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The Calculus of Higher-Level Rules, Propositional Quantification, and the Foundational Approach to Proof-Theoretic Harmony

Abstract. We present our calculus of higher-level rules, extended with propositional quantification within rules. This makes it possible to present general schemas for introduction and elimination rules for arbitrary propositional operators and to define what it means that introductions and eliminations are in harmony with each other. This definition does not presuppose any logical system, but is formulated in terms of rules themselves. We therefore speak of a foundational (rather than reductive) account of proof-theoretic harmony. With every set of introduction rules a canonical elimination rule, and with every set of elimination rules a canonical introduction rule is associated in such a way that the canonical rule is in harmony with the set of rules it is associated with. An example given by Hazen and Pelletier is used to demonstrate that there are significant connectives, which are characterized by their elimination rules, and whose introduction rule is the canonical introduction rule associated with these elimination rules. Due to the availabiliy of higher-level rules and propositional quantification, the means of expression of the framework developed are sufficient to ensure that the construction of canonical elimination or introduction rules is always possible and does not lead out of this framework.

Keywords: Proof-theoretic semantics, Assumptions, Higher-level rules, Propositional quantification, Harmony.

1. Introduction

Both Gentzen's (1934/35) [11] calculus of natural deduction and Jaśkowski's (1934) [15] calculus of suppositions are based on what may be called the dynamic view of assumptions. According to the traditional static view, assumptions are suppositions on which all subsequent formulas in a derivation

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depend. According to the dynamic view assumptions made (or 'introduced') can be discharged (or 'eliminated') at the application of certain rules. The dynamic view of assumptions makes it possible to give implication a proof-theoretic meaning based on the idea that asserting an implication means the same as deriving its consequent from its antecedent.

This paper deals with the systematics of introduction and elimination rules in natural deduction and their relationship often described as 'harmony' (Dummett, 1973 [5]) and is therefore related to Gentzen's approach. However, it does so by using the idea of rules of higher levels, which extends the dynamic view of assumptions by not only allowing that assumptions be discharged but also that assumptions be introduced in the course (and not only at the top) of a derivation, where these assumptions are not necessarily formulas, but can be rules as well. This idea is here extended to include propositional quantification within rules, i.e. the idea that rules may universally quantify over propositions.

The idea to study propositional quantification occurs already in Jaśkowski's (1934) [15] paper on suppositions, but not in Gentzen's work on natural deduction. Jaśkowski studies propositional quantifiers before he passes on to first-order ones. However, unlike Jaśkowski, we do not use propositional quantification and propositional eigenvariables in rules in order to define propositional quantifiers, but in order to define propositional connectives. Thus one of our central claims is that propositional quantification is useful and indeed necessary to study propositional connectives from a general proof-theoretic perspective. This claim comes along with the view that introduction rules are not given priority over elimination rules, as in Gentzen's work, but that introductions and eliminations are treated on par. In fact, it will be the elimination-based approach where the idea of propositional quantification in rules develops its full power.

In Schroeder-Heister (2014b) [37] a notion of proof-theoretic harmony was proposed. There the meaning of a connective according to given introduction rules was described by a formula of second-order intuitionistic propositional logic **PL2**, and likewise for eliminaton rules. When the two formulas obtained were equivalent, introduction and elimination rules were said to be in harmony. This approach was called *reductive*, since it took the system **PL2** for granted, which means that it did not apply to operators such as conjunction, disjunction or implication, as they are already an ingredient of **PL2**[1]. Now we follow a *foundational* approach in that we shall define

[1]Since conjunction and disjunction are definable in **PL2**, only implication and propositional quantification are actually needed.

a notion of harmony between introduction and elimination rules which is exclusively defined in terms of the rules used rather than in terms of certain formulas of an external system.

Following the idea presented in Schroeder-Heister (1984b) [31], we shall define a purely structural calculus of rules, which is considered more elementary than any logical system, where these rules may now contain propositional quantification to express generality. If we just want to express, for example, that for any A and B, $A \wedge B$ can be inferred from A and B, we do not necessarily need propositional quantification. We can just use A and B as schematic letters in the rule schema

$$\frac{A \quad B}{A \wedge B} \ .$$

However, if we want to express that $\neg A$ can be inferred whenever, for any B, A entails B, we need some propositional variable-binding device. In the notation to be developed, we write such a rule two-dimensionally as

$$\frac{\begin{pmatrix} A \\ p \end{pmatrix}_p}{\neg A}$$

or linearly as

$$(A \Rightarrow_p p) \Rightarrow \neg A \ .$$

Formally this means that, in order to infer $\neg A$, it is sufficient to derive p from A and possibly further assumptions, where p must not occur free in A or any other assumption on which p depends.

When developing a general schema for elimination rules given certain introduction rules, no quantification is needed. If the introduction rules do not contain any quantification, neither does the general elimination rule, as quantification from outside can be expressed by schematic variables. However, when developing a general schema for introduction rules given certain elimination rules, we need this sort of quantification. Even without such a general schema, when defining harmony for arbitrary introduction and elimination rules, the availability of propositional quantification is crucial. Propositional quantification is a very elementary device in rule application, as it essentially relies on the proper handling of variables. However, it gives us powerful new structural means of expression.

When using second-order quantification, we are exposed to the objection of employing impredicative notions. It should be noted that our formulas do not contain any quantifiers, as quantification occurs only in rules and not in formulas. Of course, the quantified propositional variables run over propositional formulas which may contain connectives which are defined by using these rules. However, if one calls this way of defining a propositional connective impredicative, then it is impredicativity of a harmless sort. Then even the standard rule for disjunction elimination

$$\frac{A{\vee}B \quad \begin{matrix} A \\ C \end{matrix} \quad \begin{matrix} B \\ C \end{matrix}}{C}$$

would be impredicative, as the schematic letter C runs over arbitrary formulas, and thus in particular over $A{\vee}B$ itself. This sort of impredicativity is harmless, as we are using variables only in a schematic sense, corresponding to what Carnap (1931) [2] called "specific" (in contradistinction to "numeric") generality. The handling of quantified rules is described by the proper handling of variables in derivations.

In Section 2 we describe our calculus of rules as a system of natural deduction. It is essentially a pure calculus of suppositions, where the suppositions are of an extended form. As the presentation in Schroeder-Heister (1984b) [31], where rules were identified with the schema of their application, is difficult to read, we here provide a more elementary approach in which rules are expressions labelling their application in a derivation, and are not just extracted from a certain inference figure. One crucial point of our dealing with rules is the fact that rules are always *applied* in a derivation. They never occur as items that are *asserted*. Only formulas can be asserted. This *applicative behavior* is what makes rules a most fundamental entity, whose usage can be explained without recurring to logic and therefore can serve in a foundational approach to logic.

Section 3 proposes general notions of introduction and elimination rules. Introductions and eliminations are considered independent rules which are not expected to stand in any particular relation to each other. However, we define a *canonical* elimination rule given arbitrary introduction rules, and a *canonical* introduction rule given arbitrary elimination rules, which play a special role. The canonical elimination rule for given introduction rules is the uniform general elimination rule proposed in Schroeder-Heister (1984b) [31]. The canonical introduction rule for given elimination rules is a uniform general introduction rule.

In Section 4 we give a definition of harmony in terms of rules only. It is defined for any set of introductions and eliminations given. It can then be shown that the canonical elimination rule is in harmony with its associated introductions, and that the canonical introduction rule is in harmony with its associated eliminations. It is, however, crucial that our notion of harmony is not based on the construction of general canonical elimination or general canonical introduction rules. This makes our approach different from other approaches which base harmony on the canonical form of rules (normally elimination rules[2]).

The fact that the consideration of a general canonical introduction rule for given elimination rules is not just a theoretical possibility dealt with for reasons of symmetry, but has an intrinsic value, is demonstrated in Section 5 by using an example provided by Hazen and Pelletier (2014) [14]. It defines a theoretically significant (namely expressively complete) ternary connective in terms of its elimination rules, while its harmonious introduction is the canonical introduction rule.

In the final Section 6 we discuss the merits of the foundational approach proposed and outline some possible directions of further work.

2. The calculus of higher-level rules

The expressions of the languages we shall consider are built up using

- *propositional variables*, denoted by $p, q, r, \ldots$ (with and without indices),
- *logical constants*, which are
  - the *standard propositional connectives* $\rightarrow, \wedge, \vee, \bot, \top$,
  - further *propositional connectives* of various arities, denoted by c and by other symbols introduced ad hoc,
- the *rule arrow* $\Rightarrow$,
- auxiliary symbols such as commas and parentheses.

Formulas are formed in the usual way from propositional variables and propositional connectives. They are denoted by $A, B, C, \ldots$, with and without indices. A list $A_1, \ldots, A_n$ of formulas is also denoted by $\overrightarrow{A}$, a list $p_1, \ldots, p_n$ of variables by $\overrightarrow{p}$.When we use standard connectives, conjunction and disjunction are supposed to bind stronger than implication. By $A[p/B]$ we denote the substitution of B for p in A, and by $A[p_1, \ldots, p_n/B_1, \ldots, B_n]$ the simultaneous substitution of $B_1, \ldots, B_n$ for $p_1, \ldots, p_n$, respectively, in A. If $\overrightarrow{p}$ is $p_1, \ldots, p_n$ and $\overrightarrow{B}$ is $B_1, \ldots, B_n$, we also write $A[\overrightarrow{p}/\overrightarrow{B}]$, where, when using this notation, we always assume that $\overrightarrow{p}$ and $\overrightarrow{B}$ match in length. An n-ary connective c with arguments $A_1, \ldots, A_n$ is written as $c(A_1, \ldots, A_n)$.

Rules and their levels are defined as follows.

[2]Such as the programme described by Read as 'general-elimination-harmony', see Read (2010, 2014) [27, 28], Francez & Dyckhoff (2012) [10].

DEFINITION 1.

