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Popper's Structuralist Theory of Logic*

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In what follows I take a fresh look at Popper's papers on the foundations of (deductive) logic published between 1947 and 1949 (referred to as P1–P6). In my study of 1984 (Schroeder-Heister 1984b) I gave a detailed analysis of these papers including the objections raised and points made within the logic community. These objections and, in particular, the fact that Tarski refused to take a look at them[1] eventually led Popper to abandon his project of 'new foundations for logic' (title of P2) which he had started 'with much enthusiasm' (Popper 1974, p. 1095).

In Schroeder-Heister (1984b) I argued that Popper's theory can be given a coherent sense when it is read as an attempt to delineate logical from extra-logical signs, a point whose significance Popper had stressed himself, in particular in his reply to Lejewski in the Schilpp volume (Lejewski 1974; Popper 1974). I now think that the logicality aspect, though important, is not the whole story and definitely not the central point of Popper's theory.[2] Rather I shall argue that Popper puts forward a structuralist approach according to which logic is a metalinguistic theory of consequence, in terms of which logical operations are characterized. I borrow the term 'structuralist' from Koslow's monograph (1992), where such an approach is developed in much detail. Popper's view will be reconstructed against the background of Koslow's work, which represents a mature account allowing me to evaluate the merits of Popper's ideas.

By a 'structuralist theory of logic' Koslow denotes an approach that characterizes logical systems axiomatically in terms of 'implication relations'. An implication relation corresponds to a finite consequence operation in Tarski's sense, which can also be described by Gentzen-style structural rules.[3] Logical

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[1]Letter to the author of 9 July 1982.

[2]In letters to the author of 10 July and 19 August 1982, after taking notice of the logicality interpretation, Popper actually claimed that somehow laying the foundations of logic might be possible after all, in addition to the task of delineating logical signs.

[3]However, the designation 'structuralist' has nothing to do with 'structural rules', but rather with the fact that 'structures' in the model-theoretic sense are defined.

compounds are then characterized as objects satisfying certain constraints stated in terms of implication relations. For example, a conjunction C of A and B satisfies the conditions that

(i) C implies both A and B, and

(ii) C is the weakest object (weakest with respect to the given implication relation) such that (i) is fulfilled; that is, for any C', if C' implies both A and B, then C' implies C.

Koslow develops a structuralist theory in the precise metamathematical sense, which does not specify the domain of objects in any sense beyond the axioms given. Even if the domain is supposed to be a language, the structural axioms do not tell what a conjunction of A and B looks like (if there is one at all). Rather, if a language or any other domain of objects equipped with an implication relation is given, the structuralist approach may be used to *single out* logical compounds by checking their *implicational properties*. It does not postulate axioms and inference rules for a formal object language. Whether and how implication structures are realized as object languages, is left entirely open. In particular, nothing is being said about the particular inferential format used in such a realization (for example, whether it takes the form of a Hilbert-style or a Gentzen-style system).

I claim that Popper's approach, though often formulated by him in a misleading way and (by far) lacking the precision of Koslow's, is exactly of this kind. I shall try to present it as a coherent theory without going into all details of his line of argument. I also confine myself to propositional logic, as this eases my presentation and entirely suffices to give an idea of Popper's basic aims. The details of the first-order case are even less consistent and more problematic than those of the propositional one. For further issues, I refer to my 1984 paper (Schroeder-Heister 1984b). Often the details of Popper's presentation are a stumbling block to an overall understanding of his view. It is the *global* picture that makes his theory interesting, not so much the individual steps of his exposition. If one wanted to build further on his ideas, one would anyway take a better developed theory such as Koslow's as a starting point.

My paper is divided as follows. In the first section I give a brief sketch of Koslow's theory. Section 2 deals with Popper's inferential definitions, delineating them from Koslow's characterizations. In section 3, I argue that and why Popper's approach cannot be turned into a semantics and therefore is not a justification of a logical system. The concluding remarks (section 4) point to the fact that the structuralist reading of Popper's theory is well in line with his general philosophical views. The appendices illuminate other interesting aspects of Popper's work, for instance his axiomatization of structural rules and his usage of multiple succedent consequence.

1 Koslow's structuralist theory of logic

I give a brief outline of Koslow's theory as far as it is relevant for the reconstruction of Popper's approach, restating it in my terminology and notation, which in some cases is more explicit than Koslow's, and in many respects more 'formalistic'. As indicated, Koslow (1992) characterizes logical systems as structures in the metamathematical (model-theoretic) sense rather than as syntactically specified systems with axioms and inference rules. The domains of his structures are sets of objects for which a (finite) consequence relation is available. Koslow himself speaks of 'implication relations'. I shall speak of 'deducibility relations', keeping 'implication' from now on for the propositional connective (which may occur both in the metalanguage and in the object languages considered). The reader should bear in mind that 'deducibility' is here considered an abstract term that is not bound to the concept of deduction in formal languages.

A *deducibility structure* $\langle \mathcal{D}, \vdash \rangle$ consists of a non-empty domain $\mathcal{D}$ and a relation '$\vdash$' between finite subsets and elements of $\mathcal{D}$, called a *deducibility relation*, which satisfies the following conditions:

$$\begin{array}{l} \Gamma \cup \{A\} \vdash A \\ \Gamma \vdash A \;\&\; \Delta \cup \{A\} \vdash B \;\Rightarrow\; \Gamma \cup \Delta \vdash B, \end{array}$$

where capital Greek letters stand for subsets of $\mathcal{D}$ and capital Latin letters for elements of $\mathcal{D}$, and where & and $\Rightarrow$ are metalinguistic conjunction and implication, with & binding stronger than $\Rightarrow$. I use the common abbreviations, leaving out the set brackets to the left of $\vdash$, and writing $\Gamma, A \vdash B$ for $\Gamma \cup \{A\} \vdash B$ and $\Gamma, \Delta \vdash A$ for $\Gamma \cup \Delta \vdash A$.

