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INVERSION BY DEFINITIONAL REFLECTION AND THE ADMISSIBILITY OF LOGICAL RULES

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Abstract. The *inversion principle* for logical rules expresses a relationship between introduction and elimination rules for logical constants. Hallnäs & Schroeder-Heister (1990, 1991) proposed the *principle of definitional reflection*, which embodies basic ideas of inversion in the more general context of clausal definitions. For the context of admissibility statements, this has been further elaborated by Schroeder-Heister (2007). Using the framework of definitional reflection and its admissibility interpretation, we show that, in the sequent calculus of minimal propositional logic, the left introduction rules are admissible when the right introduction rules are taken as the definitions of the logical constants and vice versa. This generalizes the well-known relationship between introduction and elimination rules in natural deduction to the framework of the sequent calculus.

§1. Inversion principle. The idea of inverting logical rules can be found in a well-known remark by Gentzen: "The introductions are so to say the 'definitions' of the symbols concerned, and the eliminations are ultimately only consequences hereof, what can approximately be expressed as follows: In eliminating a symbol, the formula concerned – of which the outermost symbol is in question – may only 'be used as that what it means on the ground of the introduction of that symbol'."[1] The inversion principle itself was formulated by Lorenzen (1955) in the general context of rule-based systems and is thus not restricted to logical rules. It is based on the idea that if we have certain defining rules for some α, for example,

$$\frac{\beta^1_1, \ldots, \beta^1_{n_1}}{\alpha} \quad \ldots \quad \frac{\beta^k_1, \ldots, \beta^k_{n_k}}{\alpha}$$

then a rule with premiss α and conclusion γ

$$\frac{\alpha}{\gamma}$$

is justified if for each *defining condition* Γ_i of α, where $\Gamma_i = \beta^i_1, \ldots, \beta^i_{n_i}$, the rule

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[1] Our translation of: "Die Einführungen stellen sozusagen die 'Definitionen' der betreffenden Zeichen dar, und die Beseitigungen sind letzten Endes nur Konsequenzen hiervon, was sich etwa so ausdrücken läßt: Bei der Beseitigung eines Zeichens darf die betreffende Formel, um deren äußerstes Zeichen es sich handelt, nur 'als das benutzt werden, was sie auf Grund der Einführung dieses Zeichens bedeutet'." Gentzen (1935, p. 189).

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$$\frac{\Gamma_i}{\gamma}$$

is justified (cf. Schroeder-Heister, 2007, 2008). Formulated as a natural deduction-style rule schema, this corresponds to the pattern

$$\frac{\alpha \qquad \begin{array}{c}\beta^1_1, \ldots, \beta^1_{n_1}\\ \vdots \\ \gamma\end{array} \quad \cdots \quad \begin{array}{c}\beta^k_1, \ldots, \beta^k_{n_k}\\ \vdots \\ \gamma\end{array}}{\gamma}$$

which is related to the formulation of generalized elimination rules in natural deduction as proposed by Prawitz (1965), Schroeder-Heister (1984), and von Plato (2001). However, we will not work in a natural deduction-style framework with assumptions that may be discharged, but in a sequent-style framework, as this allows for an elegant treatment of admissibility statements.

Whereas in the context of natural deduction, the idea of inversion has been intensively discussed following Prawitz's (1965) adaptation of Lorenzen's term 'inversion principle' to explicate Gentzen's remark,[2] there has been no comparable investigation of the relationship between right introduction and left introduction rules of the sequent calculus. The present investigation is a step toward filling this gap. We show that the left introduction rules are inverses of the right introduction rules in the sense of being admissible given the right introduction rules as primitive. In order to show this admissibility, we use the principle of definitional reflection proposed by Hallnäs and Schroeder-Heister[3] in the admissibility interpretation given to it by Schroeder-Heister (2007). We can also show the converse direction, namely, that the right introduction rules are admissible given the left introduction rules as primitive. This is due to the inherent symmetry of the sequent calculus. This symmetry is not present in natural deduction, which makes the formulation of an inversion principle based on elimination rules rather than introduction rules quite difficult.

In this paper, we confine ourselves to minimal propositional logic, that is, to the single succedent sequent calculus for conjunction, disjunction, and implication. In Section 2 we present and discuss the version of definitional reflection we are going to use in our investigation. Section 3 explains the admissibility interpretation of definitional reflection. In Section 4, which is the core part of our investigation, we show the mutual admissibility of right introduction and left introduction rules. As our metalinguistic framework we use a calculus of so-called *f-sequents*, whose antecedents and succedents consist of object-linguistic sequents, called *o-sequents*, where f-sequents are interpreted as admissibility statements concerning o-sequents. The concluding Section 5 discusses the role played by the structural rules at the object-linguistic and the metalinguistic level.

§2. Definitions and definitional reflection. In the following, lowercase Greek letters denote atoms. Expressions of the form $\alpha \Leftarrow \beta^i_1, \ldots, \beta^i_{n_i}$ are called *definitional clauses*, for

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[2] See, for example, the discussion in Negri & von Plato (2001, p. 6f.), where unfortunately Lorenzen is not mentioned, though he is explicitly referred to in Prawitz's (1965) discussion. For a discussion of the relationship of proof-theoretic notions of validity proposed in proof-theoretic semantics and admissibility notions underlying Lorenzen's inversion principle compare Moriconi & Tesconi (2008) and Schroeder-Heister (2008).

[3] Compare Hallnäs (1991), Hallnäs & Schroeder-Heister (1990, 1991), and Schroeder-Heister (1993).

$n \geq 0$, where the *body* $\beta^i_1, \ldots, \beta^i_{n_i}$ is the defining condition of the *head* α. A certain finite set $\mathcal{D}$ of these clauses

$$\mathcal{D} \begin{cases} \alpha_p \Leftarrow \Gamma_p \\ \quad \vdots \\ \alpha_s \Leftarrow \Gamma_s \end{cases}$$

is a *definition*, where each $\Gamma_i = \beta^i_1, \ldots, \beta^i_{n_i}$ is the body of the ith clause, for $p \leq i \leq s$. We observe that clauses may have an empty body. In that case, the atom defined does not depend on any condition. Such clauses are also called *facts*. Let

$$\mathcal{D}_{qr} \begin{cases} \alpha \Leftarrow \Gamma_q \\ \quad \vdots \\ \alpha \Leftarrow \Gamma_r \end{cases}$$

be a subcollection of $\mathcal{D}$ containing all clauses in $\mathcal{D}$ with same head α. These clauses are the *defining clauses* of α with respect to definition $\mathcal{D}$, and $\mathcal{D}(\alpha)$ is used to represent the set of defining conditions of α, that is, $\mathcal{D}(\alpha) = \{\Gamma_q, \ldots, \Gamma_r\}$. Since here we are concerned with logical constants and their role in propositions, atoms in definitional clauses will be taken as propositions.

The consequences of α will be those which are at the same time consequence of each body of $\mathcal{D}_{qr}$. This is what Hallnäs & Schroeder-Heister (1990, 1991) call the *principle of definitional reflection* ($\mathcal{D}\vdash$). This principle can be stated by means of a sequent calculus inference:

$$(\mathcal{D}\vdash)\ \frac{\Delta, \Gamma_q \vdash \gamma \quad \ldots \quad \Delta, \Gamma_r \vdash \gamma}{\Delta, \alpha \vdash \gamma} \qquad \text{(definitional reflection)}$$

It then takes the form of a left introduction rule for atoms α defined by definitional clauses with bodies $\Gamma_q, \ldots, \Gamma_r$, and it is thus a way of stating the inversion principle for definitions.[4] This principle complements the *principle of definitional closure* ($\vdash\mathcal{D}$), for each body $\Gamma_i = \beta^i_1, \ldots, \beta^i_{n_i}$ in $\mathcal{D}(\alpha)$:

$$\frac{\Delta \vdash \beta^i_1 \quad \ldots \quad \Delta \vdash \beta^i_{n_i}}{\Delta \vdash \alpha}\ (\vdash\mathcal{D}) \qquad \text{(definitional closure)}$$

It is the right introduction rule for atoms α under defining condition $\Gamma_i = \beta^i_1, \ldots, \beta^i_{n_i}$. We say that those two principles are part of the framework of the respective definition. They are principles that can be used to reason about the definition. It is natural to extend this framework by other sequent calculus inferences.

Indeed, when both principles are added as inference principles for atoms to a given logical system L, we obtain an extended system $L(\mathcal{D})$, which is a definitional logic based on definition $\mathcal{D}$. If the reasoning principles of L present a symmetry pattern similar to the left/right pattern of sequent calculus systems, then this symmetry pattern is preserved in $L(\mathcal{D})$. The definitional clauses are then the basis for sequent-style right introduction and

[4] Notice that this rule is nonmonotonic in the sense that it has to be altered if $\mathcal{D}$ is extended with a further clause for α.—The idea of inversion does not have to be restricted to logic or logical constants only but can be used for many kinds of definitions. It is then used as a general principle of definitional reasoning (cf. Hallnäs, 1991, Hallnäs & Schroeder-Heister, 1990, 1991, and Schroeder-Heister, 1993). Nonetheless, here we present it taking assertions or propositions as atoms, since our main interest is in definitions of logical constants, and each definitional clause can be interpreted as relating assertions.

left introduction inferences. We assume for our present purposes that the underlying system L consists of the structural inferences *identity* (Id), *thinning* (Thin), and *cut* (Cut):[5]

$$\text{(Id)}\ \frac{}{A \vdash A} \qquad \text{(Thin)}\ \frac{\Delta \vdash A}{B, \Delta \vdash A} \qquad \text{(Cut)}\ \frac{\Delta \vdash C \quad C, \Sigma \vdash A}{\Delta, \Sigma \vdash A}$$

where A, B, and C are variables for formulas. The assumed framework is arguably acceptable from a constructivist point of view, at least inasmuch as the principles of definitional closure and definitional reflection are.

