Conservativeness and uniqueness

by

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IN THIS PAPER we shall try to show that the notions of conservativeness and uniqueness are in a certain sense dual to each other. These notions are analogous to the notions considered by Belnap in [1], and similar notions are also mentioned in requirements which it is customary to connect with mathematical definitions (cf., for example, [11], Chapter 8).

In Section 1, after introducing preliminary definitions and notation, we shall state what we understand by conservativeness and uniqueness. The first of these notions is fairly well known in mathematical logic (Belnap claims in [1] that it stems from Post), whereas the second seems to be less well known, and we shall make a few historical remarks on it. Next, in Section 2, we shall introduce examples of expressions which are not conservative but unique, and the other way round, and we shall state what intuitively induces us to think that duality can be found. After that, in Section 3, we shall make precise a certain limited context in which two special notions of conservativeness and uniqueness can be shown dual to each other. Still, we assume this context and the notions involved are general enough to be of interest. It is possible that in contexts where our results are not applicable an analogous duality between conservativeness and uniqueness can be established. At the end of this paper, in Section 4, we shall consider this possibility.

1

Let L be a language in which a constant α, of an arbitrary syntactic category, does not occur, and let S be a formal system formulated in L with a set of postulates, i. e. axioms, axiom-schemata and rules. The system S_α will be obtained by extending L with α, and by

adding to the postulates of S some postulates in which α occurs. The system S is as general as possible: we even leave open the possibility that the set of postulates of S is empty, and hence that there are no theorems in S. But we assume that S_α properly extends S and that there are theorems in S_α in which α occurs. If S_α is a given system in L with α, then S_{α^*} will be the system in L with α^* obtained by replacing everywhere α by α^* in the postulates of S_α. (Of course, α and α^* are of the same syntactic category, and differ only "graphically".)

We shall use $A, B, C, \ldots, A_1, \ldots$ as schematic letters for formulae of L, or L with α. Schemata of the form $A(\alpha)$ will be used for those formulae of L with α in which α occurs at least once. Expressions of the form $\vdash_S A$ will mean that A is a theorem of S. Capital Greek letters will be used for sets of formulae. Expressions of the form $\Phi \vdash_S A$ will mean that A is provable in the system obtained from S by taking the formulae in Φ as additional axioms, whereas expressions of the form $\Phi \vdash_S \Psi$ will mean that for *every* A in Ψ we have $\Phi \vdash_S A$. When $\{A\}$, or $\Phi \cup \{A\}$, or $\Phi \cup \Psi$, occur on the left or right-hand side of $\vdash_S$ we shall write respectively A, or Φ, A, or Φ, Ψ. It is easy to check the following properties of $\vdash_S$:

(i) $\Phi \vdash_S \Phi$

(ii) $\Phi \vdash_S \Psi \Rightarrow \Phi, \Pi \vdash_S \Psi$

(iii) $(\Phi \vdash_S \Psi \text{ and } \Pi, \Psi \vdash_S \Sigma) \Rightarrow \Pi, \Phi \vdash_S \Sigma$

(iv) $\Phi \vdash_S \Psi \Rightarrow \Phi \vdash_{S_\alpha} \Psi$

(v) $\Phi \vdash_{S_\alpha} \Psi \Rightarrow \Phi^* \vdash_{S_{\alpha^*}} \Psi^*$

where Φ^* and Ψ^* are obtained from Φ and Ψ respectively by substituting uniformly α^* for α.

As a matter of fact, $\vdash_S$ described as above is not the only relation which we could use to present the results of this paper. Other relations would do as well. For example, we can require that $\Phi \vdash_S A$ holds only if in the derivation of A some primitive rules of S are not applied to formulae which depend on the hypotheses in Φ (this is the way Universal Generalization can be restricted in first-order

logic). However, it is essential that these other relations satisfy (i)—(v).

If we want to claim that the postulates of S_α in which α occurs give a certain kind of definition of α, i. e., a characterization of α, and not also of something in the old language L, it is natural to require that:

(1)   The system S_α is a *conservative extension* of S, i. e., for every A in L, if $\vdash_{S_\alpha} A$, then $\vdash_S A$.