Given a deducibility structure $\langle \mathcal{D}, \vdash \rangle$, an n-place logical operation H over $\langle \mathcal{D}, \vdash \rangle$ is a function that associates with any n-tuple of objects a (possibly empty) set of objects, which are all interdeducible, more formally,

$$H : \mathcal{D}^n \longrightarrow \mathcal{P}(\mathcal{D})$$

such that $A \dashv\vdash B$ holds for all $A, B \in H(A_1, \ldots, A_n)$. Furthermore, it is assumed that H is invariant with respect to interdeducibility, that is to say that $H(A_1, \ldots, A_i, \ldots, A_n) = H(A_1, \ldots, A'_i, \ldots, A_n)$, if $A_i \dashv\vdash A'_i$, and $H(A_1, \ldots, A_n)$ is closed under interdeducibility, that is, $A \in H(A_1, \ldots, A_n)$ is implied by $A \dashv\vdash B$ and $B \in H(A_1, \ldots, A_n)$. Since a logical operation is unique up to interdeducibility, the set $H(A_1, \ldots, A_n)$ may be identified with any of its elements, if it is non-empty.

Let $\langle s_1, \ldots, s_m \rangle$ be a signature, that is, an m-tuple of non-negative integers (corresponding to the arities of logical operations considered). A *logic structure* of signature $\langle s_1, \ldots, s_m \rangle$ is then given as $\langle \mathcal{D}, \vdash, H_1, \ldots, H_m \rangle$ such that

(i) $\langle \mathcal{D}, \vdash \rangle$ is a deducibility structure

(ii) Each H_i $(1 \leq i \leq m)$ is an s_i-place logical operation over $\langle \mathcal{D}, \vdash \rangle$.

An *inferential characterization* of an n-place logical operation H is a (metalinguistic) formula $\mathfrak{A}(A, A_1, \ldots, A_n)$ with at most $n+1$ variables such that

$$(1.3) \qquad A \in H(A_1, \ldots, A_n) \quad \Leftrightarrow \quad \mathfrak{A}(A, A_1, \ldots, A_n).$$

This means in particular that the first argument of $\mathfrak{A}$ is uniquely determined up to interdeducibility, that is, for any $A \in H(A_1, \ldots, A_n)$, the *uniqueness condition*

$$(1.4) \qquad (\forall B)(B \dashv\vdash A \quad \Leftrightarrow \quad \mathfrak{A}(B, A_1, \ldots, A_n))$$

is satisfied, which is equivalent to

$$\mathfrak{A}(B, A_1, \ldots, A_n) \ \& \ \mathfrak{A}(B', A_1, \ldots, A_n) \ \Rightarrow \ B \dashv\vdash B'$$

plus

$$\mathfrak{A}(B, A_1, \ldots, A_n) \ \& \ B \dashv\vdash B' \ \Rightarrow \ \mathfrak{A}(B', A_1, \ldots, A_n).$$

If there is an inferential characterization $\mathfrak{A}$ of H, H is said to be *inferentially characterized.* Conversely, if an $\mathfrak{A}$ is given such that (1.2) holds, then (1.1) defines a logical operation H such that $\mathfrak{A}$ is an inferential characterization of H. A logic structure $\langle \mathcal{D}, \vdash, H_1, \ldots, H_m \rangle$ is called *inferentially characterized*, if with each logical operation H_i an inferential characterization $\mathfrak{A}_i$ is associated. In what follows, I shall always assume that logic structures are inferentially characterized.

So far, the syntactic form of $\mathfrak{A}$ has been left unspecified. If nothing else is said, I assume that $\mathfrak{A}$ is built up from deducibility '$\vdash$' as the only predicate by using first-order logical constants.

It is characteristic of the structuralist approach that, even if the domain $\mathcal{D}$ consists of syntactically specified expressions of some object language rather than arbitrary objects, the results of logical operations need not have a standard form. If $\mathcal{D}$ contains for every A_1 and A_2 a conjunction A of A_1 and A_2, then A need not have a form like $A_1 \wedge A_2$. However, if $H_\wedge$ is the logical operation of conjunction, and if conjunctions of two expressions always exist, then $\mathcal{D}$ can easily be extended by introducing $A_1 \wedge A_2$ as a standard expression for a conjunction of A_1 and A_2 by just adding $A_1 \wedge A_2$ to $H_\wedge(A_1, A_2)$. It is obvious that this extension is conservative. In this way, any logical operation H available in a language can be represented explicitly by a logical constant in a conservative extension, provided the following *existence condition* is satisfied for H:

$$(1.5) \qquad (\forall A_1, \ldots, A_n)(\exists A)\mathfrak{A}(A, A_1, \ldots, A_n).$$

This condition is not necessarily fulfilled. It is easy to construct deducibility structures where, for example, disjunctions do not always exist (see Koslow 1992, p. 118).

In Koslow's theory inferential characterizations $\mathfrak{A}(A, A_1, \ldots, A_n)$ have a specific syntactic form. They provide inferential conditions corresponding to elimination rules in natural deduction and then require A to be the weakest object satisfying these conditions. Formally, the inferential characterizations for the four standard operations of intuitionistic logic (conjunction, disjunction, implication, and negation) can be stated as follows.

$$
\begin{array}{lcll}
\mathfrak{A}_\wedge(A, A_1, A_2) & :\Leftrightarrow & \quad A \vdash A_1 \;\&\; A \vdash A_2 & \text{elimination} \\
 & & \& \; (\forall B)(B \vdash A_1 \;\&\; B \vdash A_2 \;\Rightarrow\; B \vdash A) & \text{minimality}
\end{array}
$$

('A is a conjunction of A_1 and A_2')

$$
\begin{array}{lcll}
\mathfrak{A}_\vee(A, A_1, A_2) & :\Leftrightarrow & \quad (\forall C)(A_1 \vdash C \;\&\; A_2 \vdash C \;\Rightarrow\; A \vdash C) & \text{elimination} \\
 & & \& \; (\forall B)((\forall C)(A_1 \vdash C \;\&\; A_2 \vdash C \;\Rightarrow\; B \vdash C) & \\
 & & \quad \Rightarrow\; B \vdash A) & \text{minimality}
\end{array}
$$

('A is a disjunction of A_1 and A_2')

$$
\begin{array}{lcll}
\mathfrak{A}_\rightarrow(A, A_1, A_2) & :\Leftrightarrow & \quad A, A_1 \vdash A_2 & \text{elimination} \\
 & & \& \; (\forall B)(B, A_1 \vdash A_2 \;\Rightarrow\; B \vdash A) & \text{minimality}
\end{array}
$$

('A is an implication between A_1 and A_2')

$$
\begin{array}{lcll}
\mathfrak{A}_\neg(A, A_1) & :\Leftrightarrow & \quad (\forall C)(A, A_1 \vdash C) & \text{elimination} \\
 & & \& \; (\forall B)((\forall C)(B, A_1 \vdash C) \;\Rightarrow\; B \vdash A) & \text{minimality}
\end{array}
$$

