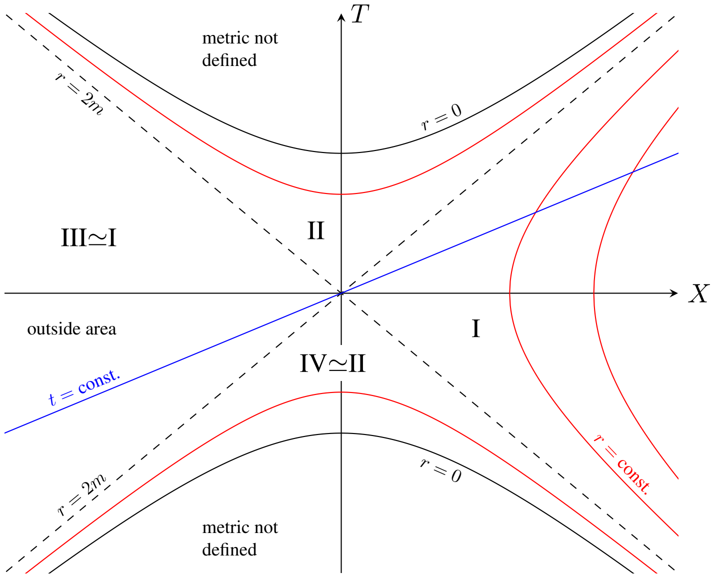


Figure 3.1: Kruskal diagram.

The next thing we want to discuss is the parameter m in the lapse function N of the Pseudo-Schwarzschild solution. It is well known that this parameter represents the ADM-mass in the four-dimensional Schwarzschild solution (see for example [Wald, 1984]). However, it is not obvious whether the same is valid for the Pseudo-Schwarzschild solution. Both, the Schwarzschild and the Pseudo-Schwarzschild solution, degenerate to the Minkowski solution if $m = 0$. Furthermore, both solutions only admit a black hole horizon if m is positive (as we will prove in the next section), which makes the negative m case less interesting.

Proposition 3.1.6 Let $(\mathfrak{L}^3, \mathfrak{g})$ denote the Pseudo-Schwarzschild solution and let the parameter m be positive. Then the stress-energy-momentum tensor T corresponding to $(\mathfrak{L}^3, \mathfrak{g})$ does not satisfy the weak energy condition.

Proof. It is sufficient to find a timelike vector field $X \in \Gamma(T\mathfrak{L}^3)$ such that $(X, X) < 0$. We choose $X := \partial_t$, then it holds

$$\mathfrak{g}(X, X) = \mathfrak{g}(\partial_t, \partial_t) = -N^2 < 0,$$

hence X is timelike. Propositions 3.1.3 and 3.1.5 give

$$(X, X) = (\partial_t, \partial_t) = \kappa(\partial_t, \partial_t) = \kappa \frac{-mN^2}{r^3} < 0.$$

□

This also rules out the dominant energy condition for the Pseudo-Schwarzschild solution.

3.2 Submanifolds in Pseudo-Schwarzschild

The Pseudo-Schwarzschild solution is a static spacetime. We will work out some properties of its canonical time slices $\{t = \text{const}\} \cong \{t = 0\}$. After that we will discuss black hole horizons and photon spheres.

Proposition 3.2.1 The spatial Pseudo-Schwarzschild solution (M^2, g) of mass $m \in \mathbb{R}$ is conformally flat. More precisely, there exists a smooth function $u \in C^\infty(M^2)$ such that $g = u^4\,\delta$, where δ denotes the flat metric. The conformal factor is given as $u(s) = 1 + m/2s$.

Proof. First we note that the flat metric in spherical coordinates is given as $\delta = \mathrm{d}s^2 + s^2\,\mathrm{d}\varphi^2$. The equation we want to solve is

$$N^{-2}(r)\,\mathrm{d}r^2 + r^2\,\mathrm{d}\varphi^2 = u^4(s)\Big(\mathrm{d}s^2 + s^2\,\mathrm{d}\varphi^2\Big),$$

where $s = s(r)$. Equating coefficients gives

$$r^2\,\mathrm{d}\varphi^2 = u^4(s)s^2\,\mathrm{d}\varphi^2 \qquad \Longleftrightarrow \qquad r = u^2(s)s \tag{3.3}$$

and

$$\begin{aligned} & N^{-2}(r)\,\mathrm{d}r^2 = u^4(s)\,\mathrm{d}s^2 \\ \Longleftrightarrow \qquad & N^{-2}(r) \qquad = u^4(s)\left(\frac{\mathrm{d}s}{\mathrm{d}r}\right)^2 \\ \Longleftrightarrow \qquad & N^{-1}(r) \qquad = u^2(s)s'(r), \qquad \text{for } s'(r) > 0. \end{aligned}$$

