GENERALIZED RULES FOR QUANTIFIERS AND THE COMPLETENESS OF THE INTUITIONISTIC OPERATORS $\&, \vee, \supset, \lambda, \forall, \exists$.

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In this paper, we develop a proof-theoretic framework for the treatment of arbitrary quantifiers binding m variables in n formulas. In particular, we motivate a schema for introduction and elimination rules for such quantifiers based on a concept of 'derivation' that allows rules as assumptions which may be discharged. With respect to this schema, the system of the standard operators of intuitionistic quantifier logic turns out to be complete.

1. GENERAL INTRODUCTION

This paper, which is a sequel to [18], deals with a generalization of natural deduction systems. The calculi of natural deduction as developed by Gentzen [7] and investigated by Prawitz [13] have at least two distinctive features: Firstly they present a conceptualization of reasoning from assumptions in allowing that assumptions may be discharged in the course of a derivation, and secondly they contain a certain systematics in that the rules governing the logical signs are split up into introduction (I) and elimination (E) rules for each sign. Both aspects are especially important for intuitionistic logic. As to the first, the possibility of discharging assumptions directly admits a derivational interpretation of implication as opposed to the truth-functional one (the term 'derivativ' is due to Schmidt [16]): $\alpha\supset\beta$ means that β can be derived from α as is made obvious by the $\supset$I rule

$$\frac{\begin{matrix}[\alpha]\\ \beta\end{matrix}}{\alpha\supset\beta} \quad .$$

Concerning the second, the I and E rules for the intuitionistic system show a certain symmetry or duality in the sense that they can be considered inverses of each other (cf. Prawitz [13]) while in the classical case this symmetry is at least partly lost. For example, the absurdity rule

$$\frac{\lambda}{\alpha}$$

M.M. Richter, E. Börger, W. Oberschelp, B. Schinzel, W. Thomas (Eds.), Computation and Proof Theory. Proceedings of the Logic Colloquium held in Aachen, July 18-23, 1983, Part II. Berlin: Springer Lecture Notes in Mathematics, Vol. 1104, 1984.

can be conceived as the elimination rule of the 0-place operator λ and can be justified from the fact that there is no I rule for λ , whereas the corresponding classical rule of indirect proof

$$\frac{\begin{matrix}[\alpha\supset\lambda]\\ \lambda\end{matrix}}{\alpha}$$

cannot be so justified. Thus it is not surprizing that recent attempts at building up a proof-theoretical semantics for intuitionistic logic ('proof' here regarded not purely syntactically but in the traditional philosophical sense as the foundation of a proposition) are mainly based on systems of natural deduction (cf. Dummett [6], Prawitz [14]).

The extension of intuitionistic natural deduction which is proposed in the following, concerns both aspects. We shall first of all define notions of 'inference rule' as well as of 'derivation', according to which not only formulas but also rules themselves are allowed as assumptions which may be discharged by the application of rules. It seems quite natural to use as a hypothesis that one can <u>pass over</u> from derived formulas to other ones, and not only that one may <u>start</u> with a certain formula as a hypothesis. This can of course be achieved by formulas too; for instance, $\alpha\supset\beta$ as an assumption allows the transition from α to β by means of modus ponens. However, this is a <u>result</u> of our investigations which are intended to start from an intuitively plausible concept of 'derivation' which does not presuppose any specific inference rules. Derivations in a formal system can be considered derivations in a system without any basic rules, if <u>all</u> rules which are used are counted as assumptions. This is just the way derivations will be introduced: A derivation is defined as an arbitrary finite tree of pairs of formulas and individual variables, indicating the eigenvariables of inferences, together with a discharge function, indicating where an assumption is discharged (this is a generalization of Prawitz' notion of a discharge function in [13]). Those rules which justify such a tree as a derivation and which can be decoded from the tree are then the assumptions of this derivation, and the undischarged assumptions can be considered either assumptions on which the end-formula <u>depends</u> or applications of basic rules (if there are any). So the concept of a derivation is defined independently of the question of which assumptions are <u>ad hoc</u>, i.e. are ones on which a formula in a derivation depends, and which belong to the considered <u>framework</u>, i.e. are applications of basic rules - in the same way as in the usual concept of natural deduction it does not affect the intrinsic structure of a derivation (but only the set of formulas on which it depends) whether a certain formula is an axiom or an assumption. (Gentzen's system NI of natural deduction for intuitionistic logic does not contain any axiom but only proper rules, but this is a specific feature of this particular calculus and not of the type of calculus of which NI is a representative; for the classical system NK, e.g., Gentzen proposes the 'tertium non datur' as an axiom schema.)

The generalized notions of 'rule' and 'derivation' will then be used to make the often stated 'symmetry' or 'harmony' between the I and E rules for logical operators more explicit. This is done by providing a general schema for I and E rules of an arbitrary operator which captures the I and E rules for all standard intuitionistic connectives &,v, ⊃,⅄,∀,∃, and by a metalinguistic characterization in terms of derivability which justifies this general schema. For this purpose the vocabulary of our language is based on the generalized notion of a 'quantifier' or 'operator' which binds m individual variables in n formula-arguments, thus including the usual existential and universal quantifiers (m=1, n=1) and n-ary sentential connectives (m=0). The justification of this schema is based on the idea that from the conclusion of an introduction rule it should follow exactly what can be concluded from all its possible premisses (and what will be called the 'common content with respect to certain eigenvariables' of these premisses). In the case of $\exists x\alpha$, for example, all that follows from every substitution instance $\alpha[x|t]$ of α should be a consequence of $\exists x\alpha$. This is guaranteed by the $\exists$E rule

$$\frac{\exists x\alpha \quad \begin{matrix}[\alpha]\\ \beta\end{matrix}}{\beta}$$ (x not free in β nor in an assumption besides α on which β depends)

which states that whatever follows from α also follows from $\exists x\alpha$ where the eigenvariable x in a derivation from α is understood universally, i.e. representing all substitution instances of the derivation.

This idea is closely connected with the concept of rules as assumptions, because if one wants to speak of what follows from the premisses of an I rule, one must have representatives of these premisses which can serve as assumptions (if they are not simply formulas without dischargeable assumptions and eigenvariable conditions). Rules are very suitable for that purpose. For instance, the rule $\alpha \Rightarrow \beta$ ('from α you may infer β') represents the premiss (including the dischargeable assumption) of ⊃I and the rule $\Rightarrow_x \alpha$ ('for arbitrary t you may infer $\alpha[x|t]$') the premiss (including the eigenvariable condition) of ∀I. This is generalized in such a way that arbitrary finite lists of arbitrarily complex rules can be premisses of I rules and therefore assumptions of minor premisses of E rules.

This general conception of I and E rules for generalized operators suggests as a technical question whether a certain set of operators is complete with respect to this conception (in that sense of 'completeness' one uses when speaking of 'functional completeness' in classical sentential logic). It will be shown that the six standard intuitionistic operators $\&, \vee, \supset, \curlywedge, \forall, \exists$ suffice to explicitly define every other operator which falls under our general schema and that, therefore, all that can be formulated by use of rules as assumptions can be expressed with their help. This shows that rules as assumptions are superfluous once we have the standard operators at our disposal (but this is an insight for which the concept of such rules is necessary!).

2. RULES AND QUANTIFIERS

In the sentential case, as treated in [18], a rule was defined to be an arbitrary formula tree growing upwards, whose height was called its level. So the general form of a rule was

$$\frac{\dfrac{\Gamma_1}{\beta_1} \quad \ldots \quad \dfrac{\Gamma_n}{\beta_n}}{\alpha}$$

written linearly

$$\langle\Gamma_1 \Rightarrow \beta_1, \ldots, \Gamma_n \Rightarrow \beta_n\rangle \Rightarrow \alpha$$

where α and the β_i are formulas, Γ_i are lists (i.e. linear graphic arrangements) of rules (where formulas are special cases of rules, viz. rules of level 1). The intended meaning of such a rule, underlying the definition of a derivation and the derivability of a formula α from a list of rules Δ, was: If, for all i ($1 \leq i \leq n$), β_i has been derived from Γ_i and additional assumptions Δ_i (i.e., rules of Γ_i and Δ_i can have been used in the derivation of β_i), one may immediately infer α and consider it derived from $\Delta_1, \ldots, \Delta_n$ alone (i.e. the Γ_i may be discharged by the application of the rule).

The standard form for I and E rules for arbitrary n-ary sentential operators S, according to which, roughly speaking, the E rule allows one to establish all that is implied by the premisses of all I rules, was

$$(\text{S-I}) \qquad \frac{\Phi_1(\underline{A})}{S\underline{A}} \quad \ldots \quad \frac{\Phi_m(\underline{A})}{S\underline{A}}$$

$$(\text{S-E}) \qquad \frac{S\underline{A} \quad \dfrac{\Phi_1(\underline{A})}{B} \quad \ldots \quad \dfrac{\Phi_m(\underline{A})}{B}}{B}$$

where $\underline{A}$ is a list $A_1 \ldots A_n$ of different schematic letters for formulas, B a schematic letter for formulas different from $A_1, \ldots, A_n$ and the $\Phi_i(\underline{A})$ systems of rule schemata containing at most $A_1, \ldots, A_n$ as schematic letters. The rule schemata for the standard intuitionistic connectives were

$$(\&I) \quad \frac{A \quad B}{A\&B} \qquad\qquad (\&E) \quad \frac{A\&B \quad \dfrac{A \quad B}{C}}{C}$$

$$(\vee I) \quad \frac{A}{A\vee B} \quad \frac{B}{A\vee B} \qquad\qquad (\vee E) \quad \frac{A\vee B \quad \dfrac{A}{C} \quad \dfrac{B}{C}}{C}$$

$$(\supset I) \quad \frac{\dfrac{A}{B}}{A\supset B} \qquad\qquad (\supset E) \quad \frac{A\supset B \quad \dfrac{\dfrac{A}{B}}{C}}{C}$$

$$[\text{no } \curlywedge I] \qquad\qquad (\curlywedge E) \quad \frac{\curlywedge}{A}$$

Here $\supset$E, which can be shown to be equivalent to modus ponens (cf. [18], Lemma 4.4), is an example of a rule schema of level 4 (whereas usual natural deduction systems only contain rules of levels ≤ 3).

