Celebrating 90 Years of Gödel's Incompleteness Theorems
Nürtingen (Germany), 2021

Missing Proofs and the Provability of Consistency

Sergei Artemov

CUNY Graduate Center

July 7, 2021

Abstract

We argue that there is a class of widely used and readily formalizable arithmetical proofs of universal properties which are not accounted for in the traditional unprovability of consistency analysis.

On this basis, we offer a mathematical proof of consistency for Peano Arithmetic PA and demonstrate that this proof is formalizable in PA. This refutes the wide spread belief that there exists no consistency proof of a system that can be formalized in the system itself.

Gödel's Second Incompleteness theorem yields that PA cannot derive the consistency formula $\mathsf{Con}_{\mathsf{PA}}$. This does not interfere with our formalized proof of PA-consistency which is not a derivation of the consistency formula $\mathsf{Con}_{\mathsf{PA}}$.

Hilbert consistency program and Gödel theorems

In the 1920s, Hilbert outlined a program of establishing consistency of formal mathematical theories by trusted means. The consistency property for a theory T is that:

no finite sequence of formulas is a derivation of a contradiction in T.

In the base case when T is Peano Arithmetic PA, Gödel's Second Incompleteness Theorem, G2, states that the arithmetical formula $\mathrm{Con}_{\mathrm{PA}}$, in which consistency property of PA is internalized using numerical codes of sequences, is not derivable in PA (given that PA is consistent).

Together with the widely accepted **Formalization Principle**:

> *any rigorous reasoning within the postulates of* PA *can be formalized as a derivation in* PA*,*

this suggests that PA-consistency cannot be established by means of PA. Popular wisdom thus concludes that Hilbert's program was refuted by G2.

Yet, neither Hilbert nor Gödel accepted this conclusion.

Hilbert

Hilbert (Grundlagen der Mathematik, 1934):[1]

> *"... the view, ... that certain recent results of Gödel show that my proof theory can't be carried out, has been shown to be erroneous. In fact that result shows only that one must exploit the finitary standpoint in a sharper way for the farther reaching consistency proofs."*

[1]This English translation is quoted from S. Feferman "Lieber Herr Bernays!, Lieber Herr Gödel! Gödel on finitism, constructivity and Hilbert's program," Dialectica, Vol. 62, No.2 (2008), pp. 179–203

Gödel

Gödel himself challenges Formalization Principle. In his original G2 paper, "On formally undecidable propositions ...," 1931, Gödel writes

> *... it is conceivable that there exist finitary proofs that cannot be expressed in the formalism of [our basic system].*

There are reliable indications that late Gödel remained skeptic w.r.t. the impossibility consequences of G2. Gerald Sacks recalled Gödel saying, in 1961 – 1962, that some type of revival of Hilbert's consistency program would eventually become feasible[2]. Gödel *"did not think"* the objectives of Hilbert's consistency program *"were erased"* by the Incompleteness Theorem, and Gödel believed it left Hilbert's program

> *"very much alive and even more interesting than it initially was."*[3]

---

[2]We thank Dan Willard for bringing this to our attention.

[3]G. Sacks. Reflections on Gödel. The Thomas and Yvonne Williams Symposia for the Advancement of Logic, Philosophy, and Technology, Lecture at the University of Pennsylvania, April 11, 2007 (available on YouTube).

Von Neumann and the Impossibility Paradigm

Upon presenting the Albert Einstein Award in 1951, John von Neumann credited Gödel, among other things, with proving that

> *... no such system [which permits ... a rigorous and exhaustive description, in terms of modern logic] can its freedom from inner contradictions be demonstrated with the means of the system itself.*

Such interpretation has been elevated to a major foundational paradigm (which we call here **the Impossibility Paradigm**):

> *"there exists no consistency proof of a system that can be formalized in the system itself"* (Encyclopædia Britannica).

The Impossibility Paradigm, IP, is usually regarded as a quintessential roadblock on the way of Hilbert's consistency program.

We provide a mathematical proof of PA**-consistency and formalize this proof in** PA**. This refutes the Impossibility Paradigm and thus reopens the door to the investigation of Hilbert's program.**

This talk has three sections

I. (Preliminary) We argue that “popular wisdom” concerning G2 and the Impossibility Paradigm is poorly founded.

II. (Mathematical) We provide a direct mathematical proof of consistency of PA by means of PA. Namely,

> **for any** PA-**derivaton** S**, we find a** PA-**definable invariant**
>
> $$\mathcal{I}_S$$
>
> **and establish in arithmetic that for each** φ **in** S**,** $\mathcal{I}_S(\varphi)$ **holds,** $\mathcal{I}_S(0{=}1)$ **does not hold, hence** $(0{=}1)$ **does not occur in** S**.**

Furthermore, we naturally formalize this consistency proof in PA.

