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Section 5: Philosophical Logic
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INVERSION PRINCIPLES AND THE COMPLETENESS OF INTUITIONISTIC NATURAL DEDUCTION SYSTEMS

Some influential semantical conceptions for intuitionistic logic, in particular those of M. Dummett and D. Prawitz, consider the introduction rules (I-rules) for logical operators to be 'canonical' rules which give a meaning to these operators; the elimination rules (E-rules) are then justified with respect to a semantics depending on I-rules. The often stated harmony between I- and E-rules suggests that one might reverse this procedure, i.e. choose the E-rules to be canonical and justify the I-rules with respect to a semantics depending on E-rules. In the following, which represents an attempt in this direction, we shall define a concept of validity based on E-rules. It can then be shown that all I-rules are valid, and conversely that all valid rules are derivable in intuitionistic logic. In this sense intuitionistic logic is complete. This approach is dual to that proposed in [7] where the inversion principle, as formulated by Prawitz [3,4], is generalized to a notion of validity based on I-rules, with respect to which the completeness of intuitionistic logic could be established. So the present approach formulates an 'inverted' inversion principle.

The Inversion Principle In Lorenzen [2], the inversion principle is treated as a principle to establish the admissibility of rules. A rule R is called admissible in a calculus K, if its addition to the inference rules of K, yielding an extended calculus K+R, does not enlarge the set of formulas derivable in K, i.e. for each formula D, if $\vdash_{K+R} D$, then $\vdash_{K} D$. Here the 'if...then' is understood constructively, i.e. there must be an effective procedure eliminating each application of R in a derivation of D. The inversion principle is applied in such cases where the premises of R can be derived in K only by application of certain inference rules $R_1,\ldots,R_n$ of K: then we know that a derivation of the premises of R in K contains a derivation of at least some of the premises of $R_1,\ldots,R_n$; if we know furthermore that for each i ($1 \leq i \leq n$) the step from the premises of R_i to the conclusion of R is admissible, we can infer the admissibility of R. (For a precise description see [1]). The main application of the inversion principle within formal logic is the justification of the $\wedge$-, $\vee$-, and $\exists$-E-rules as admissible rules in every calculus K having the $\wedge$-, $\vee$-, $\exists$-I-rules as the only inference rules making it possible to infer conjunctions, disjunctions and existential quantifications. As can easily be seen, the admissibility concept and thus the inversion principle in Lorenzen's version does not work for derivations from assumptions. If we defined R to be admissible in K if for all finite sets of assumptions Γ and all formulas D: if $\Gamma \vdash_{K+R} D$ then $\Gamma \vdash_{K} D$, then each admissible rule R would be derivable: Taking Γ to be the set of premises of R and D to be its conclusion, $\Gamma \vdash_{K+R} D$ would be trivially fulfilled, thus $\Gamma \vdash_{K} D$ would hold.

Following some remarks of Gentzen, Prawitz used in [3,4] a somewhat different inversion principle to describe the relation between I- and E-rules of natural deduction systems: if the major premise of an E-rule is derived using an I-rule in the last step, this derivation already 'contains', together with derivations of the minor premises of the E-rule, a derivation of the conclusion of the E-rule. This relation is made explicit in the reduction steps and normalization procedures stated by Prawitz. Such an inversion principle obviously does not allow the elimination of an E-rule R from all derivations in C+R, where C is the (canonical) part of an intuitionistic natural deduction system having only I-rules as inference rules. But we can formulate it in a way that makes it closely related to Lorenzen's inversion principle: Define for an E-rule R a derivation in C+R to be a derivation which applies R only if its major premise is the conclusion of an application of an I-rule. Then it holds in fact for all E-rules R that for all sets of assumptions Γ and all formulas D: if $\Gamma \vdash_{C+R} D$, then $\Gamma \vdash_{C} D$. The difference to Lorenzen's inversion principle is that in calculi without assumptions the major premise of an E-rule can be derived only by using an I-rule in the last step where this fact must be required in the case of calculi with assumptions. On the one hand this weakens the inversion principle, but on the other hand it makes it possible to treat → in this framework (which was not possible in Lorenzen [2]).