And if by "characterization" we mean "complete characterization", it is also natural to require that:

(2)   The constant α is *unique* in S_α, i. e., if we extend S_α with the postulates of S_{α^*}, we must be able to show that in the resulting system $S_{\alpha\alpha^*}$ in the enriched language with both α and α^*, the constants α and α^* are synonymous.

N. B. For the schematic letters of an axiom-schema, or of a rule of S_α, we can substitute in $S_{\alpha\alpha^*}$ the expressions of the language with both α and α^*.

The notion of synonymity invoked in (2), which should correspond to intersubstitutability *salva* provability, can be defined in various ways. In the following definition, and also later, $A(\alpha^*)$ will be the result of substituting everywhere α^* for α in $A(\alpha)$:

The constans α and α^* are *synonymous* in $S_{\alpha\alpha^*}$ iff for every $A(\alpha)$ we have $A(\alpha) \vdash_{S_{\alpha\alpha^*}} A(\alpha^*)$.

It is easily seen that if for every $A(\alpha)$ we have $A(\alpha) \vdash_{S_{\alpha\alpha^*}} A(\alpha^*)$, then for every $A(\alpha^*)$ we have $A(\alpha^*) \vdash_{S_{\alpha\alpha^*}} A(\alpha)$.

Other notions of synonymity need not involve *uniform* substitution as ours does. With nonuniform substitution instead of $A(\alpha) \vdash_{S_{\alpha\alpha^*}} A(\alpha^*)$ we can use the weaker condition $\vdash_{S_{\alpha\alpha^*}} A(\alpha) \Rightarrow \vdash_{S_{\alpha\alpha^*}} A(\alpha^*)$—with uniform substitution this last implication is trivial. (Although our notion of synonymity is based on uniform substitution, the effect of nonuniform substitutions can in many contexts be obtained by uniform ones. If a variable β of the syntactic category of α is

available, and if some other natural conditions are satisfied, in order to obtain $A(\alpha, \alpha) \vdash_{S_{\alpha\alpha^*}} A(\alpha, \alpha^*)$ we could use $A(\beta, \alpha) \vdash_{S_{\alpha\alpha^*}} A(\beta, \alpha^*)$, and then substitute α for β.)

A notion of uniqueness similar to ours was considered by Belnap in [1], and also by Smiley in [10]. Smiley does not use the term "uniqueness", but "functional dependence". He shows that in the Heyting propositional calculus conjunction (as well as disjunction and negation) is functionally dependent, whereas implication is functionally independent. This functional independence, i. e., non-uniqueness, of implication is connected with the failure of the Deduction Theorem in the Heyting propositional calculus extended with $\rightarrow^*$ and postulates for $\rightarrow^*$. Let $S_\rightarrow$ be a standard Hilbert-type axiomatization of the Heyting propositional calculus based on $\rightarrow$, $\wedge$, $\vee$ and $\neg$, with axiom-schemata, and *modus ponens* as sole rule (see, for example, [6], pp. 82, 101). If $S_{\rightarrow\rightarrow^*}$ is the extension of $S_\rightarrow$ with the postulates of $S_{\rightarrow^*}$, we have $A \rightarrow^* B, A \vdash_{S_{\rightarrow\rightarrow^*}} B$ and $A \rightarrow B, A \vdash_{S_{\rightarrow\rightarrow^*}} B$, but neither $A \rightarrow^* B \vdash_{S_{\rightarrow\rightarrow^*}} A \rightarrow B$ nor $A \rightarrow B \vdash_{S_{\rightarrow\rightarrow^*}} A \rightarrow^* B$, as can be shown using the matrices of [10], p. 430 (note that here $\vdash_{S_{\rightarrow\rightarrow^*}}$ coincides with the usual relation of deducibility from hypotheses). In an attempted proof of the Deduction Theorem for $\rightarrow$, the step that would be blocked would be the one from $\Phi \vdash_{S_{\rightarrow\rightarrow^*}} A \rightarrow (B \rightarrow^* C)$ and $\Phi \vdash_{S_{\rightarrow\rightarrow^*}} A \rightarrow B$ to $\Phi \vdash_{S_{\rightarrow\rightarrow^*}} A \rightarrow C$. Similarly, if $S_\rightarrow$ is a standard Hilbert-type axiomatization of the classical propositional calculus based on $\rightarrow$ and $\neg$, with axiom-schemata, and *modus ponens* as sole rule, with the help of the matrices