In order to treat quantifiers within a related framework, we must first have a concept of substitution of (individual) terms for (individual) variables at our disposal. Furthermore we must be able to express the fact that a rule holds for all substitutions of a variable by a term, as e.g. in the case of $\exists$I, and to express eigenvariable conditions, as e.g. in the case of $\exists$E. The latter include restrictions concerning the 'additional' assumptions, i.e. those assumptions on which the premisses of an application of a rule depend but which cannot be discharged by an application of that rule. These restrictions cannot be dealt with by an appropriate choice of the formulas which occur in the rule itself - at least as long as one works in a natural deduction framework where 'additional' assumptions are not made explicit in the formulation of a rule.

Our proposal is to introduce a certain kind of universal quantification into a rule. We define a variable-formula-pair (VF-pair) to be a sign $\langle \underline{x},\alpha\rangle$ consisting of a list of distinct variables $\underline{x}$ and a formula α, and define a rule of level n to be a tree of VF-pairs (growing upwards) of height n. The general schema of a rule then becomes

$$\frac{\dfrac{\Gamma_1}{\langle \underline{x}_1,\beta_1\rangle} \quad \cdots \quad \dfrac{\Gamma_n}{\langle \underline{x}_n,\beta_n\rangle}}{\langle \underline{x},\alpha\rangle} \qquad (1)$$

where the Γ_i are lists of rules. If $\underline{x}$ is the empty list, we simply write α instead of $\langle \underline{x},\alpha\rangle$; so formulas are special kinds of level-1-rules. The variables $\underline{y}$ of a VF-pair $\langle \underline{y},\gamma\rangle$ are considered bound in γ and in the formulas of all VF-pairs above $\langle \underline{y},\gamma\rangle$. This is made more obvious in our linear notation of (1):

$$\langle \Gamma_1 \Rightarrow_{\underline{x}_1} \beta_1,\ldots,\Gamma_n \Rightarrow_{\underline{x}_n} \beta_n\rangle \Rightarrow_{\underline{x}} \alpha \quad .$$

A list of variables $\underline{y}$ in $\ldots\Rightarrow_{\underline{y}}\ldots$ universally quantifies $\ldots\Rightarrow\ldots$ with respect to $\underline{y}$ in a certain sense. Variants of rules, resulting by relabelling such bound variables and adding or omitting vacuous quantifications, can then be defined in the obvious way, as can the substitution $[\underline{x}|\underline{t}]$ of appropriate lists $\underline{t}$ of terms for lists $\underline{x}$ of variables in rules (for precise definitions see section 3).

The intended meaning of (1) can then be stated as follows: For any variant

$$\frac{\dfrac{\Gamma'_1}{\langle \underline{y}_1,\beta'_1\rangle} \quad \cdots \quad \dfrac{\Gamma'_n}{\langle \underline{y}_n,\beta'_n\rangle}}{\langle \underline{y},\alpha'\rangle}$$

of (1) and any appropriate list of terms $\underline{t}$: If for each i $(1 \leq i \leq n)$ $\beta'_i[\underline{y}|\underline{t}]$ has been derived from $\Gamma'_i[\underline{y}|\underline{t}]$ and additional assumptions Δ_i not containing $\underline{y}_i$ free, then one may immediately infer $\alpha'[\underline{y}|\underline{t}]$ and consider it derived from $\Delta_1,\ldots,\Delta_n$ alone (i.e. the $\Gamma'_i[\underline{y}|\underline{t}]$ may be discharged by the application of (1)).

If, for example, α is a formula containg only x as a free variable and β a formula without free variables,

$$\frac{\alpha}{\langle x,\exists x\alpha\rangle} \qquad \frac{\langle x,\alpha\rangle}{\forall x\alpha} \qquad \frac{\exists x\alpha \qquad \dfrac{\alpha}{\langle x,\beta\rangle}}{\beta} \qquad \frac{\forall x\alpha \qquad \dfrac{\langle x,\alpha\rangle}{\langle x,\beta\rangle}}{\beta}$$

are instances of ∃I, ∀I, ∃E and ∀E. The last rule is equivalent to the usual

$$\frac{\forall x\alpha}{\langle x,\alpha\rangle} \quad ,$$

but falls under a standard form for E rules.

In the following section we define in detail the notion of a derivation of formulas from rules as assumptions, following the intended meaning of a rule as stated above. This theory is applied directly to a language for quantifiers in a generalized sense. Such a quantifier is considered to be an operator which gives a formula from n_1 variables and n_2 formulas, where n_1 and n_2 are natural numbers. (n_1,n_2) is then called its <u>type</u>. If S is a quantifier of type (n_1,n_2) and $x_1,\dots,x_{n_1}$ are distinct variables and $\alpha_1,\dots,\alpha_{n_2}$ formulas, then $Sx_1\dots x_{n_1}\alpha_1\dots\alpha_{n_2}$ is a formula, in which free occurrences of $x_1,\dots,x_{n_1}$ in $\alpha_1,\dots,\alpha_{n_2}$ become bound. If $n_1 = 0$, S is a sentential operator. In the following we shall simply speak of operators instead of quantifiers in the generalized sense.

<u>Historical Remark</u>. Our notion of a quantifier or operator of type (n_1, n_2) corresponds to the notion of a variable-binding operator of degree $(1,n_1,0,n_2)$ in Kalish/Montague [8], i.e. a variable-binding operator without terms as arguments or values. Borkowski [3] considers only quantifiers of type $(1,n_2)$. Concepts of rules of higher levels with bound variables can be found in Lorenzen [10] and Prawitz [12,15]. Both approaches differ from the one presented here in that they consider the consequence relation to be a relation between rules and allow - in our terminology - iteration of ⇒ to the right in the linear notation of a rule so that a rule does not have a tree structure. Furthermore, Lorenzen's theory is based on the concept of admissibility whereas we take the concept of a formula being <u>derived from</u> assumptions to be primary and not the concept of a formula being <u>derivable if</u> certain assumptions are <u>derivable</u>.

3. THE LANGUAGE. DERIVATIONS FROM RULES

When we speak of a <u>list</u>, we mean a (possibly empty) linear graphic arrangement of symbols which are called its <u>members</u>, i.e. a sequence in the graphic sense, not in the sense of an abstract mathematical entity. Its number of members is called its <u>length</u>. Analogously, trees are conceived as graphic objects.

As basic signs we assume to be given:

(i) Denumerably many (individual) variables (syntactical variables for them: 'x', 'y', 'z', for lists of distinct variables: '$\underline{x}$', '$\underline{y}$', '$\underline{z}$', all with and without ' and indices).

(ii) Denumerably many (individual) terms, forming a (not necessarily proper) superset of the set of variables (syntactical variables for them: 't', for lists of terms: '$\underline{t}$', both with and without ' and indices).

(iii) For each number of argument places denumerably many predicate letters (syntactical variables: 'P', with and without indices).

(iv) Finitely or denumerably many *operators*, each with an associated pair (n_1,n_2) of natural numbers as its type (syntactical variable: 'S').

Atomic formulas are of the form $P\underline{t}$, where the length of $\underline{t}$ is equal to the arity of P. *Formulas* are atomic formulas and signs $S\underline{x}\underline{\alpha}$ where, if S is of type (n_1,n_2), $\underline{x}$ is of length n_1 and $\underline{\alpha}$ is a list of formulas of length n_2. In the case of binary sentential operators we may write $(\alpha_1 S\alpha_2)$, where outer brackets can be omitted. *Parts* of a formula β are β itself and the parts of members of $\underline{\alpha}$ if β is $S\underline{x}\underline{\alpha}$. Syntactical variables for formulas: 'α', 'β', 'γ', for lists of formulas: '$\underline{\alpha}$', '$\underline{\beta}$', '$\underline{\gamma}$', all with and without ' and indices.

The members of $\underline{x}$ in $S\underline{x}\underline{\alpha}$ are considered binding corresponding occurrences of variables in the members of $\underline{\alpha}$. All elements of $\underline{\alpha}$ belong to the scope of $\underline{x}$. So free and bound (occurrences of) variables in formulas can be defined as usual. $\underline{x}$ is free in $\underline{\alpha}$ if for each member x of $\underline{x}$, x is free in $\underline{\alpha}$. t is free for x in α if x does not occur free in α within the scope of a variable which occurs also in t. $\underline{t}$ is free for $\underline{x}$ in α if $\underline{x}$ and $\underline{t}$ are of the same length n and for each i $(1\leq i\leq n)$, the i-th member t_i of $\underline{t}$ is free for the i-th member x_i of $\underline{x}$ in α. $\alpha[\underline{x}|\underline{t}]$ is defined if $\underline{t}$ is free for $\underline{x}$ in α and is the result of simultaneously substituting the free occurrences of x_i in α by t_i $(1\leq i\leq n)$ if $\underline{x}$ is $x_1\ldots x_n$ and $\underline{t}$ is $t_1\ldots t_n$. $\underline{\alpha}[\underline{x}|\underline{t}]$ is defined if for all members β of $\underline{\alpha}$, $\beta[\underline{x}|\underline{t}]$ is defined, and is the result of forming $\beta[\underline{x}|\underline{t}]$ for all β.

Rules of level n were already defined in § 2 (see (1)) as finite trees of height n of VF-pairs $\langle\underline{x},\alpha\rangle$ called their *elements*, where $\underline{x}$ binds corresponding occurrences of variables in α and in formulas above $\langle\underline{x},\alpha\rangle$, and where, if $\underline{x}$ is empty, α is identified with $\langle\underline{x},\alpha\rangle$. *Parts* of a rule ρ are ρ itself and the rules $\rho_1,\ldots,\rho_n$ if

$$\frac{\rho_1 \quad \cdots \quad \rho_n}{\langle\underline{x},\alpha\rangle}$$

is a part of ρ (i.e., the parts of ρ are ρ and all subtrees of ρ). Proper parts of ρ are parts of ρ different from ρ. Syntactical variables for rules: 'ρ', for lists of rules: 'Δ', 'Γ', all with and without ' and indices.

