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FREGE'S SEQUENT
CALCULUS

Abstract Frege's logical system as developed in the "Grundgesetze der Arithmetik" can be regarded as a sequent calculus of a specific form.

1 FREGE-HILBERT VS. JAŚKOWSKI-GENTZEN CALCULI

The type of logical calculus developed by Frege in his **Begriffsschrift** [4] and later used by Hilbert in his proof-theoretic investigations is normally distinguished from that proposed by Gentzen [6] and Jaśkowski [8]. As this difference concerns to a great extent the handling of consequence and implication, we can confine ourselves to implicational logic. In Frege-Hilbert-style calculi theorems are derived from a variety of axioms by means of the single rule of modus ponens. (Frege has an additional substitution rule, which we can disregard for the point we want to make.) Derivations from assumptions play a subordinate rule. In fact, Frege does not use them at all. He explicitly rejects the idea of making assumptions as a separate form of judgement, arguing that the dependency on assumptions is exclusively expressed by implications. His derivations always establish implicational sentences such as



is just a sentence-building operator. Introduction and elimination rules for implication can then be given as follows:

$$(\rightarrow I) \frac{\Gamma, A \Rightarrow B}{\Gamma \Rightarrow A \rightarrow B} \quad (\rightarrow E) \frac{\Gamma \Rightarrow A \quad \Delta \Rightarrow A \rightarrow B}{\Gamma, \Delta \Rightarrow B}$$

We throughout assume that the right side of a sequent consists of a single sentence, whereas the left side consists of a list of sentences, which may be empty. We always speak of the “right side” and “left side” of a sequent, as we will use the terms “antecedent” and “succedent” in connection with Frege implications rather than with sequents. For the point we want to make, the basic ingredient of the sequent calculus is the distinction between sequent arrow and implication and the presence of the implication introduction rule ($\rightarrow I$), which in a sense reduces the logical meaning of implication to the structural meaning of the sequent arrow. So the fundamental feature of the sequent calculus is its **two-layer** structure with respect to implication. Such a sequent calculus just needs the identity axiom

$$(Id) \Gamma, A \Rightarrow A$$

as an initial sequent to start with. All other primitive inferences are rules rather than axioms. Thus the sequent calculus has the one-axiom/many-rules feature, whereas Frege–Hilbert–systems have the one-rule/many-axioms feature.

Normally a Frege implication (1) is translated into modern notation as follows:

$$A \rightarrow (B \rightarrow (C \rightarrow (D \rightarrow E)))^1$$

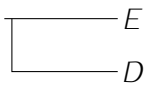
This is certainly justified looking at the explanation Frege’s gives for his symbolism in the **Begriffsschrift** [4] and at the usage he makes of it in that text. From this point of view, the **Begriffsschrift** uses a special notation for iterated implicational sentences, where its two-dimensional shape is the price Frege pays for avoiding parentheses.

¹ A recent edition [2] of the **Grundgesetze der Arithmetik** is throughout based on this translation.

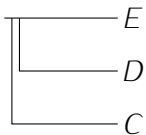
However, if we look at the usage and terminology made of this notation in the **Grundgesetze der Arithmetik** [3] and the inference rules given there, an alternative interpretation suggests itself, which leads to reading Frege implications as sequents and the calculus of the **Grundgesetze** as a sort of sequent calculus. This was first and independently observed by von Kutschera [9].

2 ANTECEDENTS AND SUCCEDENT OF A FREGE IMPLICATION

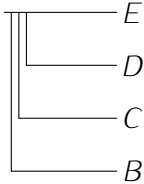
Given a Frege implication of the form (1), Frege distinguishes between its "Oberglied" and its "Unterglieder". Literally translated, "Oberglied" means "upper member" and "Unterglieder" means "lower members" (plur.), where these terms obviously refer to the fact that the "Oberglied" always occurs above the "Unterglieder". Note that the "Oberglied" is always a single sentence, whereas the "Unterglieder" may be multiple sentences. In the following I use the term "succedens" for the "Oberglied" and the term "antecedents" for the "Unterglieder" of a Frege implication, thus using Hertz's terms that Gentzen adopted to denote the left and right sides of sequents. In the Frege implication (1) the sentence E can be viewed as its succedent, and the sentences A, B, C and D as its antecedents. However, without changing the sentence (1), we may alternatively regard the implication



as its succedent and the sentences A, B and C as its antecedents, or, as a further alternative, the sentence



as its succedent and the sentences A and B as its antecedents, or the sentence



as its succedent and the sentence A as its antecedent. As a limiting case we might even consider (1) to be succedent of itself, without there being any antecedent.

