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Sektion 1: Logik

On the notion of *assumption* in logical systems

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1. The asymmetry between assumptions and assertions

When we *assume* a formula A in a logical derivation, we mean that A as well as subsequent formulae inferred using A *depend* on A. In certain types of logical calculi, especially natural deduction systems, assumptions can also be *discharged*, i.e., the dependence on certain assumptions can be removed. This happens, for example, when an implication $A \rightarrow B$ is inferred given a derivation of B which depends on A. The introduction of an assumption A is normally *unspecific* in the sense that there are no restrictions as to the form of A or the context in which A occurs. In principle, just any formula A can serve as an assumption.

This is different with *assertions* made in a derivation. There is, of course, an unspecific way of asserting A, viz., when A is asserted as depending on itself as an assumption. But there are normally also many *specific* ways of asserting a formula, depending on its form or its context. Any introduction inference in natural deduction gives an example for that: We can assert $A \wedge B$ given derivations of both A and B, we can assert $\exists x A(x)$ given a derivation of A(t) for some t, and so on. Even the elimination inferences constitute a specific way of making assertions, where "specific" now applies to the premisses and therefore to the context in which the assertion is made: We can assert A(t) given a derivation of $\forall x A(x)$, we can assert C given derivations of $A \vee B$, of C depending on A, and of C depending on B, and so on. There is a variety of specific inference rules for making assertions, but just a single unspecific rule for making assumptions.

2. Removing the asymmetry: Natural-deduction-style sequent calculus

I claim that this asymmetry should be removed. There is no reason why assertions should be better off in logic than assumptions. In any case it is interesting to see what conceptual insights we gain from considering a more symmetric system. Fortunately, such a system is at hand in the form of the sequent calculus. By "sequent calculus", I mean the symmetric sequent calculus with introductions both on the right and on the left side of the sequent sign ("$\Rightarrow$" in our notation), not sequent-style natural deduction with introductions and eliminations only on its right side. If we transform this calculus into natural deduction format, we can read the left introduction rules as *specific* assumption introduction rules. For example, the ($\wedge \Rightarrow$)- rule

$$\frac{\Gamma, A \Rightarrow C}{\Gamma, A \wedge B \Rightarrow C}$$

allows one *to introduce* $A \wedge B$ *as an assumption* from which C can be inferred (in a context Γ), given that C can be inferred from the assumption A (in the context Γ), and the (∃ ⇒)-rule

$$\frac{\Gamma, A(y) \Rightarrow C}{\Gamma, \exists x A(x) \Rightarrow C}$$

allows one *to introduce* $\exists x A(x)$ *as an assumption*, from which C can be inferred (in a context Γ), given that C can be inferred from the assumption A(y) (in the context Γ, modulo certain eigenvariable conditions), etc.

In the resulting natural deduction system, which might be called a *natural-deduction-style sequent calculus*, major premisses of elimination rules are only allowed to occur in top position, i.e., as assumptions introduced in a *specific* way. Besides that we still have the *unspecific* way of introducing assumptions (and assertions) by means of just assuming (and at the same time asserting) a formula A, which corresponds to initial sequents $A \Rightarrow A$ in the sequent calculus. Obviously, now the situation is completely symmetric with respect to assumptions and assertions: both of them can be introduced either in a *specific* way (by applying an inference rule governing the main operator of the assumption) or in an *unspecific* or *trivial* way (by just stating them). Correspondingly, we shall speak of *specific* and *unspecific* (or *trivial*) *assumptions*.

There have been some proof-theoretic investigations of such systems (e.g., by von Plato 2001), and there have also been strong extensions of similar systems beyond pure logic in theories of definitional reflection (e.g., by Hallnäs 1990 and Schroeder-Heister 1993), but their philosophical significance has not been fully appreciated so far.