('A is a negation of A_1')

As I am only illustrating the method of inferential characterizations, I do not consider here parametric versions, which also take into account additional assumptions ('contexts'), which may be present in deducibility statements. For example, with such parameters, the elimination condition in $\mathfrak{A}_\vee$ should be

$$(\forall \Gamma)(\forall C)(\Gamma, A_1 \vdash C \;\&\; \Gamma, A_2 \vdash C \Rightarrow \Gamma, A \vdash C).$$

It can easily be seen that $\mathfrak{A}_\wedge, \mathfrak{A}_\vee, \mathfrak{A}_\rightarrow, \mathfrak{A}_\neg$ are in fact inferential characterizations of corresponding operations $H_\wedge, H_\vee, H_\rightarrow$ and $H_\neg$, respectively, as the uniqueness condition (1.2) is fulfilled. This is due to the minimality requirements (see Koslow 1992, pp. 81ff.)

Inferentially characterized logical operations in Koslow's sense have the following two remarkable properties: They are (i) stable and (ii) distinct (see Koslow 1992, p. 10).

Ad (i): Suppose H is inferentially characterized by $\mathfrak{A}$ with respect to some deducibility structure $\langle \mathcal{D}, \vdash \rangle$. Let $\langle \mathcal{D}', \vdash' \rangle$ be a deducibility structure extending $\langle \mathcal{D}, \vdash \rangle$, that is, $\mathcal{D} \subseteq \mathcal{D}'$ and $\vdash \;\subseteq\; \vdash'$. Let $\mathfrak{A}'$ result from $\mathfrak{A}$ by replacing $\vdash$ with $\vdash'$ throughout. Then H is called *stable* with respect to $\langle \mathcal{D}', \vdash' \rangle$ if

$$(\forall A, A_1, \ldots, A_n \in \mathcal{D})(\mathfrak{A}(A, A_1, \ldots, A_n) \;\Leftrightarrow\; \mathfrak{A}'(A, A_1, \ldots, A_n)),$$

that is, objects that are conjunctions, disjunctions, implications, and negations in $\langle \mathcal{D}, \vdash \rangle$ remain conjunctions, disjunctions, implications, and negations, when $\langle \mathcal{D}, \vdash \rangle$ is extended to $\langle \mathcal{D}', \vdash' \rangle$. The following can easily be shown: suppose that $\langle \mathcal{D}', \vdash' \rangle$ is a conservative extension $\langle \mathcal{D}, \vdash \rangle$, that is, $\mathcal{D} \subseteq \mathcal{D}'$, $\vdash \subseteq \vdash'$, and

$$(\forall \Delta \subseteq \mathcal{D})(\forall A \in \mathcal{D})(\Delta \vdash' A \ \Rightarrow \ \Delta \vdash A).$$

Then H is stable with respect to $\langle \mathcal{D}', \vdash' \rangle$. In short, *inferentially characterized logical operations are stable with respect to conservative extensions* (see Koslow 1992, pp. 10, 31). This holds for inferential characterizations of logical operations in general. It does not depend on their specific syntactic form, in particular not on the minimality conditions.

Ad (ii): *Distinctness* of inferentially characterized logical operations H_1 and H_2 of arities n and $n+m$, respectively, over $\langle \mathcal{D}, \vdash \rangle$ simply means that they are different, that is, that there are $A_1, \ldots, A_{n+m} \in \mathcal{D}$ such that

$$H_1(A_1, \ldots, A_n) \neq H_2(A_1, \ldots, A_{n+m}).$$

Whether H_1 and H_2 are distinct, depends on their inferential characterizations. Koslow shows that the standard logical operations $H_\wedge, H_\vee, H_\rightarrow$ and $H_\neg$ are pairwise distinct *if the underlying deducibility structure* $\langle \mathcal{D}, \vdash \rangle$ *is non-trivial*, that is, if not every B is deducible from every Δ (Koslow 1992, pp. 10, 151-153).

An approach developed by the author (Schroeder-Heister 1984a) is similar to Koslow's in certain respects. It characterizes logical constants as expressing the *common content* of systems of conditions. This essentially means that from logical compounds exactly those sentences should be derivable that can be derived from the premises of their introduction rules. This again means that elimination rules are taken as a basis, and the compounds are weakest sentences having the power of major premises of elimination rules.[4] This idea was developed in the syntactic setting of a calculus of rules of higher levels, not in a metalinguistic structuralist theory like Koslow's. However, it could easily be used to extend Koslow's structuralist approach by generalized elimination principles for arbitrary logical operations.

2 Popper's theory

Popper develops a structuralist theory of logic based on deducibility structures (§ 2.1) and inferential characterizations of logical constants (§ 2.2), whose internal form is less restricted than Koslow's (§ 2.3).

[4]The *major premise* of an elimination rule is the formula from which a logical operator is eliminated. If, for instance, a step of $\rightarrow$-elimination (*modus ponens*) leads from $A \rightarrow B$ and A to B, then $A \rightarrow B$ is its major premise and A its minor premise. See Prawitz (1965) for this terminology.

2.1 Popper's deducibility structures

Popper bases the structures he investigates on the principles

$$\Gamma, A \vdash A$$
$$(\Gamma \vdash B_1 \,\&\, \cdots \,\&\, \Gamma \vdash B_n) \Rightarrow (B_1, \ldots, B_n \vdash C \Rightarrow \Gamma \vdash C)$$

called 'reflexivity' and 'generalized transitivity', respectively (see P1, p. 278; P2, pp. 198-200). He uses the symbols '/' and '//' for what are here denoted by '$\vdash$' and '$\dashv\vdash$'. Normally he does not regard antecedents (= left-hand sides) of deducibility statements as sets, but has explicit rules for permutation and contraction, a point that can be disregarded here. Principles like reflexivity and generalized transitivity are called 'absolutely valid', because they do not refer to logical connectives in some object language. They are independent of the distinction between logical and extra-logical signs (P1, p. 279). These principles are the same as those given by Hertz (1923) as the basis of his system. Hertz was the first to formulate what Gentzen (1935) later called 'structural rules'. Gentzen immediately built on Hertz when developing his sequent calculus (see Schroeder-Heister 2002). It is not clear how well acquainted Popper was with Gentzen's work, when he wrote his papers.[5] At that time (the late 1940s), Gentzen systems did not yet belong to the basic inventory of logic. In any case it is remarkable that Popper realized the significance of structural principles as the basis of logical reasoning. Even twelve years after Gentzen's thesis this was not a commonplace.