III. (Foundational) We embed these findings into the current metamathematical studies.

Section I

Historical and mathematical context.

Peano Arithmetic

Peano Arithmetic, PA, is a formal first-order theory containing 0, functions $'$ (successor), $+$, $\times$, and the usual recursive identities for these functions. *Numerals* are terms

$$0, 0', 0'', 0''', \ldots.$$

PA contains *Induction Principle*: for each formula $\varphi(x)$,

$$[\varphi(0) \wedge \forall x(\varphi(x) \rightarrow \varphi(x'))] \rightarrow \forall x \varphi(x).$$

Primitive recursive (p.r.) functions are representable in PA: we can assume that terms for all p.r. functions are present in the language of PA along with the defining recursive conditions. If for a p.r. function f it holds that $f(n) = m$, then PA proves this fact

$$\mathsf{PA} \vdash f(n) = m.$$

Consequently, any p.r. relation $R(x_1, \ldots, x_k)$ is naturally represented in PA as well, and if $R(n_1, \ldots, n_k)$ holds then

$$\mathsf{PA} \vdash R(n_1, \ldots, n_k).$$

Proof and provability predicates

Let

$$Proof(u, v) \text{ (or "} u\!:\!v \text{" for short)}$$

denote the standard p.r. proof predicate in PA stating

u is a code of a PA-*proof of a formula having code v.*

In particular, *p is a* PA-*proof of* φ iff $\ulcorner p \urcorner\!:\!\ulcorner \varphi \urcorner$, with $\ulcorner X \urcorner$ denoting the Gödel number of X. We omit notation "$\ulcorner\ \urcorner$" when safe. So,

$$p \text{ is a PA-proof of } \varphi \quad \Leftrightarrow \quad p\!:\!\varphi.$$

Furthermore, we define

$$Provable(v) \text{ as } \exists u(u\!:\!v).$$

We also extend the notational convention, to reading

$$\Box\varphi \text{ as } Provable(\ulcorner \varphi \urcorner).$$

As usual, $\Box\varphi(x)$ is $Provable[\varphi^{\bullet}(x)]$ where $\varphi^{\bullet}(x)$ is a natural p.r. term which for any n returns $\ulcorner \varphi(n) \urcorner$.

The PA-consistency formula $\mathrm{Con}_{\mathrm{PA}}$ is $\neg\Box\bot$ where $\bot$ is $0=1$.

A strict Formalization Principle and IP

The Impossibility Paradigm tacitly uses not the familiar Formalization Principle, but rather a **strict Formalization Principle, sFP**:

> *Suppose a property $\mathcal{P}$ is formalized as a* PA*-formula $F_{\mathcal{P}}$. Then any arithmetical proof of $\mathcal{P}$ yields a* PA*-derivation of $F_{\mathcal{P}}$.*

sFP requires two inputs:

i) a formalization of a property $\mathcal{P}$ as an arithmetical formula $F_{\mathcal{P}}$;

ii) an informal arithmetical proof of $\mathcal{P}$.

sFP promises a formalization of (ii) which is a PA-derivation of (i).

sFP does not appear justified: a mathematical proof of $\mathcal{P}$ may be incompatible with an *a priori* choice of $F_{\mathcal{P}}$. We will show this happening.

Mathematical proof should precede formalization

The PA-consistency definition

no finite sequence of formulas is a PA*-derivation of* $\perp$.

is a mathematical statement about syntactic objects - finite sequences.

The approach "internalize consistency as a PA-sentence, e.g. $\mathsf{Con}_{\mathsf{PA}}$, and establish its provability in PA" does not itself yield the consistency of PA since an inconsistent theory vacuously proves anything.

A way to prove the consistency of PA by means of PA would be

to find a mathematical proof of PA**-consistency and *then* formalize this proof in** PA.

Consistency proof via the standard model

Let ω be the standard model of arithmetic (the set $\{0, 1, 2, 3, \ldots\}$ with the usual operations $+$ and $\times$). Let D be a derivation in PA. Then all formulas in D are true in ω.