$\rightarrow$	1	2	3	4
1	1	2	3	4
2	1	1	1	1
3	1	1	1	1
4	1	1	1	1

$\rightarrow^*$	1	2	3	4
1	1	2	3	4
2	1	1	3	3
3	1	2	1	2
4	1	1	1	1

	$\neg$
1	4
2	1
3	1
4	1

where 1 is designated, we can show that $\rightarrow$ is not unique in $S_\rightarrow$, and that the Deduction Theorem fails in $S_{\rightarrow\rightarrow^*}$.

In formal systems where an analogue of the Deduction Theorem

is assumed as primitive, like in Gentzen-type sequent-systems or natural deduction systems, intuitionistic and classical implication can easily be shown unique ([3], §9, [5]).

Smiley does not seem to take uniqueness as a desirable property for a constant in a system. He proposes uniqueness as a counterpart to Padoa's criterion for dependence. From [8] one can get necessary, but not sufficient, conditions for *definitional* dependence (i. e., sufficient, but not necessary, conditions for definitional independence), and necessary and sufficient conditions for *functional* dependence (or independence). This view is also expressed by McKinsey in [7]. To show that a constant is unique means to show that it is somehow dependent on the system in which it occurs. Smiley demonstrates that this does not mean that it is explicitly definable in terms of other expressions of the system. Roughly speaking, if in S_α the constant α is synonymous with a β from L, then α and α^* will be synonymous in $S_{\alpha\alpha^*}$, provided α remains synonymous with β in $S_{\alpha\alpha^*}$. So, explicit definability gives uniqueness. But if α and α^* are synonymous in $S_{\alpha\alpha^*}$, this does not mean that there must be a β from L synonymous with α in S_α, as the analogy with Beth's Definability Theorem might induce us to think.

Various notions of synonymity and uniqueness, like those mentioned in (2) and in the comments after (2), were also studied and compared in [2] (§§22, 76–85). What seems to come out of this work is that in many contexts and for many purposes all these notions will give equivalent results. In [2] (§85) it was briefly suggested that conservativeness and uniqueness are somehow dual.

2

Now we shall give examples of constants which are nonconservative and unique, and the other way round. Let S be a standard Hilbert-type axiomatization of the classical propositional calculus based on $\rightarrow$, $\wedge$, $\vee$ and $\neg$, with axiom-schemata, and *modus ponens* as sole rule (see, for example, [6], p. 82), and let S_β be S enlarged with the axiom-schemata

β1.1. $A\rightarrow(A\beta B)$ β1.2. $B\rightarrow(A\beta B)$
β2.1. $(A\beta B)\rightarrow A$ β2.2. $(A\beta B)\rightarrow B$.

The connective β behaves like disjunction in β1.1 and β1.2, and like conjunction in β2.1 and β2.2. It is in this respect analogous to Prior's connective "tonk" of [9]. It is easy to show that S_β is not a conservative extension of S ($A\rightarrow B$ is provable in S_β, and hence S_β is not only nonconservative, but also inconsistent). On the other hand, β is unique in S_β ($A(\beta)\rightarrow A(\beta^*)$ is provable in $S_{\beta\beta^*}$, since this system is also inconsistent).

Let now S be as above, and let S_γ be S enlarged with the axiom-schemata

γ1. $(C\rightarrow A)\rightarrow((C\rightarrow B)\rightarrow(C\rightarrow(A\gamma B)))$
γ2. $(A\rightarrow C)\rightarrow((B\rightarrow C)\rightarrow((A\gamma B)\rightarrow C))$.

The connective γ behaves like conjunction in γ1, and like disjunction in γ2. It is easy to show that $S\gamma$ is a conservative extension of S (interpret γ as conjunction, or as disjunction). On the other hand, γ is not unique in S_γ (show that $p\gamma q \vdash_{S_{\gamma\gamma^*}} p\gamma^* q$ does not hold by interpreting γ as disjunction, and γ^* as conjunction). Note that these examples with β and γ don't depend essentially on S being an axiomatization of the classical propositional calculus: S could as well be an axiomatization of the Heyting propositional calculus.