It is obvious that reflexivity and generalized transitivity are equivalent to the principles governing deducibility structures in the sense of § 1. So it may be said that, like Koslow, Popper starts with deducibility structures $\langle \mathcal{D}, \vdash \rangle$, in terms of which logical operations are explained. Popper even tries to axiomatize '$\vdash$' in such a way that reflexivity and generalized transitivity become derivable, as discussed in Appendix 1. Popper also occasionally considers deducibility statements whose succedent (= right-hand side) may contain more than one formula, or even none at all — see Appendix 2.

At this point a remark about the principles governing deducibility relations as compared with structural rules in Gentzen's sense is appropriate. Formally, the *principle*

$$\Gamma \vdash A \;\&\; \Delta \cup \{A\} \vdash B \;\Rightarrow\; \Gamma \cup \Delta \vdash B$$

is to be distinguished from the *rule* of Cut

$$\frac{\Gamma \vdash A \quad \Delta, A \vdash B}{\Gamma, \Delta \vdash B}$$

[5] Both Hertz and Gentzen are only very briefly mentioned at a few places in Popper's papers. One might even suspect that it was Bernays who drew his attention to Hertz and Gentzen, when his papers had been essentially completed. That his 'main intention was to simplify logic by developing what has been called by others "natural deduction" ' (Popper 1974, p. 1096) appears to be a later re-interpretation, which is only partially true, because, as will be seen, it is not the development of a particular object-linguistic deduction system or form of deduction that Popper is aiming at.

in sequent-style systems. The latter one uses '$\vdash$' as an object-linguistic operator, whereas both Koslow's and Popper's theories are entirely metalinguistic, without any preference for a particular object-linguistic style of reasoning. Nevertheless the close relationship between structural rules and principles for deducibility or consequence is obvious, and one might use sequent-style rules as representations of deducibility principles, and vice versa, keeping their fundamental distinctness in mind.[6]

2.2 Inferential definitions

Popper now proceeds by presenting what he calls 'inferential definitions' of logical constants. The inferential definitions for conjunction $\wedge$, disjunction $\vee$, implication $\to$ and classical negation $\sim$ run as follows:[7]

$$(2.1) \qquad A \dashv\vdash A_1 \wedge A_2 \quad \Leftrightarrow \quad (\forall C)(A \vdash C \Leftrightarrow A_1, A_2 \vdash C)$$

$$(2.2) \qquad A \dashv\vdash A_1 \vee A_2 \quad \Leftrightarrow \quad (\forall C)(A \vdash C \Leftrightarrow A_1 \vdash C \ \&\ A_2 \vdash C)$$

$$(2.3) \qquad A \dashv\vdash A_1 \to A_2 \quad \Leftrightarrow \quad (\forall C)(C \vdash A \Leftrightarrow C, A_1 \vdash A_2)$$

$$(2.4) \qquad A \dashv\vdash {\sim}A_1 \quad \Leftrightarrow \quad (\forall C, D)(A, C \vdash A_1 \Rightarrow A, C \vdash D \ \&\ C \vdash A_1).$$

So the general form of an inferential definition for an n-place connective $J(A_1, \ldots, A_n)$ is

$$(2.5) \qquad A \dashv\vdash J(A_1, \ldots, A_n) \quad \Leftrightarrow \quad \mathfrak{A}_J(A, A_1, \ldots, A_n),$$

where $\mathfrak{A}_J$ is an expression in the metalanguage with no more than $n+1$ open places. It is obvious that (2.5) brings Popper's inferential definitions already close to the way inferential characterizations were introduced in § 1.

Although Popper's terminology is quite unfortunate, calling (2.1)–(2.4) *explicit definitions* of logical constants (P2, p. 218), he makes it clear at many places that he does not want to define particular expressions of some object language. Instead he insists that he wants a purely metalinguistic characterization of the logical operations independently of whether they exist syntactically. The following quotations, in which the emphasis is Popper's own, are from P2, p. 208, ibidem, P3, p. 564, P2, p. 207, and P3, p. 562.

> ... we say that $a \wedge b$ is a conjunction of a and b, rather than that it is *the* conjunction of a and b.
>
> ... what we have defined is not so much the conjunction of a and b but the precise *logical force* (or the logical import) of any statement c that is equal in force to a conjunction of a and b.

---

[6] Actually, at many places, Popper calls principles such as generalized transitivity 'rules' (see P3, p. 565, and P2, *passim*). In a footnote on Gentzen (P4, p. 52), he emphasizes that the *metalinguistic* character of his own theory distinguishes it from Gentzen's.

[7] Again I use my terminology and omit, as Popper himself does, parametric context formulas. The examples are from P2, p. 218.

> ...we do not define, e.g., conjunction, but rather the logical force of conjunction.
>
> We do not even assume that the language we are discussing — the language to which our statements $a, b, c, \dots$ belong — possesses a special sign for linking statements into conjunctions.
>
> Our theory is completely metalinguistic.

This suggests that rather than read the logical constants $\wedge$, $\vee$, $\rightarrow$, $\sim$ in (2.1)–(2.4) as particular term-forming operators, one should interpret $A \dashv\vdash A_1 \wedge A_2$, $A \dashv\vdash A_1 \vee A_2$, $A \dashv\vdash A_1 \rightarrow A_2$, $A \dashv\vdash \sim A_1$ in Koslow's sense as expressing the availability of a set of terms A of equal logical force. One would then write

$$\begin{array}{lll} A \in H_{\wedge}(A_1, A_2) & \text{for} & A \dashv\vdash A_1 \wedge A_2, \\ A \in H_{\vee}(A_1, A_2) & \text{for} & A \dashv\vdash A_1 \vee A_2, \\ A \in H_{\rightarrow}(A_1, A_2) & \text{for} & A \dashv\vdash A_1 \rightarrow A_2, \\ A \in H_{\sim}(A_1) & \text{for} & A \dashv\vdash \sim A_1, \end{array}$$

and in general

$$A \in H_J(A_1, \dots, A_n) \;\; \text{for} \;\; A \dashv\vdash J(A_1, \dots, A_n).$$

Thus one arrives at inferential definitions whose general (outer) form

$$A \in H_J(A_1, \dots, A_n) \quad \Leftrightarrow \quad \mathfrak{A}_J(A, A_1, \dots, A_n) \tag{2.6}$$

corresponds exactly to (1.1) of § 1. Only the internal form of the inferential characterizations $\mathfrak{A}_J$ is different (see below § 2.3).[8]

That Popper intended something of this kind is clear from passages such as the following (P2, p. 214, Popper's emphasis), where he describes an implication-like expression (written by Popper as $a > b$) as

> a *variable name* of any statement which stands in a certain logical relationship to the two statements a and b.