- Every formula A is a rule of level 0.
- For every formula A and every (possibly empty) list of propositional variables $p_1, \ldots, p_m$ ($m \geq 0$), the expression $\Rightarrow_{p_1, \ldots, p_m} A$ is a rule of level 0. The variables $p_1, \ldots, p_m$ are bound in A. If m is 0, i.e., if the list of variables is empty, then $\Rightarrow A$ is identified with A.
- If $R_1 \ldots, R_n$ are rules ($n \geq 1$), whose maximal level is ℓ, A a formula and $p_1, \ldots, p_m$ ($m \geq 0$) a (possibly empty) list of propositional variables, then $(R_1, \ldots, R_n \Rightarrow_{p_1, \ldots, p_m} A)$ is a rule of level $\ell + 1$. The variables $p_1, \ldots, p_m$ are bound in $R_1, \ldots, R_n, A$. If m is 0, i.e., if the list of variables is empty, we write $(R_1, \ldots, R_n \Rightarrow A)$.

Outer parentheses are omitted, if no misreading of rules can occur. Rules are denoted by R and R', with and without indices. Finite lists of rules are denoted by $\Gamma, \Delta, \ldots$, with and without primes and indices. We can thus write rules in the form $\Delta \Rightarrow_{p_1, \ldots, p_m} A$. Obviously, the general form of a rule is

$$(\Gamma_1 \Rightarrow_{\overrightarrow{q_1}} B_1), \ldots, (\Gamma_n \Rightarrow_{\overrightarrow{q_n}} B_n) \Rightarrow_{\overrightarrow{p}} A \ , \tag{I}$$

where $n \geq 0$ and $\overrightarrow{q_1}, \ldots, \overrightarrow{q_m}, \overrightarrow{p}$ are (possibly empty) lists of propositional variables. The variables occurring as indices to the rule arrow $\Rightarrow$ are bound in the premisses and the conclusion of the rule, so that the usual restrictions concerning substitutions apply.[3] When substituting a formula A for a variable p in a rule R, we always presuppose that A can be freely substituted for p in R, i.e., that A does not contain any variable q such that p occurs in R in the range of a rule arrow with index q.

The intended meaning of a rule of form (I) is the following: *For any* $\overrightarrow{p}$*: Suppose, for each* i ($1 \leq i \leq m$)*, we have derived* B_i *from* Γ_i*, where this derivation is schematic in* $\overrightarrow{q_i}$*; then we may pass over to* A*.* That the derivation of B_i from Γ_i is schematic in $\overrightarrow{q_i}$ will be expressed by an eigenvariable condition. That the rule can be applied for any $\overrightarrow{p}$ will be expressed by allowing for arbitrary substitutions of lists of formulas for $\overrightarrow{p}$. According to this reading the variables occurring as indices to the rule arrow $\Rightarrow$ function as universal quantifiers. If such variables are present, we speak of *quantified* (higher-level) rules, or of (higher-level) rules *with quantification.*

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[3] This notation, according to which indices to the rule arrow bind variables, is due to Lorenzen (1955) [17], corresponding to the notation for universally quantified implications in Principia Mathematica (Whitehead & Russell (1910) [40], Vol. I, Intro., Ch.I), where it was adopted from Peano.

Formally the intended meaning of a rule will be explained by giving a schema according to which a rule of the form (I) is applied in a derivation.

First we define structural derivations which are derivations generated without any primitive rule of inference. They are defined for any language, whether it contains logical constants or not (in the latter case propositional variables are the only formulas). In such a derivation a rule of form (I) can be applied as an *assumption rule*, i.e., by using it as an assumption on which the subsequent derivation depends. In standard natural deduction, there is just a single purely structural way of assuming something, namely by setting a formula

$$A$$

as an assumption to start with. In our framework, assumptions are always *rules*. Thus, we would write the assumption of A as

$$\dfrac{}{A}\,A$$

expressing that we are introducing the rule A as an assumption, by means of which we can infer A. Obviously, this is just a notational variant of just writing A. However, unlike standard natural deduction, we can extend this notation to arbitrarily complex rules as assumptions. For example,

$$\dfrac{\begin{matrix}\vdots & & \vdots\\ B_1 & \ldots & B_n\end{matrix}}{A}\;B_1,\ldots,B_n \Rightarrow A$$

expresses that we are introducing the rule $B_1,\ldots,B_n \Rightarrow A$ as an *assumption rule*, by applying it to $B_1,\ldots,B_n$ in order to obtain A. Here we are introducing an assumption not at the top, but in the course of a derivation, after having derived its premisses $B_1,\ldots,B_n$. More complicated rules allow one, by introducing them as assumptions, to discharge previously introduced assumption rules, so that subsequent formulas no longer depend on it. In the simplest case a level-0 rule, i.e. a formula, is discharged, as in the following example:

$$1\,\dfrac{\begin{matrix} & \dfrac{}{C}\,[C]^1\\ \vdots & \vdots\\ B_1 & B_2\end{matrix}}{A}\;B_1,(C \Rightarrow B_2) \Rightarrow A \quad .$$

Here the level-2 rule $B_1,(C \Rightarrow B_2) \Rightarrow A$ is introduced as an assumption allowing one to pass over from B_1 and B_2 to A, whereby the level-0 rule C is discharged, which is indicated by the square brackets around C and the numeral 1, telling at which step C is discharged. In the following more complicated case, a level-1 rule is discharged:

$$
{\scriptstyle 1}\,\dfrac{\begin{matrix}\vdots \\ B_1\end{matrix} \qquad \begin{matrix}\dfrac{\begin{matrix}\vdots\\ D_1\end{matrix}\quad\begin{matrix}\vdots\\ D_2\end{matrix}}{C}\;[D_1,D_2\Rightarrow C]^1 \\ \vdots \\ B_2\end{matrix}}{A}\; B_1,((D_1,D_2\Rightarrow C)\Rightarrow B_2)\Rightarrow A \quad .
$$

Here the level-3 rule $B_1,((D_1,D_2\Rightarrow C)\Rightarrow B_2)\Rightarrow A$ is introduced as an assumption allowing one to pass over from B_1 and B_2 to A, whereby a previous application of the level-1 rule $D_1,D_2\Rightarrow C$ is discharged, as indicated by the numeral 1. This level-1 rule $D_1,D_2\Rightarrow C$ had been used to pass over from D_1 and D_2 to C.[4]

Using quantification we further generalize this idea. The derivations stated so far were given for particular formulas $A,B_1,B_2,\ldots$. Considering them as schematic letters for arbitrary formulas, the above derivations become derivation schemas. However, in this sense we just have quantification from outside, similar to the understanding of a mathematical equation $x+y=y+x$ as its universal closure. By attaching variables to the rule arrow $\Rightarrow$, we can consider specific ways of generalizing derivations. For example,

$$
{\scriptstyle 1}\,\dfrac{\begin{matrix}\vdots \\ B_1\end{matrix} \qquad \begin{matrix}\dfrac{\begin{matrix}\vdots\\ D_1\end{matrix}\quad\begin{matrix}\vdots\\ D_2\end{matrix}}{C}\;[D_1,D_2\Rightarrow C]^1 \\ \vdots \\ B_2\end{matrix}}{A}\; q_1,((s_1,s_2\Rightarrow r)\Rightarrow q_2)\Rightarrow_{pq_1q_2rs_1s_2} p
$$

gives us a derivation, in which the rule $q_1,((s_1,s_2\Rightarrow r)\Rightarrow q_2)\Rightarrow_{pq_1q_2rs_1s_2} p$ is introduced as an assumption. The variables p,q_1,q_2,r,s_1,s_2, which are attached as an index to the rule arrow, are understood universally. This means that they must be replaced with specific formulas, when the rule is applied. Here they are replaced with A,B_1,B_2,C,D_1,D_2, respectively. Now we further generalize this idea by considering index variables to rule arrows

[4] What is discharged are always specific *applications* of a rule rather than the rule itself, which is similar to standard natural deduction, where formula *occurrences* rather than formulas themselves are discharged. However, when the context is clear, we often speak more sloppily of rules as being discharged.

within the left side of another rule:

$$1\ \dfrac{\begin{array}{c}\vdots\\ B_1\end{array} \qquad \begin{array}{c}\dfrac{\begin{array}{cc}\vdots & \vdots\\ q_2 & D_2\end{array}}{C}\,[q_2, D_2 \Rightarrow C]^1 \\ \vdots \\ q_2\end{array}}{A}\ q_1, ((q_2, D_2 \Rightarrow r) \Rightarrow_{q_2} q_2) \Rightarrow_{pq_1r} p \quad .$$

Here the assumption rule $q_1, ((q_2, D_2 \Rightarrow r) \Rightarrow_{q_2} q_2) \Rightarrow_{pq_1r} p$ is universally quantified only with respect to p, q_1, r. These variables are replaced with A, B_1, C, respectively. Furthermore, the index q_2 to the rule arrow of the second premiss $(q_2, D_2 \Rightarrow r) \Rightarrow_{q_2} q_2$ of this rule indicates that q_2 is an eigenvariable, which is not allowed to occur in any assumption beyond $q_2, D_2 \Rightarrow C$ in the right branch of the derivation. An example in which all rule arrows of a level-3 assumption rule are quantified (i.e., receive an index), is the following:

$$1\ \dfrac{\begin{array}{c}\vdots\\ B_1\end{array} \qquad \begin{array}{c}\dfrac{\begin{array}{cc}\vdots & \vdots\\ q_2 & D_2\end{array}}{C}\,[q_2, D_2 \Rightarrow_r r]^1 \\ \vdots \\ q_2\end{array}}{A}\ q_1, ((q_2, D_2 \Rightarrow_r r) \Rightarrow_{q_2} q_2) \Rightarrow_{pq_1} p \quad .$$

Here the rule $q_1, ((q_2, D_2 \Rightarrow_r r) \Rightarrow_{q_2} q_2) \Rightarrow_{pq_1} p$, which is introduced as an assumption, is quantified with respect to p and q_1. These variables are replaced with A and B_1, respectively. In the application of this rule, the variable q_2 is functioning as an eigenvariable, which is not allowed to occur in any open assumption above the second premiss q_2 apart from $q_2, D_2 \Rightarrow_r r$. This latter assumption, which was introduced and applied by substituting C for r, is discharged as this step.