- Base case - axioms of PA and logical postulates – is secured.
- The induction step: all logical rules in D are truth-preserving.

Inconsistency, e.g., $0 = 1$, is not true, hence cannot occur in D. QED

Though convincing, this is not a proof by means of PA because it uses the notion *true in standard model*, which is not itself expressible in PA.

So, the problem is not that of finding a proof of PA-consistency, there are many, but rather it is that of finding a proof of PA-consistency *formalizable in* PA.

Complete Induction and selector proofs

Example 1. Consider the property of *Complete Induction*, $\mathcal{CI}$, in PA:

$$\text{if for all } x \text{ "}\forall y < x\ \psi(y) \text{ implies } \psi(x)\text{," then } \forall x \psi(x),$$

and its **textbook proof**: take an arbitrary ψ, apply the usual induction to $\varphi(x) = \forall y < x\ \psi(y)$ to get the $\mathcal{CI}$ statement $\mathcal{CI}(\psi)$ for ψ. QED.

This proof is a **selector proof** consisting of selecting a PA-derivation for each instance of $\mathcal{CI}$. It is easily formalizable as a PA-derivation p of

$$\forall x[s(x){:}CI^{\bullet}(x)], \quad \text{with} \tag{1}$$

$CI^{\bullet}(x)$ a p.r. term which given $n = \ulcorner \psi \urcorner$ computes $CI^{\bullet}(n) = \ulcorner \mathcal{CI}(\psi) \urcorner$;

$s(x)$ a primitive recursive **"selector"** term which for any code of a formula ψ computes the code of a derivation in PA of $\mathcal{CI}(\psi)$;

p a PA-proof ("**verifier**") of (1).

Note: *Complete Induction* cannot be represented by a single formula in PA hence sFP does not cover Example 1.

Selector proofs are ubiquitous: Tautologies

Example 2. Take one of de Morgan's Laws[4]: *for any formulas* X, Y,

$$(\neg X \vee \neg Y) \to \neg(X \wedge Y). \tag{2}$$

How do we prove de Morgan's Law in arithmetic? For given X, Y, we find the standard logical derivation $D(X, Y)$ of (2) in PA.

This is a *selector proof* which **builds an individual** PA-**derivation for each instance of de Morgan's Law**. It is formalized in PA as

$$\mathsf{PA} \vdash \forall x, y[s(x, y){:}dML^{\bullet}(x, y)].$$

Here $dML^{\bullet}(x, y)$ *is an obvious p.r. term such that*

$$dML^{\bullet}(\ulcorner X \urcorner, \ulcorner Y \urcorner) = \ulcorner(\neg X \vee \neg Y) \to \neg(X \wedge Y)\urcorner.$$

The selector $s(x, y)$ is a p.r. term s.t. $s(\ulcorner X \urcorner, \ulcorner Y \urcorner) = \ulcorner D(X, Y)\urcorner$.

[4] Any tautology will do too.

Iterated Consistency

Moshe Vardi asked (in 2021) what was wrong with this following proof.

Example 3. Consider theories:

$$\mathsf{PA}_0 = \mathsf{PA}, \qquad \mathsf{PA}_{i+1} = \mathsf{PA}_i + \mathsf{Con}_{\mathsf{PA}_i} \qquad \mathsf{PA}^\omega = \bigcup_{i\in\omega} \mathsf{PA}_i.$$

We prove **consistency of** PA^ω **in** PA^ω as follows.

Let D be a derivation in PA^ω and let i be the largest index of $\mathsf{Con}_{\mathsf{PA}_i}$'s occurring in D. Then D is a derivation in PA_{i+1}, and $\mathsf{Con}_{\mathsf{PA}_{i+1}}$ – one of the postulates of PA^ω – implies that D does not contain $\bot$. QED.

This is a selector proof formalizable in PA by a selector $s(x)$ which given n computes the (code of) PA^ω-derivation of "*n is not a proof of $\bot$*":

$$\mathsf{PA} \vdash \forall x[s(x){:}^\omega \neg x{:}^\omega \bot]^5.$$

Moral: **this is a consistency proof of PA^ω by means of PA^ω, which is not, however, a derivation of the consistency formula $\mathsf{Con}_{\mathsf{PA}^\omega}$.**

[5] Here "${:}^\omega$" is the p.r. proof predicate for PA^ω.