So neither S_β, nor S_γ, give a characterization of the constants they introduce. For a characterization we should require both conservativeness and uniqueness, as Belnap in [1] has done in answering Prior.

What seems to come out of a consideration of the constants β and γ is that the requirements of conservativeness and uniqueness are somehow dual to each other. This is not at all clear from the definitions alone of these two notions. Roughly speaking, our examples with β and γ show that if the *sufficient conditions* in the old language for a formula with the new constant are strictly weaker than the *necessary conditions* in the old language for this formula, then the constant in question is nonconservative and unique. This is the case with β, which has sufficient conditions for disjunction and necessary conditions for conjunction. On the other hand, if the

sufficient conditions are strictly stronger than the necessary conditions—this is the case with γ, which has sufficient conditions for conjunction and necessary conditions for disjunction—then the constant in question is conservative and nonunique.

The constant β above is trivially unique in S_β, since $S_{\beta\beta^*}$ is inconsistent. An example of uniqueness with nonconservativeness, which does not fall into inconsistency, is provided by the following. Let S be a standard Hilbert-type axiomatization of the Heyting propositional calculus based on $\rightarrow$, $\wedge$, $\vee$ and $\neg$, with axiom-schemata, and *modus ponens* as sole rule, and let S_δ be the extension of S with the axiom-schemata

δ1. $(A\rightarrow A)\rightarrow\delta A$
δ2. $\delta A\rightarrow(A\vee\neg A)$.

The unary connective δ is unique in S_δ, since in $S_{\delta\delta^*}$ we have $\delta A\rightarrow\delta^* A$ and $\delta^* A\rightarrow\delta A$, and $S_{\delta\delta^*}$ is closed under replacement of equivalents. The system S_δ is not a conservative extension of S, but it is a consistent system (interpreting δA by $A\rightarrow A$, it becomes classical propositional logic). As before, we can interpret this example by saying that the sufficient conditions for δA in the old language are strictly weaker than the necessary conditions for δA in the old language ($A\vee\neg A$ implies $A\rightarrow A$, but not the other way round).

Let S be again the Heyting propositional calculus, and let S_ε be S enlarged with the axiom-schemata

ε1. $(A\vee\neg A)\rightarrow\varepsilon A$
ε2. $\varepsilon A\rightarrow(A\rightarrow A)$.

(It is clear that ε2 is redundant, since it follows from $B\rightarrow(A\rightarrow A)$.) The system S_ε is a conservative extension of S (interpret εA as $A\rightarrow A$), but ε is not unique in S_ε (show that $\varepsilon A \vdash_{S_{\varepsilon\varepsilon^*}} \varepsilon^* A$ does not hold by interpreting εA as $A\rightarrow A$, and $\varepsilon^* A$ as $A\vee\neg A$). Now the sufficient conditions for εA in the old language are strictly stronger than the necessary conditions.

The examples above involve propositional connectives. However,

the results on the duality between conservativeness and uniqueness of the following sections, apply to constants of an arbitrary syntactic category, and not only to connectives.

3

In the rest of this paper we shall try to make precise, in a certain limited context, this duality between conservativeness and uniqueness. The first limitation of our context is brought by not working with conservativeness and uniqueness as introduced by (1) and (2), but with two more restricted notions, which we proceed to define:

(1.1)   The system S_α is a *strictly conservative* extension of S iff for every Φ and Ψ which are sets of formulae of L, if $\Phi \vdash_{S_\alpha} \Psi$, then $\Phi \vdash_S \Psi$.

(1.2)   The constant α is *strictly unique* in S_α iff for every $A(\alpha)$ there is a set Φ of formulae of L such that $A(\alpha) \vdash_{S_\alpha} \Phi$ and $\Phi \vdash_{S_\alpha} A(\alpha)$.

It is clear that strict conservativeness entails conservativeness in the sense of (1). The converse would also hold with a kind of Deduction Theorem, or something similar, guaranteeing first that for every A of Ψ there is a B in L such that $\Phi \vdash_{S_\alpha} A$ implies $\vdash_{S_\alpha} B$, and second that $\vdash_S B$ implies $\Phi \vdash_S A$.