Albeit our characterization above used atoms unspecified, variables should be admitted in order to strengthen definitions: even a representation of minimal propositional logic by clauses requires the use of variables and substitution of variables. For substitutions σ of variables by terms (in a term structure) the principle of definitional closure is

$$\frac{\Delta \vdash (\beta_1^i)^\sigma \quad \ldots \quad \Delta \vdash (\beta_{n_i}^i)^\sigma}{\Delta \vdash \alpha^\sigma}\ (\vdash \mathcal{D})$$

and the principle of definitional reflection is

$$(\mathcal{D} \vdash)\ \frac{\{\Delta, (\Gamma_i)^\sigma \vdash \gamma \mid \delta \Leftarrow \Gamma_i \in \mathcal{D} \text{ and } \alpha = \delta^\sigma\}}{\Delta, \alpha \vdash \gamma}$$

where for the correct handling of variables by means of substitution we have to observe the following proviso:

> For any substitution σ of variables by terms, the application of definitional reflection is restricted to the cases where $\mathcal{D}(\alpha^\sigma) \subseteq (\mathcal{D}(\alpha))^\sigma$.

In other terms, the set of defining conditions of α^σ should be a subset of the set of defining conditions obtained by applying the substitution σ to the defining conditions of α.[6]

One consequence of the proviso is that definitional reflection cannot be applied if for α^σ definitional clauses would have to be taken into account that are not relevant for α. For example, for the definition consisting of the two definitional clauses $\alpha(t) \Leftarrow$ and $\alpha(x) \Leftarrow \beta$ the sequent $\alpha(x) \vdash \beta$ would be derivable from $\beta \vdash \beta$ by $(\mathcal{D} \vdash)$ if the proviso is not respected (for $\sigma = [x/t]$ we have $\mathcal{D}(\alpha(x)^\sigma) = \mathcal{D}(\alpha(t)) = \{\beta^\sigma, \top\} = \{\beta, \top\} \nsubseteq (\mathcal{D}(\alpha(x)))^\sigma = \{\beta^\sigma\} = \{\beta\}$), but $\alpha(t)$ can be obtained in the definition while β cannot. Another consequence is that definitional reflection cannot be applied to clauses having variables in the body which are not in the head. For example, for the definition consisting of the two definitional clauses $\beta(t) \Leftarrow$ and $\alpha(t') \Leftarrow \beta(x)$ the sequent $\alpha(t') \vdash \beta(t')$ would be derivable from $\beta(t') \vdash \beta(t')$ by $(\mathcal{D} \vdash)$ if the proviso is not respected (for $\sigma = [x/t']$ we have $\mathcal{D}(\alpha(t')^\sigma) = \mathcal{D}(\alpha(t')) = \{\beta(x)\} \nsubseteq (\mathcal{D}(\alpha(t')))^\sigma = \{\beta(x)^\sigma\} = \{\beta(t')\}$), but $\alpha(t')$ can be obtained in the definition while $\beta(t')$ cannot. Hence, the proviso is not a restriction on definitions, but only a condition for the applicability of definitional reflection.

If for a given definition $\mathcal{D}$ the set of defining conditions for an atom α (i.e., the set $\mathcal{D}(\alpha)$) is understood as being the set of those Γ_i for which the clauses $\alpha \Leftarrow \Gamma_i$ are substitution instances of definitional clauses in $\mathcal{D}$, then the principle of definitional reflection can again be stated as it was done for unspecified atoms, but with the proviso that $\mathcal{D}(\alpha^\sigma) \subseteq (\mathcal{D}(\alpha))^\sigma$

5 We abstain from using *exchange* and *contraction* in order to avoid unnecessary syntactical detail. The left side of a sequent can be interpreted as a set composed of the formulas listed.

6 This proviso is part of the formulation of definitional reflection proposed in Hallnäs & Schroeder-Heister (1990, 1991). For other variants of definitional reflection and their relationship to the inversion principle compare Schroeder-Heister (2007).

for any substitution σ. In what follows, the principle of definitional reflection will be used in this form.

As an example for the application of the principles of definitional reflection and definitional closure consider the following definition for assertions involving conjunctions and disjunctions, where A and B are variables for formulas:

$$\mathcal{D}_1 \begin{cases} A \wedge B \Leftarrow A, B \\ A \vee B \Leftarrow A \\ A \vee B \Leftarrow B. \end{cases}$$

Relative to $\mathcal{D}_1$, the instances of definitional reflection are

$$(\mathcal{D}_1 \vdash)\ \frac{\Delta, A, B \vdash C}{\Delta, A \wedge B \vdash C}$$

for conjunction and

$$(\mathcal{D}_1 \vdash)\ \frac{\Delta, A \vdash C \qquad \Delta, B \vdash C}{\Delta, A \vee B \vdash C}$$

for disjunction. They correspond to the sequent calculus left introduction rules for conjunction and disjunction. The instances of definitional closure are

$$\frac{\Delta \vdash A \qquad \Delta \vdash B}{\Delta \vdash A \wedge B}\ (\vdash \mathcal{D}_1)$$

for conjunction and

$$\frac{\Delta \vdash A}{\Delta \vdash A \vee B}\ (\vdash \mathcal{D}_1) \qquad\qquad \frac{\Delta \vdash B}{\Delta \vdash A \vee B}\ (\vdash \mathcal{D}_1)$$

for disjunction. They correspond to the sequent calculus right introduction rules for conjunction and disjunction.

§3. Admissibility and logical constants.

A *definitional tree* for a given definition $\mathcal{D}$ is a production of the root α from (possibly empty) leaves $\beta_1, \ldots, \beta_n$ by applications of definitional clauses in $\mathcal{D}$. If there is such a tree, then α is *producible* in $\mathcal{D}$ from $\beta_1, \ldots, \beta_n$, short: $\beta_1, \ldots, \beta_n \Vdash_{\mathcal{D}} \alpha$. We speak of *closed definitional trees* if every leaf is empty, that is, if every branch is started by the application of a fact. For example, if the definition $\mathcal{D}$ is given by the clauses (1) $\beta_1 \Leftarrow$, (2) $\beta_2 \Leftarrow \beta_1$, (3) $\beta_3 \Leftarrow \beta_1$, and (4) $\alpha \Leftarrow \beta_2, \beta_3$, then, for example, the following closed definitional tree

$$\cfrac{\cfrac{\cfrac{}{\beta_1}\,(1)}{\beta_2}\,(2) \qquad \cfrac{\cfrac{}{\beta_1}\,(1)}{\beta_3}\,(3)}{\alpha}\,(4)$$

can be constructed from $\mathcal{D}$, and α is thus producible in $\mathcal{D}$, that is, $\Vdash_{\mathcal{D}} \alpha$. An *open definitional tree* is a definitional tree which has at least one leaf that is not empty.

Concerning a given rule R and a given definition $\mathcal{D}$, rule R is *admissible* in $\mathcal{D}$ if the relation of being producible for $\mathcal{D}$ is not enlarged by adding R to $\mathcal{D}$, yielding the extended system $\mathcal{D} + R$. Let $\Vdash_{\mathcal{D}+R} \alpha$ denote the producibility of α in definition $\mathcal{D}$ with added rule R. Then R is admissible in $\mathcal{D}$, if for every α the implication

$$\text{if } \Vdash_{\mathcal{D}+R} \alpha, \text{ then } \Vdash_{\mathcal{D}} \alpha$$

holds.[7] A rule $R = (\alpha \Leftarrow \beta_1, \ldots, \beta_n)$ is *derivable* in a definition $\mathcal{D}$ if there is an open definitional tree from $\beta_1, \ldots, \beta_n$ to α, that is, if $\beta_1, \ldots, \beta_n \Vdash_{\mathcal{D}} \alpha$ holds. Every derivable rule is admissible, but not vice versa.