3. Keeping apart specific and unspecific assumptions

I do not only want to propagate the view that assumptions deserve equal rights as compared to assertions. I should also like to draw certain philosophical consequences from the distinction between specific and unspecific assumptions, when they are treated in a different way. Specific assumptions are introduced according to their meanings whereas unspecific assumptions are just stated without special regard. Therefore one might argue that they have to be kept apart. For this to achieve I see three possible strategies:

(1) We require that any assumption which can in principle be introduced in a specific way, i.e., for which a specific assumption introduction rule is available, *must not* be introduced in an unspecific way, i.e. as a trivial assumption. In standard logical systems this just means that only atomic formulae can function as trivial assumptions, which in the sequent calculus corresponds to the restriction often imposed that in initial sequents $A \Rightarrow A$ the formula A has to be atomic. In general, this is a kind of well-foundedness condition on assumptions. If there is a specific assumption introduction rule for A, then A can only be assumed via that rule, which presupposes that certain other propositions occurring in the premisses of that rule have already been assumed, and so on. Trivial assumptions represent, so to speak, the base case of this chain. (This approach corresponds to a principle proposed for an extension of logic programming by P. Kreuger, see Schroeder-Heister 1994.)

(2) We disallow contracting different occurrences of the same formula A to a single A, if the two occurrences originate from different sorts of assumptions (i.e. one from a specific assumption and the other one from a trivial one). Here, in natural deduction format, contraction means discharging more than one occurrence of the same formula at the same time. However, it is technically difficult to make precise what

"originate" should mean. A clearcut case is only given if one occurrence of A is specific whereas the other one is not. The case of a logically complex A, with subexpressions originating from different sorts of assumptions, needs special consideration.

(3) We prohibit contraction at all, i.e. we use contraction-free logic. Although this is a very crude way of keeping different sorts of assumptions distinct, which is definitely not fully satisfying, our reasoning concerning the notion of assumption gives at least some partial philosophical justification for contraction-free systems, which for different purposes have been considered in various areas.

4. Application to antinomies

As an application I consider circular reasoning as it arises in connection with antinomies. Normally, the main step in antinomies is to derive, for a certain formula A, (i) $\neg A$ from A, and (ii) A from $\neg A$ (for example by taking A to be $R \in R$ for the Russell set R in naïve set theory). Then, in pure logic, we proceed as follows to derive a contradiction: (i) yields $\neg A$, and with (ii) we also obtain A. However, if we apply our programme of keeping specific and unspecific assumptions apart, the following happens, depending on which strategy we choose.

Ad (1): We cannot derive (i), as there are rules for specifically assuming A (in the case of Russell's antinomy: rules for introducing $\in$), which cannot be applied because their premisses cannot be assumed.

Ad (2): Given (i), we cannot derive $\neg A$, as in the derivation of $\neg A$ from (i), we have to use A as an unspecific assumption to be contracted with the specific assumption A in the derivation of (i).

Ad (3): Given (i), we cannot derive $\neg A$, as contraction is blocked anyway. This is an *a fortiori* consequence of the previous case.

Whereas strategy (1) presents a fresh look at antinomies based on the well-foundedness of assumption rules, strategies (2) and (3) challenge the logical step from the circular formula $A \leftrightarrow \neg A$ to the outright contradiction $A \wedge \neg A$ or to absurdity $\perp$ (in intuitionistic or minimal logic, of course). It should be remarked that, even without any restriction concerning assumptions and contraction, the natural deduction derivation from $A \leftrightarrow \neg A$ to $\perp$ (i.e., the derivation of $\neg(A \leftrightarrow \neg A)$ in propositional logic) has peculiar features and is by no means trivial (see Ekman 1998).

This is no solution to the antinomy problem (if there is a problem at all), but it illuminates certain logical, and especially proof-theoretic, aspects of circular reasoning which have not been studied very deeply so far. I conjecture that the phenomena mentioned are not restricted to particular antinomies such as Russell's but that something similar happens with most, if not all, mathematical and semantical antinomies.

References:

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