We say that a list of variables does not occur free in a rule if none of its members occurs free in the rule. So '$\underline{t}$ is free for $\underline{x}$ in ρ' is defined in the obvious way, and $\rho[\underline{x}|\underline{t}]$ as well. $\underline{t}$ is free for $\underline{x}$ in a list of rules Δ if $\underline{t}$ is free for $\underline{x}$ in all its members. $\Delta[\underline{x}|\underline{t}]$ is defined memberwise. When making an assertion about an $\alpha[\underline{x}|\underline{t}]$, we shall understand this assertion to be restricted to those $\underline{t}$ such that $\alpha[\underline{x}|\underline{t}]$ is defined.

A variant of ρ results from ρ by re-ordering lists of variables, omitting or adding vacuous quantifications or re-labelling the variables $\underline{x}$ of a variable-formula-pair, i.e. by once or more often replacing a part $\Delta \Rightarrow_{\underline{x}} \alpha$ of ρ by (i) $\Delta \Rightarrow_{\underline{x}_1} \alpha$ where $\underline{x}_1$ contains the same variables as $\underline{x}$ in a different order, (ii) $\Delta \Rightarrow_{\underline{x}_1} \alpha$ where $\underline{x}_1$ results form $\underline{x}$ by omitting or adding variables not occurring free in $\Delta \Rightarrow \alpha$, (iii) $\Delta[\underline{x}|\underline{y}] \Rightarrow_{\underline{y}} \alpha[\underline{x}|\underline{y}]$ provided $\underline{y}$ is free for $\underline{x}$ in $\Delta \Rightarrow \alpha$ and no member of $\underline{y}$ is free in $\Delta \Rightarrow_{\underline{y}} \alpha$.

A subrule of ρ results by arbitrarily often performing one of the following transformations, starting with ρ:
(i) Transforming a rule into a variant.
(ii) Specializing of variables $\underline{x}$ in a rule $\Delta \Rightarrow_{\underline{x}} \alpha$ to $\underline{t}$, yielding $\Delta[\underline{x}|\underline{t}] \Rightarrow \alpha[\underline{x}|\underline{t}]$.
(iii) Transforming a rule $\langle \Gamma_1 \Rightarrow_{\underline{x}_1} \beta_1, \ldots, \Gamma_n \Rightarrow_{\underline{x}_n} \beta_n \rangle \Rightarrow_{\underline{x}} \alpha$ into $\langle \Gamma'_1 \Rightarrow_{\underline{x}_1} \beta_1, \ldots, \Gamma'_n \Rightarrow_{\underline{x}_n} \beta_n \rangle \Rightarrow_{\underline{x}} \alpha$ where Γ'_i $(1 \leq i \leq n)$ results from Γ_i by omitting, duplicating, re-ordering members of Γ_i and replacing members of Γ_i by subrules of them.
If ρ' is a subrule of ρ, then, according to the intended meaning of a rule, an application of ρ' can be conceived as an application of ρ as well.

In the sentential case ([18]), a derivation was considered a pair (T, f) consisting of a finite formula tree T and a discharge function f, i.e. a function defined on the set of all formula occurrences of T such that f(α) is either α or a formula occurrence below α, indicating where the rule with conclusion α is discharged (it remains undischarged if f(α) is the lowermost formula of T). In the quantifier case we also have to make explicit the eigenvariables related to an inference step. For that purpose one could define an assignment of eigenvariables e for T to be a function which associates a list of variables $\underline{x}$ with each formula occurrence α of T besides the lowermost one, having the intended meaning that the substitution of terms $\underline{t}$ for $\underline{x}$ is blocked above α, i.e. that the inference step with α as one of its premisses need not remain valid if $\underline{x}$ is substituted by a list of terms $\underline{t}$. A derivation could then be conceived of as a triple (T,e,f) consisting of a finite formula tree T, an assignment of eigenvariables e for T and a discharge function f for T.

For technical purposes however it is easier to consider VF-pairs $\langle\underline{x},\alpha\rangle$ instead of assignments of eigenvariables to formulas and to let a discharge function operate on VF-pairs instead of formulas. So we define a <u>discharge function</u> for a finite tree T of VF-pairs to be a function f defined on the set of all elements of T such that $f(\langle\underline{x},\alpha\rangle)$ is either $\langle\underline{x},\alpha\rangle$ or a VF-pair below $\langle\underline{x},\alpha\rangle$. A <u>derivation</u> is a pair (T,f) consisting of a finite tree T of VF-pairs, in whose lowermost element the list of variables is empty, and a discharge function f for T. Obviously, this approach is equivalent to the previously sketched one using an assignment e of eigenvariables, and everything which follows could be translated into the former approach (take $\underline{x}$ in $\langle\underline{x},\alpha\rangle$ to be $e(\alpha)$ and identify $f(\langle\underline{x},\alpha\rangle)=f(\langle\underline{y},\beta\rangle)$ with $f(\alpha)=f(\beta)$). With our latter definitions, however, we can immediately take over the notions defined for rules (i.e. trees of VF-pairs): Derivations (T,f) and (T',f') are called <u>variants</u> of each other, if T and T' are variants of each other and $f(\langle\underline{x},\alpha\rangle)$ and $f'(\langle\underline{x}',\alpha'\rangle)$ are corresponding elements of T and T' whenever $\langle\underline{x},\alpha\rangle$ and $\langle\underline{x}',\alpha'\rangle$ are corresponding elements of T and T'. <u>Substitution</u> in derivations is defined as follows: If $\underline{t}$ is free for $\underline{x}$ in T, then $(T,f)[\underline{x}|\underline{t}]$ is defined to be the derivation $(T[\underline{x},\underline{t}],f')$ where $f'(\langle\underline{y},\alpha\rangle[\underline{x}|\underline{t}])= f(\langle\underline{y},\alpha\rangle)[\underline{x}|\underline{t}]$.

The rules (= the assumptions!) which are used in a derivation and thus justify its inference steps are not considered part of the derivation itself. Rather, they are assigned to it by a function g associating with each VF-pair $\langle\underline{x},\alpha\rangle$ a rule ρ in such a way that $\langle\underline{x},\alpha\rangle$ may be considered the result of an application of ρ (or of a rule of which ρ is a subrule). This coincides with the usual view that comments stating which rule is applied in a particular step belong to the metalanguage. Such a rule assignment g is determined to a great extent by the derivation (T,f): it has to be chosen in accordance with the intended meaning of a discharge function which is, that by the application of the rule one of whose premisses is $f(\langle \underline{x},\alpha\rangle)$, the application of the rule whose conclusion is $\langle \underline{x},\alpha\rangle$ is discharged; moreover with the intended meaning of the eigenvariables $\underline{x}$ of VF-pairs $\langle \underline{x},\alpha\rangle$ of T which is that they become bound in all rules which have been applied above $\langle \underline{x},\alpha\rangle$ but not yet discharged above $\langle \underline{x},\alpha\rangle$. So we define:

g is a rule assignment for a derivation (T,f), if g associates a rule $g(\langle \underline{x},\alpha\rangle)$ with each VF-pair $\langle \underline{x},\alpha\rangle$ of T such that:
If $\langle \underline{x},\alpha\rangle$ occurs in T as

$$\frac{\begin{matrix}\vdots \\ \langle \underline{x}_1,\beta_1\rangle\end{matrix} \quad \cdots \quad \begin{matrix}\vdots \\ \langle \underline{x}_n,\beta_n\rangle\end{matrix}}{\langle \underline{x},\alpha\rangle},$$

then

$$g(\langle \underline{x},\alpha\rangle) \quad = \frac{\dfrac{\Gamma_1}{\langle \underline{x}_1,\beta_1\rangle} \quad \cdots \quad \dfrac{\Gamma_n}{\langle \underline{x}_n,\beta_n\rangle}}{\langle \underline{y},\alpha\rangle}$$

where for all i $(1\leq i\leq n)$, Γ_i contains all rules $g(\langle \underline{z},\gamma\rangle)$ for all $\langle \underline{z},\gamma\rangle$ such that $f(\langle \underline{z},\gamma\rangle)=\langle \underline{x}_i,\beta_i\rangle$, and where $\underline{y}$ contains all variables belonging to a VF-pair which equals or is below $\langle \underline{x},\alpha\rangle$ and is properly above $f(\langle \underline{x},\alpha\rangle)$ in T. This definition contains as a limiting case the occurrence of $\langle \underline{x},\alpha\rangle$ as a top VF-pair.

There are only finitely many rule assignments for a given derivation, since they can differ only in the order of the members of $\underline{y}$ and of the Γ_i. In particular, if g and g' are rule assignments for (T,f) and $\langle \underline{x},\alpha\rangle$ is an element of T, then $g(\langle \underline{x},\alpha\rangle)$ and $g'(\langle \underline{x},\alpha\rangle)$ are subrules of each other.

So far we have defined what a derivation looks like and how to find rules which justify the inference steps, but not on which assumptions a derivation or its lowermost formula depends. For that purpose one has to consider the undischarged assumptions of a derivation (T,f) with respect to a rule assignment g for (T,f), which, according to the intended meaning of f, are defined to be the rules assigned by g to those VF-pairs $\langle \underline{x},\alpha\rangle$ of T for which $f(\langle \underline{x},\alpha\rangle)$ is the lowermost formula of T. We define an assumption system for (T,f) to be a list Γ of rules such that for a given rule assignment g for (T,f) each undischarged assumption is a subrule of a member of Γ (this definition is independent of the choice of g!). Now a derivation (T,f) with α as the lowermost formula of T and Γ as an assumption system for (T,f) can be considered a derivation of α from Γ.

However, some members of Γ may be basic rules of a calculus, i.e. belong to the given framework and are not assumed ad hoc within a derivation, and therefore should not be counted as something on which a derived formula depends. So we define: Let a set R of rules which are distinguished as basic rules be given. Then a derivation (T,f) is a derivation of α from Δ in R (i.e., in the calculus having R as its set of basic rules), if α is the lowermost formula of T and there is an assumption system Γ for (T,f) such that each member of Γ belongs either to Δ or is a subrule of an element of R. This definition includes as a limiting case that R is empty, i.e. no basic rules are given. As can easily be seen, the concept of a derivation from Δ in R is decidable, if 'subrule of a rule of R' is decidable. α is derivable from Δ in R ('$\Delta \vdash_R \alpha$'), if there is a derivation of α from Δ in R. We write '$\vdash$' instead of '$\vdash_R$' if a statement is independent of a specific choice of basic rules or if it is obvious what is meant.