The principles of definitional reflection and definitional closure can be interpreted as principles for admissibility (cf. Schroeder-Heister, 2007) if sequents $\beta_1, \ldots, \beta_n \vdash \alpha$ are interpreted as stating the admissibility of rules $\alpha \Leftarrow \beta_1, \ldots, \beta_n$ relative to a given definition $\mathcal{D}$. For definitional reflection

$$(\mathcal{D}\vdash)\ \frac{\Delta, \Gamma_q \vdash \gamma \quad \ldots \quad \Delta, \Gamma_r \vdash \gamma}{\Delta, \alpha \vdash \gamma}$$

consider the rule $\gamma \Leftarrow \alpha, \Delta$ which corresponds to the conclusion of definitional reflection. In this case, α was produced by a rule $\alpha \Leftarrow \beta_1^i, \ldots, \beta_{n_i}^i$, for some i, in the last step, and $\beta_1^i, \ldots, \beta_{n_i}^i$ were produced in previous steps (likewise for Δ). Thus, if the rules $\gamma \Leftarrow \Gamma_i, \Delta$, that is, $\gamma \Leftarrow \beta_1^i, \ldots, \beta_{n_i}^i, \Delta$ (corresponding to the premisses of definitional reflection) are admissible, then the rule corresponding to the conclusion of definitional reflection is admissible as well since all consequences γ producible from $\beta_1^i, \ldots, \beta_{n_i}^i, \Delta$ should be consequences of α. For definitional closure

$$\frac{\Delta \vdash \beta_1^i \quad \ldots \quad \Delta \vdash \beta_{n_i}^i}{\Delta \vdash \alpha}\ (\vdash\mathcal{D})$$

consider the rule $\alpha \Leftarrow \beta_1^i, \ldots, \beta_{n_i}^i$ of the given definition $\mathcal{D}$. Since the rules $\beta_1^i \Leftarrow \Delta, \ldots, \beta_{n_i}^i \Leftarrow \Delta$ (corresponding to the premisses of definitional closure) are assumed to be admissible and α can be produced by using $\alpha \Leftarrow \beta_1^i, \ldots, \beta_{n_i}^i$, the rule $\alpha \Leftarrow \Delta$ (corresponding to the conclusion of definitional closure) is admissible as well by the construction of definitional trees.[8]

Inside the logical framework, using the above principles of definitional reflection and definitional closure, we can derive sequents representing natural deduction rules if the turnstile '$\vdash$' is interpreted as an inference bar. The derivation of the sequent representing the natural deduction conjunction introduction rule is

$$\frac{(\text{Thin})\ \dfrac{(\text{Id})\ \overline{A \vdash A}}{A, B \vdash A} \qquad (\text{Thin})\ \dfrac{(\text{Id})\ \overline{B \vdash B}}{A, B \vdash B}}{A, B \vdash A \wedge B}\ (\vdash\mathcal{D}_1)$$

and the sequents representing the conjunction elimination rules are derived as follows:

$$(\mathcal{D}_1\vdash)\ \frac{(\text{Thin})\ \dfrac{(\text{Id})\ \overline{A \vdash A}}{A, B \vdash A}}{A \wedge B \vdash A} \qquad\qquad (\mathcal{D}_1\vdash)\ \frac{(\text{Thin})\ \dfrac{(\text{Id})\ \overline{B \vdash B}}{A, B \vdash B}}{A \wedge B \vdash B}$$

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[7] In a constructive framework as Lorenzen's, we would expect that this implication is established by giving a *procedure* by means of which every application of R can be eliminated from derivations in $\mathcal{D} + R$. Lorenzen here speaks of an 'elimination procedure' (cf. Lorenzen, 1955, section 3).

[8] It should be emphasized that the admissibility interpretation of definitional reflection used in the present investigation is a particular interpretation of this principle. More general interpretations include partial inductive definitions, which allow for implications in the bodies of clauses and where definitions are not necessarily well founded (cf. Hallnäs, 1991).

Finally, the derivations of the sequents representing disjunction introduction rules are

$$\dfrac{\dfrac{}{A \vdash A}\,(\mathrm{Id})}{A \vdash A \vee B}\,(\vdash\mathcal{D}_1) \qquad\qquad \dfrac{\dfrac{}{B \vdash B}\,(\mathrm{Id})}{B \vdash A \vee B}\,(\vdash\mathcal{D}_1)$$

However, a sequent representing the natural deduction disjunction elimination rule cannot be derived because the discharging of assumptions in this rule cannot be expressed by means of a simple sequent.

The preceding clauses $\mathcal{D}_1$ for logical constants were based on the natural deduction introduction rules. Maybe it is possible to define the logical constants also by means of clauses representing natural deduction elimination rules. For example, could the following be taken as a definition of conjunction?

$$\mathcal{D}_2 \begin{cases} A \Leftarrow A \wedge B \\ B \Leftarrow A \wedge B. \end{cases}$$

Here the problem is that a sequent representing conjunction introduction cannot be derived because the proviso on variables forbids the use of definitional reflection on clauses that have variables in the body, which do not occur in the head.

Having considered definitions of conjunction and disjunction, we now turn to the problem of defining implication. In natural deduction, the implication introduction rule involves discharging of assumptions, which would lead to the problem of how to capture discharging of assumptions in a clausal definition.

One possibility for characterizing implication that avoids the problem of how to discharge assumptions is to make appeal to the elimination rule and axioms. In axiomatic systems, implication can be characterized by the two following axiom schemata plus an elimination rule:

Axioms

1. $A \to (B \to A)$
2. $(A \to (B \to C)) \to ((A \to B) \to (A \to C))$

Modus Ponens

$$\frac{A \qquad A \to B}{B}$$

These schemata suffice to prove the deduction theorem, that is, implication introduction, thus providing a sufficient characterization of implication. A corresponding clausal definition would be the following:

$$\mathcal{D}_3 \begin{cases} A \to (B \to A) \Leftarrow \\ (A \to (B \to C)) \to ((A \to B) \to (A \to C)) \Leftarrow \\ B \Leftarrow A, A \to B. \end{cases}$$

However, in the last clause the variable A occurs only in the body and not in the head. Again, this violates the proviso on variables of definitional reflection.

There still remains another possibility of representing more complex objects by means of clauses and to characterize implication by means of these objects. The structure of clauses suggests that they could be interpreted as inference rules involving assertions in which the body of a clause are the premisses, and the head is the conclusion. Indeed, sequents can be seen as assertions if the sequent symbol is interpreted as a relation of *deductive consequence* holding between the antecedent and the succedent of a sequent. Clausal definitions of sequents may thus provide a way of explaining the behavior of logical constants and of implication in particular. Besides, in sequent calculus we do not have to consider the problem of the discharging of assumptions.

§4. Logical constants defined by sequents. For the clausal definition of sequents we introduce the following representation for object language sequents $\mathfrak{s}$, called *o-sequents*:

$$\begin{array}{c}\Omega\\ \triangledown\\ A\end{array}$$

We read it as: *A follows from* Ω. This should not be confused with the sequents in the framework, which from now on will be called *f-sequents* and are expressed with the turnstile '$\vdash$'.[9] Finite sets of o-sequents will be denoted by $\mathfrak{S}$.

Our aim is to represent minimal propositional logic, that is, sequent calculus for minimal propositional logic. This is why o-sequents have exactly one formula at the bottom; it corresponds to the succedent of sequents. What is written on top (corresponding to the antecedent of sequents) is either a (possibly empty) finite multiset of formulas or a comma-separated list of such sets, the comma representing multiset union.[10] Singletons are written without braces. The proposed notation resembles schematic representations of natural deduction derivations. Therefore, the triangle '$\triangledown$' can be used either to represent the relation of deductive consequence or to represent derivations in natural deduction. We concentrate on the first reading, where the sequent symbol '$\triangledown$' represents the relation of deductive consequence.[11]

The basic properties of the relation of deductive consequence are the following: First, any formula A is a deductive consequence of itself. Second, if A is a consequence of Ω, then A is a consequence of $\Omega \cup B$. Third, the relation of logical consequence is usually assumed to be transitive (at least, it is so in minimal logic). These properties of the usual deductive consequence relation are captured in sequent calculus by the inferences *identity*, *thinning*, and *cut*. The following rules (o-Id), (o-Thin), and (o-Cut) express these properties for o-sequents and are added to the framework:[12]

$$\dfrac{}{\vdash \begin{array}{c}A\\ \triangledown\\ A\end{array}}\,(\text{o-Id}) \qquad \dfrac{\mathfrak{S} \vdash \begin{array}{c}\Omega\\ \triangledown\\ A\end{array}}{\mathfrak{S} \vdash \begin{array}{c}\Omega, B\\ \triangledown\\ A\end{array}}\,(\text{o-Thin}) \qquad \dfrac{\mathfrak{S}_1 \vdash \begin{array}{c}\Omega\\ \triangledown\\ C\end{array} \quad \mathfrak{S}_2 \vdash \begin{array}{c}C, \Psi\\ \triangledown\\ A\end{array}}{\mathfrak{S}_1, \mathfrak{S}_2 \vdash \begin{array}{c}\Omega, \Psi\\ \triangledown\\ A\end{array}}\,(\text{o-Cut})$$

We use framework rules instead of definitional clauses corresponding to the structural rules for o-sequents because we are interested in inversion for logical rules only, and definitional clauses for structural rules would add nothing to the definition of logical constants. If

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9 For example, $\begin{array}{c}\Omega\\ \triangledown\\ A\end{array}, \begin{array}{c}\Omega\\ \triangledown\\ B\end{array} \vdash \begin{array}{c}\Omega\\ \triangledown\\ A \wedge B\end{array}$ is an f-sequent having the two o-sequents $\begin{array}{c}\Omega\\ \triangledown\\ A\end{array}$ and $\begin{array}{c}\Omega\\ \triangledown\\ B\end{array}$ in the antecedent and the o-sequent $\begin{array}{c}\Omega\\ \triangledown\\ A \wedge B\end{array}$ in the succedent.

10 This way, we do not need *exchange* for antecedents, that is, for the top of o-sequents.

11 As pointed out by Prawitz (1965, p. 90f.), the sequent calculus can be conceived as a meta-calculus for the deductive consequence relation in a corresponding calculus of natural deduction. Therefore, the sequent symbol stands for a deductive consequence relation and sequents can be understood as assertions about this relation.

12 Multisets require *contraction*. We do not give it here as a rule because it is not needed in the following.

definitional clauses were given instead, they would have to be restricted to definitional closure.