The following definition of a *derivation* makes this fully precise.

Definition 2. For a level-0 rule A (i.e. a formula),

$$\dfrac{}{A}\, A$$

is a derivation of A depending on $\{A\}$.

For a level-0 rule $\Rightarrow_{\overrightarrow{p}} A$ and any list $\overrightarrow{C}$ of formulas,

$$\dfrac{}{A[\overrightarrow{p}/\overrightarrow{C}]} \Rightarrow_{\overrightarrow{p}} A$$

is a derivation of $A[\overrightarrow{p}/\overrightarrow{C}]$ depending on $\{\Rightarrow_{\overrightarrow{p}} A\}$.

Now consider a level-$(\ell+1)$ rule $(\Gamma_1 \Rightarrow_{\overrightarrow{q_1}} B_1), \ldots, (\Gamma_n \Rightarrow_{\overrightarrow{q_n}} B_n) \Rightarrow_{\overrightarrow{p}} A$. Suppose that, for each i $(1 \leq i \leq n)$ and each $\overrightarrow{C}$, a derivation

$$\begin{array}{c} \Gamma_i[\overrightarrow{p}/\overrightarrow{C}] \\ \vdots \\ B_i[\overrightarrow{p}/\overrightarrow{C}] \end{array}$$

of $B_i[\overrightarrow{p}/\overrightarrow{C}]$ depending on a set Σ of rules is given, where Σ may contain some or all of the rules in $\Gamma_i[\overrightarrow{p}/\overrightarrow{C}]$. Suppose furthermore that the variables in $\overrightarrow{q_i}$ do not occur in any element of Σ apart from the elements of $\Gamma_i[\overrightarrow{p}/\overrightarrow{C}]$. Then

$$k\,\frac{\begin{array}{c}[\ \Gamma_1[\overrightarrow{p}/\overrightarrow{C}]\ ]^k \\ \vdots \\ B_1[\overrightarrow{p}/\overrightarrow{C}]\end{array} \quad \ldots \quad \begin{array}{c}[\ \Gamma_n[\overrightarrow{p}/\overrightarrow{C}]\ ]^k \\ \vdots \\ B_n[\overrightarrow{p}/\overrightarrow{C}]\end{array}}{A[\overrightarrow{p}/\overrightarrow{C}]}\,(\Gamma_1 \Rightarrow_{\overrightarrow{q_1}} B_1), \ldots, (\Gamma_n \Rightarrow_{\overrightarrow{q_n}} B_n) \Rightarrow_{\overrightarrow{p}} A \tag{II}$$

is a derivation of $A[\overrightarrow{p}/\overrightarrow{C}]$ depending on $\Sigma'_1 \cup \ldots \cup \Sigma'_n$, where Σ'_i results from Σ_i by deleting any number of elements of $\Gamma_i[\overrightarrow{p}/\overrightarrow{C}]$. The elements deleted ('discharged') are put into square brackets, and the brackets are linked with a fresh number k to the inference line at which they are discharged.

A derivation of A from Σ is a derivation of A depending on a subset of Σ. If there is a derivation of A from Σ, then A is derivable from Σ, symbolically: $\Sigma \vdash A$.

This notion of derivation and derivability is only based on the intended meaning of rules and not relative to a particular language and to given primitive rules of inference. In this sense one might speak of *structural derivations* and *structural derivability*.

If certain rules are distinguished as primitive rules of inference in a formal system K, then derivability in K is defined as follows.

DEFINITION 3. A derivation of A from Σ in K is a (structural) derivation of A from some set $\Sigma \cup \Sigma'$, such that Σ' only contains primitive rules of K. For derivability in K, we write, as usual, $\Sigma \vdash_K A$.

For primitive rules we also use a two-dimensional notation, which is often better readable than the 'official' one-dimensional one. Instead of

$$(\Gamma_1 \Rightarrow_{\overrightarrow{q_1}} B_1), \ldots, (\Gamma_n \Rightarrow_{\overrightarrow{q_n}} B_n) \Rightarrow_{\overrightarrow{p}} A,$$

where $\overrightarrow{p}$ comprises *all* variables free in $(\Gamma_1 \Rightarrow_{\overrightarrow{q_1}} B_1), \ldots, (\Gamma_n \Rightarrow_{\overrightarrow{q_n}} B_n) \Rightarrow A$, we also write:

$$\frac{\begin{pmatrix}\Gamma_1\\ B_1\end{pmatrix}_{\overrightarrow{q_1}} \quad \ldots \quad \begin{pmatrix}\Gamma_n\\ B_n\end{pmatrix}_{\overrightarrow{q_n}}}{A},$$

where the parentheses can be omitted when $\overrightarrow{q_n}$ is empty. Our proviso concerning the variables $\overrightarrow{p}$ means in effect that at the inference line all variables free above or below become bound. In other words, we only consider primitive inference rules without free variables. This convention is fully appropriate and sufficient for the context of this paper. Note that in our two-dimensional notation for rules there is no need to use any square brackets to indicate that assumptions can be discharged, since *all* assumptions displayed may be discharged.

An example of a formal system is intuitionistic propositional logic **PL**, which has the following primitive rules of inference:

$(\wedge\,\mathrm{I})$	$p, q \Rightarrow_{pq} p\wedge q$	$(\wedge\,\mathrm{E})$	$p\wedge q \Rightarrow_{pq} p \qquad p\wedge q \Rightarrow_{pq} q$
$(\vee\,\mathrm{I})$	$p \Rightarrow_{pq} p\vee q \qquad q \Rightarrow_{pq} p\vee q$	$(\vee\,\mathrm{E})$	$p\vee q, (p \Rightarrow r), (q \Rightarrow r) \Rightarrow_{pqr} r$
$(\rightarrow\mathrm{I})$	$(p \Rightarrow q) \Rightarrow_{pq} p \rightarrow q$	$(\rightarrow\mathrm{E})$	$(p \rightarrow q), p \Rightarrow_{pq} q$
$(\bot\,\mathrm{I})$	(none)	$(\bot\,\mathrm{E})$	$\bot \Rightarrow_p p$
$(\top\,\mathrm{I})$	$\Rightarrow \top$	$(\top\,\mathrm{E})$	(none) ,

where for absurdity introduction and triviality elimination we could have used instead the rules

$$(\bot\,\mathrm{I})' \quad (\Rightarrow_p p) \Rightarrow \bot \qquad\qquad (\top\,\mathrm{E})' \quad \top, p \Rightarrow_p p \quad .$$

In two-dimensional notation these rules look more common:

$$(\wedge\,\mathrm{I}) \quad \frac{p \quad q}{p\wedge q} \qquad\qquad (\wedge\,\mathrm{E}) \quad \frac{p\wedge q}{p} \qquad \frac{p\wedge q}{q}$$

$$(\vee\,\mathrm{I}) \quad \frac{p}{p\vee q} \qquad \frac{q}{p\vee q} \qquad\qquad (\vee\,\mathrm{E}) \quad \frac{p\vee q \quad \begin{matrix}p\\ r\end{matrix} \quad \begin{matrix}q\\ r\end{matrix}}{r}$$

$$(\rightarrow\mathrm{I}) \quad \frac{\begin{matrix}p\\ q\end{matrix}}{p \rightarrow q} \qquad\qquad (\rightarrow\mathrm{E}) \quad \frac{p \rightarrow q \quad p}{q}$$

$$(\bot\,\mathrm{I}) \quad \text{(none)} \qquad\qquad (\bot\,\mathrm{E}) \quad \frac{\bot}{p}$$

$$(\top\,\mathrm{I}) \quad \top \qquad\qquad (\top\,\mathrm{E}) \quad \text{(none)}$$

with the alternative forms

$$(\bot\,\mathrm{I})' \quad \frac{\left(\, p \,\right)_p}{\bot} \qquad\qquad (\top\,\mathrm{E})' \quad \frac{\top \quad p}{p}$$

of absurdity introduction and triviality elimination.

We have defined derivations from rules as assumptions. We have not defined, what it means to derive a rule. What is derived are always formulas. Rules only occur as labels on the right side of inference lines. It is possible to extend the calculus such as to allow for the explicit introduction of rules as statements that can be derived. We do not follow this idea here (see Schroeder-Heister, 1987 [32]), but define the derivability of a rule as a metalinguistic abbreviation. This is a conceptual decision, as we want to base our system on the applicative behavior of rules and not on some more general notion of implication (see the discussion in Section 6).

Definition 4. A rule of the form (I) is derivable, if

$$(\Gamma_1 \Rightarrow_{\overrightarrow{q_1}} B_1), \ldots, (\Gamma_n \Rightarrow_{\overrightarrow{q_n}} B_n) \vdash A\ . \tag{III}$$

More generally, a rule $\Gamma \Rightarrow_{\overrightarrow{p}} A$ is derivable from a set Σ of rules, if $\Sigma, \Gamma \vdash A$, provided that no variable of $\overrightarrow{p}$ is free in Σ. (If a variable of $\overrightarrow{p}$ is free in Σ, we could instead require that $\Sigma, \Gamma[\overrightarrow{p}/\overrightarrow{q}] \vdash A[\overrightarrow{p}/\overrightarrow{q}]$ for an appropriate list of fresh variables $\overrightarrow{q}$.)

When a rule R is derivable from Σ, we also write, as a metalinguistic abbreviation, $\Sigma \vdash R$.

Obviously, (III) establishes that the conclusion of the rule (I) is derivable from its premisses. Conversely, we obtain (III) by using (I) as an additional assumption, i.e. we can show

$$((\Gamma_1 \Rightarrow_{\overrightarrow{q_1}} B_1), \ldots, (\Gamma_n \Rightarrow_{\overrightarrow{q_n}} B_n) \Rightarrow_{\overrightarrow{p}} A), (\Gamma_1 \Rightarrow_{\overrightarrow{q_1}} B_1), \ldots, (\Gamma_n \Rightarrow_{\overrightarrow{q_n}} B_n) \vdash A \tag{IV}$$

Thus (IV) can be seen as expressing the reflexivity statement between rules: From each rule R this very same rule R can be derived: $R \vdash R$. Using this notation, we can also prove transitivity in the sense that $\Sigma \vdash R$ and $\Sigma, R \vdash A$ implies $\Sigma \vdash A$. We skip the proofs of these two facts here and note them as results[5]

[5] For the case without quantifiers, the proofs of reflexivity and transitivity can be found in Schroeder-Heister (1984b) [31].