Concerning the set of basic rules R our only restriction is that basic rules contain no free variables (this will be crucial for the important lemma 3.1(iv)). When writing a basic rule in the form

$$\frac{\Delta}{\alpha}$$

we mean the rule

$$\frac{\Delta}{\langle \underline{x},\alpha\rangle}$$

where $\underline{x}$ contains all variables free in Δ or α (in a certain standard order, for the sake of uniqueness). This convention is important if we write basic rules schematically, since then different instances of a schema have different lists $\underline{x}$ of variables of that kind.

Lemma 3.1:

(i) If ρ_1 is a subrule of ρ_2, then $\rho_1[\underline{x}|\underline{t}]$ is a subrule of $\rho_2[\underline{x}|\underline{t}]$.

(ii) Let (T',f') be a variant of a derivation (T,f) and g be a rule assignment for (T,f). Then there is a rule assignment g' for (T',f') such that for each $\langle \underline{x}',\alpha'\rangle$ of T' which corresponds to $\langle \underline{x},\alpha\rangle$ in T, $g'(\langle \underline{x}',\alpha'\rangle)$ is a variant of $g(\langle \underline{x},\alpha\rangle)$.

(iii) Let (T,f) be a derivation. Let $\underline{t}$ be free for $\underline{x}$ in T. If g is a rule assignment for (T,f), then g' is a rule assignment for $(T,f)[\underline{x}|\underline{t}]$ where $g'(\langle\underline{y},\alpha\rangle[\underline{x}|\underline{t}])=g(\langle\underline{y},\alpha\rangle)[\underline{x}|\underline{t}]$.

(iv) If $\Gamma\vdash\alpha$, then for all $\underline{x}$, $\underline{t}$: $\Gamma[\underline{x}|\underline{t}]\vdash\alpha[\underline{x}|\underline{t}]$.

Proof: (i) - (iii) follow straightforward from our definitions. (iv) follows by use of (i) - (iii): Consider a derivation (T,f) of α from Γ, choose a variant (T',f') of (T,f) in such a way that $\underline{x}$ is free for $\underline{t}$ not only in Γ and α but also in T, and apply (ii), (iii) and (i) (note that basic rules contain no free variables).

A rule $\Delta\Rightarrow_{\underline{x}}\gamma$ is called derivable from Γ in R, if for all its variants $\Delta'\Rightarrow_{\underline{y}}\gamma'$ and for all $\underline{t}$ such that $(\Delta'\Rightarrow\gamma')[\underline{y}|\underline{t}]$ has the form $\langle\Gamma_1\Rightarrow_{\underline{x}_1}\beta_1,\ldots,\Gamma_n\Rightarrow_{\underline{x}_n}\beta_n\rangle\Rightarrow\alpha$, the following holds for all $\Delta_1,\ldots,\Delta_n$: If, for each i $(1\leq i\leq n)$: $\Gamma,\Delta_i,\Gamma_i \vdash_R \beta_i$, where no variable of $\underline{x}_i$ occurs free in Δ_i or Γ, then $\Gamma,\Delta_1,\ldots,\Delta_n \vdash_R \alpha$. This definition follows the intended meaning we have given to a rule (see § 2).

We shall use $\Gamma\vdash\Delta\Rightarrow\gamma$ as an abbreviation for $\Gamma,\Delta\vdash\gamma$. $\Gamma\vdash\Delta\Rightarrow_{\underline{x}}\gamma$ expresses that for a variant $\Delta'\Rightarrow_{\underline{y}}\gamma'$ of $\Delta\Rightarrow_{\underline{x}}\gamma$ such that no variable of $\underline{y}$ is free in Γ: $\Gamma,\Delta'\vdash\gamma'$. By lemma 3.1 (iv), this then holds for any variant of this kind. Furthermore this is, again by lemma 3.1 (iv), equivalent to the statement that for all $\underline{t}$: $\Gamma,\Delta'[\underline{y}|\underline{t}]\vdash\gamma'[\underline{y}|\underline{t}]$, where $\Delta'\Rightarrow_{\underline{y}}\gamma'$ is any variant of $\Delta\Rightarrow_{\underline{x}}\gamma$ (without restriction). Our notation $\Gamma\vdash\rho$, which suggests that ρ is derivable from Γ, but is not defined in this way, will be justified by lemma 3.3 which is based on lemma 3.2.

Lemma 3.2:

(i) $\rho\vdash\rho$ (i.e. $\Delta\Rightarrow_{\underline{x}}\alpha$, $\Delta\vdash\alpha$).

(ii) If $\Delta\vdash\rho$ and $\Delta,\rho\vdash\gamma$ then $\Delta\vdash\gamma$.

Proof: See the proofs of lemmata 3.4 and 3.5 in [18]. Only a few additions are necessary which deal with bound variables.

Lemma 3.3: $\Gamma\vdash\rho$ iff ρ is derivable from Γ.

Proof: Let ρ be $\Delta\Rightarrow_{\underline{x}}\gamma$. $\Gamma\vdash\rho$ means that for all variants $\Delta'\Rightarrow_{\underline{y}}\gamma'$ of ρ and all $\underline{t}$ we have $\Gamma,\Delta'[\underline{y}|\underline{t}]\vdash\gamma'[\underline{y}|\underline{t}]$. Let this be of the form:

$$\Gamma,\Gamma_1\Rightarrow_{\underline{x}_1}\beta_1,\ldots,\Gamma_n\Rightarrow_{\underline{x}_n}\beta_n\vdash\alpha \quad . \tag{2}$$

If we have for all i ($1 \le i \le m$): $\Gamma, \Delta_i, \Gamma_i \vdash \beta_i$ where no variable of $\underline{x}_i$ occurs free in Γ or Δ_i, we obtain, since this can be written as $\Gamma, \Delta_i \vdash \Gamma_i \Rightarrow_{\underline{x}_i} \beta_i$, by (2) and lemma 3.2 (ii) (n-fold application): $\Gamma, \Delta_1, \ldots, \Delta_n \vdash \alpha$. Conversely, since by lemma 3.2 (i) it holds that for all i ($1 \le i \le n$): $\Gamma_i \Rightarrow_{\underline{x}_i} \beta_i, \Gamma_i \vdash \beta_i$, the derivability of ρ from Γ implies (2).

By this lemma we may use $\Gamma \vdash \rho$ as an alternative formulation of the derivability of ρ from Γ. $\Delta \vdash \Gamma$ means that $\Delta \vdash \rho_i$ for all i ($1 \le i \le n$) if Γ is the list $\rho_1 \ldots \rho_n$. As a limiting case, $\Delta \vdash \Gamma$ is considered to be true if Γ is empty. $\Delta \dashv\vdash \Gamma$ means that $\Delta \vdash \Gamma$ and $\Gamma \vdash \Delta$.

Lemma 3.4: Let $\alpha \dashv\vdash \beta$. Let ρ contain $\langle \underline{x}, \alpha \rangle$ as an element and let ρ' be the result of replacing this element by $\langle \underline{x}, \beta \rangle$. Then $\rho \dashv\vdash \rho'$.

Proof: Let ρ_1 be the part of ρ whose lowermost element is $\langle \underline{x}, \alpha \rangle$ (linearly written: $\Delta \Rightarrow_{\underline{x}} \alpha$), and let ρ_1' be $\Delta \Rightarrow_{\underline{x}} \beta$. From $\Delta \Rightarrow_{\underline{x}} \alpha, \Delta \vdash \alpha$ and $\alpha \vdash \beta$ it follows that $\Delta \Rightarrow_{\underline{x}} \alpha \vdash \Delta \Rightarrow_{\underline{x}} \beta$; analogously $\Delta \Rightarrow_{\underline{x}} \beta \vdash \Delta \Rightarrow_{\underline{x}} \alpha$. So we have $\rho_1 \dashv\vdash \rho_1'$. If ρ_1 is identical with ρ, nothing remains to be shown. Let ρ_1 occur as a proper part of a part ρ_2 of ρ of the form $\langle \ldots \rho_1 \ldots \rangle \Rightarrow_{\underline{y}} \gamma$ and let ρ_2' be $\langle \ldots \rho_1' \ldots \rangle \Rightarrow_{\underline{y}} \gamma$. Then $\rho_2, \ldots, \rho_1, \ldots \vdash \gamma$ by lemma 3.2 (i) and $\rho_2, \ldots, \rho_1', \ldots \vdash \gamma$ by lemma 3.2 (ii), i.e. $\rho_2 \vdash \rho_2'$. In the same way we obtain $\rho_2' \vdash \rho_2$. Repeated application of this procedure yields $\rho \dashv\vdash \rho'$.

4. BASIC RULES FOR OPERATORS

We shall define a standard form for schematically given basic rules (more precisely, I and E rules) for operators. For this purpose we assume to be given:

(i) Denumerably many schematic letters to be instantiated by formulas (syntactical variables for them: 'A', 'B', 'C', for lists of distinct letters: '$\underline{A}$', '$\underline{B}$', '$\underline{C}$', all with and without indices).

(ii) For each list of distinct schematic letters for formulas $\underline{A}$, denumerably many schematic letters to be instantiated by variables which do not occur (neither free nor bound) in a member of the list of formulas by which $\underline{A}$ is instantiated (syntactical variables for them: '$X_{\underline{A}}$', '$Y_{\underline{A}}$', '$Z_{\underline{A}}$' where 'X', 'Y', 'Z' may have an index, for lists of distinct letters: '$\underline{X}_{\underline{A}}$', '$\underline{Y}_{\underline{A}}$', '$\underline{Z}_{\underline{A}}$', for lists of distinct letters of the kind $Y_{1\underline{A}_1} \ldots Y_{n\underline{A}_n}$: '$\underline{U}$', '$\underline{V}$'). If $\underline{A}$ is empty these schematic letters can be instantiated by any variable (syntactical variables in that case: 'X', 'Y', 'Z', '$\underline{X}$', '$\underline{Y}$', '$\underline{Z}$', with and without indices).

A schematic letter $X_{\underline{A}}$ for nonempty $\underline{A}$ can be instantiated only if $\underline{A}$ is instantiated at the same time. - As for variables, we use '$x_{\underline{\alpha}}$' or '$\underline{x}_{\underline{\alpha}}$' in order to express that x or $\underline{x}$ does not occur in $\underline{\alpha}$.