4.1. Admissibility of left introduction rules. Next we are going to show that having clauses for right introduction rules, that is, clauses for the introduction of a logical constant in the bottom of an o-sequent, we can derive framework sequents of the form $\mathfrak{S} \vdash \mathfrak{s}$ representing the left introduction rules, that is, clauses for the introduction of a logical constant in the top of an o-sequent, inside the logical framework by using definitional reflection.[13] We thereby show that each left introduction rule is admissible if the corresponding right introduction rule is given by definition.

We do this by using the following context-independent versions of definitional reflection

$$(\mathcal{D}\vdash)\ \frac{\Delta_q, \Gamma_q \vdash \gamma \quad \dots \quad \Delta_r, \Gamma_r \vdash \gamma}{\Delta_q, \dots, \Delta_r, \alpha \vdash \gamma}$$

and definitional closure

$$\frac{\Delta_1 \vdash \beta_1 \quad \dots \quad \Delta_n \vdash \beta_n}{\Delta_1, \dots, \Delta_n \vdash \alpha}\ (\vdash\mathcal{D})$$

together with the structural rules for f-sequents (Id), (Thin), (Cut)[14] and the structural rules for o-sequents (o-Id), (o-Thin), (o-Cut) already presented. The resulting framework is equivalent to the framework with context-sharing versions, as can be seen by considering

$$(\mathcal{D}\vdash)\ \frac{(\text{Thin})\ \dfrac{\Delta_q, \Gamma_q \vdash \gamma}{\Delta_q, \dots, \Delta_r, \Gamma_q \vdash \gamma} \quad \dots \quad (\text{Thin})\ \dfrac{\Delta_r, \Gamma_r \vdash \gamma}{\Delta_q, \dots, \Delta_r, \Gamma_r \vdash \gamma}}{\Delta_q, \dots, \Delta_r, \alpha \vdash \gamma}$$

for definitional reflection and

$$\frac{(\text{Thin})\ \dfrac{\Delta_1 \vdash \beta_1}{\Delta_1, \dots, \Delta_n \vdash \beta_1} \quad \dots \quad (\text{Thin})\ \dfrac{\Delta_n \vdash \beta_n}{\Delta_1, \dots, \Delta_n \vdash \beta_n}}{\Delta_1, \dots, \Delta_n \vdash \alpha}\ (\vdash\mathcal{D})$$

for definitional closure (double lines indicating multiple applications of structural inferences). The context-sharing versions are then special cases of the context-independent versions since the antecedents of f-sequents are interpreted as sets.[15]

In what follows, we first present the definition of the right introduction rule together with the definitional reflection for this definition in the most general form. Then the framework

[13] We maintain sequent calculus terminology; that is, we speak of 'right introduction rules' respectively 'left introduction rules'. Although for the o-sequents used here, it would be more appropriate to speak of 'bottom introduction rules' respectively 'top introduction rules'.

[14] That is, the formulas respectively sets of formulas occurring in (Id), (Thin), and (Cut) are now o-sequents respectively sets of o-sequents.

[15] Proof systems for admissible rules have been investigated and developed, for example, by Rybakov (1997), Iemhoff (2001, 2003), Jeřábek (2008), and Iemhoff & Metcalfe (2009). This research tradition is concerned with characterizing the rules that are admissible for a given logic (e.g., intuitionistic or modal logic). In contrast, we are dealing here with the question of whether by means of a general principle, such as definitional reflection, we can justify certain inference rules (e.g., the left introduction rules, if the right introduction rules are assumed as given) by showing them to be admissible. Apart from the inversion principle, we do not discuss here further admissibility principles proposed by Lorenzen (1955), who coined the terms 'admissible' ('zulässig') and 'admissibility' ('Zulässigkeit') and developed a theory of admissible rules for arbitrary atomic systems (cf. Schroeder-Heister, 2008).

sequent for the corresponding left introduction rule is given and its derivation is shown. Note that in the derivations the definitional reflection for the respective definition has not to be used in its most general form.

4.1.1. Definition of right conjunction and admissibility of left conjunction. The right conjunction introduction rule is given by the following definition $\mathcal{D}_\wedge$, which provides the means for obtaining the left conjunction introduction rule by definitional reflection ($\mathcal{D}_\wedge \vdash$):

$$\mathcal{D}_\wedge \left\{ \begin{array}{c}\Omega\\ \triangledown\\ A\wedge B\end{array} \Leftarrow \begin{array}{c}\Omega\\ \triangledown\\ A\end{array}, \begin{array}{c}\Omega\\ \triangledown\\ B\end{array} \right. \qquad\qquad (\mathcal{D}_\wedge \vdash)\ \dfrac{\mathfrak{S}, \begin{array}{c}\Omega\\ \triangledown\\ A\end{array}, \begin{array}{c}\Omega\\ \triangledown\\ B\end{array} \vdash \mathfrak{s}}{\mathfrak{S}, \begin{array}{c}\Omega\\ \triangledown\\ A\wedge B\end{array} \vdash \mathfrak{s}}$$

The f-sequent for the left conjunction introduction rule is

$$\begin{array}{c}\Theta, A\\ \triangledown\\ C\end{array} \vdash \begin{array}{c}\Theta, A\wedge B\\ \triangledown\\ C\end{array} \quad \text{respectively} \quad \begin{array}{c}\Theta, B\\ \triangledown\\ C\end{array} \vdash \begin{array}{c}\Theta, A\wedge B\\ \triangledown\\ C\end{array}$$

and its derivation is:

$$\dfrac{\dfrac{}{\vdash \begin{array}{c}A\wedge B\\ \triangledown\\ A\wedge B\end{array}}\text{(o-Id)} \qquad (\mathcal{D}_\wedge\vdash)\dfrac{(\text{Thin})\dfrac{\dfrac{(\text{Id})\dfrac{}{\begin{array}{c}A\wedge B\\ \triangledown\\ A\end{array} \vdash \begin{array}{c}A\wedge B\\ \triangledown\\ A\end{array}} \qquad (\text{Id})\dfrac{}{\begin{array}{c}\Theta, A\\ \triangledown\\ C\end{array} \vdash \begin{array}{c}\Theta, A\\ \triangledown\\ C\end{array}}}{\begin{array}{c}A\wedge B\\ \triangledown\\ A\end{array}, \begin{array}{c}\Theta, A\\ \triangledown\\ C\end{array} \vdash \begin{array}{c}\Theta, A\wedge B\\ \triangledown\\ C\end{array}}\text{(o-Cut)}}{\begin{array}{c}\Theta, A\\ \triangledown\\ C\end{array}, \begin{array}{c}A\wedge B\\ \triangledown\\ A\end{array}, \begin{array}{c}A\wedge B\\ \triangledown\\ B\end{array} \vdash \begin{array}{c}\Theta, A\wedge B\\ \triangledown\\ C\end{array}}}{\begin{array}{c}\Theta, A\\ \triangledown\\ C\end{array}, \begin{array}{c}A\wedge B\\ \triangledown\\ A\wedge B\end{array} \vdash \begin{array}{c}\Theta, A\wedge B\\ \triangledown\\ C\end{array}}}{\begin{array}{c}\Theta, A\\ \triangledown\\ C\end{array} \vdash \begin{array}{c}\Theta, A\wedge B\\ \triangledown\\ C\end{array}}\text{(Cut)}$$

Likewise for $\begin{array}{c}\Theta, B\\ \triangledown\\ C\end{array} \vdash \begin{array}{c}\Theta, A\wedge B\\ \triangledown\\ C\end{array}$.

4.1.2. Definition of right disjunction and admissibility of left disjunction. The right disjunction introduction rule is given by the following definition $\mathcal{D}_\vee$, which provides the means for obtaining the left disjunction introduction rule by definitional reflection ($\mathcal{D}_\vee \vdash$):

$$\mathcal{D}_\vee \left\{ \begin{array}{l} \begin{array}{c}\Omega\\ \triangledown\\ A\vee B\end{array} \Leftarrow \begin{array}{c}\Omega\\ \triangledown\\ A\end{array} \\[2ex] \begin{array}{c}\Omega\\ \triangledown\\ A\vee B\end{array} \Leftarrow \begin{array}{c}\Omega\\ \triangledown\\ B\end{array} \end{array} \right. \qquad\qquad (\mathcal{D}_\vee \vdash)\ \dfrac{\mathfrak{S}_1, \begin{array}{c}\Omega\\ \triangledown\\ A\end{array} \vdash \mathfrak{s} \qquad \mathfrak{S}_2, \begin{array}{c}\Omega\\ \triangledown\\ B\end{array} \vdash \mathfrak{s}}{\mathfrak{S}_1, \mathfrak{S}_2, \begin{array}{c}\Omega\\ \triangledown\\ A\vee B\end{array} \vdash \mathfrak{s}}$$