Substituting (3.3) in this last equation, we find the ODE

$$\frac{s'(r)}{s(r)} = (N(r)r)^{-1},$$

with the solution

$$s(r) = (r(N(r)+1) - m) \cdot C, \qquad C \in \mathbb{R} \setminus \{0\}.$$

Again, substituting (3.3) in this solution, we obtain

$$u^2(s) = \frac{\left(1 + \frac{Cm}{s}\right)^2}{2C} \qquad \Longleftrightarrow \qquad u(s) = \frac{1 + \frac{Cm}{s}}{\sqrt{2|C|}}.$$

Considering the fall-off condition, i.e. $u(s) \to 1$ for $r \to \infty$, the constant C is uniquely determined as $C = 1/2$. This concludes in

$$u(s) = 1 + \frac{m}{2s}.$$

□

Remark 3.2.2 The three-dimensional Pseudo-Schwarzschild solution $(L^3,{}^3g)$ can be written in isotropic coordinates as

$$-N^2(r)\,\mathrm{d}t^2 + N^{-2}(r)\,\mathrm{d}r^2 + r^2\,\mathrm{d}\varphi^2 = -f^2(s)\,\mathrm{d}t^2 + u^4(s)\Big(\mathrm{d}s^2 + s^2\,\mathrm{d}\varphi^2\Big).$$

The conformal factor $u(s)$ is given as in Proposition 3.2.1 and the new lapse function $f(s)$ is given as

$$f(s) = \frac{1 - m/2s}{1 + m/2s}. \tag{3.4}$$

Proposition 3.2.1 shows that the spatial parts of both, the Schwarzschild and the Pseudo-Schwarzschild solution, behave similar in the sense of being conformally flat with the exact same conformal factor. This indicates that they could behave similar among more characteristics. But the following holds:

Proposition 3.2.3 The spatial Pseudo-Schwarzschild solution (M^2, g) of positive mass m has negative scalar curvature: ${}^g\!R = -2m/r^3 < 0$.

Proof. In order to calculate the scalar curvature, we need to calculate the components of the Ricci tensor:

$$\mathrm{Ric}_{rr} = -\frac{m}{r^3 N^2} \qquad\qquad \mathrm{Ric}_{\varphi\varphi} = -\frac{m}{r}.$$

Thus, the scalar curvature is

$$\begin{aligned} {}^g\!R = g_{ij}^{ij} &= g_{ii}^{ii} \\ &= N^2\Big(-\frac{m}{r^3 N^2}\Big) + \frac{1}{r^2}\Big(-\frac{m}{r}\Big) \\ &= -\frac{m}{r^3} - \frac{m}{r^3} \\ &= \frac{-2m}{r^3} < 0. \end{aligned}$$

□

A similarity to the four dimensional analogue is the black hole horizon.

Proposition 3.2.4 The Pseudo-Schwarzschild solution possesses a black hole horizon.

Proof. We show that, in isotropic coordinates, $\partial M^2 = \mathbb{S}_{m/2}$ is a black hole horizon. First, we observe that the lapse function (3.4) vanishes on ∂M^2:

$$f(s)\Big|_{s=m/2} = f\Big(\frac{m}{2}\Big) = \frac{0}{2} = 0.$$

Next, in order to look at the mean curvature H of $\partial M^2 \hookrightarrow M^2$, we calculate the second fundamental form h. We observe the normal vector ν of ∂M^2 is given as $\nu = u^{-2}(s)\partial_s$. Now we can make use of Proposition 1.6.4:

$$\begin{aligned}
h(\partial_\varphi, \partial_\varphi) &= -g\Big({}^2\nabla_{\partial_\varphi}\partial_\varphi, \nu\Big) \Big|_{s=m/2} \\
&= -u^{-2}(s)\; g\Big({}^2\nabla_{\partial_\varphi}\partial_\varphi, \partial_s\Big) \Big|_{s=m/2} \\
&= -u^{-2}(s)\; {\Gamma_{\varphi\varphi}}^s \; \underbrace{g(\partial_s, \partial_s)}_{=u^4(s)} \Big|_{s=m/2} \\
&= -u^2(s)\; {\Gamma_{\varphi\varphi}}^s \Big|_{s=m/2}.
\end{aligned}$$