LEMMA 1. *For any formula A, rule R and set of rules Σ:*
(i) $R \vdash R$
(ii) *If* $\Sigma \vdash R$ *and* $\Sigma, R \vdash A$ *then* $\Sigma \vdash A$.

Corresponding to the notation for the derivability of rules, we use, as a notational convention,

$$\Gamma \Rightarrow_{\overrightarrow{p}} R$$

to stand for

$$\Gamma, \Delta \Rightarrow_{\overrightarrow{p}\,\overrightarrow{q}} A$$

if R has the form $\Delta \Rightarrow_{\overrightarrow{q}} A$, provided that the lists of variables $\overrightarrow{p}$ and $\overrightarrow{q}$ are disjoint.

3. Introduction and elimination rules

We now present a general schema for introduction rules for n-ary propositional connectives, as well as a general schema for elimination rules for such connectives. These schemas are not related to each other, but are independent. They represent what, in an intuitively plausible way, should count as introduction and elimination rules. Afterwards, we shall define the *canonical* elimination rule corresponding to arbitrary introduction rules, and, conversely, the *canonical* introduction rule corresponding to arbitrary elimination rules. The canonical elimination rule is what in Schroeder-Heister (1984b) [31] was called the *generalized* elimination rule for n-ary propositional connectives. The canonical elimination rule is related to given introduction rules in a harmonious way, and the canonical introduction rule is related to given elimination rules in a harmonious way. The notion of harmony is defined independently of the relationship between introductions and canonical eliminations or between eliminations and canonical introductions, even though these relationships represent a particularly important kind of harmony. This makes our approach to harmony differ from common approaches that define harmony (usually of eliminations in relation to introductions) through the canonical form of rules (normally some schema for generalized elimination rules). Unlike previous publications, we now define harmony directly in terms of the rules used to define a connective rather than via a translation into a logical system of second-order intuitionistic propositional logic. In this sense the current approach is foundational rather than reductive. As explained in the discussion in Section 6, this is considered to be a substantial gain from the conceptual point of view. We shall deal with propositional operators only, i.e. with n-ary connectives defined by means of rules that use propositional quantification of the form $\Rightarrow_{\overrightarrow{p}}$.

Suppose c is an n-ary connective. We write $c(\overrightarrow{p})$ for $c(p_1, \ldots, p_n)$. Then an introduction rule for c has the following form:

$$(c\,\mathrm{I}) \qquad (\Gamma_1 \Rightarrow_{\overrightarrow{q_1}} B_1), \ldots, (\Gamma_m \Rightarrow_{\overrightarrow{q_m}} B_m) \Rightarrow_{\overrightarrow{p}\,\overrightarrow{q}} c(\overrightarrow{p})\ . \tag{V}$$

Here $\overrightarrow{q}$ contains all propositional variables free in $(\Gamma_1 \Rightarrow_{\overrightarrow{q_1}} B_1), \ldots, (\Gamma_m \Rightarrow_{\overrightarrow{q_m}} B_m)$ apart from those in $\overrightarrow{p}$. This notation is supposed to cover the case where, for some or all i, Γ_i and/or $\overrightarrow{q_i}$ are lacking. As a limiting case, m can be 0, in which case the rule has no premisses such as the introduction rule $(\top\,\mathrm{I})$ for triviality. To avoid notational overhead, we often write (V) as

$$\Delta \Rightarrow_{\overrightarrow{p}\,\overrightarrow{q}} c(\overrightarrow{p})$$

where, according to our conventions, Δ stands for a list of rules. In two-dimensional notation, the rule (V) is written as

$$\frac{\begin{pmatrix}\Gamma_1 \\ B_1\end{pmatrix}_{\overrightarrow{q_1}} \quad \ldots \quad \begin{pmatrix}\Gamma_m \\ B_m\end{pmatrix}_{\overrightarrow{q_m}}}{c(\overrightarrow{p})}\ . \tag{VI}$$

For the sake of simplicity, we assume that above the inference line *no* connective is allowed to occur, which means in particular that $B_1, \ldots, B_m$ are propositional variables. One might consider definitional chains according to which a connective can rely on other connectives for which rules have been given beforehand, for example in the definition of negation in terms of absurdity:

$$\frac{\begin{matrix} p \\ \bot \end{matrix}}{\neg p}\ ,$$

or, in even more advanced settings, an introduction rule in which $c(\overrightarrow{p})$ depends on itself in a self-referential way — cases, which we do not discuss here (see Section 6).

The idea behind the schema (V)/(VI) is obvious: The introduction rule for c should allow one to infer $c(\overrightarrow{p})$ from a given list of premisses in a uniform way. The introduction rules $(\wedge\,\mathrm{I})$, $(\vee\,\mathrm{I})$, $(\to\mathrm{I})$, $(\bot\,\mathrm{I})'$ and $(\top\,\mathrm{I})$ are all of the form (V). As one often wants to associate not exactly one, but two or more, or, as a limiting case, even no introduction rule with c, we speak of the *list of introduction rules associated with* c or of the c*-introductions*:

$$\begin{cases} \Delta_1 & \Rightarrow_{\overrightarrow{p}\,\overrightarrow{q_1}} & c(\overrightarrow{p}) \\ & \vdots & \\ \Delta_k & \Rightarrow_{\overrightarrow{p}\,\overrightarrow{q_k}} & c(\overrightarrow{p}) \end{cases} . \tag{VII}$$

Here k can be any number ≥ 0. If $k = 0$, this list is empty, which covers the case of $(\bot\, \mathrm{I})$.

Given a list of introduction rules of the form (VII), the *canonical elimination rule* for c is the following rule:

$$c(\overrightarrow{p}), (\Delta_1 \Rightarrow_{\overrightarrow{q_1}} r), \ldots, (\Delta_k \Rightarrow_{\overrightarrow{q_k}} r) \;\Rightarrow_{\overrightarrow{p}\,r}\; r \tag{VIII}$$

where r is a fresh variable not occurring in $\overrightarrow{p}, \overrightarrow{q_1}, \ldots, \overrightarrow{q_k}$. In two-dimensional notation this rule reads as

$$\frac{c(\overrightarrow{p}) \quad \begin{pmatrix} \Delta_1 \\ r \end{pmatrix}_{\overrightarrow{q_1}} \quad \ldots \quad \begin{pmatrix} \Delta_k \\ r \end{pmatrix}_{\overrightarrow{q_k}}}{r} . \tag{IX}$$

The canonical elimination rule for c says that everything that can be derived from each introduction premiss of $c(\overrightarrow{p})$ can be derived from $c(\overrightarrow{p})$ itself. It is a single rule of a special form which is harmonious with the given c-introductions in the sense to be specified in Section 4. For example, $(\vee\, \mathrm{E})$ is the canonical elimination rule for disjunction given $(\vee\, \mathrm{I})$ as its introduction rules, $(\bot\, \mathrm{E})$ is the canonical elimination rule for absurdity given the empty list $(\bot\, \mathrm{I})$ as its introduction, and $(\top\, \mathrm{E})'$ is the canonical elimination rule for triviality given $(\top\, \mathrm{I})$ as its introduction. The canonical elimination rules for conjunction and implication with $(\wedge\, \mathrm{I})$ and $(\rightarrow \mathrm{I})$ as their introductions are given by their higher-level forms

$$\frac{p \wedge q \qquad \begin{matrix} p\ ,\ q \\ r \end{matrix}}{r} \quad \text{and} \quad \frac{p \rightarrow q \qquad \begin{matrix} p \Rightarrow q \\ r \end{matrix}}{r} \quad \text{, respectively.}$$

The canonical elimination rule for absurdity corresponding to the introduction rule $(\bot\, \mathrm{I})'$ has the form

$$\frac{\bot \qquad \begin{matrix} \Rightarrow_p p \\ r \end{matrix}}{r} .$$

Next we define the general form of an elimination rule for c as follows:

$$c(\overrightarrow{p}), (\Gamma_1 \Rightarrow_{\overrightarrow{q_1}} B_1), \ldots, (\Gamma_\ell \Rightarrow_{\overrightarrow{q_\ell}} B_\ell) \ \Rightarrow_{\overrightarrow{p}\,\overrightarrow{q}} C \tag{X}$$

where $\overrightarrow{q}$ comprises all variables beyond $\overrightarrow{p}$ which are free in $\Gamma_1, \ldots, \Gamma_\ell, B_1, \ldots, B_\ell, C$. In two-dimensional notation, this rule reads as:

$$\frac{c(\overrightarrow{p}) \qquad \begin{pmatrix}\Gamma_1 \\ B_1\end{pmatrix}_{\overrightarrow{q_1}} \quad \ldots \quad \begin{pmatrix}\Gamma_\ell \\ B_\ell\end{pmatrix}_{\overrightarrow{q_\ell}}}{C} \ . \tag{XI}$$

As with the introduction rules, we confine ourselves to the case where this rule contains besides the displayed occurrence of c no other occurrence of a connective, which in particular means that $B_1, \ldots, B_\ell, C$ must be propositional variables. The idea behind this general schema is that an elimination rule for c states that together with certain additional premisses, whose form is not restricted, a certain conclusion can be drawn from $c(\overrightarrow{p})$. As a limiting case, we allow for the possibility that ℓ is 0, i.e. that there are no additional ('minor') premisses. Obviously, the elimination rules ($\wedge$ E), ($\vee$ E), ($\rightarrow$ E), ($\bot$ E) and ($\top$ E)$'$ are all of the form (XI). The canonical elimination rule (IX) is of this form as well. Note, however, that our schema (XI) is more general than the canonical elimination rule, as $B_1, \ldots, B_\ell$ need not be equal and can differ from C. It is thus able to cover, e.g., the elimination rules ($\wedge$ E) for conjunction and ($\rightarrow$ E) for implication, whereas, of the standard connectives, the canonical rule (IX) only covers disjunction, absurdity and triviality (with the redundant form ($\top$ E)$'$).