The f-sequent for the left disjunction introduction rule is

$$\begin{smallmatrix}\Theta, A\\ \triangledown\\ C\end{smallmatrix}\ ,\ \begin{smallmatrix}\Lambda, B\\ \triangledown\\ C\end{smallmatrix}\ \vdash\ \begin{smallmatrix}\Theta, \Lambda, A\vee B\\ \triangledown\\ C\end{smallmatrix}$$

and its derivation is:

$$\dfrac{\dfrac{}{\vdash \begin{smallmatrix}A\vee B\\ \triangledown\\ A\vee B\end{smallmatrix}}(\text{o-Id}) \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{\begin{smallmatrix}A\vee B\\ \triangledown\\ A\end{smallmatrix} \vdash \begin{smallmatrix}A\vee B\\ \triangledown\\ A\end{smallmatrix}}(\text{Id}) \quad \dfrac{}{\begin{smallmatrix}\Theta, A\\ \triangledown\\ C\end{smallmatrix} \vdash \begin{smallmatrix}\Theta, A\\ \triangledown\\ C\end{smallmatrix}}(\text{Id})}{\begin{smallmatrix}A\vee B\\ \triangledown\\ A\end{smallmatrix}, \begin{smallmatrix}\Theta, A\\ \triangledown\\ C\end{smallmatrix} \vdash \begin{smallmatrix}\Theta, A\vee B\\ \triangledown\\ C\end{smallmatrix}}(\text{o-Cut})}{\begin{smallmatrix}\Theta, A\\ \triangledown\\ C\end{smallmatrix}, \begin{smallmatrix}A\vee B\\ \triangledown\\ A\end{smallmatrix} \vdash \begin{smallmatrix}\Theta, \Lambda, A\vee B\\ \triangledown\\ C\end{smallmatrix}}(\text{o-Thin}) \quad \dfrac{\dfrac{\dfrac{}{\begin{smallmatrix}A\vee B\\ \triangledown\\ B\end{smallmatrix} \vdash \begin{smallmatrix}A\vee B\\ \triangledown\\ B\end{smallmatrix}}(\text{Id}) \quad \dfrac{}{\begin{smallmatrix}\Lambda, B\\ \triangledown\\ C\end{smallmatrix} \vdash \begin{smallmatrix}\Lambda, B\\ \triangledown\\ C\end{smallmatrix}}(\text{Id})}{\begin{smallmatrix}A\vee B\\ \triangledown\\ B\end{smallmatrix}, \begin{smallmatrix}\Lambda, B\\ \triangledown\\ C\end{smallmatrix} \vdash \begin{smallmatrix}\Lambda, A\vee B\\ \triangledown\\ C\end{smallmatrix}}(\text{o-Cut})}{\begin{smallmatrix}\Lambda, B\\ \triangledown\\ C\end{smallmatrix}, \begin{smallmatrix}A\vee B\\ \triangledown\\ B\end{smallmatrix} \vdash \begin{smallmatrix}\Theta, \Lambda, A\vee B\\ \triangledown\\ C\end{smallmatrix}}(\text{o-Thin})}{\begin{smallmatrix}\Theta, A\\ \triangledown\\ C\end{smallmatrix}, \begin{smallmatrix}\Lambda, B\\ \triangledown\\ C\end{smallmatrix}, \begin{smallmatrix}A\vee B\\ \triangledown\\ A\vee B\end{smallmatrix} \vdash \begin{smallmatrix}\Theta, \Lambda, A\vee B\\ \triangledown\\ C\end{smallmatrix}}(\mathcal{D}_{\vee}\vdash)}{\begin{smallmatrix}\Theta, A\\ \triangledown\\ C\end{smallmatrix}, \begin{smallmatrix}\Lambda, B\\ \triangledown\\ C\end{smallmatrix} \vdash \begin{smallmatrix}\Theta, \Lambda, A\vee B\\ \triangledown\\ C\end{smallmatrix}}(\text{Cut})$$

4.1.3. Definition of right implication and admissibility of left implication. The right implication introduction rule is given by the following definition $\mathcal{D}_{\to}$, which provides the means for obtaining the left implication introduction rule by definitional reflection $(\mathcal{D}_{\to}\vdash)$:

$$\mathcal{D}_{\to}\left\{ \begin{smallmatrix}\Omega\\ \triangledown\\ A\to B\end{smallmatrix} \Leftarrow \begin{smallmatrix}\Omega, A\\ \triangledown\\ B\end{smallmatrix}\right. \qquad\qquad (\mathcal{D}_{\to}\vdash)\ \dfrac{\mathfrak{S},\ \begin{smallmatrix}\Omega, A\\ \triangledown\\ B\end{smallmatrix}\ \vdash_{\mathfrak{s}}}{\mathfrak{S},\ \begin{smallmatrix}\Omega\\ \triangledown\\ A\to B\end{smallmatrix}\ \vdash_{\mathfrak{s}}}$$

The f-sequent for the left implication introduction rule is

$$\begin{smallmatrix}\Theta\\ \triangledown\\ A\end{smallmatrix}\ ,\ \begin{smallmatrix}\Lambda, B\\ \triangledown\\ C\end{smallmatrix}\ \vdash\ \begin{smallmatrix}\Theta, \Lambda, A\to B\\ \triangledown\\ C\end{smallmatrix}$$

and its derivation is:

$$\dfrac{\dfrac{}{\vdash \begin{smallmatrix}A\to B\\ \triangledown\\ A\to B\end{smallmatrix}}(\text{o-Id}) \qquad \dfrac{\dfrac{\dfrac{}{\begin{smallmatrix}\Theta\\ \triangledown\\ A\end{smallmatrix} \vdash \begin{smallmatrix}\Theta\\ \triangledown\\ A\end{smallmatrix}}(\text{Id}) \quad \dfrac{\dfrac{}{\begin{smallmatrix}A\to B, A\\ \triangledown\\ B\end{smallmatrix} \vdash \begin{smallmatrix}A\to B, A\\ \triangledown\\ B\end{smallmatrix}}(\text{Id}) \quad \dfrac{}{\begin{smallmatrix}\Lambda, B\\ \triangledown\\ C\end{smallmatrix} \vdash \begin{smallmatrix}\Lambda, B\\ \triangledown\\ C\end{smallmatrix}}(\text{Id})}{\begin{smallmatrix}A\to B, A\\ \triangledown\\ B\end{smallmatrix}, \begin{smallmatrix}\Lambda, B\\ \triangledown\\ C\end{smallmatrix} \vdash \begin{smallmatrix}\Lambda, A\to B, A\\ \triangledown\\ C\end{smallmatrix}}(\text{o-Cut})}{\begin{smallmatrix}\Theta\\ \triangledown\\ A\end{smallmatrix}, \begin{smallmatrix}\Lambda, B\\ \triangledown\\ C\end{smallmatrix}, \begin{smallmatrix}A\to B, A\\ \triangledown\\ B\end{smallmatrix} \vdash \begin{smallmatrix}\Theta, \Lambda, A\to B\\ \triangledown\\ C\end{smallmatrix}}(\text{o-Cut})}{\begin{smallmatrix}\Theta\\ \triangledown\\ A\end{smallmatrix}, \begin{smallmatrix}\Lambda, B\\ \triangledown\\ C\end{smallmatrix}, \begin{smallmatrix}A\to B\\ \triangledown\\ A\to B\end{smallmatrix} \vdash \begin{smallmatrix}\Theta, \Lambda, A\to B\\ \triangledown\\ C\end{smallmatrix}}(\mathcal{D}_{\to}\vdash)}{\begin{smallmatrix}\Theta\\ \triangledown\\ A\end{smallmatrix}, \begin{smallmatrix}\Lambda, B\\ \triangledown\\ C\end{smallmatrix} \vdash \begin{smallmatrix}\Theta, \Lambda, A\to B\\ \triangledown\\ C\end{smallmatrix}}(\text{Cut})$$

4.2. Remarks. The definitional clause for right conjunction introduction $\mathcal{D}_\wedge$ has been formulated with shared context Ω because the use of independent contexts would violate the proviso on variables (viz. $\mathcal{D}(\mathfrak{s}^\sigma) \subseteq (\mathcal{D}(\mathfrak{s}))^\sigma$ for any substitution σ) of definitional reflection. A violation would occur for independent contexts irrespective of whether the top of o-sequents is taken as set, multiset, or list. The definitional clause for right conjunction introduction with independent contexts Ω and Ψ would be

$$\mathcal{D}'_\wedge \left\{ \begin{array}{c}\Omega,\Psi\\ \triangledown\\ A\wedge B\end{array} \Leftarrow \begin{array}{c}\Omega\\ \triangledown\\ A\end{array}, \begin{array}{c}\Psi\\ \triangledown\\ B\end{array}\right. .$$

Then for multisets and $\sigma = [\Omega/A, \Psi/A]$ we would have

$$\begin{aligned}\mathcal{D}\left(\left[\begin{array}{c}\Omega,\Psi\\ \triangledown\\ A\wedge B\end{array}\right]^\sigma\right) &= \mathcal{D}\left(\begin{array}{c}A,A\\ \triangledown\\ A\wedge B\end{array}\right) = \left\{\begin{array}{c}\\ \triangledown\\ A\end{array}, \begin{array}{c}A,A\\ \triangledown\\ B\end{array}; \begin{array}{c}A,A\\ \triangledown\\ A\end{array}, \begin{array}{c}\\ \triangledown\\ B\end{array}; \begin{array}{c}A\\ \triangledown\\ A\end{array}, \begin{array}{c}A\\ \triangledown\\ B\end{array}\right\} \\ &\supset \left(\mathcal{D}\left(\begin{array}{c}\Omega,\Psi\\ \triangledown\\ A\wedge B\end{array}\right)\right)^\sigma = \left\{\begin{array}{c}\Omega\\ \triangledown\\ A\end{array}, \begin{array}{c}\Psi\\ \triangledown\\ B\end{array}; \begin{array}{c}\Psi\\ \triangledown\\ A\end{array}, \begin{array}{c}\Omega\\ \triangledown\\ B\end{array}\right\}^\sigma = \left\{\begin{array}{c}A\\ \triangledown\\ A\end{array}, \begin{array}{c}A\\ \triangledown\\ B\end{array}\right\}.\end{aligned}$$

For sets and for lists a similar situation obtains.