This Christoffel symbol is easy to calculate:

$$\begin{aligned}
{\Gamma_{\varphi\varphi}}^s\Big|_{s=m/2} &= -\frac{1}{2}\, g^{ss}\, (\partial_s\, g_{\varphi\varphi}) \Big|_{s=m/2} \\
&= -\frac{1}{2}u^{-4}(s)\Big(2su^4(s) - 4s^2u^3(s)\frac{m}{2s^2}\Big) \Big|_{s=m/2} \\
&= -s + mu^{-1}(s) \Big|_{s=m/2} \\
&= -\frac{m}{2} + \frac{m}{2} \\
&= 0.
\end{aligned}$$

Hence, we obtain

$$h(\partial_\varphi, \partial_\varphi) = -u^2(s)\; {\Gamma_{\varphi\varphi}}^s \Big|_{s=m/2} = 0 \qquad \Longrightarrow \qquad H \equiv 0.$$

□

Proposition 3.2.4 basically shows that the Schwarzschild and the Pseudo-Schwarzschild solution both share a very similar horizon. This leads to the question if the same is valid for photon surfaces and spheres. In [Foertsch et al., 2003], a lot of analysis of photon surfaces is done. In particular, the authors worked out examples of photon surfaces in the Pseudo-

Schwarzschild solution, which, in fact, is one of the reasons this bachelor thesis exists. As well as the horizon, the photon sphere in Pseudo-Schwarzschild behaves the same way as it does in Schwarzschild.

Theorem 3.2.5 Let $(\mathfrak{L}^3, \mathfrak{g})$ be the $(2+1)$-dimensional Pseudo-Schwarzschild solution. Then the set $P^2 = \{r = 3m\}$ is a photon sphere in $\mathfrak{L}^3$.

Proof. The proof works the exact same way as the proof of Theorem 2.2.1. $\square$

3.3 Construction with Pseudo-Schwarzschild

In this section we discuss whether the same or a similar construction as in the proof of Theorem 2.3.2 is possible in $(2+1)$ dimensions. This question essentially breaks down to the major differences of two instead of three space dimensions and their consequences.

In Section 1.5, we defined the ADM-mass for dimensions greater than two. However, from a purely formal point of view, the formula is applicable also for $n = 2$. Brute force calculation in isotropic coordinates gives

$$\begin{aligned}
m_{ADM} &= \frac{1}{2(n-1)\omega_{n-1}} \lim_{s\to\infty} \int_{\mathbb{S}_s} \sum_{i,j} \left(\partial_i {}^2g_{ij} - \partial_j {}^2g_{ii}\right) \nu^j \mathrm{d}\xi(s) \\
&= \frac{1}{4\pi} \lim_{s\to\infty} \int_{\mathbb{S}_s} \sum_{i,j} 4\left(1+\frac{m}{2s}\right)^3 \left(-\frac{m}{2s^2}\right) \left[\delta_{ij}\frac{x^i}{s} - \delta_{ii}\frac{x^j}{s}\right] \frac{x^j}{s} \mathrm{d}\xi(s) \\
&= \frac{-m}{2\pi} \lim_{s\to\infty} \frac{1}{s^2}\left(1+\frac{m}{2s}\right)^3 \underbrace{\int_{\mathbb{S}_s} \frac{-1}{s^2}\left(\left(x^1\right)^2 + \left(x^2\right)^2\right)\mathrm{d}\xi(s)}_{=-2\pi s} \\
&= m \lim_{s\to\infty} \frac{1}{s}\left(1+\frac{m}{2s}\right)^3 \\
&= 0.
\end{aligned}$$

What is the reason for this result? The Pseudo-Schwarschild solution has the very same conformal factor as the Schwarzschild solution. This means the powers of the radial component follows the same asymptotic, however, the area element of a one-dimensional sphere has the 'wrong' power to cancel the terms of the conformal factor.