To avoid notational overhead, we write (X) also as

$$c(\overrightarrow{p}) \Rightarrow_{\overrightarrow{p}} R$$

where R stands for $(\Gamma_1 \Rightarrow_{\overrightarrow{q_1}} B_1), \ldots, (\Gamma_\ell \Rightarrow_{\overrightarrow{q_\ell}} B_\ell) \ \Rightarrow_{\overrightarrow{q}} C$, in accordance with the notational conventions made at the end of Section 2. As one often wants to associate not exactly one, but two or more, or, as a limiting case, even no elimination rule with c, we speak of the *list of elimination rules associated with* c or of the *c-eliminations*

$$\left\{ \begin{array}{lll} c(\overrightarrow{p}) & \Rightarrow_{\overrightarrow{p}} & R_1 \\ & \vdots & \\ c(\overrightarrow{p}) & \Rightarrow_{\overrightarrow{p}} & R_k \end{array} \right. \ . \tag{XII}$$

Here k can be any number ≥ 0. If $k = 0$, this list is empty, which covers the case of triviality elimination ($\top$ E). With respect to this notation for elimination rules, the rules R_i are sometimes referred to as the *conclusions* of the elimination rules for c.

Given a list of elimination rules of the form (XII), the *canonical introduction rule* for c is the following rule:

$$R_1, \ldots, R_k \ \Rightarrow_{\overrightarrow{p}} c(\overrightarrow{p}) \ . \tag{XIII}$$

If R_i has the form $\Gamma_i \Rightarrow_{\overrightarrow{q_i}} B_i$ we obtain, in two-dimensional notation, the following schema:

$$\frac{\begin{pmatrix}\Delta_1 \\ B_1\end{pmatrix}_{\overrightarrow{q_1}} \quad \ldots \quad \begin{pmatrix}\Delta_k \\ B_k\end{pmatrix}_{\overrightarrow{q_k}}}{c(\overrightarrow{p})} \ . \tag{XIV}$$

The canonical introduction rule for c says that the conclusions of all eliminations taken together suffice to derive $c(\overrightarrow{p})$. It is a single rule of a special form which is harmonious with the given c-introductions in a sense to be specified in Section 4. In any case it is a legitimate introduction rule for c following the schema (V)/(VI).

For example, ($\wedge$ I) is the canonical introduction rule for conjunction given ($\wedge$ E) as its elimination rules. Similarly, ($\rightarrow$ I) is the canonical introduction rule for implication given modus ponens ($\rightarrow$ E) as its elimination rule. Also ($\bot$ I)$'$ is the canonical introduction rule for absurdity given ($\bot$ E) as its elimination rule, and ($\top$ I) is the canonical introduction rule for triviality assuming that its list of eliminations ($\top$ E) is empty. The canonical introduction rule for disjunction given ($\vee$ E) as its elimination rule has the form

$$\frac{\begin{pmatrix}(p \Rightarrow r), (q \Rightarrow r) \\ r\end{pmatrix}_r}{p \vee q} \ .$$

As another example, let the elimination rules for a 3-place connective $\oslash$ be given as

$$\frac{\oslash(p_1, p_2, p_3) \quad \begin{matrix} p_1 \\ r \end{matrix} \quad \begin{matrix} p_2 \\ r \end{matrix}}{r} \qquad \frac{\oslash(p_1, p_2, p_3)}{p_3} \ , \tag{XV}$$

then the corresponding harmonious introduction rule according to (XIV) is

$$\frac{\begin{pmatrix}(p_1 \Rightarrow r), (p_2 \Rightarrow r) \\ r\end{pmatrix}_r \quad p_3}{\oslash(p_1, p_2, p_3)} \ . \tag{XVI}$$

As well as disjunction, this example shows that in order to formulate the canonical introduction rule to given elimination rules, we need the machinery of rules with quantified variables, even though in these cases the given elimination rules do not contain quantified variables beyond $p_1, \ldots, p_n$. This is a crucial point. In order to generate introduction rules from elimination rules in a canonical way, propositional quantification at the rule level is indispensable.

We fix our conventions by a formal definition.

DEFINITION 5. Suppose c is an n-ary connective.

Every rule of the form (V), in two-dimensional notation (VI), is called an introduction rule for c.

Given a list of introduction rules for c of the form (VII), the rule (VIII), in two-dimensional notation (IX), is called the canonical elimination rule corresponding to this list of introduction rules.

Every rule of the form (X), in two-dimensional notation (XI), is called an elimination rule for c.

Given a list of elimination rules for c of the form (XII), the rule (XIII), in two-dimensional notation (XIV), is called the canonical introduction rule corresponding to this list of elimination rules.

4. Harmony

We have defined what an introduction and an elimination rule should look like. We have not said anything about the possible relationship between them, in particular the relation of a perfect fit called "harmony". For example, the introduction rule

$$\frac{p}{p \bullet q}$$

and the elimination rule

$$\frac{p \bullet q}{q}$$

of a binary *tonk*-like operator $\bullet$ (see Prior, 1960 [26]) are instances of our schema for introductions (VI) and eliminations (XI), respectively. However, there is no harmonious relationship between these rules in the sense that the elimination rule can be considered to be appropriate for the given introduction rule, since, by first introducing $\bullet$ and then eliminating it, we are able to derive any B from any A, which trivializes the system.

Similarly, the introduction rule

$$\frac{p \qquad q}{p \mathbin{\bigcirc\!\!\!\!\wedge}\, q}$$

and the elimination rule

$$\frac{p \owedge q}{p}$$

of a binary operator $\owedge$ are instances of our schema for introductions (VI) and eliminations (XI), respectively. However, there is no harmonious relationship between these rules in the sense that the introduction rule can be considered to be appropriate for the given elimination rule, since from the result of eliminating $\owedge$ we cannot introduce $\owedge$ again. This means that potential information contained in the proposition $A \owedge B$ is irretrievably lost when eliminating $\owedge$, if A is different from B. Each of these two connectives misses out on one of the two criteria we shall consider to be the basic ingredients of harmony. The first we call the *criterion of reduction*, the second one the *criterion of recovery.* Rougly speaking, the reduction criterion says that an introduction followed by an elimination is redundant, and the recovery criterion says that eliminations followed by introductions can restore what we started with.

We shall say that given introduction and given elimination rules are in harmony with each other, when these two criteria are met. In particular, this will be the case for the canonical elimination rule in relation to given introduction rules, as well as for the canonical introduction rule in relation to given elimination rules. However, we do not *define* harmony by reference to the canonical elimination or introduction rules, but *demonstrate* that these canonical rules satisfy our notion of harmony. In other words, the idea of a canonical elimination rule for given introduction rules, or of a canonical introduction rule for given elimination rules is not needed to define harmony. Thus our approach differs from many approaches discussed in the literature, where harmony is defined by using canonical rules, where usually canonical elimination rules are considered (see footnote 2). It also differs from approaches such as Prawitz's that justify inference rules via a notion of validity (see the discussion in Section 6 on that point).

As to the criterion of reduction, when eliminating c, we should not be able obtain more than what we needed to introduce c. This corresponds to Prawitz's (1965) [23] idea of a reduction step: The introduction of c followed by its elimination can be seen as a detour that should be eliminable. In our framework, we describe this as follows. Suppose that for c a list of introduction rules of the form (VII) and a list of elimination rules of the form (XII) are given. Then we require that from Δ_i we can derive R_j for all i, j:

$$\Delta_i \vdash R_j \text{ for all } i, j \quad (1 \leq i \leq m, 1 \leq j \leq k) , \tag{XVII}$$

i.e., from every sufficient condition for c we must be able to derive every consequence of c. Note that, in (XVII), $\vdash$ refers to derivability without any primitive rule, i.e., R_j must be derived from the rules in Δ_i alone.

Definition 6. A pair, consisting of a list of introduction rules of the form (VII) and a list of elimination rules of the form (XII), satisfies the criterion of reduction, if (XVII) holds.

As to the criterion of recovery, when eliminating c, we should not loose any information. This means that the consequences of c should suffice to get back to c. This does not necessarily mean that we should be able to obtain from the consequences $R_1, \ldots, R_k$ of c the premisses Δ_i of one of the introduction rules for c. As the introduction rules for c are alternative possibilities to arrive at $c(p_1, \ldots, p_n)$, each of them can be stronger than their conclusion $c(p_1, \ldots, p_n)$. For example, in the case of disjunction, the premisses of each introduction rule for $p_1 \vee p_2$, namely p_1 and p_2, are each stronger than $p_1 \vee p_2$. Instead we require that the introduction rules for c allow us to restore $c(\overrightarrow{p})$ from the consequences which $c(\overrightarrow{p})$ has according to its elimination rules. Formally this means that from $R_1, \ldots, R_k$ (the consequences of $c(\overrightarrow{p})$) and $(\Delta_1 \Rightarrow_{\overrightarrow{p}\,\overrightarrow{q_1}} c(\overrightarrow{p})), \ldots, (\Delta_m \Rightarrow_{\overrightarrow{p}\,\overrightarrow{q_m}} c(\overrightarrow{p}))$ (the introduction rules for c), we must be able to derive $c(\overrightarrow{p})$:

$$R_1, \ldots, R_k, (\Delta_1 \Rightarrow_{\overrightarrow{p}\,\overrightarrow{q_1}} c(\overrightarrow{p})), \ldots, (\Delta_m \Rightarrow_{\overrightarrow{p}\,\overrightarrow{q_m}} c(\overrightarrow{p})) \vdash c(\overrightarrow{p}) \; . \qquad \text{(XVIII)}$$

Again, in (XVIII), $\vdash$ refers to derivability without any primitive rule, i.e. the right side of (XVIII) should be derivable from the rules stated on its left side. This is due to the fact that we have put the introduction rules for c as assumptions on the left side of the turnstile.