He even reads (2.1) as 'A is a conjunction of A_1 and A_2 iff ...', and so on (see P2, p. 206). This makes it clear that using compounds such as $A_1 \wedge A_2$, $A_1 \rightarrow A_2$, and so on, as syntactically specified terms is prone to misunderstandings. Unfortunately, Popper often uses them in that way, for example by substituting $A_1 \wedge A_2$ for A in (2.1) to obtain certain inference rules (see P3, p. 565).

The reaction from the reviewers of Popper's papers was only natural: as a fundamental objection, they saw hidden existence assumptions in inferential

[8] For the sake of terminological precision it should be pointed out that for Popper an inferential definition is an expression of the form (2.5), whereas an inferential characterization in my (or Koslow's) sense is just its right hand side (that is, an $\mathfrak{A}$). Therefore I say that Popper's inferential definitions *make use of* or *provide* inferential characterizations.

definitions,[9] which is, of course, true, if conjunctions and other combinations are considered terms of a particular form. There are far too many misleading formulations by Popper which support this reading.[10]

Koslow's set notation (2.6) avoids all these problems. However, when rephrasing (2.5) as (2.6), it must be explicitly demanded, as it is demanded in the presentation of Koslow's theory (see (1.2)), that $\mathfrak{A}(A, A_1, \ldots, A_n)$ characterize A up to interdeducibility. This uniqueness condition is built into the left hand side of Popper's inferential definitions.

Thus my proposal is to read Popper's inferential definitions (2.5) as providing inferential characterizations of logical operations in the sense of § 1, for which both (1.1) and (1.2) hold. In other words, Popper defines logic structures $\langle \mathcal{D}, \vdash, H_1, \ldots, H_m \rangle$, where $H_1, \ldots, H_m$ are logical operations inferentially characterized by certain $\mathfrak{A}_1, \ldots, \mathfrak{A}_m$. This is a neat view of Popper's aim with a maximum of support in his papers.

2.3 The internal form of Popper's inferential characterizations and the logicality problem

The basic contrast to Koslow is that Popper imposes no specific restriction on the form of an $\mathfrak{A}(A, A_1, \ldots, A_n)$ except that it guarantees the uniqueness of A. As the different patterns of the right hand sides of (2.1)–(2.4) indicate, $\mathfrak{A}$ may take various distinct forms. For example, (2.1) and (2.2) can be read as saying that conjunction and disjunction are *strongest* elements of the domain (with respect to $\vdash$) such that the standard introduction rules for them are valid, (2.3) says that an implication is a *weakest* element such that the elimination rule (*modus ponens*) holds, and (2.4) says that the rule of *self-affirmation*

$$\frac{\begin{array}{c}[\sim A_1]\\ A_1\end{array}}{A_1}$$

(a variant of the classical *reductio* rule) as well as the *contradiction* rule

$$\frac{A_1 \quad \sim A_1}{C}$$

are valid without maximality or minimality requirement. Only (2.3) corresponds to Koslow's idea (elimination principle plus minimality condition). An

[9] Most clearly Kleene (1947/1948) and Hasenjaeger (1949). Other reviews were by Curry (1948/1949) and McKinsey (1948) (all not very favourable). For more details see Schroeder-Heister (1984b).

[10] There is a single place, in P3, p. 569, where Popper claims that existence postulates have to be 'added' when applying the results to some specific object language, which means that existence of operators is not presupposed. However, this is not enough to prevent readers from seeing implicit existence assumptions in his using terms such as $A \wedge B$, $A \vee B$, and so on.

alternative definition for classical negation is given by

$$(2.7) \qquad A \dashv\vdash {\sim}A_1 \quad \Leftrightarrow \quad (\forall C)(A, A_1 \vdash C \;\&\; (A, C \vdash A_1 \Rightarrow C \vdash A_1))$$

(P1, p. 282; P2, p. 220, note), which is a variant of (2.4). A further definition of classical negation uses deducibility statements with multiple succedents, which essentially is a characterization by using the *excluded middle* (see Appendix 2).

As an inferential definition for intuitionistic negation, Popper proposes

$$(2.8) \qquad A \dashv\vdash \neg A_1 \quad \Leftrightarrow \quad (\forall C)(A, A_1 \vdash C \;\&\; (A_1, C \vdash A \Rightarrow C \vdash A)),$$

which is obtained from (2.7) by interchanging A and A_1 in the right conjunct (see P1, p. 282, note, and P2, p. 220, note). This corresponds to the fact that instead of classical self-affirmation only its intuitionistic counterpart

$$\frac{\begin{array}{c}[A_1]\\ \neg A_1\end{array}}{\neg A_1}$$

(*'self-denial'* — a variant of the intuitionistic *reductio* rule) is expected to hold. The right hand side of (2.8) is equivalent to $\mathfrak{A}_\neg(A, A_1)$ as defined by Koslow (see § 1).[11]

This all indicates that the form of $\mathfrak{A}$ is not restricted in any particular way.[12] This is confirmed also by the fact that Popper deals with classical negation extensively, which is not possible in frameworks with $\mathfrak{A}$ having Koslow's restricted form, which naturally leads to intuitionistic logic. Therefore, in Popper, $\mathfrak{A}$ is just a metalinguistic logical formula with '$\vdash$' as the only extra-logical constant.

It is important to see that the uniqueness restriction is not as innocent as it perhaps seems to be at first glance. It excludes certain characterizations of well-known operations, which one would normally consider logical. Take the negation $\neg_w$ (in the following called *weak negation*), which in natural deduction is characterized by the self-denial rule

$$\frac{\begin{array}{c}[A_1]\\ \neg_w A_1\end{array}}{\neg_w A_1}$$

[11]In fact, Popper himself (P5, p. 111) considers a version corresponding to Koslow's.