It is obvious that the distinction between succedent and antecedents corresponds to introducing some sort of sequent arrow, in particular as Frege uses this distinction only at the outermost level, i.e., as he never uses this distinction for an embedded implication such as (2). In this sense we can speak of a sequent calculus present in the **Grundgesetze**. However, as the decomposition of a sentence into succedent and antecedents is not uniquely determined by the syntax of the sentence but can be carried out in different ways, this sequent calculus is metalinguistically specified, not in terms of a specific syntax for sequents. This approach resembles the procedure followed by Schütte [10] in his systems based on positive and negative, or right and left parts of sentences, which allows him to read a sequent calculus structure into sentences, which, by their pure syntax, do not contain any division between positive and negative or between right and left parts. Frege explicitly says that the division into antecedents and succedent is a matter of consideration ("Auffassung") and thus a way of metalinguistically dealing with sentences. He thus avoids to introduce any explicit syntactical consequence sign which would be in need of justification given his aversion against any form of consequence operator beyond implication.

If we make the distinction between antecedents and succedent explicit as a syntactical distinction expressed by a sequent arrow, then we would arrive at a sequent calculus containing the rules

$$(\rightarrow I) \frac{\Gamma, A \Rightarrow B}{\Gamma \Rightarrow A \rightarrow B} \quad (\rightarrow R) \frac{\Gamma \Rightarrow A \rightarrow B}{\Gamma, A \Rightarrow B}$$

governing implication. The left rule is the **implication introduction** rule well-known from from Gentzen and Jaśkowski, whereas the right rule is its inverse, which we call the **implication removal** rule. This sequent-style formulation allows one to move sentences from the left to the right side of the sequent sign and vice versa. It turns the matter of metalinguistic consideration, which determines which sentence is succedens and which sentences are the antecedents of a Frege implication, into a syntactical distinction between the right and left side of a sequent. What is regarded as the succedent can then be displayed as the right side of a sequent by applying a sufficient number of (\rightarrow I) and (\rightarrow R) steps².

3 STRUCTURAL RULES

Based on the distinction between antecedents and succedent of a sentence, in his **Grundgesetze** Frege [3] presents inference rules that almost literally correspond to structural rules in Gentzen's sense, namely exchange, contraction and cut. In his list of inference rules in §48 of the **Grundgesetze**, he presents a rule of exchange by permitting that antecedents of an implication can be arbitrarily interchanged. In sequent-style notation this corresponds to the rule:

$$(\text{Exch}) \frac{\Gamma \Rightarrow B}{\Gamma' \Rightarrow B}$$

where Γ' stands for an arbitrary permutation of Γ . A rule of contraction is presented by permitting that a sentence which occurs several times as an antecedent within the same sentence need only be written once. In sequent-style notation this corresponds to

$$(\text{Contr}) \frac{\Gamma[A, \dots, A] \Rightarrow B}{\Gamma[A] \Rightarrow B}$$

where $\Gamma[A, \dots, A]$ stands for a list of sentences containing several occurrences of A and $\Gamma[A]$ for a list obtained by removing all but one of

² The notational advantage of the **Begriffsschrift** notation and its relation to importation and exportation laws in conjunction-implication logic, which correspond to (\rightarrow I) and (\rightarrow R), was first pointed out by Thiel [13].