Definition 7. A pair, consisting of a list of introduction rules of the form (VII) and a list of elimination rules of the form (XII), satisfies the criterion of recovery, if (XVIII) holds.

In the literature these two criteria, or criteria related to them, run under different names. As examples we mention Belnap (1962) [1], Zucker and Tragesser (1978) [41] and Dummett (1991) [6]. The reduction criterion corresponds to "conservativeness" in the sense of Belnap. However, our criterion is a weaker, and more local, condition, which does not say anything about the global conservativeness of certain rules[6]. Zucker and Tragesser

[6] As a local condition, our reduction criterion applies to non-wellfounded phenomena, something that would be ruled out by Belnap's global conservativeness requirement. See Section 6.

speak of "validity", Dummett speaks of "harmony". The second criterion corresponds to Belnap's requirement of "uniqueness", because getting from the consequences of c back to c can be seen as deriving an identical copy c' of c from c, which is exactly what uniqueness says[7]. Zucker and Tragesser speak of "implicit definability", which is, of course, related to uniqueness (see Došen & Schroeder-Heister, 1988 [4]). Dummett uses the term "stability".

We speak of harmony, when both criteria are met. Our terminology differs from Dummett's (1976, 1991) [5, 6], who tends to use "harmony" for rules satisfying only the first criterion (or some global extension of it, corresponding to conservativeness). For us "harmony" denotes the perfect match between introductions and eliminations.

Definition 8. A pair, consisting of a list of introduction rules of the form (VII) and a list of elimination rules of the form (XII), is in harmony, if it satisfies both the criterion of reduction and the criterion of recovery, i.e., if (XVII) and (XVIII) hold.

We also say that the elimination rules are in harmony with the introduction rules or vice versa, or that we have harmonious elimination rules given certain introduction rules or vice versa.

It is almost trivial that the standard connectives with the usual introduction and elimination rules are in harmony with each other.

Lemma 2. *The pairs consisting of* $(\wedge\,\mathrm{I})$ *and* $(\wedge\,\mathrm{E})$*,* $(\vee\,\mathrm{I})$ *and* $(\vee\,\mathrm{E})$*,* $(\rightarrow\mathrm{I})$ *and* $(\rightarrow\mathrm{E})$*,* $(\bot\,\mathrm{I})'$ *[or* $(\bot\,\mathrm{I})$*] and* $(\bot\,\mathrm{E})$*,* $(\top\,\mathrm{I})$ *and* $(\top\,\mathrm{E})'$ *[or* $(\top\,\mathrm{E})$*] are in harmony.*

Proof. For conjunction we have to verify

$$p_1, p_2 \vdash p_1 \quad p_1, p_2 \vdash p_2 \qquad p_1, p_2, (p_1, p_2 \Rightarrow p_1 \wedge p_2) \vdash p_1 \wedge p_2 \ ,$$

for implication

$$p_1 \Rightarrow p_2, p_1 \vdash p_2 \qquad p_1 \Rightarrow p_2, ((p_1 \Rightarrow p_2) \Rightarrow p_1 \rightarrow p_2) \vdash p_1 \rightarrow p_2 \ ,$$

for disjunction

$$\begin{aligned} &p_i, (p_1 \Rightarrow r), (p_2 \Rightarrow r) \vdash r \quad (i = 1, 2) \\ &((p_1 \Rightarrow r), (p_2 \Rightarrow r) \Rightarrow_r r), (p_1 \Rightarrow p_1 \vee p_2), (p_2 \Rightarrow p_1 \vee p_2) \vdash p_1 \vee p_2 \ , \end{aligned}$$

[7] However, in contexts not considered here, in which the premisses of introductions or the conclusions of eliminations may contain logical constants, recovery can be weaker than uniqueness. Recovery is related to what Naibo and Petrolo (2014) [18] discuss as "deducibility of identicals".

for absurdity with $(\bot\,\mathrm{I})'$

$$(\Rightarrow_p p) \vdash p \qquad (\Rightarrow_p p), ((\Rightarrow_p p) \Rightarrow \bot) \vdash \bot\ ,$$

for absurdity with $(\bot\,\mathrm{I})$

$$\text{[vacuous]} \qquad (\Rightarrow_p p) \vdash \bot\ ,$$

for triviality with $(\top\,\mathrm{E})'$

$$p \vdash p \qquad (p \Rightarrow_p p), (\Rightarrow \top) \vdash \top$$

and for triviality with $(\top\,\mathrm{E})$

$$\text{[vacuous]} \qquad (\Rightarrow \top) \vdash \top\ . \qquad \square$$

Given introduction rules of the form (VII) for c, we can easily verify that the canonical elimination rule (VIII) for c is in harmony with the introduction rules. Conversely, given elimination rules of the form (XII) for c, the canonical introduction rule (XIII) for c is in harmony with the elimination rules.

LEMMA 3. (i) *The pair consisting of a list of c-introduction rules of the form* (VII) *and the corresponding canonical c-elimination rule is in harmony.*

(ii) *The pair consisting of a list of c-elimination rules of the form* (XII) *and the corresponding canonical c-introduction rule is in harmony.*

PROOF. For (i) we must show

$$\begin{array}{l}\Delta_i, (\Delta_1 \Rightarrow_{\overrightarrow{q_1}} r), \ldots, (\Delta_m \Rightarrow_{\overrightarrow{q_m}} r) \vdash r \quad (1 \leq i \leq m)\\ ((\Delta_1 \Rightarrow_{\overrightarrow{q_1}} r), \ldots, (\Delta_m \Rightarrow_{\overrightarrow{q_m}} r) \Rightarrow_r r), (\Delta_1 \Rightarrow_{\overrightarrow{q_1}\,\overrightarrow{p}} c(\overrightarrow{p})), \ldots,\\ (\Delta_m \Rightarrow_{\overrightarrow{q_1}\,\overrightarrow{p}} c(\overrightarrow{p})) \vdash c(\overrightarrow{p})\ .\end{array}$$

For (ii) we must show

$$\begin{array}{l}R_i \vdash R_i \quad (1 \leq i \leq k)\\ R_1, \ldots, R_k, (R_1 \Rightarrow_{\overrightarrow{p}} c(\overrightarrow{p})), \ldots, (R_k \Rightarrow_{\overrightarrow{p}} c(\overrightarrow{p})) \vdash c(\overrightarrow{p})\ .\end{array}$$

These assertions follow immediately from Lemma 1. $\square$

We do not claim that harmony, or, what would be sufficient, satisfaction of the criterion of reduction implies normalization. The latter is a global property affecting the derivations of a whole system, whose normalization depends on the particular formulation of introduction and elimination rules and would normally need additional reductions such as permutative ones. Our criterion of reduction, as a more local property, secures normalization only in certain circumstances. However, if we take introduction rules together with the canonical elimination rule, then we can prove normalization[8], and similarly, if we take elimination rules together with the canonical introduction rule[9]. The uniqueness of connectives follows immediately from the satisfaction of the criterion of recovery.

5. A non-trivial example of harmony due to Hazen and Pelletier

In their contribution to this issue, Hazen and Pelletier (2014) [14, Section 3.4] give introduction and elimination rules for a ternary connective ★ using what in our terminology are higher-level rules with propositional quantification. In our formalism their rules for ★ can, in two-dimensional notation, be stated as follows:

★-introduction rule

$$\frac{\dfrac{p_1, (p_2 \Rightarrow_q q)}{p_3} \quad \begin{pmatrix} p_2, p_3 \\ q \end{pmatrix}_q \quad \begin{pmatrix} (p_2, p_3 \Rightarrow_q q), ((p_2 \Rightarrow_q q) \Rightarrow p_3), (p_1 \Rightarrow r), (p_2 \Rightarrow r) \\ r \end{pmatrix}_r}{\bigstar(p_1, p_2, p_3)}$$

★-elimination rules

$$\frac{\bigstar(p_1, p_2, p_3) \quad p_1 \quad \begin{pmatrix} p_2 \\ q \end{pmatrix}_q}{p_3}$$

$$\frac{\bigstar(p_1, p_2, p_3) \quad p_2 \quad p_3}{q}$$

$$\frac{\bigstar(p_1, p_2, p_3) \quad \begin{pmatrix} p_2, p_3 \\ q \end{pmatrix}_q \quad \dfrac{p_2 \Rightarrow_q q}{p_3} \quad \dfrac{p_1}{r} \quad \dfrac{p_2}{r}}{r}$$

[8]Without propositional variables this is spelled out in all detail in Schroeder-Heister (1981) [29] (in German).

[9]To spell out this normalization theorem would be an interesting exercise.

It is obvious that the ★-introduction rule is the canonical introduction rule corresponding to the ★-elimination rules. According to Lemma 3 this means that ★-introduction and ★-elimination rules are in harmony with each other.

From our perspective, this is a significant example of a connective that is characterized by its elimination rules, and whose introduction rule is the canonical one complementing the elimination rules in order to achieve harmony. It shows that the characterization of connectives by means of elimination rules is not just a theoretical possibility discussed for reasons of symmetry but useful in certain cases. At the same time it shows how important higher-level rules and propositional quantification at the rule level are, because without them, neither the elimination rules themselves nor the canonical introduction rule could be formulated. The significance of ★ derives from the fact that $\bigstar(p_1, p_2, p_3)$ is equivalent to $(p_1 \vee p_2) \leftrightarrow (p_3 \leftrightarrow \neg p_2)$, a formula, of which Došen (1985) [3] could show that it represents a Sheffer (i.e., expressively complete) connective for intuitionistic propositional logic. Using propositional quantification at the rule level, Hazen and Pelletier have given an explicit definition of this Sheffer connective in terms of harmonious introduction and elimination rules.