<u>Formula schemata</u> are defined as follows: Each schematic letter for formulas is a formula schema. For all A, $\underline{U}$ and (not necessarily distinct) $X_{1\underline{A}_1},\ldots,X_{n\underline{A}_n}$, where $\underline{U}$ is of length n and all $\underline{A}_i$ ($1 \le i \le n$) contain A (i.e., the $X_{i\underline{A}_i}$ must not be instantiated by variables occurring in the instance of A), $A[\underline{U}|X_{1\underline{A}_1}\ldots X_{n\underline{A}_n}]$ is a formula schema. (.[.|.] is here a sign and not an operation!) If S is an operator of type (n_1,n_2), $\underline{U}$ is of length n_1 and $F_1,\ldots,F_{n_2}$ are formula schemata, then $S\underline{U}F_1\ldots F_{n_2}$ is a formula schema. All occurrences of schematic letters of $\underline{U}$ in $S\underline{U}F_1\ldots F_{n_2}$ are called bound. A <u>rule schema</u> is a finite tree of pairs $\langle\underline{U},F\rangle$ where F is a formula schema. A linear notation for rule schemata is defined in the same way as for rules. Occurrences of schematic letters of $\underline{U}$ in $\langle\underline{U},F\rangle$ and above $\langle\underline{U},F\rangle$ in the considered rule schema are called bound.

The <u>instantiation</u> of formula/rule schemata to formulas/rules is defined as follows: Replace schematic letters for formulas A by formulas α and different schematic letters $X_{\underline{A}}$ for variables by different variables $x_{\underline{\alpha}}$ not occurring in the instance $\underline{\alpha}$ of $\underline{A}$. $\alpha[\underline{y}|z_1\ldots z_n]$, when resulting from $A[\underline{U}|X_{1\underline{A}_1}\ldots X_{n\underline{A}_n}]$ is then always defined and can be evaluated, since no z_i occurs in α because of the restriction on the $X_{i\underline{A}_i}$. A formula/rule schema is called <u>derivable</u> from a set of basic rules R iff all its instances are derivable from the empty list of assumptions in R.

<u>Remark</u>. Whereas on the level of formula/rule schemata .[.|.] is a sign, on the level of formulas/rules .[.|.] is an operation to be evaluated. So the procedure of instantiating a schema includes the evaluation of .[.|.] conceived as a metalinguistic substitution operation. This way of dealing with substitution could have been avoided by treating quantifiers not as variable-binding operators but as operators which are applied to λ-terms. Then we would have had to add rules of λ-conversion to the basic rules.

As syntactical variables we use 'F' for formula schemata, 'R' for rule schemata, 'Φ' for lists of rule schemata (all with and without indices). If $\underline{U}$ and $\underline{V}$ have no schematic letter in common, we write '$F(\underline{U},\underline{V},\underline{A})$', '$R(\underline{U},\underline{V},\underline{A})$', '$\Phi(\underline{U},\underline{V},\underline{A})$' to indicate that F, R and Φ contain no other schematic letters for variables than those of $\underline{U}$ and $\underline{V}$ and no other schematic letters for formulas than those of $\underline{A}$ (but possibly fewer). If $\underline{U}$, $\underline{V}$, $\underline{A}$

can be instantiated by $\underline{x}$, $\underline{y}$, $\underline{\alpha}$, then '$F(\underline{x},\underline{y},\underline{\alpha})$', '$R(\underline{x},\underline{y},\underline{\alpha})$', '$\Phi(\underline{x},\underline{y},\underline{\alpha})$', respectively, denote the formula, rule or list of rules which is the result of this instantiation.

We motivate our standard form for I and E rules for an operator by referring to the common content of lists of rules $\Gamma_1,\ldots,\Gamma_m$ with respect to a list of variables $\underline{x}$. It is defined as follows: The common content of $\Gamma_1,\ldots,\Gamma_m$ with respect to $\underline{x}$ in R is the set of all ρ such that for all $\underline{t}$ and for all i $(1\leq i\leq m)$: $\Gamma_i[\underline{x}|\underline{t}]\vdash_R \rho$. Whereas in the propositional case the common content was the finite intersection of contents of lists of rules (see [18]), the common content with respect to a list of variables can be considered to be an infinite intersection of contents of lists of rules. The limiting case m=0 leads to intuitionistic logic since it allows one to interpret the absurdity sign.

Similar to [18], we assume that with each operator S of type (n_1,n_2) lists of rule schemata $\Phi_1(\underline{X},\underline{Y}_{\underline{A}},\underline{A}),\ldots,\Phi_m(\underline{X},\underline{Y}_{\underline{A}},\underline{A})$ $(m\geq 0)$ are associated where $\underline{X}$ is of length n_1 and $\underline{A}$ of length n_2. It is required that all operators can be ordered in a sequence S_1, $S_2,\ldots$ in such a way that in the lists associated with an operator S_k at most the operators S_j for $j<k$ occur. Furthermore, the $\Phi_i(\underline{X},\underline{Y}_{\underline{A}},\underline{A})$ $(1\leq i\leq m)$ must fulfil the condition that letters of $\underline{Y}_{\underline{A}}$ only occur bound. This is because instances of Φ_i should not contain free variables beyond those in the corresponding instance of $S\underline{X}\underline{A}$, save variables by which $\underline{X}$ is instantiated. The reason for writing '$\underline{Y}_{\underline{A}}$' instead of '$\underline{Y}$' is that variables free in an instance of $S\underline{X}\underline{A}$ should be free in the corresponding instance of Φ_i. In other words, instances $\Phi_i(\underline{x},\underline{y},\underline{\alpha})$ and $S\underline{x}\underline{\alpha}$ of Φ_i and $S\underline{X}\underline{A}$ can, with respect to free variables, differ only in that variables of $\underline{x}$ are free in $\Phi_i(\underline{x},\underline{y},\underline{\alpha})$ but bound in $S\underline{x}\underline{\alpha}$. S is called a $\bot$-operator if m=0, i.e., if no list of rules (not even the empty list) is associated with S.

We require that the set R_S of basic rules for S be a minimal set with respect to derivability which satisfies the condition that for all $\underline{x}$, $\underline{y}_{\underline{\alpha}}$, $\underline{\alpha}$ (where $\underline{x}$ is of length n_1 and $\underline{\alpha}$ of length n_2), $S\underline{x}\underline{\alpha}$ expresses the common content of $\Phi_1(\underline{x},\underline{y}_{\underline{\alpha}},\underline{\alpha}),\ldots,\Phi_m(\underline{x},\underline{y}_{\underline{\alpha}},\underline{\alpha})$ with respect to $\underline{x}$ in R, i.e.

(*) for all $\underline{x}$, $\underline{y}_{\underline{\alpha}}$, $\underline{\alpha}$ and for all ρ: $S\underline{x}\underline{\alpha}\vdash_R\rho$ iff for all $\underline{t}$ and all i $(1\leq i\leq m)$: $\Phi_i(\underline{x},\underline{y}_{\underline{\alpha}},\underline{\alpha})[\underline{x}|\underline{t}]\vdash_R\rho$.

By 'minimal with respect to derivability' we mean that if (*) holds for a certain R, then for each $\rho\in R_S$: $\vdash_R\rho$. Obviously, different minimal sets R and R' are interderivable in the sense that for each $\rho\in R'$: $\vdash_R\rho$, and for each $\rho \in R$: $\vdash_{R'} \rho$.

We shall show that R_S can be chosen as the set of all instances of I and E rule schemata of the following standard form.

$$\text{S-I} \qquad \frac{\Phi_1(\underline{X},\underline{Y}_{\underline{A}},\underline{A})}{S\underline{X}\underline{A}} \quad \cdots \quad \frac{\Phi_m(\underline{X},\underline{Y}_{\underline{A}},\underline{A})}{S\underline{X}\underline{A}}$$

Linear notation: $\Phi_i(\underline{X},\underline{Y}_{\underline{A}},\underline{A}) \Rightarrow S\underline{X}\underline{A}$ $\quad (1 \leq i \leq m)$

$$\text{S-E} \qquad \frac{S\underline{X}\underline{A} \qquad \dfrac{\Phi_1(\underline{X},\underline{Y}_{\underline{A}},\underline{A})}{\langle \underline{X}, B[\underline{X}|\underline{Z}_B]\rangle} \qquad \dfrac{\Phi_m(\underline{X},\underline{Y}_{\underline{A}},\underline{A})}{\langle \underline{X}, B[\underline{X}|\underline{Z}_B]\rangle}}{B[\underline{X}|\underline{Z}_B]}$$

where B is a schematic letter different from those in $\underline{A}$, and $\underline{Z}_B$ and $\underline{X}$ have no schematic letter in common, i.e., they are instantiated by different variables.

Linear notation: $\langle S\underline{X}\underline{A}, \Phi_1(\underline{X},\underline{Y}_{\underline{A}},\underline{A}) \Rightarrow_{\underline{X}} B[\underline{X}|\underline{Z}_B], \ldots, \Phi_m(\underline{X},\underline{Y}_{\underline{A}},\underline{A}) \Rightarrow_{\underline{X}} B[\underline{X}|\underline{Z}_B]\rangle$
$\Rightarrow B[\underline{X}|\underline{Z}_B]$.

One should remember that according to a convention stated in § 3 all variables are bound in instances of rule schemata of this form, so that for appropriate $\underline{z}$ containing all variables which are free in $\Phi_i(\underline{x},\underline{y}_{\underline{\alpha}},\underline{\alpha})$,

$$\frac{\Phi_i(\underline{x},\underline{y}_{\underline{\alpha}},\underline{\alpha})}{\langle \underline{z}, S\underline{x}\underline{\alpha}\rangle}$$

would be an instance of S-I.

Theorem 4.1: (*) holds iff S-I and S-E are derivable in R.