Thus the proviso would be violated and the f-sequent corresponding to left conjunction introduction would not be derivable. The formulation of right conjunction introduction with shared contexts is, however, not an undue restriction since the rule for right conjunction introduction with independent contexts can be shown to be admissible by deriving the corresponding f-sequent with definitional closure on $\mathcal{D}_\wedge$:

$$\dfrac{\dfrac{(\mathrm{Id})\,\dfrac{}{\begin{array}{c}\Omega\\ \triangledown\\ A\end{array} \vdash \begin{array}{c}\Omega\\ \triangledown\\ A\end{array}}}{\begin{array}{c}\Omega\\ \triangledown\\ A\end{array} \vdash \begin{array}{c}\Omega,\Psi\\ \triangledown\\ A\end{array}}\,(\text{o-Thin}) \qquad \dfrac{(\mathrm{Id})\,\dfrac{}{\begin{array}{c}\Psi\\ \triangledown\\ B\end{array} \vdash \begin{array}{c}\Psi\\ \triangledown\\ B\end{array}}}{\begin{array}{c}\Psi\\ \triangledown\\ B\end{array} \vdash \begin{array}{c}\Omega,\Psi\\ \triangledown\\ B\end{array}}\,(\text{o-Thin})}{\begin{array}{c}\Omega\\ \triangledown\\ A\end{array}, \begin{array}{c}\Psi\\ \triangledown\\ B\end{array} \vdash \begin{array}{c}\Omega,\Psi\\ \triangledown\\ A\wedge B\end{array}}\,(\vdash\mathcal{D}_\wedge)$$

For right disjunction introduction $\mathcal{D}_\vee$ and right implication introduction $\mathcal{D}_\to$ the question of whether to use shared or independent contexts does not arise.

Although the definitional clauses for the right introduction rules were treated separately above, they can be combined to yield the definition of minimal propositional logic. Since the o-sequents have only one formula in the bottom, only the defining conditions for that formula have to be considered in applications of definitional reflection. Inversion by definitional reflection for the given right introduction rules is in this sense local.

4.3. Admissibility of right introduction rules. The reason for not taking the eliminations to define conjunction and implication (see Section 3) was a problem with the proviso, but in using o-sequents we avoided this problem.

Definitional clauses that are in accordance with the proviso can be given also for the left introduction rules. And the following derivations show admissibility of the right introduction rules by using definitional reflection on those clauses.

4.3.1. Definition of left conjunction and admissibility of right conjunction. The left conjunction introduction rule is given by the following definition $\mathcal{D}^{\wedge}$, which provides the means for obtaining the right conjunction introduction rule by definitional reflection $(\mathcal{D}^{\wedge}\vdash)$:

$$\mathcal{D}^{\wedge}\left\{\begin{array}{l} \begin{array}{c}\Omega, A\wedge B\\ \triangledown\\ C\end{array} \Leftarrow \begin{array}{c}\Omega, A\\ \triangledown\\ C\end{array} \\[2ex] \begin{array}{c}\Omega, A\wedge B\\ \triangledown\\ C\end{array} \Leftarrow \begin{array}{c}\Omega, B\\ \triangledown\\ C\end{array}\end{array}\right. \qquad (\mathcal{D}^{\wedge}\vdash)\ \frac{\mathfrak{S}_1, \begin{array}{c}\Omega, A\\ \triangledown\\ C\end{array} \vdash \mathfrak{s} \qquad \mathfrak{S}_2, \begin{array}{c}\Omega, B\\ \triangledown\\ C\end{array} \vdash \mathfrak{s}}{\mathfrak{S}_1, \mathfrak{S}_2, \begin{array}{c}\Omega, A\wedge B\\ \triangledown\\ C\end{array} \vdash \mathfrak{s}}$$

The f-sequent for the right conjunction introduction rule is

$$\begin{array}{c}\Theta\\ \triangledown\\ A\end{array}, \begin{array}{c}\Lambda\\ \triangledown\\ B\end{array} \vdash \begin{array}{c}\Theta, \Lambda\\ \triangledown\\ A\wedge B\end{array}$$

and its derivation is:

$$\dfrac{\dfrac{}{\vdash \begin{array}{c}A\wedge B\\ \triangledown\\ A\wedge B\end{array}}\,(\text{o-Id}) \qquad \dfrac{\begin{array}{c} \dfrac{\begin{array}{c}\dfrac{}{\begin{array}{c}\Theta\\ \triangledown\\ A\end{array} \vdash \begin{array}{c}\Theta\\ \triangledown\\ A\end{array}}\,(\text{Id}) \quad \dfrac{}{\begin{array}{c}A\\ \triangledown\\ A\wedge B\end{array} \vdash \begin{array}{c}A\\ \triangledown\\ A\wedge B\end{array}}\,(\text{Id})\end{array}}{\begin{array}{c}\Theta\\ \triangledown\\ A\end{array}, \begin{array}{c}A\\ \triangledown\\ A\wedge B\end{array} \vdash \begin{array}{c}\Theta\\ \triangledown\\ A\wedge B\end{array}}\,(\text{o-Cut}) \\ \hline\hline \begin{array}{c}\Theta\\ \triangledown\\ A\end{array}, \begin{array}{c}A\\ \triangledown\\ A\wedge B\end{array} \vdash \begin{array}{c}\Theta, \Lambda\\ \triangledown\\ A\wedge B\end{array}\end{array}\,(\text{o-Thin}) \quad \begin{array}{c} \dfrac{\begin{array}{c}\dfrac{}{\begin{array}{c}\Lambda\\ \triangledown\\ B\end{array} \vdash \begin{array}{c}\Lambda\\ \triangledown\\ B\end{array}}\,(\text{Id}) \quad \dfrac{}{\begin{array}{c}B\\ \triangledown\\ A\wedge B\end{array} \vdash \begin{array}{c}B\\ \triangledown\\ A\wedge B\end{array}}\,(\text{Id})\end{array}}{\begin{array}{c}\Lambda\\ \triangledown\\ B\end{array}, \begin{array}{c}B\\ \triangledown\\ A\wedge B\end{array} \vdash \begin{array}{c}\Lambda\\ \triangledown\\ A\wedge B\end{array}}\,(\text{o-Cut}) \\ \hline\hline \begin{array}{c}\Lambda\\ \triangledown\\ B\end{array}, \begin{array}{c}B\\ \triangledown\\ A\wedge B\end{array} \vdash \begin{array}{c}\Theta, \Lambda\\ \triangledown\\ A\wedge B\end{array}\end{array}\,(\text{o-Thin})}{\begin{array}{c}\Theta\\ \triangledown\\ A\end{array}, \begin{array}{c}\Lambda\\ \triangledown\\ B\end{array}, \begin{array}{c}A\wedge B\\ \triangledown\\ A\wedge B\end{array} \vdash \begin{array}{c}\Theta, \Lambda\\ \triangledown\\ A\wedge B\end{array}}\,(\mathcal{D}^{\wedge}\vdash)}{\begin{array}{c}\Theta\\ \triangledown\\ A\end{array}, \begin{array}{c}\Lambda\\ \triangledown\\ B\end{array} \vdash \begin{array}{c}\Theta, \Lambda\\ \triangledown\\ A\wedge B\end{array}}\,(\text{Cut})$$

4.3.2. Definition of left disjunction and admissibility of right disjunction. The left disjunction introduction rule is given by the following definition $\mathcal{D}^{\vee}$, which provides the means for obtaining the right disjunction introduction rule by definitional reflection $(\mathcal{D}^{\vee}\vdash)$:

$$\mathcal{D}^{\vee}\left\{ \begin{array}{c}\Omega, A\vee B\\ \triangledown\\ C\end{array} \Leftarrow \begin{array}{c}\Omega, A\\ \triangledown\\ C\end{array}, \begin{array}{c}\Omega, B\\ \triangledown\\ C\end{array}\right. \qquad (\mathcal{D}^{\vee}\vdash)\ \frac{\mathfrak{S}, \begin{array}{c}\Omega, A\\ \triangledown\\ C\end{array}, \begin{array}{c}\Omega, B\\ \triangledown\\ C\end{array} \vdash \mathfrak{s}}{\mathfrak{S}, \begin{array}{c}\Omega, A\vee B\\ \triangledown\\ C\end{array} \vdash \mathfrak{s}}$$