To show that strict uniqueness entails uniqueness in the sense of (2), we use the fact that $\Phi \vdash_{S_\alpha} A(\alpha)$ implies, by (v), $\Phi \vdash_{S_{\alpha^*}} A(\alpha^*)$, which with $A(\alpha) \vdash_{S_\alpha} \Phi$, (iii) and (iv) gives $A(\alpha) \vdash_{S_{\alpha\alpha^*}} A(\alpha^*)$. To see that uniqueness does not always entail strict uniqueness consider the following counterexample. We can show that $\neg$ is unique in the classical propositional calculus $S_\neg$. On the other hand, it is not strictly unique in $S_\neg$, since for no set Φ of formulae without $\neg$ we can have $\Phi \vdash_{S_\neg} p \wedge \neg p$ (a set of formulae without $\neg$ is always satisfiable by assigning truth to every propositional letter). Although uniqueness and strict uniqueness differ, in many natural cases the notion of uniqueness we shall have to deal with will be strict

uniqueness. Quite often, we shall reach $A(\alpha) \vdash_{S_{\alpha\alpha^*}} A(\alpha^*)$ by "interpolating" some formulae Φ of L between $A(\alpha)$ and $A(\alpha^*)$, since in $S_{\alpha\alpha^*}$ we don't necessarily assume postulates connecting α and α^*. (If in S_α there are schemata in the postulates in which α occurs, then it might be possible to connect α and α^* in $S_{\alpha\alpha^*}$ by substituting α^* in these schemata; still, this need not absolve us from looking for an interpolated set Φ.) Strict uniqueness corresponds to an abstract notion of definability (cf. [4]).

Next we give the following definitions, in which Γ, Δ, Φ and Ψ are sets of formulae of L, and A is a formula in which α occurs:

$$G_A = \{\Gamma \mid \Gamma \vdash_{S_\alpha} A \text{ and } \forall \Phi(\Phi \vdash_{S_\alpha} A \Rightarrow \Phi \vdash_S \Gamma)\}$$
$$D_A = \{\Delta \mid A \vdash_{S_\alpha} \Delta \text{ and } \forall \Phi, \Psi(\Phi, A \vdash_S \Psi \Rightarrow \Phi, \Delta \vdash_S \Psi)\}.$$

Intuitively G_A is the set of weakest sufficient conditions in L for A, and D_A is the set of strongest necessary conditions in L for A. Here "weakest" and "strongest" are understood with respect to S. The slight asymmetry between G_A and D_A which consists in having both Φ and Ψ for D_A, and only Φ for G_A, should be ascribed to the asymmetry between premises and conclusions in deductions: there are possibly many premises, but only one conclusion. It is easy to check that alternatively we could have introduced G_A and D_A as follows:

$$G_A = \{\Gamma \mid \forall \Phi(\Phi \vdash_{S_\alpha} A \Leftrightarrow \Phi \vdash_S \Gamma)\}$$
$$D_A = \{\Delta \mid \forall \Phi, \Psi(\Phi, A \vdash_{S_\alpha} \Psi \Leftrightarrow \Phi, \Delta \vdash_S \Psi)\}.$$

Roughly speaking, a Γ replaces A in S as a conclusion, and a Δ replaces A in S as a premise.

In the following lemma we show that the members of G_A, or D_A, are interdeducible:

Lemma: (a) $(\forall \Gamma_1, \Gamma_2 \in G_A) (\Gamma_1 \vdash_S \Gamma_2 \text{ and } \Gamma_2 \vdash_S \Gamma_1)$.

(b) $(\forall \Delta_1, \Delta_2 \in D_A) (\Delta_1 \vdash_S \Delta_2 \text{ and } \Delta_2 \vdash_S \Delta_1)$.

Proof: Suppose $\Gamma_1, \Gamma_2 \in G_A$. Then we have that $(\Gamma_1 \vdash_{S_\alpha} A \text{ and } (\Gamma_2 \vdash_{S_\alpha} A$

$\Rightarrow \Gamma_2 \vdash_S \Gamma_1$)) *and* ($\Gamma_2 \vdash_{S_\alpha} A$ *and* ($\Gamma_1 \vdash_{S_\alpha} A \Rightarrow \Gamma_1 \vdash_S \Gamma_2$)), from which (a) follows. For (b) we proceed analogously.