Proof: Let arbitrary $\underline{x},\underline{y}_{\underline{\alpha}},\underline{\alpha}$ be given such that $\Phi_i(\underline{x},\underline{y}_{\underline{\alpha}},\underline{\alpha})$ is defined for every i $(1 \leq i \leq m)$. Taking ρ to be $S\underline{x}\underline{\alpha}$ we obtain from (*) for all i and $\underline{t}$ $(1 \leq i \leq m)$: $\Phi_i(\underline{x},\underline{y}_{\underline{\alpha}},\underline{\alpha})[\underline{x}|\underline{t}] \vdash S\underline{x}\underline{\alpha}$, in particular $\Phi_i(\underline{x},\underline{y}_{\underline{\alpha}},\underline{\alpha}) \vdash S\underline{x}\underline{\alpha}$. Thus $\Phi_i(\underline{x},\underline{y}_{\underline{\alpha}},\underline{\alpha}) \Rightarrow_{\underline{z}} S\underline{x}\underline{\alpha}$ is derivable where $\underline{z}$ contains all variables free in $\Phi_i(\underline{x},\underline{y}_{\underline{\alpha}},\underline{\alpha})$. -

By lemma 3.2 (i) we have for all i $(1 \leq i \leq m)$ if $\underline{x}$ is not free in β:

$\Phi_1(\underline{x},\underline{y}_{\underline{\alpha}},\underline{\alpha}) \Rightarrow_{\underline{x}} \beta, \ldots, \Phi_m(\underline{x},\underline{y}_{\underline{\alpha}},\underline{\alpha}) \Rightarrow_{\underline{x}} \beta, \Phi_i(\underline{x},\underline{y}_{\underline{\alpha}},\underline{\alpha}) \vdash \beta$,

therefore by lemma 3.1 (iv) for all $\underline{t}$:

$\Phi_1(\underline{x},\underline{y}_{\underline{\alpha}},\underline{\alpha}) \Rightarrow_{\underline{x}} \beta, \ldots, \Phi_m(\underline{x},\underline{y}_{\underline{\alpha}},\underline{\alpha}) \Rightarrow_{\underline{x}} \beta, \Phi_i(\underline{x},\underline{y}_{\underline{\alpha}},\underline{\alpha})[\underline{x}|\underline{t}] \vdash \beta$, which is the same as

$\Phi_i(\underline{x},\underline{y}_{\underline{\alpha}},\underline{\alpha})[\underline{x}|\underline{t}] \vdash \langle\Phi_1(\underline{x},\underline{y}_{\underline{\alpha}},\underline{\alpha})\Rightarrow_{\underline{x}}\beta,\ldots,\Phi_m(\underline{x},\underline{y}_{\underline{\alpha}},\underline{\alpha})\Rightarrow_{\underline{x}}\beta\rangle\Rightarrow\beta$.

From (*) we obtain

$S\underline{x}\underline{\alpha} \vdash \langle\Phi_1(\underline{x},\underline{y}_{\underline{\alpha}},\underline{\alpha})\Rightarrow_{\underline{x}}\beta,\ldots,\Phi_m(\underline{x},\underline{y}_{\underline{\alpha}},\underline{\alpha})\Rightarrow_{\underline{x}}\beta\rangle\Rightarrow\beta$

which is the same as

$S\underline{x}\underline{\alpha},\Phi_1(\underline{x},\underline{y}_{\underline{\alpha}},\underline{\alpha})\Rightarrow_{\underline{x}}\beta,\ldots,\Phi_m(\underline{x},\underline{y}_{\underline{\alpha}},\underline{\alpha})\Rightarrow_{\underline{x}}\beta \vdash \beta$.

Thus $\langle S\underline{x}\underline{\alpha},\Phi_1(\underline{x},\underline{y}_{\underline{\alpha}},\underline{\alpha})\Rightarrow_{\underline{x}}\beta,\ldots,\Phi_m(\underline{x},\underline{y}_{\underline{\alpha}},\underline{\alpha})\Rightarrow_{\underline{x}}\beta\rangle\Rightarrow_{\underline{z}}\beta$

is derivable, where $\underline{z}$ contains all variables free in a $\Phi_i(\underline{x},\underline{y}_{\underline{\alpha}},\underline{\alpha})$ $(1\le i\le m)$ or β. -

Conversely, from $\Phi_i(\underline{x},\underline{y}_{\underline{\alpha}},\underline{\alpha}) \vdash S\underline{x}\underline{\alpha}$ follows $\Phi_i(\underline{x},\underline{y}_{\underline{\alpha}},\underline{\alpha})[\underline{x}|\underline{t}] \vdash S\underline{x}\underline{\alpha}$ by lemma 3.1 (iv), thus together with $S\underline{x}\underline{\alpha}\vdash\rho$: $\Phi_i(\underline{x},\underline{y}_{\underline{\alpha}},\underline{\alpha})[\underline{x}|\underline{t}]\vdash\rho$.-

Assume $\Phi_i(\underline{x},\underline{y}_{\underline{\alpha}},\underline{\alpha})[\underline{x}|\underline{t}]\vdash\rho$ for all i $(1\le i\le m)$ and all $\underline{t}$.

Choose $\underline{z}_1,\underline{z}_2$ in such a way that $\underline{x}$, $\underline{z}_1$, $\underline{z}_2$ have no variable in common, no variable of $\underline{z}_1$ occurs free in $\Phi_i(\underline{x},\underline{y}_{\underline{\alpha}},\underline{\alpha})$ for any i $(1\le i\le m)$, no variable of $\underline{z}_2$ occurs free in ρ, $\rho[\underline{x}|\underline{z}_1][\underline{z}_1|\underline{x}]$ is ρ and $\Phi_i(\underline{x},\underline{y}_{\underline{\alpha}},\underline{\alpha})[\underline{x}|\underline{z}_2][\underline{z}_2|\underline{x}]$ is $\Phi_i(\underline{x},\underline{y}_{\underline{\alpha}},\underline{\alpha})$ for all i $(1\le i\le m)$. Then we obtain $\Phi_i(\underline{x},\underline{y}_{\underline{\alpha}},\underline{\alpha})[\underline{x}|\underline{z}_2]\vdash\rho$ for all i $(1\le i\le m)$.

Therefore by two applications of theorem 3.1 (iv):

$\Phi_i(\underline{x},\underline{y}_{\underline{\alpha}},\underline{\alpha})\vdash\rho[\underline{x},\underline{z}_1]$ for all i $(1\le i\le m)$,

thus for a variant $\Gamma'\Rightarrow_{\underline{z}}\beta'$ of $\rho[\underline{x}|\underline{z}_1]$ such that $\underline{z}$ and $\underline{x}$ have no variable in common and $\underline{z}$ is not free in any $\Phi_i(\underline{x},\underline{y}_{\underline{\alpha}},\underline{\alpha})$:

$\Phi_i(\underline{x},\underline{y}_{\underline{\alpha}},\underline{\alpha}),\Gamma'\vdash\beta'$, thus

$\Gamma'\vdash\Phi_i(\underline{x},\underline{y}_{\underline{\alpha}},\underline{\alpha})\Rightarrow_{\underline{x}}\beta'$.

By S-E and lemma 3.2 (ii):

$S\underline{x}\underline{\alpha},\Gamma'\vdash\beta'$, thus

$S\underline{x}\underline{\alpha}\vdash\rho[\underline{x}|\underline{z}_1]$, thus by lemma 3.1 (iv):

$S\underline{x}\underline{\alpha}\vdash\rho$.

If one takes R_S to contain exactly the instances of S-I and S-E, then S-I and S-E are trivially derivable in R_S, so (by the theorem) R_S fulfils (*). Conversely, if R satisfies (*) then (by the theorem) S-I and S-E are derivable in R, i.e. R_S is a minimal set satisfying (*).

Basic rule schemata for the standard quantifiers ∀,∃ which are of type (1,1), have the form:

$$\forall I \quad \frac{\langle X,A\rangle}{\forall XA} \qquad\qquad \forall E \quad \frac{\forall XA \qquad \dfrac{\langle X,A\rangle}{\langle X,B[X|Z_B]\rangle}}{B[X|Z_B]}$$

lin.: $\langle\Rightarrow_X A\rangle\Rightarrow\forall XA$ lin.: $\langle\forall XA,\langle\Rightarrow_X A\rangle\Rightarrow_X B[X|Z_B]\rangle\Rightarrow B[X|Z_B]$

∃I $\dfrac{A}{\exists XA}$ ∃E $\dfrac{\exists XA \qquad \dfrac{\overline{A}}{\langle X, B[X|Z_B]\rangle}}{B[X|Z_B]}$

lin.: $\langle A\rangle \Rightarrow \exists XA$ lin.: $\langle \exists XA, \langle A\rangle \Rightarrow_X B[X|Z_B]\rangle \Rightarrow B[X|Z_B]$

∀E is equivalent to the usual ∀ elimination rule which has the form

$$\frac{\forall XA}{A} \quad . \tag{3}$$

For letting x,α be arbitrary, β not containing x free, then

$$\cfrac{\cfrac{\overline{\forall x\alpha}}{\langle x,\alpha\rangle}}{\beta}$$

is a derivation of β for which a rule assignment is given by: $g(\forall x\alpha) = \Rightarrow_x \forall x\alpha$, $g(\langle x,\alpha\rangle) = \langle \forall x\alpha\rangle \Rightarrow \alpha$, and $g(\beta) = \langle \Rightarrow_x \alpha\rangle \Rightarrow \beta$. Since $\Rightarrow_x \forall x\alpha$ and $\langle \Rightarrow_x \alpha\rangle \Rightarrow \beta$ are subrules of $\forall x\alpha$ and $\langle \Rightarrow_x \alpha\rangle \Rightarrow_x \beta$ respectively, we have a derivation of β from $\forall x\alpha$, $\langle \Rightarrow_x \alpha\rangle \Rightarrow_x \beta$ and an instance of (3). Conversely, taking the instance $\langle \forall x\alpha, \langle \Rightarrow_x \alpha\rangle \Rightarrow_x \alpha[x|y_\alpha]\rangle \Rightarrow_{\underline{z}} \alpha[x|y_\alpha]$ of ∀E where $\underline{z}$ contains all variables which are free in $\alpha[x|y_\alpha]$, we obtain $\langle \forall x\alpha, \langle \Rightarrow_x \alpha\rangle \Rightarrow \alpha[x|y_\alpha]\rangle \Rightarrow_{\underline{z}} \alpha[x|y_\alpha]$ as a variant and $\langle \forall x\alpha, \langle \Rightarrow_x \alpha\rangle \Rightarrow \alpha\rangle \Rightarrow \alpha$ as a subrule; since $\Rightarrow_x \alpha \vdash \overline{\alpha}$ holds trivially, we obtain $\forall x\alpha \vdash \alpha$ by use of ∀E.