The f-sequent for the right disjunction introduction rule is

$$\begin{array}{c}\Theta\\ \triangledown\\ A\end{array} \vdash \begin{array}{c}\Theta\\ \triangledown\\ A\vee B\end{array} \quad \text{respectively} \quad \begin{array}{c}\Theta\\ \triangledown\\ B\end{array} \vdash \begin{array}{c}\Theta\\ \triangledown\\ A\vee B\end{array}$$

and its derivation is:

$$
\dfrac{
\dfrac{}{\begin{array}{c}\triangledown\\A\end{array}\vdash\begin{array}{c}\triangledown\\A\end{array}}\,\text{(o-Id)}
\qquad
(\mathcal{D}^{\vee}\vdash)\,\dfrac{
(\text{Thin})\,\dfrac{
\dfrac{
(\text{Id})\,\dfrac{}{\begin{array}{c}\Theta\\\triangledown\\A\end{array}\vdash\begin{array}{c}\Theta\\\triangledown\\A\end{array}}
\qquad
(\text{Id})\,\dfrac{}{\begin{array}{c}A\\\triangledown\\A\vee B\end{array}\vdash\begin{array}{c}A\\\triangledown\\A\vee B\end{array}}
}{\begin{array}{c}\Theta\\\triangledown\\A\end{array},\begin{array}{c}A\\\triangledown\\A\vee B\end{array}\vdash\begin{array}{c}\Theta\\\triangledown\\A\vee B\end{array}}\,\text{(o-Cut)}
}{\begin{array}{c}\Theta\\\triangledown\\A\end{array},\begin{array}{c}A\\\triangledown\\A\vee B\end{array},\begin{array}{c}B\\\triangledown\\A\vee B\end{array}\vdash\begin{array}{c}\Theta\\\triangledown\\A\vee B\end{array}}
}{\begin{array}{c}\Theta\\\triangledown\\A\end{array},\begin{array}{c}A\vee B\\\triangledown\\A\vee B\end{array}\vdash\begin{array}{c}\Theta\\\triangledown\\A\vee B\end{array}}
}{\begin{array}{c}\Theta\\\triangledown\\A\end{array}\vdash\begin{array}{c}\Theta\\\triangledown\\A\vee B\end{array}}\,\text{(Cut)}
$$

Note: the left premise of (Cut) is $\dfrac{}{\vdash\begin{array}{c}A\vee B\\\triangledown\\A\vee B\end{array}}$ (o-Id).

Likewise for $\begin{array}{c}\Theta\\\triangledown\\B\end{array}\vdash\begin{array}{c}\Theta\\\triangledown\\A\vee B\end{array}$.

4.3.3. Definition of left implication and admissibility of right implication. The left implication introduction rule is given by the following definition $\mathcal{D}^{\to}$, which provides the means for obtaining the right implication introduction rule by definitional reflection $(\mathcal{D}^{\to}\vdash)$:

$$
\mathcal{D}^{\to}\left\{\begin{array}{c}\Omega,A\to B\\\triangledown\\C\end{array}\Leftarrow\begin{array}{c}\Omega\\\triangledown\\A\end{array},\begin{array}{c}\Omega,B\\\triangledown\\C\end{array}\right.
\qquad\qquad
(\mathcal{D}^{\to}\vdash)\,\dfrac{\mathfrak{S},\begin{array}{c}\Omega\\\triangledown\\A\end{array},\begin{array}{c}\Omega,B\\\triangledown\\C\end{array}\vdash\mathfrak{s}}{\mathfrak{S},\begin{array}{c}\Omega,A\to B\\\triangledown\\C\end{array}\vdash\mathfrak{s}}
$$

The f-sequent for the right implication introduction rule is

$$
\begin{array}{c}\Theta,A\\\triangledown\\B\end{array}\vdash\begin{array}{c}\Theta\\\triangledown\\A\to B\end{array}
$$

and its derivation is:

$$
\dfrac{
\dfrac{}{\vdash\begin{array}{c}A\to B\\\triangledown\\A\to B\end{array}}\,\text{(o-Id)}
\qquad
(\mathcal{D}^{\to}\vdash)\,\dfrac{
\dfrac{
(\text{Id})\,\dfrac{}{\begin{array}{c}\triangledown\\A\end{array}\vdash\begin{array}{c}\triangledown\\A\end{array}}
\qquad
\dfrac{
(\text{Id})\,\dfrac{}{\begin{array}{c}\Theta,A\\\triangledown\\B\end{array}\vdash\begin{array}{c}\Theta,A\\\triangledown\\B\end{array}}
\qquad
(\text{Id})\,\dfrac{}{\begin{array}{c}B\\\triangledown\\A\to B\end{array}\vdash\begin{array}{c}B\\\triangledown\\A\to B\end{array}}
}{\begin{array}{c}\Theta,A\\\triangledown\\B\end{array},\begin{array}{c}B\\\triangledown\\A\to B\end{array}\vdash\begin{array}{c}\Theta,A\\\triangledown\\A\to B\end{array}}\,\text{(o-Cut)}
}{\begin{array}{c}\Theta,A\\\triangledown\\B\end{array},\begin{array}{c}\triangledown\\A\end{array},\begin{array}{c}B\\\triangledown\\A\to B\end{array}\vdash\begin{array}{c}\Theta\\\triangledown\\A\to B\end{array}}\,\text{(o-Cut)}
}{\begin{array}{c}\Theta,A\\\triangledown\\B\end{array},\begin{array}{c}A\to B\\\triangledown\\A\to B\end{array}\vdash\begin{array}{c}\Theta\\\triangledown\\A\to B\end{array}}
}{\begin{array}{c}\Theta,A\\\triangledown\\B\end{array}\vdash\begin{array}{c}\Theta\\\triangledown\\A\to B\end{array}}\,\text{(Cut)}
$$

4.4. Remarks. The definitional clauses for the left introduction rules were again treated separately. If they were combined to yield a definition of minimal propositional logic by left introduction rules, then the top of the o-sequents would have to be restricted to lists of formulas instead of multisets of formulas. Otherwise, the proviso of definitional reflection would be violated and the f-sequents representing the right introduction rules would not be admissible. The restriction to lists has the effect that in applications of definitional reflection only that one definitional clause (respectively the two clauses in the case of conjunction) has to be used that contains the logical constant in question in the rightmost formula in the top of its head o-sequent. The restriction to lists demands an additional rule for *exchange* (o-Ex) and a rule for *contraction* (o-Contr) in the top of o-sequents:[16]

$$\frac{\mathfrak{S} \vdash \begin{array}{c}\Omega, A, B, \Psi\\ \triangledown\\ C\end{array}}{\mathfrak{S} \vdash \begin{array}{c}\Omega, B, A, \Psi\\ \triangledown\\ C\end{array}}\ (\text{o-Ex}) \qquad \frac{\mathfrak{S} \vdash \begin{array}{c}\Omega, A, A\\ \triangledown\\ C\end{array}}{\mathfrak{S} \vdash \begin{array}{c}\Omega, A\\ \triangledown\\ C\end{array}}\ (\text{o-Contr})$$

Then, for the combined clauses with lists, the definitional reflections remain as formulated for separate clauses, and inversion by definitional reflection for the given left introduction rules is again local.

The clauses $\mathcal{D}^{\vee}$ and $\mathcal{D}^{\rightarrow}$ have been formulated with shared contexts to prevent proviso violations. However, corresponding rules with independent contexts can be shown to be admissible by deriving the f-sequents for them with definitional closure on $\mathcal{D}^{\vee}$:

$$\dfrac{\dfrac{\dfrac{}{\begin{array}{c}\Omega, A\\ \triangledown\\ C\end{array} \vdash \begin{array}{c}\Omega, A\\ \triangledown\\ C\end{array}}\,(\text{Id})}{\begin{array}{c}\Omega, A\\ \triangledown\\ C\end{array} \vdash \begin{array}{c}\Omega, \Psi, A\\ \triangledown\\ C\end{array}}\,(\text{o-Thin})+(\text{o-Ex}) \qquad \dfrac{\dfrac{}{\begin{array}{c}\Psi, B\\ \triangledown\\ C\end{array} \vdash \begin{array}{c}\Psi, B\\ \triangledown\\ C\end{array}}\,(\text{Id})}{\begin{array}{c}\Psi, B\\ \triangledown\\ C\end{array} \vdash \begin{array}{c}\Omega, \Psi, B\\ \triangledown\\ C\end{array}}\,(\text{o-Thin})+(\text{o-Ex})}{\begin{array}{c}\Omega, A\\ \triangledown\\ C\end{array}, \begin{array}{c}\Psi, B\\ \triangledown\\ C\end{array} \vdash \begin{array}{c}\Omega, \Psi, A\vee B\\ \triangledown\\ C\end{array}}\,(\vdash\mathcal{D}^{\vee})$$

respectively with definitional closure on $\mathcal{D}^{\rightarrow}$:

$$\dfrac{\dfrac{\dfrac{}{\begin{array}{c}\Omega\\ \triangledown\\ A\end{array} \vdash \begin{array}{c}\Omega\\ \triangledown\\ A\end{array}}\,(\text{Id})}{\begin{array}{c}\Omega\\ \triangledown\\ A\end{array} \vdash \begin{array}{c}\Omega, \Psi\\ \triangledown\\ A\end{array}}\,(\text{o-Thin}) \qquad \dfrac{\dfrac{}{\begin{array}{c}\Psi, B\\ \triangledown\\ C\end{array} \vdash \begin{array}{c}\Psi, B\\ \triangledown\\ C\end{array}}\,(\text{Id})}{\begin{array}{c}\Psi, B\\ \triangledown\\ C\end{array} \vdash \begin{array}{c}\Omega, \Psi, B\\ \triangledown\\ C\end{array}}\,(\text{o-Thin})+(\text{o-Ex})}{\begin{array}{c}\Omega\\ \triangledown\\ A\end{array}, \begin{array}{c}\Psi, B\\ \triangledown\\ C\end{array} \vdash \begin{array}{c}\Omega, \Psi, A\rightarrow B\\ \triangledown\\ C\end{array}}\,(\vdash\mathcal{D}^{\rightarrow})$$

16 *Contraction* is required also for multisets, but the rule was not needed before and hence not given.