For the duality results which follow we assume that for every A in which α occurs, G_A and D_A are not empty. This assumption limits the context where our results are applicable.

Theorem 1: The system S_α is a strictly conservative extension of S *iff* for every A in which α occurs $(\forall\Gamma\in G_A)\ (\forall\Delta\in D_A)\ \Gamma\vdash_S\Delta$.

Proof: For the *only if* part we proceed as follows. Let A be a formula in which α occurs. Then $(\forall\Gamma\in G_A)\ \Gamma\vdash_{S_\alpha}A$ and $(\forall\Delta\in D_A)\ A\vdash_{S_\alpha}\Delta$. Hence, using (iii), we obtain $(\forall\Gamma\in G_A)\ (\forall\Delta\in D_A)\ \Gamma\vdash_{S_\alpha}\Delta$. If now we suppose that S_α is a strictly conservative extension of S, it follows that $(\forall\Gamma\in G_A)\ (\forall\Delta\in D_A)\ \Gamma\vdash_S\Delta$. For the *if* part, suppose Φ and Ψ are sets of formulae of L such that $\Phi\vdash_{S_\alpha}\Psi$. According to our assumptions about S_α, there is an A in which α occurs such that $\vdash_{S_\alpha}A$. Using (ii) we obtain $\Phi, A\vdash_{S_\alpha}\Psi$. But this means that $(\forall\Delta\in D_A)\ \Phi, \Delta\vdash_S\Psi$. Now assuming that $(\forall\Gamma\in G_A)\ (\forall\Delta\in D_A)\ \Gamma\vdash_S\Delta$, and using the nonemptiness of D_A and (iii), we obtain $(\forall\Gamma\in G_A)\ \Phi, \Gamma\vdash_S\Psi$. Next we show that $\varnothing\in G_A$. Since $\vdash_{S_\alpha}A$, we have $\varnothing\vdash_{S_\alpha}A$, and $\forall\Phi\ (\Phi\vdash_{S_\alpha}A \Rightarrow \Phi\vdash_S\varnothing)$ is trivially satisfied (its consequent is always true, viz., if $B\in\varnothing$, then $\Phi\vdash_S B$). So, $\Phi\vdash_S\Psi$, and the Theorem follows.

Theorem 2: The constant α is strictly unique in S_α *iff* for every A in which α occurs $(\forall\Gamma\in G_A)\ (\forall\Delta\in D_A)\ \Delta\vdash_S\Gamma$.

Proof: For the *only if* part we proceed as follows. If α is strictly unique in S_α, then for every A in which α occurs there is a set Φ of formulae of L such that $A\vdash_{S_\alpha}\Phi$ and $\Phi\vdash_{S_\alpha}A$. But this means that $(\forall\Delta\in D_A)\ \Delta\vdash_S\Phi$ and $(\forall\Gamma\in G_A)\ \Phi\vdash_S\Gamma$, which, using (iii), gives $(\forall\Gamma\in G_A)\ (\forall\Delta\in D_A)\ \Delta\vdash_S\Gamma$. For the *if* part we have the following. Let A be a formula in which α occurs. Then $(\forall\Gamma\in G_A)\ \Gamma\vdash_{S_\alpha}A$ *and* $(\forall\Delta\in D_A)\ A\vdash_{S_\alpha}\Delta$. Now assuming $(\forall\Gamma\in G_A)\ (\forall\Delta\in D_A)\ \Delta\vdash_S\Gamma$, and using

the nonemptiness of G_A and D_A, (iii) and (iv), we obtain that for some Γ in G_A, both $A\vdash_{S_\alpha}\Gamma$ and $\Gamma\vdash_{S_\alpha}A$ (also for some Δ in D_A, we obtain $A\vdash_{S_\alpha}\Delta$ and $\Delta\vdash_{S_\alpha}A$). From this the Theorem follows.

Note that the nonemptiness of G_A and D_A was used only for the *if* parts of Theorems 1 and 2. In fact, for the *if* part of Theorem 1 we used only the nonemptiness of D_A for at least one theorem A of S_α.