6. Discussion

We have presented an approach to harmony, according to which the relationship between introduction and elimination rules is described without reference to any logical vocabulary. Instead we have used an extended concept of rule, namely rules of higher-levels with propositional quantification. In the terminology proposed in Schroeder-Heister (2014b) [37] this is a *foundational* approach to harmony in contradistinction to the *reductive* approach followed there, where a logical system (second-order intuitionistic propositional logic) is taken for granted and harmony is explained in terms of formulas of that system. In particular, our notion of harmony covers the standard connectives (Lemma 2), which cannot be achieved if they are taken for granted. Many other examples of connectives (such as those given in Schroeder-Heister, 2014b [37]) could be considered and dealt with in the framework put forward here. Furthermore, we formulated a canonical elimination rule corresponding to any list of introduction rules and a canonical introduction rule corresponding to any list of elimination rules. We could then show that we always achieve harmony by adding the canonical elimination to any list of introductions, or by adding the canonical introduction to any list of eliminations (Lemma 3).

If the introduction rules do not contain quantification in their premisses we can always formulate a harmonious canonical elimination rule without having to resort to propositional quantification, by just using schematic variables. However, when proceeding the other way round, we need to use explicit propositional quantification, as schematic letters in elimination rules become quantified variables in the premisses of the canonical introduction. Thus propositional quantification in rules is indispensable, if we start from eliminations and want to generate appropriate introductions. That starting from eliminations is not just a theoretical option is shown by the example of Hazen and Pelletier (Section 5), which presents a significant connective characterized in terms of elimination rules.

The usage of propositional quantification in rules, which goes beyond the idea of rules of higher levels, also gives us a certain closure property for the expressive means for defining connectives. This can be seen as follows. Suppose we start with introduction rules of level 1 or 2, i.e. rules which do not discharge any assumption (such as disjunction introduction) or enable the discharge of formulas as assumptions (such as implication introduction). Then the corresponding canonical elimination rule is of level 2 or 3, respectively, i.e., the level increases by one. For example, while disjunction introduction ($\vee$ I) is a level-1 rule, the canonical rule of disjunction elimination ($\vee$ E) is a level-2 rule. Analogously, while the introduction rules

$$(\star\ \mathrm{I})\quad \frac{\begin{matrix}p_1\\ p_2\end{matrix}}{\star(p_1,p_2,p_3)} \qquad \frac{p_3}{\star(p_1,p_2,p_3)}$$

for a ternary connective $\star$[10] are of maximum level 2, the canonical elimination rule corresponding to $\star$-introduction, which has the form

$$(\star\ \mathrm{E})\quad \frac{\star(p_1,p_2,p_3) \qquad \begin{matrix}p_1 \Rightarrow p_2\\ r\end{matrix} \qquad \begin{matrix}p_3\\ r\end{matrix}}{r},$$

is a rule of level 3. Once we are considering level-3 elimination rules, there is no reason to disallow level-3 introduction rules, which would then lead to level-4 canonical elimination rules etc. Thus we need the full range of rules of all finite levels to formulate the systematics of introductions and corresponding canonical eliminations.

In Olkhovikov and Schroeder-Heister (2014a) [19] it is shown that this rise of level cannot be avoided, i.e. that there is no harmonious elimination

[10] Note that $\star$ is different from Hazen and Pelletier's connective $\bigstar$ discussed in Section 5.

rule for $\star$ of level lower than 3, and in Olkhovikov and Schroeder-Heister (2014b) [20] it is proved that this result extends to any higher level, i.e. that for each level n there is an example of level-n introduction rules without harmonious elimination rules of level n or below. Now let us start with an elimination rule of level 2 such as

$$(\circ\ \mathrm{E})\ \ \frac{\circ(p_1, p_2, p_3) \qquad \begin{array}{c} p_1 \\ p_2 \end{array}}{p_3}$$

for a ternary connective $\circ$. Then its harmonious introduction rule

$$(\circ\ \mathrm{I})\ \ \frac{\begin{array}{c} p_1 \Rightarrow p_2 \\ p_3 \end{array}}{\circ(p_1, p_2, p_3)}$$

is a level-3 rule, where again this level rise cannot be avoided, neither for $\circ$ nor for certain other connectives with eliminations of a given higher level n (Olkhovikov and Schroeder-Heister, 2014a, 2014b [19, 20]). Moreover, the canonical introduction rule needs to use propositional quantification, if the given elimination rules use schematic variables, for example the canonical introduction rule (XVI) for the connective $\oslash$ with the elimination rules (XV). Once we are considering level-3 introduction rules with schematic variables, there is no reason to disallow level-3 elimination rules, which would then lead to canonical introduction rules of level 4 with quantification, and so on. Therefore, if we allow for the means of expression that we need to formulate canonical elimination or canonical introduction rules to be used also in the (non-canonical) rules we start with, we need the full range of higher-level rules with propositional quantification. Rules with propositional quantification are not a strange system, but result as a very natural extension of standard rules and lie at the heart of rule-based systems already at the propositional level. From this point of view Jaśkowski (1934) [15] had the right intuition when he considered propositional quantification in his assumption calculus even before he considered first-order quantification.

We have called our approach foundational, as it analyzes the general systematics of introduction and elimination rules in terms of (quantified) higher-level rules rather than in terms of formulas. However, it might be questioned whether our foundational approach is so much different from a reductive approach that uses formulas of second-order intuitionistic propositional logic **PL2** instead. There is an obvious translation from rules into formulas built up by using conjunction, implication and propositional quantification. For example, the rule

$$((p, q \Rightarrow r), p, q) \Rightarrow_{pqr} r$$

is translated into the formula

$$\forall pqr(((p\wedge q\rightarrow r)\wedge p\wedge q)\rightarrow r)\ ,$$

and the rule

$$((p, s \Rightarrow_q q) \Rightarrow r) \Rightarrow_{rs} r$$

is translated into the formula

$$\forall rs((\forall q(p\wedge s\rightarrow q)\rightarrow r)\rightarrow r)\ .$$

From these examples the reader can easily construct a general definition of the translation. If R^* is the translation of the rule R, it can be shown that whenever

$$R_1, \ldots, R_n \vdash A$$

holds in the calculus of quantified higher-level rules, then

$$R_1^*, \ldots, R_n^* \vdash_{\mathbf{PL2}} A^*$$

and vice versa. This is due to the fact that the calculus of higher-level rules and **PL2** have corresponding inference principles, in particular the application of rules as corresponding to modus ponens. Thus we can easily model everything for which we used quantified higher-level rules by using **PL2**. In this sense the calculus of rules and **PL2** correspond to each other. However, there is still a reason why the foundational approach based on (quantified higher-level) rules is more basic than **PL2**. In our schema for the application of rules (II), which lies at the heart of the foundational approach, we just rely on the *applicative behavior* of rules, i.e., on the idea that rules can only be applied (by passing from premisses to conclusion) rather than established. We later on defined what it means to derive a rule, but only as a metalinguistic abbreviation, not as a formal step. In a derivation rules can be applied, i.e., stand next to an inference line, but what is obtained at each inference step is a formula, i.e. an entity which does not contain the rule arrow $\Rightarrow$. In contradistinction to this idea, the calculus **PL2** uses the idea that implications can be applied via modus ponens, and established via implication introduction. It therefore uses a more elaborate notion of implication than the sort of implication which is represented by the rule arrow $\Rightarrow$. It is the idea of our foundational approach to single out the applicative aspect of rules as that aspect of implication which is most basic, and which is lost when one starts with full implication with introduction and elimination rules. Something similar holds for quantification. The quantification in rules is part of their applicative behavior, whereas the **PL2** rules for the universal quantifier comprise both introductions and eliminations. Thus, in a sense, our foundational rule-based approach singles out the elimination aspect of implication and universal quantification, which is logically expressed by elimination rules. Something similar can be said of conjunction, which in the rule context is handled by singling out comma-separated formulas, which corresponds to conjunction elimination. If we take implication to be the most fundamental connective and the rule arrow as representing the *modus ponens* aspect of implication, the slogan of our foundational approach should be: *In Defence of Modus Ponens.* The logic of rules is the logic of *modus ponens governed* implication.[11]

We finish by mentioning some possible directions of further work:

1. Our approach can be applied to circular and non-wellfounded phenomena. The definitions of the criteria of reduction and recovery, and of harmony can, for example, be applied if we allowed for self-referential introduction and elimination rules. For example, the introduction and elimination rules for the (nullary) connective $\lightning$

$$\frac{\begin{pmatrix} \lightning \\ p \end{pmatrix}_p}{\lightning} \qquad \frac{\lightning \qquad \lightning}{p}$$

are in harmony with each other, since both reduction and recovery criteria are met. This reaffirms the fact that our criteria are local rather than global criteria, as globally, these rules lead to a contradiction. It will be interesting to generalize our notion of harmony in a systematic way beyond the well-founded case (considered here for reasons of simplicity) to analyze, for example, paradoxical reasoning. Self-referential definitions of constants with canonical elimination rules have, for example, been considered by Hallnäs (1991) [12], Hallnäs and Schroeder-Heister (1991/92) [13] and Read (2010) [27].

2. We have used propositional quantification in rules in order to characterize propositional connectives. The next step would be to also consider introduction and elimination rules for propositional and first-order

[11] Therefore we do not think that modus ponens can be replaced throughout with its indirect variant proposed by Dyckhoff (1988) [7], Tennant (1992, 2002) [38, 39], Lopez-Escobar (1999) [16] and von Plato (2001) [22], or by the canonical (higher-level) elimination rule proposed in Schroeder-Heister (1984b) [31] (see the discussion in Schroeder-Heister, 2014a [36]). Even though these approaches have their proof-theoretic merits in many contexts, they cannot supersede modus ponens as the most basic and primitive aspect of implication, a conclusion also reached by Dyckhoff (2009, 2014) [8, 9].

quantifiers. We have refrained from this here, as it would have opened up a new field only marginally relevant to the point we wanted to make. Considering arbitrary quantifiers in formulas means to enter the area of generalized quantifiers, which requires a more sophisticated way of dealing with schematic variables. Without propositional quantification, for an approach based on a generalized canonical elimination rule, this idea is worked out in Schroeder-Heister (1984a) [30].