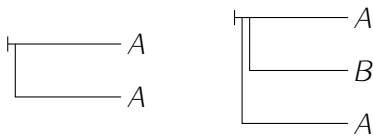
these occurrences. A rule of cut is presented by saying that if a sentence A occurs in one sentence as the succedent and in another one as an antecedent, then we may pass over to a sentence, whose succedent is that of the second sentence, and whose antecedents are those of the two sentences with the exception of A . In sequent-style notation this corresponds to

$$(\text{Cut}) \frac{\Gamma \Rightarrow A \quad \Delta, A \Rightarrow B}{\Gamma, \Delta \Rightarrow B}$$

In his formulation of the rule, Frege allows for implicit exchange and contraction, so the order of antecedents in premiss and conclusion does not matter. The limiting case in which Γ is empty, is formulated by Frege as a rule of its own.

4 AXIOMS

Frege’s only implicational axioms are the following two:



In sequent-style notation they may be rephrased as

$$A \Rightarrow A \quad A, B \Rightarrow A$$

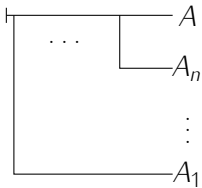
The first sequent is simple identity. From the second sequent we can generate the identity sequent (Id) by iterated application of cut. By allowing Γ to be empty, we may consider (Id) to be a faithful representation of Frege’s implicational axioms. This concludes the presentation of the implicational fragment of Frege’s system³. We thus consider the sequent-style representation of Frege’s implication calculus to consist

³ As an additional rule, Frege has the contraction of two horizontals into one. However, the dealing with the horizontal is only implicit in modern notation. As long as we stay outside the type-theoretical realm, we may consider the horizontal to be just the trivial one-place identity connective.

of (Id), (Exch), (Contr), (Cut), (\rightarrow I) and (\rightarrow R) as primitive axioms and rules of inference.

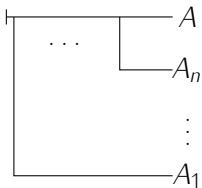
5 FORMAL EQUIVALENCE

Let the **Frege counterpart** of a sequent $A_1, \dots, A_n \Rightarrow A$ be the Frege implication



Then any derivation in the sequent calculus based on (Id), (Exch), (Contr), (Cut), (\rightarrow I) and (\rightarrow R) yields a derivation in Frege's system. We just need to replace every sequent with its Frege counterpart and delete all applications of (\rightarrow I) and (\rightarrow R) (whose premiss and conclusion have identical Frege counterparts).

Conversely, by writing Frege implications



as sequents $A_1, \dots, A_i \Rightarrow A_{i+1} \rightarrow (\dots \rightarrow (A_n \rightarrow A) \dots)$ (with $\Rightarrow A_1 \rightarrow (\dots \rightarrow (A_n \rightarrow A) \dots)$ being a limiting case), a derivation in the sequent calculus based on (Id), (Exch), (Contr), (Cut), (\rightarrow I) and (\rightarrow R) is obtained from any derivation in Frege's system. To cope with the ambiguity of dividing an implicational sentence at a particular place A_i into

antecedents and succedent, it may be necessary to insert applications of $(\rightarrow I)$ and $(\rightarrow R)$.

6 FREGE AND NATURAL DEDUCTION

We have translated Frege's implicational calculus into a sequent system with an explicit cut rule. In this system we have, as a rule eliminating implication, the implication removal rule $(\rightarrow R)$. It can easily be seen that the implication elimination rule of natural deduction, i.e. **modus ponens** $(\rightarrow E)$, is equivalent to $(\rightarrow R) + (\text{Cut})$. Thus we could equivalently rephrase Frege's system as a sequent-style natural deduction system based on $(\rightarrow I)$ and $(\rightarrow E)$. Given the prominence the rule of **modus ponens** has in the *Begriffsschrift*, this interpretation is quite plausible. It is also very coherent from the modern point of view, where one does not normally choose $(\rightarrow R)$ as a primitive rule of inference governing implication. However, looking carefully at the formulation Frege uses in the *Grundgesetze* to describe the rule of inference that detaches a formula, it turns out that (Cut) is a translation more faithful to his wordings than modus ponens $(\rightarrow E)$. Frege always speaks **pluraliter** of **any** antecedent of a sentence which can be removed by a rule of inference, which clearly corresponds to the idea of cut. One should nevertheless be aware of the fact that this question cannot be formally settled, as an application of (Cut) and an application of sequent-style modus ponens $(\rightarrow E)$ look identical when syntactically translated into inference steps in Frege's system⁴.