Findings so far

1. Selector proofs are legitimate ways of reasoning widely used in mathematical practice. In metamathematics of PA, due to limitations of the language, selector proofs are ubiquitous.
2. Selector proofs from Examples 1,2,3 are standard mathematical arguments, naturally formalizable "as is" in PA.
3. None of Examples 1,2,3 is covered by sFP. In Examples 1,2, a single-formula presentation of the property $\mathcal{P}$ does not exist. In Example 3, an *a priori* formula presentation $F_{\mathcal{P}}$ of $\mathcal{P}$ is $\mathrm{Con}_{\mathrm{PA}^{\omega}}$, but it has nothing to do with the given selector proof of $\mathcal{P}$.
4. This indicates a loophole in sFP and the Impossibility Paradigm:

some formalizable arithmetical arguments, selector proofs, are not accounted for.

Selector proofs for a traditional logician, L

L: *Each of Examples 1,2,3 yields an infinite series of finite* PA*-derivations. Such "infinitary proofs" are not legitimate proof objects in* PA.

Our response: This "infinitary proofs" observation should not be used as an excuse to avoid their fair formalization in PA.

> *Many "infinite" mathematical objects can be represented in* PA *in a finite form. A function on natural numbers is an infinite set of pairs. However,* PA *represents p.r. functions in a finite form as definable terms and works with them normally.*

The same has been done with "infinite" selector proofs from Examples 1,2,3. These proofs enjoy natural finite formalization in PA.

Note: **selector proofs do not require new derivations in** PA. We just have to accept the obvious and recognize existing PA-derivations

$$\mathsf{PA} \vdash \forall x[s(x){:}\varphi(x)]$$

as natural formalizations of selector proofs of $\varphi(n)$ for $n = 0, 1, 2, \ldots$.

Further discussion on selector proofs

L: *Can we just say that "a scheme is provable in* PA*" means "each instance of a scheme is provable in* PA*"?*

Our response: This does not work. Such simplistic reading ignores the question of **why** each instance of a scheme is provable in PA an answer to which could require tools from outside PA. Without controlling this issue, each **true** Π^0_1-sentence $\forall x \varphi(x)$ becomes **provable** in PA as a scheme.

> *This approach would recognize the standard model proof of "D is consistent for each* PA*-derivation D" as a proof in* PA *despite the observable fact that it uses tools from outside* PA*.*

More discussion on selector proofs

L: *Then "a scheme is provable in* PA*" should mean*

"PA *proves that each instance of a scheme is provable in* PA." (3)

Our response: This is a step toward selector proofs. A possible idea to formalize this intuition is to define "scheme $Q(u)$ is provable in PA" as

$$\mathrm{PA} \vdash \forall x \Box Q^{\bullet}(x) \tag{4}$$

Provability predicate "$\Box$" represents proofs implicitly which makes (4) dependent on consistency assumptions: without assuming ω-consistency[6] we cannot even prove that (4) yields $\mathrm{PA} \vdash Q(u)$ for each u.

Selector proofs avoid this deficiency by using explicit selector function "$s(x)$" instead of implicit "$\Box$":

$$\mathrm{PA} \vdash \forall x [s(x){:}Q^{\bullet}(x)] \tag{5}$$

PA proves that (5) yields $\mathrm{PA} \vdash Q(u)$ for each u (below).

[6] **1**-consistency suffices.

The Logician's Dilemma

Logicians face a conceptual choice:

1. One might **exclude selector proofs**[7] **from consideration** and thus give up the claim that formal proofs represent all of mathematical reasoning.

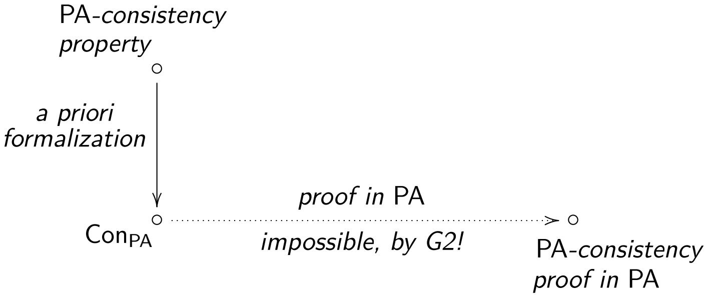
   Imagine the ensuing scandal should the logical world admit that Induction, Tautologies (e.g., De Morgan's Laws) weren't provable in arithmetic. Moreover, that Consistency wasn't provable for the same bureaucratic reason: selector proofs are not included.