[12]Perhaps except that it starts with a universal quantifier; see P3, p. 564. There Popper states the general form of inferential definitions as

$$a \dashv\vdash \text{the definiendum} \Leftrightarrow (\text{for every} \ldots : \; (\ldots))$$

But even here this universal quantifier is not considered a matter of principle, but just a description of the particular forms of inferential definitions given in the sequel to this remark. — In P6, he explicitly discusses the possibility of characterizing logical compounds as strongest or weakest statements for which certain introduction or elimination rules, respectively, hold. Apart from the fact that he considers these options as two alternatives and not opting, as Koslow does, for one of them, it is clear from the context that he does not insist on such a form.

alone (without the contradiction rule), and whose inferential characterization is

$$\mathfrak{A}_w(A, A_1) :\Leftrightarrow (\forall C)(A_1, C \vdash A \Rightarrow C \vdash A).$$

That the uniqueness requirement is not fulfilled for $\mathfrak{A}_w(A, A_1)$ can be seen as follows: suppose $\mathfrak{A}_\neg(A, A_1)$ is now the inferential characterization of intuitionistic negation in Popper's sense (the right side of (2.8)), that is,

$$\mathfrak{A}_\neg(A, A_1) \;:\Leftrightarrow\; (\forall C)(A, A_1 \vdash C \;\&\; (A_1, C \vdash A \Rightarrow C \vdash A)).$$

Suppose $\mathfrak{A}_t(A, A_1)$ is defined as

$$\mathfrak{A}_t(A, A_1) :\Leftrightarrow (\forall C)(C \vdash A),$$

which is the inferential characterization of a unary trivial truth operation. Obviously, both $\mathfrak{A}_\neg$ and $\mathfrak{A}_t$ satisfy uniqueness. Furthermore, it is clear that

$$\mathfrak{A}_\neg(A, A_1) \;\Rightarrow\; \mathfrak{A}_w(A, A_1)$$

as well as

$$\mathfrak{A}_t(A, A_1) \;\Rightarrow\; \mathfrak{A}_w(A, A_1).$$

Therefore, if for some A_1 in the given deducibility structure there are both an intuitionistic negation $\neg A_1$ and a truth operator tA_1 available, then both $\mathfrak{A}_w(\neg A_1, A_1)$ and $\mathfrak{A}_w(tA_1, A_1)$ hold. This means that, if $\mathfrak{A}_w$ is in fact an inferential characterization, for which uniqueness holds,

$$\neg A_1 \dashv\vdash tA_1$$

is valid, which trivializes the deducibility structure. Since deducibility structures, which, for some A_1, contain both $\neg A_1$ and tA_1, can easily be constructed, $\mathfrak{A}_w$ is not an inferential characterization. In fact, there is no inferential characterization for $\neg_w$, if the language considered is non-trivial. If everything is deducible in the language, then, of course, every $\mathfrak{A}$ is an inferential characterization.[13]

Observations such as these motivated me, in Schroeder-Heister (1984b), to interpret Popper's theory as a theory of logicality. According to this idea, an n-ary operation J is counted as logical if an inferential characterization

[13] This is an adaptation of an argument given by Popper for Johansson's minimal negation $\neg_j$ (see P5, p. 117, and Schroeder-Heister 1984b, pp. 101ff.). I chose $\neg_w$ rather than the slightly stronger (and more common) $\neg_j$, as for $\neg_w$ a formula $\mathfrak{A}_w(A, A_1)$ can be given that at least looks like an inferential characterization, whereas for minimal negation, which in natural deduction is characterized by the intuitionistic *reductio* rule

$$\frac{\begin{array}{cc}[A_1] & [A_1] \\ C & \neg_j C\end{array}}{\neg_j A_1},$$

not even that is possible.

$\mathfrak{A}_J(A, A_1, \dots, A_n)$ in Popper's sense is available for it. Weak negation would then not be logical, whereas classical negation would be so, and so on. This interpretation has strong support from some remarks in Popper's papers (for example, P1, p. 286) and in particular from later remarks in the Schilpp volume (Popper 1974, p. 1096). I even argued that one might consider those inferential characterizations $\mathfrak{A}$ as fundamental that contain only the metalinguistic operators of implication ($\Rightarrow$), conjunction (&), and universal quantification ($\forall$), as they suffice to formulate virtually all inferential characterizations. On that view, positive logic with implication, conjunction, and universal quantification would lie at the heart of deductive reasoning, representing some sort of basic logic, in terms of which other operations can be characterized. I still think this view can be upheld. However, it runs short of what Popper really intended. According to my reading now, Popper's message is that logical operations can be structurally characterized up to interdeducibility in terms of deducibility, using first-order metalinguistic means only. That this might at the same time indicate how to tackle the logicality problem, should be regarded only as an (important) side aspect.

3 Structuralist theory versus semantics

The title of his main philosophical paper on deduction: 'New Foundations for Logic' (P2), may suggest that Popper is aiming at a *justification* of the logical laws. Nothing is further from the truth. His structuralist theory, as I have reconstructed it, is a tool for metalinguistically *describing* logical theories, but never for *justifying* them. A justification would have to develop some sort of semantics of the logical operations and then determine which inferences or consequences are valid, and which are not valid, with respect to this semantics. The aim of the structuralist approach is to deal with a domain $\mathcal{D}$, for which a deducibility relation '$\vdash$' is given, and to characterize the operations available in $\langle \mathcal{D}, \vdash \rangle$. There is no *normative* task involved, which sometimes there is in semantics, as a semantics can distinguish between good and bad inferences and therefore between the right and wrong logic. The structuralist approach can distinguish only between candidates and non-candidates for logical operations. So Popper's talking of 'foundations' cannot be understood as justificationist.

Why is Popper's theory not suitable as a semantics? One might argue that an inferential characterization $\mathfrak{A}_J(A, A_1, \dots, A_n)$ of some constant J could be used to introduce it explicitly into the language considered, provided there is some A such that $\mathfrak{A}_J(A, A_1, \dots, A_n)$ holds, which might then be called $J(A_1, \dots, A_n)$. However, this is not what Popper aims at.