Examples of further operators are those whose I rule schemata have the following form, where the types of the operators are mentioned on the right:

$\langle\langle A\rangle \Rightarrow_X B\rangle \Rightarrow \Pi XAB$ (1,2)

$\langle A, B\rangle \Rightarrow \Sigma XAB$ (1,2)

$\langle \Rightarrow_Y A\rangle \Rightarrow \exists\forall XYA$ (2,1)

$\langle \Rightarrow_{Z_A} A[X_2X_3|Z_AZ_A]\rangle \Rightarrow LX_1X_2X_3A$

$\langle \Rightarrow_{Z_A} A[X_1X_3|Z_AZ_A]\rangle \Rightarrow LX_1X_2X_3A$ (3,1)

$\langle \Rightarrow_{Z_A} A[X_1X_2|Z_AZ_A]\rangle \Rightarrow LX_1X_2X_3A$

$\langle \Rightarrow_{Z_A} A[XY|Z_AZ_A]\rangle \Rightarrow DXYA$ (2,1)

$\langle\langle A\rangle \Rightarrow_{XY_A} A[X|Y_A]\rangle \Rightarrow IXA$ (1,1)

$\langle A, \langle A, A[X|Z_{AB}]\rangle \Rightarrow_{XZ_{AB}} B[Y|Z_{AB}]\rangle \Rightarrow JXYAB$ (2,2)

$\langle\langle A[X_2|Y_A], A[X_1X_2|Y_AZ_A]\rangle \Rightarrow_{X_1Y_AZ_A} A[X_2|Z_A]\rangle \Rightarrow TX_1X_2A$ (2,1)

Corresponding E rules are determined uniquely by the I rules. The ΠE rule, for instance, is

$<\Pi xAB, <<A>\Rightarrow_x B>\Rightarrow_x C[x|z_C]>\Rightarrow C[x|z_C]$.

Remark. At the meeting 'Konstruktive Mengenlehre und Typentheorie' (München 1980) Per Martin-Löf presented a version of his intuitionistic set theory in which the E rule for his operator Π is structurally similar to the above ΠE rule, and could be considered a rule of level 4 in our sense. He formulates it as

$$\frac{c\in\Pi(A,B) \qquad \begin{matrix}[y(x)\in B(x)\quad (x\in A)]\\ d(y)\in C(\lambda(y))\end{matrix}}{F(c,d)\in C(c)}$$

which in our notation for rules would have to be written as

$$\frac{c\in\Pi(A,B) \qquad \dfrac{\dfrac{x\in A}{<x,y(x)\in B(x)>}}{<y,d(y)\in C(\lambda(y))>}}{F(c,d)\in C(c)}$$

or linearly

$<c\in\Pi(A,B), <<x\in A>\Rightarrow_x y(x)\in B(x)>\Rightarrow_y d(y)\in C(\lambda(y))>\Rightarrow F(c,d)\in C(c)$.

Apart from the usage of a new kind of assumption, however, this similarity only concerns certain basic ideas about the relationship between I rules and E rules, and not those specific features of Martin-Löf's system which make it behave differently from other formalizations of intuitionistic mathematics. In order to justify its rules in detail, one would need further principles, in particular about the way logical rules, i.e. rules concerning propositions, are part of a type or set theory. In the published versions of his system (e.g. [11]) Martin-Löf gives an equivalent ΠE rule which is of level 2 (for a description of Martin-Löf's theory as a formal system see [1]). - Zucker and Tragesser [20] work, when treating the completeness of the standard intuitionistic operators for quantifier logic, within the framework of Martin-Löf's theory. Their general schema for I rules (whose premisses are conceived as trees growing downwards) is intended to capture all kinds of operators which may be introduced in this framework. Zucker and Tragesser give, however, almost no motivation for their schema, not even an example. Similarities between their approach (when restricted to first-order logic) and the one presented here I can only suspect.

Theorem 4.2 (Replacement theorem): Assume $\beta \dashv\vdash \beta'$. Let β occur as a part of α, and let α' be the result of replacing this part of α with β'. Then $\alpha \dashv\vdash \alpha'$. Furthermore, if $<\underline{x},\alpha>$ is an element of a member of Γ and Γ' results by replacing this element with $<\underline{x},\alpha'>$, then $\Gamma \dashv\vdash \Gamma'$.

Proof: From lemma 3.4 in almost the same way as in the propositional case (see [18], theorem 4.6).

Theorem 4.3 (Relabelling of bound operator variables): $S\underline{x}\underline{\alpha} \dashv\vdash S\underline{z}(\underline{\alpha}[\underline{x}|\underline{z}])$ if $\underline{z}$ is not free in $\underline{\alpha}$.

Proof: Induction on k where S is S_k, i.e. the k-th member in the sequence of all operators. If S is S_1 and has I and E rules of the general form stated above, then the $\Phi_i(\underline{X},\underline{Y}_{\underline{A}},\underline{A})$ do not contain any operator. From the S-I rule we have

$$\Phi_i(\underline{x},\underline{y}_{\underline{\alpha}},\underline{\alpha}) \vdash S\underline{x}\underline{\alpha} \quad \text{for all } i \ (1 \le i \le m) \quad , \tag{4}$$

and by forming a variant of $\Phi_i(\underline{x},\underline{y}_{\underline{\alpha}},\underline{\alpha})$ and applying lemma 3.1 (iv) we obtain

$$\Phi_i(\underline{z},\underline{y}_{\underline{\alpha}[\underline{x}|\underline{z}]},\underline{\alpha}[\underline{x}|\underline{z}]) \vdash S\underline{x}\underline{\alpha} \quad \text{for all } i \ (1 \le i \le m) \quad . \tag{5}$$

Application of S-E yields

$$S\underline{z}(\underline{\alpha}[\underline{x}|\underline{z}]) \vdash S\underline{x}\underline{\alpha} \quad .$$

The converse follows analogously.
If S is S_k for $k>1$, then the situation differs only in that the $\Phi_i(\underline{X},\underline{Y}_{\underline{A}},\underline{A})$ may contain S_j for $j<k$. So we have, in order to pass from (4) to (5), to apply additionally the induction hypothesis and the replacement theorem 4.2.

5. THE COMPLETENESS OF THE STANDARD INTUITIONISTIC OPERATORS

We can now show that all operators given by I and E rules of our standard form are explicitly definable in terms of $\&,\vee,\supset,\curlywedge,\forall,\exists$. We assume that the calculus we are considering already contains these operators and their basic rules and that they form the first 6 members $S_1,\ldots,S_6$ of the enumeration of operators. $\forall\underline{x}\alpha$ and $\exists\underline{x}\alpha$ are used as abbreviations for $\forall x_1\ldots\forall x_n\alpha$ and $\exists x_1\ldots\exists x_n\alpha$ if $\underline{x}$ is $x_1\ldots x_n$, and similarly for schemata.

We associate with each formula schema F, rule schema R and list of rule schemata Φ a formula schema F^*, R^* and Φ^* which contains at most $\&,\supset,$ $\curlywedge,\forall$ as operators besides operators occurring in F, R, Φ: F^* is F. R^* is $(R_1{}^*\&\ldots\&R_n{}^*)\supset F$ if R is $\langle R_1,\ldots,R_n\rangle \Rightarrow F$. R^* is $\forall\underline{X}((R_1{}^*\&\ldots\&R_n{}^*)\supset F)$ if R is $\langle R_1,\ldots,R_n\rangle \Rightarrow_{\underline{X}} F$. Φ^* is $R_1{}^*\&\ldots\&R_n{}^*$ if Φ is the list $R_1\ldots R_n$. Φ^* is $\curlywedge\supset\curlywedge$ if Φ is the empty list. In the same way we associate with each formula α, rule ρ, list of rules Δ a formula α^*, ρ^*, Δ^*.

Lemma 5.1: For all formulas α, rules ρ and lists of rules Δ: $\alpha \dashv\vdash \alpha^*$, $\rho \dashv\vdash \rho^*$, $\Delta \dashv\vdash \Delta^*$.

Proof: We consider only the case where a quantifier is involved. Assume

$$\langle\rho_1,\ldots,\rho_n\rangle\Rightarrow\alpha \dashv\vdash (\rho_1{}^*\&\ldots\&\rho_n{}^*)\supset\alpha \quad . \tag{6}$$

Since $\langle\rho_1,\ldots,\rho_n\rangle\Rightarrow\alpha$ is a subrule of $\langle\rho_1,\ldots,\rho_n\rangle\Rightarrow_{\underline{x}}\alpha$, we have

$$\langle\rho_1,\ldots,\rho_n\rangle\Rightarrow_{\underline{x}}\alpha \vdash (\rho_1{}^*\&\ldots\&\rho_n{}^*)\supset\alpha \quad .$$

By applying ∀I (possibly more than once), we obtain
$\langle\rho_1,\ldots,\rho_n\rangle\Rightarrow_{\underline{x}}\alpha \vdash \forall\underline{x}((\rho_1{}^*\&\ldots\&\rho_n{}^*)\supset\alpha)$.
Conversely, we have by (3), which is equivalent to ∀E, and (6):
$\forall\underline{x}((\rho_1{}^*\&\ldots\&\rho_n{}^*)\supset\alpha),\rho_1,\ldots,\rho_n \vdash \alpha$. Thus
$\forall\underline{x}((\rho_1{}^*\&\ldots\&\rho_n{}^*)\supset\alpha) \vdash \langle\rho_1,\ldots,\rho_n\rangle\Rightarrow_{\underline{x}}\alpha$.

Theorem 5.2: For each S of type (n_1,n_2) there is a formula schema $F(\underline{X},\underline{Y}_{\underline{A}},\underline{A})$ containing at most &,v,⊃,⅄,∀,∃ as operators, where $\underline{X}$ is of length n_1, $\underline{A}$ of length n_2 and letters of $\underline{Y}_{\underline{A}}$ only occur bound, such that for all $\underline{x}$, $\underline{y}_{\underline{\alpha}},\underline{\alpha}$: $S\underline{x}\underline{\alpha} \dashv\vdash F(\underline{x},\underline{y}_{\underline{\alpha}},\underline{\alpha})$.