Theorems 1 and 2 accord rather well with the intuitive considerations on the duality of conservativeness and uniqueness we had in Section 2 in connection with β and γ. In the example with β, the system S_β is inconsistent; therefore, if Φ is the set of all formulae of the propositional calculus, for every $A(\beta)$ we have that $\varnothing\in G_{A(\beta)}$ and $\Phi\in D_{A(\beta)}$. Since it is not the case that $\varnothing\vdash_S\Phi$, whereas $\Phi\vdash_S\varnothing$, we obtain that S_β is not a strictly conservative extension of S, and that β is strictly unique in S_β. In the example with γ, there are more difficulties in defining for each $A(\gamma)$ the representative members of $G_{A(\gamma)}$ and $D_{A(\gamma)}$, if there are any. But $\{p\rightarrow p\}\in D_{(p\rightarrow p)\ \gamma\ (p\rightarrow p)}$, and so Theorem 1 holds. Since it is easy to show that S_γ is a strictly conservative extension of S, we can conclude that for every A in which γ occurs $(\forall\Gamma\in G_A)\ (\forall\Delta\in D_A)\ \Gamma\vdash_S\Delta$. Next, we can show that $\{p\wedge q\}\in G_{p\gamma q}$ and $\{p\vee q\}\in D_{p\gamma q}$. For $\{p\wedge q\}\in G_{p\gamma q}$, we have that $p\wedge q\vdash_S p\gamma q$ and $\forall\Phi(\Phi\vdash_{S_\gamma}p\gamma q \Rightarrow \Phi\vdash_S p\wedge q)$ (to pass from $\Phi\vdash_{S_\gamma}p\gamma q$ to $\Phi\vdash_S p\wedge q$ just substitute everywhere $\wedge$ for γ in the derivation of $p\gamma q$ from Φ), and for $\{p\vee q\}\in D_{p\gamma q}$ we proceed analogously. Since it is not the case that $p\vee q\vdash_S p\wedge q$, we have that for some A in which γ occurs it is not the case that $(\forall\Gamma\in G_A)\ (\forall\Delta\in D_A)\ \Delta\vdash_S\Gamma$. For the *only if* part of Theorem 2 we did not assume the nonemptiness of G_A and D_A, and so it follows that γ is not strictly unique in S_γ. In considering the examples with δ and ε in the light of Theorems 1 and 2 we proceed similarly.

It might be difficult to determine for each $A(\alpha)$ whether $G_{A(\alpha)}$ and $D_{A(\alpha)}$ are nonempty. When they are nonempty, it might still be difficult to determine what their representative members are, and what relations hold between these members. But we are not sug-

gesting that Theorems 1 and 2 give practical criteria for determining conservativeness and uniqueness. The purpose is just to show that there is a certain duality between these notions.

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In some natural cases it might happen that G_A is empty. For example, if $S_\vee$ is a standard Hilbert-type axiomatization of the Heyting propositional calculus based on $\vee$, $\rightarrow$, $\wedge$ and $\neg$, then $G_{p \vee q}$ is empty. Otherwise, for some Γ in $G_{p \vee q}$ we would have $\Gamma \vdash_{S_\vee} p \vee q$ and $p \vdash_S \Gamma$ and $q \vdash_S \Gamma$, from which we obtain $\Gamma \vdash_{S_\vee} p \vee q$ and $p \vee q \vdash_{S_\vee} \Gamma$. But this is impossible, since it would follow that $p \vee q$ is definable in terms of the connectives of S. However, $D_{p \vee q}$ is not empty, since $\Delta = \{B \mid B$ is in L and $p \vee q \vdash_S B\}$ belongs to $D_{p \vee q}$. To show that, suppose $\Phi, p \vee q \vdash_{S_\vee} \Psi$; then, by the Deduction Theorem, for every B in Ψ we can find a C such that $p \vee q \vdash_{S_\vee} C$. But $C \in \Delta$, and hence, $\Delta \vdash_S C$; so, $\Phi, \Delta \vdash_S \Psi$. In general, whenever we have a Deduction Theorem, or something similar, enabling us to reason as above, D_A will not be empty.