Also, the notion of asymptotic flatness is not clear in two dimensions. Recall Equation

(1.6) in Definition 1.5.1:

$$g_{ij} = \delta_{ij} + \mathcal{O}_k\Big(r^{-p}\Big), \qquad p > (n-2)/2.$$

By setting $n = 2$ we would get $p > 0$. This means, the metric is allowed to flatten out arbitrary slowly. This seems not sufficient to call something 'asymptotically flat'.
Another hint why the two-dimensional case should be treated differently is in the proof of Corollary 1.5.3. It is shown that the ADM-mass, moreover its convergence, is deeply linked to the scalar curvature and its integral over the manifold. Now, in dimension two, a vanishing scalar curvature implies local flatness of the metric. This is not true for any higher dimension. Furthermore, for a compact two-dimensional manifold, the Gauss–Bonnet Theorem holds true. In this case, the integral of the scalar curvature is a topological invariant.

Besides the problem of a proper definition of mass, another big difference is the assumption of a vacuum solution. If we want to generate a vacuum solution in $(2+1)$ dimensions, that is $= 0$, the Riemannian curvature would vanish[1] and the whole spacetime were flat (see Remark 1.4.3). The vacuum equations for a static spacetime imply scalar flat time slices (Proposition 1.4.8). As we calculated this to be false for the Pseudo-Schwarzschild solution (Proposition 3.2.3), we know at least one of the equations

$$\begin{aligned} N \; &= \nabla^2 N \\ \Delta N &= 0 \end{aligned}$$

has to fail. We calculate

$$\begin{aligned} {}^{2g}\Delta N &= {}^{2}g^{jk}\,\partial_j\partial_k N - {}^{2}g^{jk}\,\Gamma_{jk}{}^{l}\,\partial_l N \\ &= {}^{2}g^{rr}\,\partial_r^2 N - \Big({}^{2}g^{rr}\,\Gamma_{rr}{}^{r} + {}^{2}g^{\varphi\varphi}\,\Gamma_{\varphi\varphi}{}^{r}\Big)\,\partial_r N \\ &= \frac{-m^2}{r^4 N} - \frac{2mN}{r^3} + \frac{m^2}{r^4 N} + \frac{mN}{r^3} \\ &= -\frac{m}{r^3} N \\ &= \frac{1}{2}\,{}^{2g}R\,N \neq 0. \end{aligned}$$

[1]There are non-trivial, $(2+1)$-dimensional vacuum solutions of the Einstein equations if one allows a cosmological constant, see e.g. [Banados et al., 1992].

So, by tracing the first equation, both of the static vacuum equations are not true for the Pseudo-Schwarzschild solution. Those equations played an important role in Step 3 of the proof of Theorem 2.3.2. They ensured the newly constructed manifold to be scalar flat.

What is still possible? The doubling in Step 2 is allowed for the Pseudo-Schwarzschild solution, since we have verified that it possesses a black hole horizon arising as its minimal boundary (Proposition 3.2.4). In the case of the Pseudo-Schwarzschild solution, Step 3 is also possible: Proposition 3.2.1 ensures the conformal transformed manifold to be scalar flat, since

$$ {}^2\widehat{g} = {}^2\delta = u^{-4}(s)\,{}^2g \quad \Longrightarrow \quad {}^{2\widehat{g}}R = 0. $$

In general, this is not true. The right hand side of the transformation formula (i) for the scalar curvature in two space dimensions is not necessarily zero without the conditions of the static vacuum equations (as stated above).

However, a formal inversion in the 2-sphere and, therefore, one-point compactification are still possible: Assuming the asymptotic Schwarzschildean conditions of a three-dimensional manifold for a static, two-dimensional spacetime with lapse function $N = 1 - \frac{m}{s} + \mathcal{O}_k(s^{-2})$ we find

$$ \begin{aligned} {}^2\widehat{g}_{ij} &= u^4\,{}^2g_{ij} = \left(\frac{1+N}{2}\right)^4 {}^2g_{ij} \\ &= \left(1 - \frac{m}{2s} + \mathcal{O}_k\big(s^{-2}\big)\right)^4 \left(\left(1+\frac{m}{2s}\right)^4 {}^2\delta_{ij} + \mathcal{O}_k\big(s^{-2}\big)\right) = {}^2\delta_{ij} + \mathcal{O}_k\big(s^{-2}\big) \end{aligned} $$