Proof: Induction on k where S is S_k. If S is S_1, then S is one of the operators &,v,⊃,⅄,∀,∃; so we can take F to be $S\underline{X}\underline{A}$. Let k>0. If S is a ⊥-operator, take F to be ⅄. Obviously $S\underline{x}\underline{\alpha} \dashv\vdash$ ⅄. Otherwise there are lists of rule schemata $\Phi_i(\underline{X},\underline{Y}_{\underline{A}},\underline{A})$ associated with S, containing at most operators S_j for j<k. For these S_j, there are, by induction hypothesis, formula schemata $F_j(\underline{Z},\underline{Z}_{1\underline{B}},\underline{B})$ containing at most &,v,⊃,⅄,∀,∃ as operators such that for all $\underline{z}$, $\underline{z}_{1\underline{\beta}},\underline{\beta}$
$S_j\underline{z}\underline{\beta} \dashv\vdash F_j(\underline{z},\underline{z}_{1\underline{\beta}},\underline{\beta})$,
and, if all members of $\underline{\beta}$ are members of $\underline{\alpha}$, then for all $\underline{z}$, $\underline{z}_{2\underline{\alpha}},\underline{\beta}$,
$S_j\underline{z}\underline{\beta} \dashv\vdash F_j(\underline{z},\underline{z}_{2\underline{\alpha}},\underline{\beta})$.
By application of the replacement theorem 4.2 we obtain lists $\Phi_i'(\underline{X},\underline{Y}_{\underline{A}},\underline{A})$ containing at most &,v,⊃,⅄,∀,∃ as operators such that for all $\underline{x}$, $\underline{y}_{\underline{\alpha}},\underline{\alpha}$

$$\Phi_i(\underline{x},\underline{y}_{\underline{\alpha}},\underline{\alpha}) \dashv\vdash \Phi_i'(\underline{x},\underline{y}_{\underline{\alpha}},\underline{\alpha}) \quad . \tag{7}$$

Now take $F(\underline{X},\underline{Y}_{\underline{A}},\underline{A})$ to be

$$\exists\underline{X}(\Phi_1'(\underline{X},\underline{Y}_{\underline{A}},\underline{A}))^* \vee \ldots \vee \exists\underline{X}(\Phi_m'(\underline{X},\underline{Y}_{\underline{A}},\underline{A}))^* \quad .$$

(This formula schema possibly contains vacuous quantifications which can be omitted.) Let $\underline{x}$, $\underline{y}_{\underline{\alpha}}$, $\underline{\alpha}$ be given. Since $\Phi_i(\underline{x},\underline{y}_{\underline{\alpha}},\underline{\alpha}) \vdash S\underline{x}\underline{\alpha}$ for all i (1≤i≤m), we have $(\Phi_i'(\underline{x},\underline{y}_{\underline{\alpha}},\underline{\alpha}))^* \vdash S\underline{x}\underline{\alpha}$ for all i (1≤i≤m) (by (7) and lemma 5.1). Thus by ∃E:

$\exists \underline{x}((\Phi_i'(\underline{x},\underline{y}_{\underline{\alpha}},\underline{\alpha}))^* \vdash S\underline{x}\underline{\alpha}$

and by vE:

$F(\underline{x},\underline{y}_{\underline{\alpha}},\underline{\alpha}) \vdash S\underline{x}\underline{\alpha}$.

Conversely, since for all i ($1 \le i \le m$)

$\Phi_i(\underline{x},\underline{y}_{\underline{\alpha}},\underline{\alpha}) \vdash (\Phi_i'(\underline{x},\underline{y}_{\underline{\alpha}},\underline{\alpha}))^*$ (by (7) and lemma 5.1)

and

$\Phi_i(\underline{x},\underline{y}_{\underline{\alpha}},\underline{\alpha}) \vdash F(\underline{x},\underline{y}_{\underline{\alpha}},\underline{\alpha})$ (by ∃I and vI) ,

and since $F(\underline{x},\underline{y}_{\underline{\alpha}},\underline{\alpha})$ does not contain $\underline{x}$ free, we obtain by S-E: $S\underline{x}\underline{\alpha} \vdash F(\underline{x},\underline{y}_{\underline{\alpha}},\underline{\alpha})$.

For example, the operators Π, Σ, ∃∀, L, D, I, J, T, for which I rules are given in § 4, are definable in terms of &,v,⊃,⅄,∀,∃ as follows:

ΠXAB:	∀X(A⊃B)
ΣXAB:	∃X(A&B)
∃∀XYA:	∃X∀YA
$LX_1X_2X_3A$:	$\exists X_1 \forall Z_A A[X_2X_3 \mid Z_AZ_A] \vee \exists X_2 \forall Z_A A[X_1X_3 \mid Z_AZ_A] \vee \exists X_3 \forall Z_A A[X_1X_2 \mid Z_AZ_A]$
DXYA:	$\forall Z_A A[XY \mid Z_AZ_A]$
IXA:	$\forall XY_A(A \supset A[X \mid Y_A])$
JXYAB:	$\exists XY(A \& \forall XZ_{AB}((A \& A[X \mid Z_{AB}]) \supset B[Y \mid Z_{AB}]))$
TX_1X_2A:	$\forall X_1Y_AZ_A((A[X_2 \mid Y_A] \& A[X_1X_2 \mid Y_AZ_A]) \supset A[X_2 \mid Z_A])$

J can, for instance, be used to express the relation between a one place predicate P_1 and a two place relation P_2 which holds if P_1 is satisfiable and for all x,y, if P_1x and P_1y, then P_2xy. T can be used to express the transitivity of a relation, etc.

Application of the replacement theorem 4.2 yields that for each rule ρ there is a rule ρ^+ containing at most &,v,⊃,⅄,∀,∃ as operators such that $\rho \dashv\vdash \rho^+$. Thus by lemma 5.1.:

$$\rho_1,\ldots,\rho_n \vdash \alpha \quad \text{iff} \quad \rho_1^{+*},\ldots,\rho_n^{+*} \vdash \alpha^{+*} ,$$

where $\rho_1^{+*},\ldots,\rho_n^{+*},\alpha^{+*}$ are formulas containing at most &,v,⊃,⅄,∀,∃ as operators. If we assume that

(**) every derivation of β from Δ can be transformed into a derivation of β from Δ only using rules for operators occurring in β or Δ as basic rules, we obtain:

$$\rho_1,\dots,\rho_n \vdash \alpha \quad \text{iff} \quad \rho_1^{+*},\dots,\rho_n^{+*} \vdash_I \alpha^{+*} \quad ,$$

where I denotes the ordinary natural deduction system for intuitionistic logic as presented in Prawitz [13]. Furthermore, since for formulas $\beta_1,\dots,\beta_n,\beta$ containing at most $\&,\vee,\supset,\curlywedge,\forall,\exists$, $^{+*}$ is the identity, it holds that

$$\beta_1,\dots,\beta_n \vdash_I \beta \quad \text{iff} \quad \beta_1,\dots,\beta_n \vdash \beta \quad .$$

This shows that our generalized calculus can be embedded in ordinary intuitionistic logic and vice versa. (**) can be proved, since the normalization procedures and subformula principles, as given in [13], can be taken over to our generalized calculus for logical operators. This is done for the sentential case in [17], and the quantifier case does not provide additional problems in principle.

6. CONCLUDING REMARKS

The approach presented here is designed for the interpretation of intuitionistic quantifier logic. It is natural to look for a similar interpretation of classical logic. Since the classical I and E rules do not immediately fit into the characterization given in § 4 by (*) and the general form for basic rules for operators, one has to change the underlying framework for rules, derivations etc. One way is to introduce the denial $\sim\alpha$ of a sentence α besides its assertion and to construct calculi with refutation-rules, i.e. rules which govern the denials of sentences. $\sim$, as distinguished from the operator $\neg$ of negation, is here a sign which can only occur in outermost position and must therefore not be iterated. The rules of 'reductio ad absurdum', formulated by use of $\sim$ (and not of $\neg$), must then be considered fixed basic rules which are independent of the I and E rules for logical operators (for sentential logic this is carried out in [17], cf. also [9]).

Another way is to use multiple-conclusion logic (cf. [19]), i.e., to consider derivations based on rules which may have more than one conclusion. In that case one could always have 'direct' E rules like

$$\frac{\alpha\vee\beta}{\alpha \;\; \beta} \quad .$$

However, since our general form of I and E rules is based on 'indirect' E rules as present in the usual ∨E and ∃E rules of systems of natural deduction, it is much more promising to use the system of multiple-conclusion logic developed by Boričić [2] as a starting point. This system is, roughly speaking, a natural deduction calculus for sets of formulas (understood disjunctively) as premisses and conclusions of rules, in which e.g. the rule of ∨E takes the form

$$\frac{M\cup\{A\vee B\} \quad \begin{matrix}[\{A\}] \\ N\end{matrix} \quad \begin{matrix}[\{B\}] \\ N\end{matrix}}{M\cup N} \quad .$$

It can be considered an immediate natural deduction counterpart of the classical sequent calculus which differs from the intuitionistic one by allowing more than one formula in the succedent.

Instead of using a natural deduction framework one could work with sequent systems from the beginning. Such systems have the advantage of making structural assumptions explicit. This is especially useful for the treatment of modal operators. Whereas sequent calculi are not very natural for the interpretation of 'ordinary' intuitionistic or classical logic because they are 'meta-calculi' in the sense that, to explain the sequent arrow, one seems to have to refer to a derivability relation between antecedent and succedent (cf. [13]), thus presupposing something like natural deduction, this meta-perspective is just appropriate for modal logic (for, e.g., $\Box\alpha$ should express that α can be logically derived). For such a framework Došen [4,5] introduced sequents of higher levels; their exact relationship to our rules of higher levels is still to be investigated.

Concerning our method of proving the completeness of a system of operators with respect to a general form for I and E rules of operators, an application to Martin-Löf's system, more precisely, to its 'logical' part without natural numbers, well-orderings and universes (and perhaps without propositional equality too), seems promising. As stated above, this system is based on several assumptions which go beyond that which can be dealt with by the approach presented here, but it is nevertheless possible to give a general schema for I and E rules in Martin-Löf's framework, and I conjecture that the completeness of Π, Σ, + and the finite types can be established with respect to such a schema. This could explain some of the 'systematic character' of Martin-Löf's theory, i.e. its being free from stipulations which seem arbitrary, which makes it, at least from the philosophical point of view, so attractive as a foundation for logic and mathematics.

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