Therefore, albeit the formulation of definitional clauses with shared contexts instead of independent contexts is necessary for inversion by definitional reflection, the use of shared contexts is not a restriction of the logic defined.

***4.5. Inversion versus eliminability of* (*Cut*).** Besides definitional reflection on the given definitional clauses, the structural rules for o-sequents (o-Id), (o-Thin), and (o-Cut) as well as the structural rules for f-sequents (Id), (Thin), and (Cut) had to be used both in the derivations showing admissibility of the left introduction rules and in the derivations showing admissibility of the right introduction rules.

Next, we show that (Cut) cannot be eliminated from those derivations, and is therefore not eliminable in general, if only clauses for either left or right introduction rules are given.

In order to prove that (Cut) is not eliminable in the derivations showing admissibility of the left introduction rules when definitional clauses are given only for the right introduction rules, we need the following lemma.

LEMMA 4.1. *In a* (Cut)*-free derivation of* $\vdash \begin{matrix}\Theta\\ \triangledown\\ A\end{matrix}$*, where A is atomic, A has to be an element of* Θ.

Proof. Since the derivation is (Cut)-free and the antecedent of the end f-sequent is empty, the derivation does not contain applications of (Id), (Thin) or definitional reflection. It can only contain applications of (o-Id), (o-Thin), (o-Cut), (o-Contr) and applications of definitional closure. Then A has to be in Θ, by induction on rule applications. □

THEOREM 4.2. (Cut) *is not eliminable in the derivations showing admissibility of the left introduction rules.*

Proof. If (Cut) were eliminable in the above derivations that show admissibility of the left introduction rules in their most general form, then (Cut) would be eliminable in any derivation of a special case of these rules. Let A be atomic and different from B, then the f-sequents $\vdash \begin{matrix}A\wedge B\\ \triangledown\\ A\end{matrix}$, $\vdash \begin{matrix}A\vee A\\ \triangledown\\ A\end{matrix}$, and $\vdash \begin{matrix}B, B\to A\\ \triangledown\\ A\end{matrix}$ are such special cases that are derivable by using particular cases of the admissibility derivations for the left conjunction, left disjunction, and left implication introduction rules. Thus, by Lemma 4.1, if there were a (Cut)-free derivation of them, then A would be an element in the top multisets of the respective o-sequents. But it is not. So there is no (Cut)-free derivation of these three f-sequents. As all of them are special cases of the derived f-sequents for the left introduction rules, the instances of (Cut) in the derivations showing admissibility of the left introduction rules are not eliminable. □

COROLLARY 4.3. (Cut) *is not eliminable in general if only clauses for right introduction rules are given.*

In order to prove that (Cut) is not eliminable in the derivations showing admissibility of the right introduction rules when definitional clauses are given only for the left introduction rules, we need the following two lemmas.

LEMMA 4.4. *In a* (Cut)*-free derivation of* $\vdash \begin{matrix}\Theta\\ \triangledown\\ A\end{matrix}$, Θ *cannot be empty.*

Proof. (Cut)-free derivations of this f-sequent contain only applications of (o-Id), (o-Thin), (o-Cut), (o-Contr) and applications of definitional closure. Any derivation starts with (o-Id), where Θ is not empty. And the rules (o-Thin), (o-Cut), (o-Contr) and definitional closure have a conclusion in which the top of the o-sequent in the succedent is not empty if the tops of the o-sequents in the succedents of all premisses are not empty. □

LEMMA 4.5. *In a* (Cut)*-free derivation of* $\vdash \begin{array}{c} A,\ldots,A \\ \triangledown \\ B \end{array}$*, where only and at least once the atomic formula A occurs in the top, B has to be A.*

Proof. Since the derivation is (Cut)-free and the antecedent of the end f-sequent is empty, the derivation does not contain applications of (Id), (Thin) or definitional reflection. It can contain only applications of (o-Id), (o-Thin), (o-Cut), (o-Contr) and applications of definitional closure. Then B is A by induction on rule applications, where Lemma 4.4 is used for the cases (o-Thin) and (o-Cut). □

THEOREM 4.6. (Cut) *is not eliminable in the derivations showing admissibility of the right introduction rules.*

Proof. If (Cut) were eliminable in the above derivations that show admissibility of the right introduction rules in their most general form, then (Cut) would be eliminable in any derivation of a special case of these rules. Let A be an atomic formula, then the f-sequents $\vdash \begin{array}{c} A \\ \triangledown \\ A\wedge A \end{array}$, $\vdash \begin{array}{c} A \\ \triangledown \\ A\vee B \end{array}$, and $\vdash \begin{array}{c} A \\ \triangledown \\ B\to A \end{array}$ are such special cases that are derivable by using particular cases of the admissibility derivations for the right conjunction, right disjunction, and right implication introduction rules. Thus, by Lemma 4.5, if there were a (Cut)-free derivation of them, then the bottom formula would be A. But it is not. So there is no (Cut)-free derivation of these three f-sequents. As all of them are special cases of the derived f-sequents for the right introduction rules, the instances of (Cut) in the derivations showing admissibility of the right introduction rules are not eliminable. □

COROLLARY 4.7. (Cut) *is not eliminable in general if only clauses for left introduction rules are given.*

The rule (Cut) would be eliminable, however, if the structural rules for o-sequents (o-Id), (o-Thin), (o-Cut), and (o-Contr) were added as definitional clauses to the definitions of left respectively right introduction rules instead of being given as framework rules. But then inversion by definitional reflection would fail because the bodies of the definitional clauses for the structural rules are additional defining conditions that would have to be taken into account in applications of definitional reflection. The definitional clause for (o-Cut), however, has a variable in its body that is not in the head, which violates the proviso on variables of the principle of definitional reflection, and thereby prevents the application of the principle. If not only definitional clauses for either left or right introduction rules are considered, but if the whole system of definitional clauses for left and right introduction rules is given (i.e., $\mathcal{D}_\wedge$, $\mathcal{D}_\vee$, and $\mathcal{D}_\to$ together with $\mathcal{D}^\wedge$, $\mathcal{D}^\vee$, and $\mathcal{D}^\to$), then the inversions of the respective rules are already given as definitional clauses, and inversion by definitional reflection can be dispensed with. In this case (Cut)-eliminability can be easily demonstrated.

§5. Conclusions. The inversion principle was first given in a form resembling generalized elimination rules in natural deduction. Then definitional clauses were introduced and the principle of definitional reflection was presented; it states the inversion principle for definitions. Adding definitional closure as well as the structural rules (Id), (Thin), and (Cut) yields a sequent-style framework for reasoning about definitions. The principles of definitional reflection and definitional closure can be interpreted as principles for admissibility, which opens the possibility to show admissibility of logical rules by reasoning about given definitions for logical constants in the framework. Defining logical constants by use of simple formulas led to problems, however. They were resolved by using clausal definitions for sequents in the form of o-sequents. The o-sequents can be read as assertions involving the relation of deductive consequence between the top formulas and a bottom formula. The formulas themselves can be interpreted as propositions or assertions, although we did not give a specific interpretation to them. That o-sequents involve the relation of deductive consequence is determined by the structural rules (o-Id), (o-Thin), and (o-Cut), which express the central features of the usual deductive consequence relation. These rules were added to the framework. The logical rules were then given by definitional clauses using o-sequents, that is, the logical constants of minimal propositional logic were defined in the context of the relation of deductive consequence.

For given definitions of right introduction rules the respective left introduction rules were shown to be admissible by using definitional reflection. In addition, it could be shown that for given definitions of left introduction rules the respective right introduction rules are admissible. The definitional clauses had to be formulated with shared contexts to comply with the proviso on variables of definitional reflection. Independent contexts would violate the proviso and thereby render definitional reflection inapplicable. However, the restriction to shared contexts is not a limitation of the logic defined since corresponding rules with independent contexts are admissible by definitional closure. If the definitional clauses are not treated separately but together as a logical system, then lists instead of multisets have to be used in the top of o-sequents in definitions of left introduction rules, and the rules (o-Ex) as well as (o-Contr) have to be added to the framework to handle those lists. Apart from these rules, the rules (o-Id), (o-Thin), and (o-Cut) for o-sequents and the rules (Id), (Thin), and (Cut) for f-sequents had to be used in the derivations. It was shown that (Cut) is not eliminable in any of the derivations that show the admissibility of logical rules. Thus, inversion by definitional reflection for logical rules cannot be accomplished without (Cut).

Given the admissibility results shown above, it seems questionable that the right introduction rules have any kind of privilege over the left introduction rules concerning the definition of logical constants or vice versa. Since the division into introduction rules and elimination rules in natural deduction is carried over to sequent calculus with its respective right introduction rules and left introduction rules, we can restate Gentzen's remark according to our context of sequents by saying that the left introduction rules are ultimately only consequences of the right introduction rules when the latter are taken as definitions of the logical constants concerned. As a consequence of the above results, we can furthermore complement this remark by saying that the left introduction rules are the definitions of the logical constants concerned, and the right introduction rules are ultimately only consequences hereof. The logical constants of minimal propositional logic can be defined by right introduction rules as well as by left introduction rules. If the right introduction rules are given as definitions, then the left introduction rules are consequences of them in the sense of being admissible relative to the given definitions, and if the left introduction rules are given as definitions, then the right introduction rules are consequences of them in the same sense of being admissible.

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