2. Alternatively, we could **acknowledge selector proofs together with their natural formalizations in** PA and reconsider the Impossibility Paradigm.

[7] or similar constructs

The impossible road

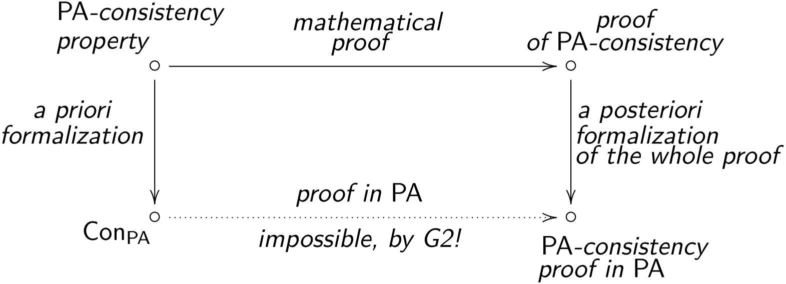
By G2, it is impossible to prove the consistency of PA in PA when we begin with an *a priori* formalization of consistency as $\mathrm{Con}_{\mathrm{PA}}$.



However, the goal of finding a PA-consistency proof formalizable in PA does not suggest starting with formalization of PA-consistency as $\mathrm{Con}_{\mathrm{PA}}$ (and hence taking the "impossible" road).

Our road

We take the alternate route of a) proving PA-consistency mathematically **and only then** b) formalizing the whole proof in PA.



The resulting formalized proof of PA-consistency is not a PA-derivation of $\mathsf{Con}_{\mathsf{PA}}$, and hence will not be precluded by G2.

Section II

Proof of PA-consistency by means of PA

The consistency proof for PA by means of PA

The proof of consistency for PA goes by two consecutive steps.

i) A direct selector proof of PA-consistency in its original form:

no PA-*derivation contains* $\perp$.

For any given PA-**derivaton** S, **we find a** PA-**definable invariant**

$$\mathcal{I}_S$$

and establish that for each φ **in** S, $\mathcal{I}_S(\varphi)$ **holds,** $\mathcal{I}_S(\perp)$ **does not hold, hence** $\perp$ **does not occur in** S[8]**.**

ii) A comprehensive formalization of (i) in PA.

---

[8] G2 prohibits having such invariant $\mathcal{I}$ *uniformly for all derivations* S, but does not rule out the possibility of having an invariant $\mathcal{I}_S$ for each S.

Partial truth definitions in PA

In the metamathematics of first-order arithmetic, there is a well-known construction called *partial truth definitions*. Namely, for each $n = 0, 1, 2, \ldots$ we build, **in a primitive recursive way**, a Σ_{n+1}-formula

$$Tr_n(x, y),$$

called *truth definition for* Σ_n*-formulas*, which satisfies natural properties of a truth predicate.

Intuitively, when φ is a Σ_n-formula and y is a sequence encoding values of the parameters in φ, then $Tr_n(\ulcorner\varphi\urcorner, y)$ defines the truth value of φ on y.

Partial truth definitions in PA

Proposition 1 (well-known classics in proof theory of PA)
For any Σ_n-formula φ, PA *naturally proves Tarksi's condition:*

$$Tr_n(\ulcorner\varphi\urcorner, y) \;\Leftrightarrow\; \varphi(y).$$

In particular, $\neg Tr_n(\ulcorner\bot\urcorner, y)$ is naturally provable, i.e., PA *proves that formula $\bot$ does not satisfy Tr_n.*

Corollary *For any axiom A of* PA *of depth $\leq n$, $Tr_n(\ulcorner A\urcorner, y)$ is provable; Tr_n supports rules of inference in* PA *for formulas of depth $\leq n$.*

Note that all proofs in Proposition 1 and its Corollary are rigorous mathematical arguments without any metamathematical assumptions about PA. The formal language of PA is used here just for bookkeeping.

A proof of consistency for PA

- Given a finite PA-derivation $\mathcal{D}$, calculate its Gödel number $n = \ulcorner \mathcal{D} \urcorner$; we can assume that all formulas from $\mathcal{D}$ have depth $\leq n$.
- Then, by induction up to the length of $\mathcal{D}$, we check that for any formula φ in $\mathcal{D}$ with parameters y, the property $Tr_n(\ulcorner \varphi \urcorner, y)$ holds. This is immediate from Corollary of Proposition 1, since each axiom from $\mathcal{D}$ satisfies Tr_n and each rule of inference respects Tr_n. So,

  Tr_n **serves as an invariant for formulas from** $\mathcal{D}$.