Prawitz' inversion principle is defined for the standard E-rules with one major premise. In Schroeder-Heister [7] it is generalized to a principle that may be used for the justification of arbitrary rules R (the E-rules being special cases thereof). The general schema for an arbitrary rule in a natural deduction system can be stated as

$$(1)\qquad \frac{\begin{matrix}\Gamma_1 & & \Gamma_n\\ \vdots\,\underline{x}_1 & \dots & \vdots\,\underline{x}_n\\ A_1 & & A_n\end{matrix}}{A}$$

where the Γ's are (possibly empty) sets of formulas, indicating the assumptions which may be discharged by application of that rule, and the $\underline{x}$'s are sets of eigenvariables to be respected. In order to formulate an inversion principle, we assume a (possibly empty) set of non-atomic assumption- and eigenvariable-free premises to be distinguished by a star, thus arriving at the schema

$$(2)\qquad \frac{\begin{matrix} & & & \Gamma_1 & & \Gamma_m\\ \vdots & \dots & \vdots & \vdots\,\underline{x}_1 & \dots & \vdots\,\underline{x}_m\\ {*A_1} & & {*A_n} & B_1 & & B_m\end{matrix}}{A}$$

Here the starred A's function like major premises in the usual E-rules which must now be written as

$$\frac{\begin{matrix}\vdots\\ {*\,A\wedge B}\end{matrix}}{A}\quad \frac{\begin{matrix}\vdots\\ {*\,A\wedge B}\end{matrix}}{B}\quad \frac{\begin{matrix} & A & B\\ \vdots & \vdots & \vdots\\ {*\,A\vee B} & C & C\end{matrix}}{C}\quad \frac{\begin{matrix}\vdots & \vdots\\ {*\,A\to B} & A\end{matrix}}{B}\quad \frac{\begin{matrix}\vdots\\ {*\,\bot}\end{matrix}}{A}\quad \frac{\begin{matrix}\vdots\\ {*\,\forall xA}\end{matrix}}{A[x/t]}\quad \frac{\begin{matrix} & A[x/y]\\ \vdots & \vdots\,y\\ {*\,\exists xA} & B\end{matrix}}{B}\quad \begin{matrix}\text{(y not free}\\ \text{in } \exists xA \text{ or B)}\end{matrix}$$

For $\mathcal{C}$ as the canonical part of the natural deduction calculus having only I-rules as inference rules, a derivation in $\mathcal{C}$+R for a rule of the form (2) is defined as applying R only if the starred premises are derived using an I-rule in the last step, i.e. the starred premises are counted as major premises in a generalized sense. We say that the inversion principle holds for R, or that R is valid, if for all Γ, D: if $\Gamma\vdash_{\mathcal{C}+R}D$, then $\Gamma\vdash_{\mathcal{C}}D$. It can be shown not only that for all rules of intuitionistic logic $\mathcal{I}$ the inversion principle holds (i.e. that they are valid), but also that all rules for which the inversion principle holds are derivable in $\mathcal{I}$; so $\mathcal{I}$ is in a certain sense complete.

Assumption Rules We allow not only formulas but also 'assumption rules' of the form $\{A_1,\dots,A_n\}\Rightarrow_{\underline{x}} A$ to be assumptions on which derivations in natural deduction calculi may depend. (Here the sets $\{A_1,\dots,A_n\}$ and/or $\underline{x}$ may be empty; in the former case the assumption rule is identified with the assumption A). Assumption rules are applied in a derivation according to the schema

$$\frac{\begin{matrix} & \vdots & & \vdots\\ \{A_1,\dots,A_n\}\Rightarrow_{\underline{x}} A & A_1[\underline{x}/\underline{t}] & \dots & A_n[\underline{x}/\underline{t}]\end{matrix}}{A[\underline{x}/\underline{t}]}$$