A strong indication for this is the fact that Popper does not consider conservativeness a criterion for selecting certain inferential characterizations, and for preferring them to others. Normally, in a semantic theory, when introducing a new constant J, the new laws should not affect what can be formulated with the 'old' vocabulary, that is the vocabulary without J. For example, if I have a theory containing J_1 and I introduce J_2, then the new laws involving J_2 should not enable me to derive new laws for J_1 alone. In this sense, the theory for J_2 and J_1 together should be a *conservative extension* of the theory for J_1 alone.

Now in his inferential definitions, Popper explicitly deals with the situation where conservativeness does not hold. He does not consider this to be problematic but rather an interesting case to be discussed, in particular with respect to the relationship between different sorts of negations, such as intuitionistic and classical ones. The general phenomenon is the following. Given two inferential definitions $\mathfrak{A}_1(A, A_1, \ldots, A_n)$ and $\mathfrak{A}_2(A, A_1, \ldots, A_n)$, suppose that

$$\mathfrak{A}_1(A, A_1, \ldots, A_n) \quad \Rightarrow \quad \mathfrak{A}_2(A, A_1, \ldots, A_n). \tag{3.1}$$

Then, owing to the uniqueness requirement, the operations characterized by $\mathfrak{A}_1$ and $\mathfrak{A}_2$ cannot be distinguished. For if both $\mathfrak{A}_1(A, A_1, \ldots, A_n)$ and $\mathfrak{A}_2(B, A_1, \ldots, A_n)$ hold, then, by (3.1), $\mathfrak{A}_2(A, A_1, \ldots, A_n)$ must hold too, which, by uniqueness, gives $A \dashv\vdash B$, which again yields $\mathfrak{A}_1(B, A_1, \ldots, A_n)$.

This means that, if an inferentially characterized operation is (seemingly) stronger than another one, in the sense that it satisfies the laws of the weaker one, then these operations cannot be distinguished; that is, the laws of the 'stronger' one hold also for the 'weaker' one, which violates conservativeness. The prominent example, which is considered by Popper, is classical in comparison with intuitionistic negation. For classical negation $\sim$, the right side of (2.4) (or alternatively (2.7)) is an inferential characterization $\mathfrak{A}_\sim$, for which uniqueness holds. As classical negation is stronger than intuitionistic negation,

$$\mathfrak{A}_\sim(A, A_1) \quad \Rightarrow \quad \mathfrak{A}_\neg(A, A_1) \tag{3.2}$$

is valid, which, according to the argument given above, yields

$$\sim A_1 \dashv\vdash \neg A_1,$$

if both classical and intuitionistic negations are available in the language considered.[14] If intuitionistic negation is introduced first, then the introduction of classical negation destroys conservativeness, as it enforces classical laws such as *excluded middle* to hold for intuitionistic negation.[15] The fact that, in the presence of classical negation, no distinct intuitionistic negation can be characterized, is not seen by Popper as a violation of any fundamental principle, but rather a significant discovery, showing that different negations cannot arbitrarily coexist. This makes good sense if Popper's basic aim is not

[14]See P5, pp. 113f. More precisely, (3.2) holds if there is a B available such that $\mathfrak{A}_\sim(B, A)$. If double negation is not available, intuitionistic negation does not necessarily collapse into classical negation.

[15]The proof in § 2.3 above, that weak negation $\neg_w$ cannot be inferentially characterized, was already an application of this sort of reasoning. There weak negation collapsed into *two* stronger operations t and $\neg$, which were mutually contradictory.

semantic foundation, but *structural description*.[16] Koslow's theory (and also my own in Schroeder-Heister 1984a) differs from Popper's in that it can in principle be turned into a semantic theory. Due to the special form of Koslow's inferential characterizations, conservativeness is bound to hold. The fact that he can prove distinctness of operators is a consequence of that. Of course, conservativeness and distinctness do not make a semantic theory yet. But the idea that inferential characterizations start with elimination rules and characterize logical compounds as the weakest sentences for which the elimination rules are valid, suggests a semantics, according to which meaning is based on elimination rules and other valid rules are justified with respect to them by minimality conditions. This would be dual to verificationist semantics in the Dummett–Prawitz tradition, which extracts meaning from introduction rules and therefore from assertibility conditions, and justifies other valid inferences by principles which come close to maximality conditions.[17]

I am not claiming that Koslow is proposing a semantics in this sense. In fact, he makes it clear throughout that his structuralist theory does not give certain logics or certain logical operations preference over others. I just want to remark that Koslow's theory with its *specific form* of inferential characterizations *might be given* a semantic reading under *certain circumstances*,[18] in contradistinction to Popper, where their form remains unspecified. Popper's theory is a *radical* structuralist theory in that just the inferential role of logical compounds is uniquely described without any further constraints. This prevents a semantic reading as there are no special features like introduction or elimination rules, *in virtue of which* certain laws are valid, as might be required by a proof-theoretic semantics. On the other hand, by dealing with syntactically specified object-linguistic operations and using a bad terminology, Popper has failed to make his intentions sufficiently clear.

This is not intended as a defence of Popper's view. Conservativeness is a principle with strong grounding, as is the distinctness of operations characterized. There are good reasons to argue that, if operations are to be inferentially characterized at all, this should be done separately and independently for each of them, that is an inferential characterization of one operation (for example, classical negation) should not substantially alter that of another one given before (for example, intuitionistic negation). Therefore Koslow's structuralist approach might be preferable at the end. In any case, Popper's radical structuralist view is a coherent approach when properly reconstructed.

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[16] Actually, in investigating various negations, some of which can be inferentially characterized whereas others cannot, and some of which can distinctly coexist, whereas others cannot (see P5, and P1, pp. 282f., note), Popper already considers the idea of combining logics, which only recently has gained significant attention (see, for example, Gabbay 1999).

[17] See Dummett (1991), Chapter 11, and Prawitz (1974), Schroeder-Heister (2006). In this tradition the dual approach based on elimination rules is occasionally mentioned as a possibility (see, for example, Prawitz 1971, Appendix A.2, and Dummett 1991, Chapter 13).

[18] Namely in cases where a syntactically specified object language is dealt with, and where the logical operations are represented in this language by syntactical operators.