- By Proposition 1, $\perp$ does not satisfy Tr_n, hence is not in $\mathcal{D}$.

Q.E.D.

This is a rigorous mathematical proof of consistency of PA. This proof uses only principles of PA. So, intuitively, this is a proof by means of PA.

Now we check that this proof is indeed step-by-step formalizable in PA.

Specifics of the consistency proof formalization

Here is a description of a primitive recursive selector $s(x)$ connecting a given $n = \ulcorner \mathcal{D} \urcorner$ with the (code of a) PA-proof $s(n)$ of $\neg n{:}\bot$.

Since n is the Gödel number of a PA-derivation $\mathcal{D}$, all formulas from $\mathcal{D}$ have depth $\leq n$. All quantifiers used in the description of the procedure are now bounded by obvious primitive recursive functions of n.

For any formula φ in $\mathcal{D}$, starting with axioms, by induction up to the length of $\mathcal{D}$, we build a PA-proof of $Tr_n(\ulcorner\varphi\urcorner, y)$. Since, by Proposition 1, PA proves $\neg Tr_n(\ulcorner\bot\urcorner, y)$, this yields a proof that $\bot$ is not in $\mathcal{D}$.

By this description, $s(x)$ is primitive recursive and PA naturally proves

$$\forall x[s(x){:}\neg x{:}\bot]. \tag{6}$$

A summary of what happened

1. We have offered a mathematical proof of Hilbert's consistency of PA in its original combinatorial format

no PA*-derivation* $\mathcal{D}$ *contains a contradiction.*

Specifically, given a derivation $\mathcal{D}$, by a well-known method, we built an invariant - arithmetical formula Tr_n with $n = \ulcorner D \urcorner$ - such that all formulas from $\mathcal{D}$ satisfy and the contradiction $\perp$ does not satisfy this invariant. This constitutes a selector proof of Hilbert's consistency of PA.

2. We formalize 1 in PA by a primitive recursive selector which given $\mathcal{D}$ returns a PA-proof of consistency for $\mathcal{D}$, and a verifier, a PA-proof that the selector does the job uniformly for all inputs $\mathcal{D}$.

Section III

Foundational analysis

How far we can go with proving consistency in PA

Let $x{:}_T\varphi$ be a shorthand for a proof predicate in a theory T[9]:

"x is a code of a proof of formula φ in T,"

and $\Box_T\varphi$ denote $\exists x(x{:}_T\varphi)$. As before, $\Box\varphi$ is $\exists x(x{:}\varphi)$.

Suppose PA selector proves consistency of a theory T. Then for some primitive recursive term $s(x)$, PA proves

$$\forall x\ s(x){:}\neg x{:}_T\bot.$$

By logical reasoning, PA then would prove

$$\forall x\ \Box\neg x{:}_T\bot$$

which was independently shown to be impossible by **Kurahashi** and **Sinclaire** for any $T \supseteq \mathrm{PA} + \mathrm{Con}_{\mathrm{PA}}$.

This indicates that PA cannot prove consistency of $\mathrm{PA} + \mathrm{Con}_{\mathrm{PA}}$ by the given method without further modifications.

[9] We drop this subscript when $T = \mathrm{PA}$.

Logician's analysis: proving schemes in PA

Let $\varphi(u)$ be an arithmetical formula with a parameter u.

> *We will skip a formal definition: think of a parameter u ranging over numerals, terms, formulas, etc. such that for each value of u, $\varphi(u)$ is a legitimate arithmetical formula.*

Such $\varphi(u)$ can be viewed as a **scheme**, notation

$$\{\varphi(u)\}.$$

For example, *Induction Principle*, *Complete Induction*, *de Morgan's Law*, *etc.*, may be regarded as schemes with the formula parameters.