3. Here we have only dealt with intuitionistic logic. This logic is most clearly related to modus ponens based implication and thus to the concept of rule. In classical, linear and many variants of substructural logics one might prefer to work in a sequent-style rather than a natural deduction framework. Some ideas towards a harmony principle in that realm have been presented in Schroeder-Heister (2013) [35].

4. The idea of harmony should be investigated in relation to monotone and partial inductive definitions (see Hallnäs, 1991 [12]). These approaches are based on introduction rules for atomic formulas or predicates as inductive clauses, using explicitly or implicitly a canonical elimination rule. It might be interesting to develop a notion of elimination clauses together with an appropriate notion of harmony, or at least develop a dual approach based on elimination clauses together with the canonical introduction rule. It will be interesting to see if such an approach makes any sense computationally, as is the case with the inductive one (e.g., in the form of logic programming). Some very rough ideas towards dualizing inductive introduction clauses have been described in Schroeder-Heister (2011, in German) [34].

5. Our criteria of reduction and recovery and our notion of harmony are restricted to introduction and elimination rules of a particular form. In proof-theoretic semantics there is a tradition coined by Prawitz (1973, 1974) [24, 25] and continued by Dummett (1991) [6] that develops a notion of validity, which is defined with respect to introduction inferences, but can be applied to *any* derivation or rule, i.e. it is not tied to inference rules of a specific form. For example, according to Prawitz's notion of validity, the rule

$$\frac{p \to (q \to r)}{q \to (p \to r)}$$

is valid, though it does not have the form of an elimination rule (see Schroeder-Heister, 2006 [33]). It should be investigated, how this notion of validity relates to our notion of harmony. One might conjecture that all valid rules can be derived if we supplement all introduction rules with harmonious elimination rules, i.e., that harmonious introductions and eliminations deductively capture everything that is valid. This means that all valid rules are derivable in intuitionistic logic, which represents a kind of completeness conjecture. Such a completeness conjecture has been put forward by Prawitz (1973, and several later publications) [24], but has recently be challenged by Piecha et al. (2014) [21].

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References

[1] Belnap, N. D. (1962) Tonk, plonk and plink, *Analysis* 22:130–134.

[2] Carnap, R. (1931) Die logizistische Grundlegung der Mathematik, *Erkenntnis* 2:91–105.

[3] Došen, K. (1985) An intuitionistic Sheffer function, *Notre Dame Journal of Formal Logic* 26:479–482.

[4] Došen, K., and P. Schroeder-Heister (1988) Uniqueness, definability and interpolation, *Jounal of Symbolic Logic* 53:554–570.

[5] Dummett, M. (1973) *Frege: Philosophy of Language*, Duckworth, London.

[6] Dummett, M. (1991) *The Logical Basis of Metaphysics*, Duckworth, London.

[7] Dyckhoff, R. (1988) Implementing a simple proof assistant, in *Workshop on Programming for Logic Teaching (Leeds, 6-8 July 1987): Proceedings 23/1988*, Centre for Theoretical Computer Science (`http://rd.host.cs.st-andrews.ac.uk/publications/LeedsPaper.pdf`), University of Leeds, pp. 49–59.

[8] Dyckhoff, R. (2009) Generalised elimination rules and harmony. Manuscript, University of St. Andrews, `http://rd.host.cs.st-andrews.ac.uk/talks/2009/GE.pdf`.

[9] Dyckhoff, R. (2014) Some remarks on proof-theoretic semantics, in T. Piecha and P. Schroeder-Heister (eds.), *Advances in Proof-Theoretic Semantics*. Berlin: Springer.

[10] Francez, N., and R. Dyckhoff (2012) A note on harmony, *Journal of Philosophical Logic* 41:613–628.

[11] Gentzen, G. (1934/35) Untersuchungen über das logische Schließen, *Mathematische Zeitschrift* 39:176–210, 405–431 (English translation in: *The Collected Papers of Gerhard Gentzen* (ed. M. E. Szabo), Amsterdam: North Holland (1969), pp. 68–131).

[12] Hallnäs, L. (1991) Partial inductive definitions, *Theoretical Computer Science* 87:115–142.

[13] Hallnäs, L., and P. Schroeder-Heister (1990/91) A proof-theoretic approach to logic programming: I. Clauses as rules. II. Programs as definitions, *Journal of Logic and Computation* 1:261–283, 635–660.

[14] Hazen, A. P., and F. J. Pelletier (2014) Gentzen and Jaśkowski natural deduction: Fundamentally similar but importantly different, *Studia Logica* (*A. Indrzejczak (ed.), Gentzen's and Jaśkowski's Heritage: 80 Years of Natural Deduction and Sequent Calculi*, this issue).

[15] Jaśkowski, S. (1934) On the rules of suppositions in formal logic, *Studia Logica* 1:5–32 (reprinted in: S. McCall (ed.), *Polish Logic 1920–1939*, Oxford 1967, pp. 232–258).

[16] López-Escobar, E. G. K. (1999) Standardizing the N systems of Gentzen', in X. Caicedo, and C. H. Montenegro (eds.), *Models, Algebras and Proofs (Lecture Notes in Pure and Applied Mathematics, Vol. 203)*, Marcel Dekker, New York, pp. 411–434.

[17] Lorenzen, P. (1955) *Einführung in die operative Logik und Mathematik*, Springer, Berlin (2nd edition 1969).

[18] Naibo, A., and M. Petrolo (2014) Are uniqueness and deducibility of identicals the same?, *Theoria*, in press.

[19] Olkhovikov, G. K., and P. Schroeder-Heister (2014a) On flattening elimination rules, *Review of Symbolic Logic* 7:60–72.

[20] Olkhovikov, G. K., and P. Schroeder-Heister (2014b) Proof-theoretic harmony and the levels of rules: General non-flattening results, in E. Moriconi, and L. Tesconi (eds.), *Second Pisa Colloquium in Logic, Language and Epistemology*, ETS, Pisa.

[21] Piecha, T., W. de Campos Sanz, and P. Schroeder-Heister (2014) Failure of completeness in proof-theoretic semantics, *Journal of Philosophical Logic*, in press.

[22] Plato, J. von (2001) Natural deduction with general elimination rules, *Archive for Mathematical Logic* 40:541–567.

[23] Prawitz, D. (1965) *Natural Deduction: A Proof-Theoretical Study*, Almqvist & Wiksell, Stockholm (Reprinted Mineola NY: Dover Publ., 2006).

[24] Prawitz, D. (1973) Towards a foundation of a general proof theory, in P. Suppes et al. (eds.), *Logic, Methodology and Philosophy of Science IV*, North-Holland, pp. 225–250.

[25] Prawitz, D. (1974) On the idea of a general proof theory. *Synthese* 27:63–77.

[26] Prior, A. N. (1960) The runabout inference-ticket, *Analysis* 21:38–39.

[27] Read, S. (2010) General-elimination harmony and the meaning of the logical constants, *Journal of Philosophical Logic* 39:557–576.

[28] Read, S. (2014) General-elimination harmony and higher-level rules, in H. Wansing (ed.), *Dag Prawitz on Proofs and Meaning.* Heidelberg: Springer.

[29] Schroeder-Heister, P. (1981) *Untersuchungen zur regellogischen Deutung von Aussagenverknüpfungen*, Doctoral dissertation, Universität Bonn [see author's homepage].

[30] Schroeder-Heister, P. (1984a) Generalized rules for quantifiers and the completeness of the intuitionistic operators $\wedge$, $\vee$, $\rightarrow$, $\bot$, $\forall$, $\exists$, in M. M. Richter, E. Börger, W. Oberschelp, B. Schinzel, and W. Thomas, (eds.), *Computation and Proof Theory. Proceedings of the Logic Colloquium held in Aachen, July 18-23, 1983, Part II. (Lecture Notes in Mathematics Vol. 1104)*, Springer, Berlin, pp. 399–426.

[31] Schroeder-Heister, P. (1984b) A natural extension of natural deduction, *Journal of Symbolic Logic* 49:1284–1300.

[32] Schroeder-Heister, P. (1987) *Structural Frameworks with Higher-Level Rules*, Habil. thesis, Universität Konstanz [see author's homepage].

[33] Schroeder-Heister, P. (2006) Validity concepts in proof-theoretic semantics, *Synthese* 148:525–571.

[34] Schroeder-Heister, P. (2011) Schluß und Umkehrschluß: Ein Beitrag zur Definitionstheorie, in C. F. Gethmann (ed.), *Lebenswelt und Wissenschaft. Kolloquienbeiträge und öffentliche Vorträge des XXI. Deutschen Kongresses für Philosophie (Essen 15. - 19.09. 2008)*, Meiner, Hamburg.

[35] Schroeder-Heister, P. (2013) Definitional Reflection and Basic Logic, *Annals of Pure and Applied Logic (Special issue, Festschrift 60th Birthday Giovanni Sambin)* 164:491–501.

[36] Schroeder-Heister, P. (2014a) Generalized elimination inferences, higher-level rules, and the implications-as-rules interpretation of the sequent calculus, in L. C. Pereira, E. H. Haeusler, and V. de Paiva (eds.), *Advances in Natural Deduction: A Celebration of Dag Prawitz's Work*, Springer, Heidelberg, pp. 1–29.

[37] Schroeder-Heister, P. (2014b) Harmony in proof-theoretic semantics: A reductive analysis, in H. Wansing (ed.), *Dag Prawitz on Proofs and Meaning*, Springer, Heidelberg.

[38] Tennant, N. (1992) *Autologic*, Edinburgh University Press, Edinburgh,.

[39] Tennant, N. (2002) Ultimate normal forms for parallelized natural deductions, *Logic Journal of the IGPL* 10:299–337.

[40] Whitehead, A. N., and B. Russell (1910) *Principia Mathematica*, Cambridge University Press, Cambridge (2nd edition 1927).

[41] Zucker, J. I., and R. S. Tragesser (1978) The adequacy problem for inferential logic, *Journal of Philosophical Logic* 7:501–516.

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