If G_A and D_A are empty, it is not possible to show the duality of conservativeness and uniqueness in the sense of Theorems 1 and 2. There is a possibility to overcome this limitation, but only at the price of introducing another assumption. We shall now investigate this possibility.

Consider the following definitions, in which Π is a set of formulae of L, B is a formula of L, and A is a formula in which α occurs:

$$P_A = \{\Pi \mid \Pi \vdash_{S_\alpha} A\}$$
$$\Sigma_A = \{B \mid A \vdash_{S_\alpha} B\}.$$

Intuitively, P_A is the set of all sufficient conditions in L for A, and Σ_A is the maximal necessary condition in L for A. The set P_A could possibly be empty in some cases, but the nonemptiness of this set is not assumed for the results which follow. The set Σ_A will never be empty if S has at least some theorems; but again, its nonemptiness is

not essential. What we must assume, it seems, is a kind of Deduction Theorem for $\vdash_{S_\alpha}$, or something similar, in order to be able to show that conservativeness implies strict conservativeness. We have mentioned the possibility of this assumption after the definition of strict conservativeness in Section 3, but in contradistinction to what we have here, we didn't make this assumption for Theorems 1 and 2. The simplest way to formulate our assumption is to say that we assume that conservativeness implies strict conservativeness. Then we can show the following theorems:

Theorem 3: The system S_α is a (strictly) conservative extension of S *iff* for every A in which α occurs $(\forall \Pi \in P_A)$ $\Pi \vdash_S \Sigma_A$.

Proof: For the *only if* part we use (iii), $\Pi \vdash_{S_\alpha} A$ and $A \vdash_{S_\alpha} \Sigma_A$ to get $\Pi \vdash_{S_\alpha} \Sigma_A$, which by (strict) conservativeness gives $\Pi \vdash_S \Sigma_A$. For the *if* part, suppose that for a B in L we have $\vdash_{S_\alpha} B$, and take an A in which α occurs such that $\vdash_{S_\alpha} A$. For this A we have that $\emptyset \in P_A$ and $B \in \Sigma_A$. Hence, from $(\forall \Pi \in P_A)$ $\Pi \vdash_S \Sigma_A$ we obtain $\emptyset \vdash_S \Sigma_A$, and so, $\vdash_S B$.

Theorem 4: The constant α is strictly unique in S_α *iff* for every A in which α occurs $(\exists \Pi \in P_A)$ $\Sigma_A \vdash_S \Pi$.

Proof: For the *only if* part we proceed as follows. Suppose that for any A in which α occurs there is a Φ in L such that $A \vdash_S \Phi$ and $\Phi \vdash_{S_\alpha} A$; then $\Phi \subseteq \Sigma_A$ and $\Phi \in P_A$. So, using (i), and perhaps (ii), we have $(\exists \Pi \in P_A)$ $\Sigma_A \vdash_S \Pi$. For the *if* part, suppose that for some Π in P_A we have $\Sigma_A \vdash_S \Pi$. Then by (iv) we obtain $\Sigma_A \vdash_{S_\alpha} \Pi$, which with $A \vdash_{S_\alpha} \Sigma_A$ and (iii) gives $A \vdash_{S_\alpha} \Pi$. Since we also have $\Pi \vdash_{S_\alpha} A$, the Theorem follows.

With Theorems 3 and 4 we have still not lifted all limitations on our results, and it might be interesting to see if there are other ways to make the duality between conservativeness and uniqueness more general. It might also be interesting to find a certain duality between conservativeness and uniqueness in a natural deduction

framework, where, roughly speaking, the role of the members of G_A would be taken by the premises of introduction rules, and the role of the members of D_A by the conclusions of elimination rules.

The results of this paper were all of a syntactical nature. However, if we interpret $\vdash_S$ semantically, they can also suggest analogous results of a more semantical nature. A somewhat different kind of duality between conservativeness and uniqueness, which bears on semantics, was already suggested by Belnap in [1]. If we assume that S is sound and complete, to establish that S_α is conservative amounts to showing that there is *at least one* α. So, the conservativeness of S_α should correspond to its soundness. On the other hand, to establish that α is unique in S_α amounts to showing that there is *at most one* α, and this seems to bear on questions related to the categoricity and completeness of S_α.

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