Scheme $\{\neg n{:}\bot\}$ with a numeral parameter n is a PA-consistency scheme

$$\mathsf{ConS}_{\mathsf{PA}}.$$

The scheme format of $\mathsf{ConS}_{\mathsf{PA}}$ follows Hilbert's understanding of a finitary general proposition as: **"a hypothetical judgment that comes to assert something when a numeral is given."**

Proofs of schemes

Definition. A **proof of scheme** $\{\varphi(u)\}$ in PA is a pair $\langle s(x), p\rangle$ where

- $s(x)$ is a primitive recursive term (**selector**),
- p is a PA-proof (**verifier**) of

$$\forall x[s(x){:}\varphi^\bullet(x)].$$

Here $\varphi^\bullet(x)$ is a natural term for a p.r. function $\varphi^\bullet(\ulcorner u\urcorner)=\ulcorner\varphi(u)\urcorner$.

Basic properties:

- proofs of schemes are finite syntactic objects,
- proofs of schemes are decidable,
- the set of provable schemes is recursively enumerable.

Formalizing consistency in PA

The mathematical definition of PA-consistency is

no PA-*derivation* S *contains* $\perp$.

is a sentence with a parameter S ranging over PA-derivations.

Gödel numbering naturally formalizes it as an **arithmetical scheme**:

$$\textit{no numeral } n \textit{ is a code of a } \mathsf{PA}\textit{-derivation containing } \perp. \quad (7)$$

In our notation, it is $\{\neg n{:}\perp\}$ which we call $\mathrm{ConS}_{\mathrm{PA}}$.

At this stage, we treat Consistency on the same grounds as other arithmetical schemes including *Tautologies, Induction, Complete Induction, etc.*: we selector prove $\mathrm{ConS}_{\mathrm{PA}}$ in the same way.
This yields a proof of Consistency formalizable in PA.

Further **internalization** of (7) to formula $\forall x(\neg x{:}\perp)$ (a.k.a. $\mathrm{Con}_{\mathrm{PA}}$) becomes irrelevant: we have found a formalizable proof of Consistency and the failures of other attempts no longer matter.

Consistency Scheme vs. Consistency Formula

We do not argue what is a "proper" arithmetical representation of PA-consistency,

$$\mathsf{Con}_{\mathsf{PA}} \quad \textit{or} \quad \mathsf{ConS}_{\mathsf{PA}}.$$

We admit both.

The consistency formula $\mathsf{Con}_{\mathsf{PA}}$ plays a pivotal role in the proof-theoretical studies. Nothing undermines this role.

The consistency scheme $\mathsf{ConS}_{\mathsf{PA}}$ refutes the Impossibility Paradigm, which has significant foundational and cognitive consequences.

Provability of schemes

A scheme $\{\varphi(u)\}$ is **instance provable in** PA if PA $\vdash \varphi(u)$, for each value of the parameter u.

The following proposition about schemes is proven by reasoning in PA.

Proposition 2. *Provable* $\Rightarrow$ *Instance Provable.*

Proof. Let selector s, verifier p be given and

$$p{:}\forall x[s(x){:}\varphi^{\bullet}(x)].$$

Given u, we want to check that PA $\vdash \varphi(u)$. By an easy transformation, find a proof q_u such that $q_n{:}s(u){:}\varphi(u)$. If $s(u){:}\varphi(u)$ holds (which is a p.r. test), $s(u)$ is a PA-derivation of $\varphi(u)$ hence PA $\vdash \varphi(u)$.

If $s(u){:}\varphi(u)$ does not hold, by completeness of PA w.r.t. primitive recursive conditions, find r such that $r{:}\neg s(u){:}\varphi(u)$. Combining q_u and r, find t such that $t{:}\bot$, which is impossible by the PA-consistency.

Corollary. *Arithmetic proves that proofs of schemes do not add new theorems to* PA. *So,* PA *with proofs of schemes is consistent and formalizes a selector proof of its own consistency.*

Provable vs. Instance Provable

Instance Provable $\not\Rightarrow$ Provable

Scheme $\{\neg n{:}\Box\bot\}$ is instance provable since $\neg n{:}\Box\bot$ is true for each $n = 0, 1, \ldots$, and hence provable in PA as a true p.r. sentence.

Suppose scheme $\{\neg n{:}\Box\bot\}$ is provable. Then

$$\mathsf{PA} \vdash \forall x \Box \neg x{:}\Box\bot. \tag{8}$$

We claim that then we would have $\mathsf{PA} \vdash \Box\Box\bot \rightarrow \Box\bot$ (which is false).