An assumption rule $\{A_1,\dots,A_n\}\Rightarrow_{\underline{x}} A$ represents on the object level the metalogical assumption that a derivation of A from $\{A_1,\dots,A_n\}$ is given whereby eventual further assumptions do not contain any variable of $\underline{x}$ free. The concept of assumption rules allows us to define the derivability of a rule of form (1) as $\{\Gamma_1\Rightarrow_{\underline{x}_1} A_1,\dots,\Gamma_n\Rightarrow_{\underline{x}_n} A_n\}\vdash A$, analogously to the usual definition of the derivability of a rule of the form $\frac{A_1 \ \ldots\ A_n}{A}$ as $\{A_1, \ldots, A_n\} \vdash A$. If Δ denotes a set of premises of a rule of form (1), Δ' is defined to be the set $\{\Gamma_1 \Rightarrow_{\underline{x}_1} A_1, \ldots, \Gamma_n \Rightarrow_{\underline{x}_n} A_n\}$. So a rule $\frac{\Delta}{A}$ is derivable if $\Delta' \vdash A$. (For a systematic treatment of assumption rules also of higher levels see [6]).

<u>An Inversion Principle Based on Elimination Rules</u> The 'harmony' between I- and E-rules has often been emphasized but usually I-rules are chosen to be canonical rules (with the exception of the approach sketched in [4, Appendix A.2] which is somewhat different from the one given here). I shall take the E-rules to be canonical and justify the I-rules by an inversion principle which treats I-rules as inverses of E-rules, dual to the path taken in [7]. This means that we have to formulate counterparts to the concepts defined there. (E.g., counterparts of the major premises of E-rules are now the conclusions of I-rules). Thus we define the canonical part C of the intuitionistic natural deduction calculus I to be the subsystem containing only the E-rules for $\wedge$, $\vee$, $\rightarrow$, $\bot$, $\forall$, $\exists$ (as stated above, but without a star). $\Gamma \vdash_C D$ is defined as usual where Γ may include assumption rules. The general form of an arbitrary rule R is

$$\frac{\begin{matrix}\Gamma_1 \\ \vdots \underline{x}_1 \\ A_1\end{matrix} \quad \ldots \quad \begin{matrix}\Gamma_n \\ \vdots \underline{x}_n \\ A_n\end{matrix}}{(*)A} \qquad \text{or shortly} \qquad \frac{\Delta}{(*)A}$$

where a non-atomic conclusion A can be starred (premisses must not be starred). A derivation in C+R is a derivation in the calculus resulting from C by addition of R as an inference rule, where, if A is starred, the conclusion of each application of R in the derivation is major premise of an application of an E-rule. We shall say that R fulfils the inversion principle or is valid if for all Γ, D: if $\Gamma \vdash_{C+R} D$, then $\Gamma \vdash_C D$. Since all E-rules of I belong to C, they are trivially valid. The I-rules of I, now to be written as

$$\frac{\begin{matrix}\vdots \\ A\end{matrix} \quad \begin{matrix}\vdots \\ B\end{matrix}}{*\ A \wedge B} \quad \frac{\begin{matrix}\vdots \\ A\end{matrix}}{*\ A \vee B} \quad \frac{\begin{matrix}\vdots \\ B\end{matrix}}{*\ A \vee B} \quad \frac{\begin{matrix}A \\ \vdots \\ B\end{matrix}}{*\ A \rightarrow B} \quad \frac{\begin{matrix}\vdots y \\ A[x/y]\end{matrix}}{*\ \forall x A} \ \begin{matrix}\text{(y not free} \\ \text{in } \forall x A)\end{matrix} \quad \frac{\begin{matrix}\vdots \\ A[x/t]\end{matrix}}{*\ \exists x A} \qquad (\bot \text{ has no I-rule})$$