4 Von Kutschera [9], who presents the same sequent system as we did above, nevertheless speaks of "Frege and natural deduction". By "natural deduction" he means a sequent-style system with single formulas on the right side of a sequent and without introduction rules operating on the left side. We prefer reserving the term "natural deduction" for a system where operators are characterized by introduction and elimination rules, which in the implicational case means that $(\rightarrow I)$ and $(\rightarrow E)$ are available as primitive rules of inference.

7 NEGATION

For completeness we add that the full system of classical implicational logic is obtained in the **Grundgesetze** by means of rules of contraposition and dilemma, which in sequent-style formulation run as follows:

$$\text{(Con)} \frac{\Gamma, A \Rightarrow B}{\Gamma, \bar{A} \Rightarrow \bar{B}} \quad \text{(Dil)} \frac{\Gamma, A \Rightarrow B \quad \Gamma, \neg A \Rightarrow B}{\Gamma \Rightarrow B}$$

Here \bar{A} stands indifferently both for $\neg\neg C$ and for C , if A is a negated formula $\neg C$ (Frege speaks of the “inversion” of the truth value of A).

Taking negation into account, it should also be mentioned that the reading of Frege implications as sequents turns out to be particularly significant, when there are no embedded implications. In this case a Frege implication is translated into a sequent of the form $A_1, \dots, A_n \Rightarrow A$, where A_1, \dots, A_n, A are literals, i.e. atoms or their negations. In this system the rules of (Cut) and (Dil) correspond to the rule of propositional resolution. Already in 1880/81 Frege [1] formulated such a system and used it to solve a well-known combinatorial problem of propositional logic originally posed by Boole. Frege's way of solving this problem can be seen as the origin of propositional resolution (see [12]).

8 FREGE AND HERTZ'S STRUCTURAL REASONING

We have interpreted Frege's system in terms of a sequent calculus which, by its very nature, distinguishes between the sequent arrow and the implication sign and supplements Frege's system with implication introduction and removal rules. However, rather than reading Frege implications as sequents in disguise and adding a structural level to the logical level, one might try to interpret Frege implications directly as structural implications without considering any logical level. This would correspond to Hertz's [7] way of proceeding, whose implicational logic is purely structural, namely solely based on the structural rules of identity, contraction, thinning and cut, without any additional implicational connective and corresponding introduction and elimination rules. This interpretation is tempting also in the case of Frege, as Frege insists on a one-layer implicational system, which has explicit structural

inferences and in particular cut as a primitive rule. A Frege implication such as (1) would then be interpreted as a sentence (“Satz”) in Hertz’s sense, which is governed by structural inference schemas. This idea also underlies the system put forward by Gentzen in his first publication of 1933 [5], which was written under the influence of Hertz’s ideas, before he proceeded with his dissertation [6], whose crucial feature is a two-layer rather than a one-layer system. We cannot pursue this idea here. The fact that Hertz’s system serves well to accommodate propositional resolution points to similarities to Frege’s approach (see [11]).

9 CONCLUSION

We do not claim that the sequent calculus presented here is the exclusive modern rephrasing of what Frege intended with his system. Frege would have almost certainly rejected the reading of his distinction between antecedents and succedent of a sentence as expressing some sort of conditional judgement that can be made explicit by means of a sequent arrow \Rightarrow . However, there still remains the Schütte-style reading of this sequent calculus, which would consider a sequent $\Gamma \Rightarrow A$ as representing the Frege implication in which A is succedens and the sentences in Γ are antecedents, rather than a syntactical entity in its own right.

Given this reading, the implication rules (\rightarrow I) and (\rightarrow R) pass from one representation of a Frege implication to another representation of the same Frege implication, so they just serve to make explicit different ways of looking at it, rather than explicitly introducing or eliminating an implication sign. In any case Frege’s distinction between various ways of apprehending an implication by splitting it into succedens and antecedents, and his basing inference rules on this distinction, shows that he is aware of fundamental structural issues that have been put forward much later by Hertz, Jaśkowski and Gentzen.

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