Indeed, reason in PA and assume $\Box\Box\bot$ i.e. $\exists x(x{:}\Box\bot)$. By a strong form of provable Σ_1-completeness, $\mathsf{PA} \vdash x{:}\Box\bot \rightarrow \Box x{:}\Box\bot$ and we would have $\exists x \Box x{:}\Box\bot$. From (8), we get $\Box\bot$.

A proof of $\mathsf{ConS_{PA}}$ which is not a proof of PA-consistency.

Consider a p.r. function $r(x)$ which given n returns a proof of $\neg n{:}_T\bot$.

Fix a formal theory T containing PA*. Given n, check whether n is a proof of $\bot$ in T. If YES, then put $r(n)$ to be a simple T-derivation of $\neg n{:}_T\bot$ from $\bot$. If NO, then use provable Σ_1-completeness and put $r(n)$ to be a constructible derivation of $\neg n{:}_T\bot$ in T. Let also p be a* PA*-proof of*

$$\forall x[r(x){:}_T\neg x{:}_T\bot].$$

One can recognize here a version of the well-known argument which proves consistency formula based on Rosser's provability predicate.

Two questions:

a) Is $\langle r, p\rangle$ a legitimate proof of scheme $\{\neg n{:}_T\bot\}$?

b) When $T = \mathsf{PA}$, whether $\langle r, p\rangle$ is a proof of PA-consistency in PA?

The answer to (a) is obviously YES when $T = \mathsf{PA}$ since $\langle r, p\rangle$ fits the definition of a proof of scheme $\{\neg n{:}\bot\}$, so $\langle r, p\rangle$ is a proof of $\mathsf{ConS_{PA}}$.

More subtleties: look at the proof

The answer to (b) is NO, and this case deserves more discussion.

Since a consistency proof of PA in PA, in the first place, should be a mathematical proof of PA-consistency, this question should be understood as whether $r(n)$, as a mathematical argument, proves that a PA-derivation n does not contain $\perp$. The answer to this question is obviously negative: $r(n)$ only tells us that if n contains $\perp$, we would still be able to offer a fake proof of $\neg n{:}\perp$. This is NOT a consistency proof.

Traditional proof theorists should not feel disappointed not to see a clean formal criterion of **what counts as a consistency proof**. After all, for decades, logicians have not had a clean formal criterion of **what counts as a consistency formula**, and have been using their informal judgements to rule out most of them.

The Impossibility Paradigm and Hilbert's program

Selector proofs are the standard mathematical tool for proving schemes of arithmetical formulas as universal propositions. Such proofs are ubiquitous in the metamathematics of PA.

> *We have demonstrated that* **there is a mathematical proof of the** PA-**consistency formalizable in** PA**.**

Technically, it has been shown that there is a selector proof of the PA-consistency scheme formalizable in PA.

This refutes the Impossibility Paradigm .

We don't know to what extent Hilbert's program of proving consistency of stronger systems by means of a trusted core is possible. But now, with a major road block removed, as Gödel said, *Hilbert's program is very much alive and even more interesting than it initially was.*

Summary

The principal contribution of this work is conceptual. We hope that showing the Impossibility Paradigm to be illusory has a general foundational value.

Thanks

Thanks to

Lev Beklemishev, Sam Buss, Walter Carnielli, John Connor, Michael Detlefsen, Jon Michael Dunn, Hartry Field, Mel Fitting, Harvey Friedman, Richard Heck, Carl Hewitt, Harold Hodes, John H. Hubbard, Reinhard Kahle, Juliette Kennedy, Dongwoo Kim, Karen Kletter, Vladimir Krupski, Taishi Kurahashi, Hirohiko Kushida, Roman Kuznets, David Makinson, Yuri Matiyasevich, Richard Mendelsohn, Eoin Moore, Larry Moss, Andrei Morozov, Anil Nerode, Elena Nogina, Vladimir Orevkov, Fedor Pakhomov, Rohit Parikh, Vincent Peluce, Brian Porter, Graham Priest, Michael Rathjen, Andrei Rodin, Chris Scambler, Luke Serafin, Sasha Shen', Richard Shore, Morgan Sinclaire, Stanislav Speranski, Thomas Studer, Moshe Vardi, Albert Visser, Dan Willard, Noson Yanofsky, and many others.

Reference

The provability of consistency
S. Artemov

arXiv preprint arXiv:1902.07404, 2019
https://arxiv.org/abs/1902.07404

Originally submitted February 19, 2019;
current version is of June 22, 2020.