can be shown to be valid by application of the standard reduction steps. So all inference rules of I are valid. Conversely, we can prove that all valid rules are derivable in I: First we state that all valid rules R without starred conclusions are derivable in C and hence in I. (Take Γ to be Δ'; then $\Delta' \vdash_{C+R} A$ holds trivially and thus $\Delta' \vdash_C A$). Secondly, if a rule R of one of the forms

$$\frac{\Delta}{*\ A \wedge B} \quad \frac{\Delta}{*\ A \vee B} \quad \frac{\Delta}{*\ A \rightarrow B} \quad \frac{\Delta}{*\ \bot} \quad \frac{\Delta}{*\ \forall x A} \quad \frac{\Delta}{*\ \exists x A}$$

is valid, then also

$$\frac{\Delta}{A} \text{ and } \frac{\Delta}{B}, \quad \frac{\Delta \quad \begin{matrix}A \\ \vdots \\ C\end{matrix} \quad \begin{matrix}B \\ \vdots \\ C\end{matrix}}{C} \text{ for all C}, \quad \frac{\Delta \quad \begin{matrix}\vdots \\ A\end{matrix}}{B}, \quad \frac{\Delta}{C} \text{ for all C}, \quad \frac{\Delta}{A[x/t]} \text{ for all t}, \quad \frac{\Delta \quad \begin{matrix}A[x/y] \\ \vdots y \\ B\end{matrix}}{B}$$

(y not free in $\exists x A$ or B) for all B, respectively, are valid and hence derivable in I. (E.g. in the third case, by replacing each application of

$$\frac{\Delta \quad \begin{matrix}\vdots \\ A\end{matrix}}{B} \quad \text{by} \quad \frac{\frac{\Delta}{A \rightarrow B} \quad \begin{matrix}\vdots \\ A\end{matrix}}{B}$$

, we obtain a derivation in C+R, from which R can be eliminated). By application of I-rules in I the derivability of R in I then follows. (E.g. in the second case, we have a derivation of $A \vee B$ from $\Delta' \cup \{A \Rightarrow A \vee B, B \Rightarrow A \vee B\}$ in I; replacing all applications of the assumption rules $A \Rightarrow A \vee B$ and $B \Rightarrow A \vee B$ by applications of the $\vee$-I-rules we obtain a derivation of $A \vee B$ from Δ'). If we denote by $\Gamma \Vdash D$ that D is derivable from Γ only by use of valid rules, we have established:

Theorem $\Gamma \Vdash D$ iff $\Gamma \vdash_{I} D$.

Remarks 1. This theorem does not include that each rule derivable in I is valid. For example the rule R

$$\dfrac{\begin{matrix} \vdots & \vdots & \vdots \\ A & B & C \end{matrix}}{{*}\,(A \wedge B) \wedge C}$$

which can be derived in I by twofold application of $\wedge$-I is not valid in our sense. Its application is not eliminable e.g. from the derivation

$$\dfrac{\dfrac{A \quad B \quad C}{(A \wedge B) \wedge C}}{A \wedge B}$$

in C+R. Thus if $\Delta' \Vdash D$, $\dfrac{\Delta}{D}$ need not be valid. Combination of valid rules does not always yield a valid rule. So the proposed inversion principle is weaker than the definitions of validity Prawitz proposed in [4,5], which are transitive in the sense that combination of valid rules always yield valid rules. A completeness proof for intuitionistic logic with respect to Prawitz' concept of validity (or a related concept) would be more informative than the one given here, but is still a desideratum.

2. We would obtain an analogous result for classical logic if we took C to include

$$\dfrac{\begin{matrix} A \to \bot \\ \vdots \\ \bot \end{matrix}}{A} \qquad \text{instead of} \qquad \dfrac{\bot}{A}\ .$$

If we wanted to give reasons for preferring intuitionistic logic to classical logic in our framework, we would have to argue for a certain choice of canonical E-rules (e.g. that major premises of E-rules must not depend on assumptions). The completeness result ifself does not provide reasons for such a preference.

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