as $s \to \infty$. A similar calculation for the doubled part of the manifold gives

$$ \begin{aligned} {}^2\widehat{g}_{ij} &= u^4\,{}^2g_{ij} = \left(\frac{1-N}{2}\right)^4 {}^2g_{ij} \\ &= \left(\frac{m}{2s} + \mathcal{O}_k\big(s^{-2}\big)\right)^4 \left(\left(1+\frac{m}{2s}\right)^4 {}^2\delta_{ij} + \mathcal{O}_k\big(s^{-2}\big)\right) = \left(\frac{m}{2s}\right)^4 {}^2\delta_{ij} + \mathcal{O}_k\big(s^{-5}\big) \end{aligned} $$

as $s \to \infty$. Let (x^i) denote the coordinates corresponding to the Schwarzschildean asymptotic. We introduce new coordinates $X^i := (2/m)^2 x^i s^{-2}$. This gives

$$ \mathrm{d}X^i = \left(\frac{2}{m}\right)^2 \left(\frac{\mathrm{d}x^i}{s^2} - \frac{2x^i}{s^3}\,\mathrm{d}s\right), \qquad \mathrm{d}s = \frac{x_j}{s}\,\mathrm{d}x^j. $$

Now we can convert our metric:

$$\begin{aligned} {}^{2}\widehat{g}\,(\partial_{X^p}, \partial_{X^q}) &= \left(\frac{m}{2s}\right)^4 {}^{2}\delta_{ij}\, \mathrm{d}x^i(\partial_{X^p})\, \mathrm{d}x^j(\partial_{X^q}) + \mathcal{O}_k\Big(s^{-5}\Big) \\ &= {}^{2}\delta_{pq} + \mathcal{O}_k\Big(s^{-1}\Big) \end{aligned}$$

as $s \to \infty$. So, precisely as in Step 3 of Theorem 2.3.2, we are allowed to glue in a point at $s = \infty$.

Step 4 is invalid for the two-dimensional case since we have neither a Positive Mass Theorem nor a sensible relation of the parameter m of the lapse function to something we could call 'mass'.

As a conclusion to this work, we present two tables to illustrate the major similarities and differences of the Schwarzschild and the Pseudo-Schwarzschild solution in a compact way.

	Schwarzschild	Pseudo-Schwarzschild
metric	${}^{4}\mathfrak{g} = -N^2\,\mathrm{d}t^2 + N^{-2}\,\mathrm{d}r^2 + r^2\,\mathrm{d}\Omega^2$	${}^{3}\mathfrak{g} = -N^2\,\mathrm{d}t^2 + N^{-2}\,\mathrm{d}r^2 + r^2\,\mathrm{d}\varphi^2$
Ricci curvature	${}^{4}\mathfrak{g} = 0$ (vacuum, $= 0$)	non-vanishing ($= \frac{c^4}{8\pi G}$)
scalar curvature	${}^{4\mathfrak{g}}R = 0$	${}^{3\mathfrak{g}}R = 0$
static	✓	✓
lapse function	$N = (1 - \frac{2m}{r})^{1/2}, \quad {}^{3g}\Delta N = 0$	$N = (1 - \frac{2m}{r})^{1/2}, \quad {}^{2g}\Delta N \neq 0$
Kruskal extension	✓	✓
photon sphere	$\{r = 3m\}$	$\{r = 3m\}$

Figure 3.2: Schwarzschild and Pseudo-Schwarzschild as a spacetime.

	Schwarzschild	Pseudo-Schwarzschild
metric	${}^3g = N^{-2}\,\mathrm{d}r^2 + r^2\,\mathrm{d}\Omega^2$	${}^2g = N^{-2}\,\mathrm{d}r^2 + r^2\,\mathrm{d}\varphi^2$
scalar curvature	${}^{^3g}R = 0$	${}^{^2g}R = -2m/r^3 < 0$
conformally flat	${}^3\delta = u^{-4}(s)\,{}^3g, \quad u(s) = 1 + \frac{m}{2s}$	${}^2\delta = u^{-4}(s)\,{}^2g, \quad u(s) = 1 + \frac{m}{2s}$
asymptotically flat	✓	?
black hole horizon	$\{r = 2m\}$	$\{r = 2m\}$
ADM-mass	m	?

Figure 3.3: Time slice in Schwarzschild and Pseudo-Schwarzschild.

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