4 Conclusion: A new view of logic?

Popper does not give new foundations for logic in the sense that he semantically justifies a logical system. Rather, the structuralist view is a descriptive approach, providing a framework in which various different logics (and even non-logics, as Koslow emphasized) can be defined as deducibility structures $\langle \mathcal{D}, \vdash \rangle$ with operations H_J that are inferentially characterized by conditions $\mathfrak{A}_J$. If providing such a framework is considered the basic task of logic, this is a new view of logic as a philosophical or mathematical discipline. Logic would then take a neutral stance towards competing logical systems, and just aim at comparing them with respect to their structural properties. This view not only is advocated by an outspoken structuralist such as Koslow, but has been present in certain areas of logic for two decades, especially in dealing with alternative logics. Examples are 'logical frameworks' in computer science describing various logics from a general (often type-theoretic) position,[19] and also the treatment of non-classical logics in terms of principles governing the consequence relation.[20] In this sense Popper's talking of 'new foundations' for logic would just be a misleading way of proposing a new orientation of logic as a descriptive rather than normative discipline.

This interpretation of Popper's logical papers is much in line with his other writings, in which he adopts a strong *anti-foundationalist* and *anti-justificationist* point of view (see Popper 1960). It would be extremely surprising if with respect to deductive logic Popper claimed just the opposite by *justifying* it and laying *new foundations* for it. Even though the logical writings were written much earlier than his papers against foundationalism, his main views have been there since *Logik der Forschung* (1935). Popper's later claim that classical logic is best suited for scientific reasoning, as it makes refutations easiest (Popper 1968, § 6 (3′), and 1970, § 4), thus fitting best with the idea of logic as the 'Organon of rational criticism' (Popper 1963, Chapter 1, appendix, paragraph (13)), also supports this view. Structural description of logical systems is one issue, the question of which logic to choose for a certain purpose is a different one, not belonging to philosophical logic any more. The latter question has to be answered by extra-logical reasons, yet on the basis of the structural evaluation of what the various possible systems can achieve.

Hence the structuralist reading, for which Koslow's theory is the best elaborated model, is not only perfectly *compatible* with Popper's general philosophical approach, but gives deductive logic a *clear-cut role* in his philosophical framework.

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[19] See Huet & Plotkin (1991) and the references therein.

[20] See Gabbay (1994). An example is the treatment of consequence relations in non-monotonic logics, as in the classic paper of Kraus, Lehmann, & Magidor (1990).

Appendix 1: The axiomatization of absolutely valid rules

In P4, Popper tries to axiomatize the deducibility predicate '$\vdash$' in such a way that the structural (= 'absolutely valid') rules for '$\vdash$' become derivable. For deducibility $\Gamma \vdash A$, where Γ is now an ordered *list* (rather than a *set*) of sentences, he gives the following axioms, called *Basis I*:

(A.1) $$B \vdash A \;\Leftrightarrow\; B, B \vdash A$$

(A.2) $$\Gamma \vdash A \;\&\; A, \Gamma \vdash C \;\Rightarrow\; \Delta, \breve{\Gamma} \vdash C$$

(A.3) $$(\forall C)(\Gamma, A \vdash C \Rightarrow \breve{\Gamma} \vdash C) \Rightarrow \Gamma \vdash A.$$

Here $\breve{\Gamma}$ results from Γ by putting its elements in reverse order, that is, if Γ is $\langle A_1, \ldots, A_n \rangle$, then $\breve{\Gamma}$ is $\langle A_n, \ldots, A_1 \rangle$. The axioms are stated in my notation, using capital Greek letters for lists of sentences and understanding that the schematic letters A, B, C, Γ, Δ are universally quantified from outside. Popper's notation is less perspicuous. He combines (A.2) and (A.3) into a single axiom which is difficult to read. Under a formalist reading of deducibility statements, Axiom (A.3) can also be formulated as:

$$\Gamma \vdash A \text{ holds, if for every } C\text{, the rule } \frac{\Gamma, A \vdash C}{\breve{\Gamma} \vdash C} \text{ is admissible.}$$

Using (A.1), (A.2), and (A.3), Popper's reasoning in P4, Part I, can then be reconstructed as a derivation of all relevant structural rules.

It is difficult to see why elementary structural rules such as permutation, thinning, or cut should and could be reduced to something even more fundamental. It is very questionable whether the principles considered by Popper as a 'basis', are philosophically basic indeed. The standard structural rules appear more plausible and clearcut than principles (A.1)–(A.3). (A.1) is a special case of contraction and expansion (the dual of contraction), (A.2) combines cut with thinning and a special form of permutation, and (A.3) is a principle, which cannot even be formulated in the form of an inference rule, stating some sort of inverse to cut.

This is even more a problem for Popper's second axiomatization in P4, Part II, which combines purely structural aspects of '$\vdash$' with properties of (object-linguistic) conjunction $\wedge$, thus violating the idea that the structural base should be independent of the logical operations available.

Appendix 2: Deducibility with multiple succedents

At some places, Popper uses deducibility statements whose succedent is a finite set of formulas (P4, pp. 51-53). However, in contradistinction to Gentzen's approach, this is not considered a primitive notion. It is defined as follows:

$$\Gamma \vdash \Delta \;:\Leftrightarrow\; (\forall C)((\forall A \in \Delta)(A \vdash C) \Rightarrow \Gamma \vdash C).$$

Actually, the definition with context formulas

$$\Gamma \vdash \Delta \;:\Leftrightarrow\; (\forall \Gamma_1)(\forall C)((\forall A \in \Delta)(A, \Gamma_1 \vdash C) \Rightarrow \Gamma, \Gamma_1 \vdash C)$$

would be more general. The definition of deducibility with multiple succedents gives Popper the possibility to define the *refutability* of a list of formulas Γ as the limiting case where the succedent Δ is empty.

Using his definition of multiple succedent deducibility, in P5 Popper gives an inferential definition of classical negation as follows (p. 112):

$$A \dashv\vdash {\sim}A_1 \;\Leftrightarrow\; (A, A_1 \vdash \;\&\; \vdash A, A_1)$$

which would have to be spelled out as

$$A \dashv\vdash {\sim}A_1 \;\Leftrightarrow\; ((\forall C)(A, A_1 \vdash C) \;\&\; (\forall D)(A \vdash D \;\&\; A_1 \vdash D \;\Rightarrow\; \vdash D)).$$

This means that classical negation is characterized by the contradiction rule and by the *classical dilemma* (which corresponds to the law of *excluded middle*), or, as Popper puts it, ‘the classical negation of *b* can be defined (as Aristotle might have defined it) as that statement which is at once contradictory and complementary to *b*’ (ibidem).

Unfortunately, Popper only briefly mentions Gentzen’s idea of multiple succedent sequents in a footnote (P4, p. 52).

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