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DIALOGICAL LOGIC FOR DEFINITIONAL REASONING AND IMPLICATIONS AS RULES*

1 Introduction

In dialogical logic, the logical constants are given a game-theoretic interpretation (see Lorenzen 1960, 1961; cf. Lorenz 1968, 1973, 2001, Lorenzen and Lorenz 1978 and Lorenzen 1982; for an overview, see Keiff 2011 and Piecha 2014). Dialogues are two-player games between a proponent and an opponent, where each of the two players can either attack claims made by the other player or defend their own claims. For example, an implication $A \to B$ can be attacked by claiming A and is defended by claiming B. This means that in order to have a winning strategy for $A \to B$, the proponent must be able to argue successfully for B depending on what the opponent can put forward in defense of A. The logical constant of implication has thus been given a certain game-theoretic or dialogical interpretation, and corresponding dialogical interpretations can be given for the other logical constants as well.

Here this approach will be extended in two directions: First, we want to make it possible that also definitions can be treated dialogically (cf. Piecha 2012). A definition is understood as a rule system which specifies the meaning of atomic assertions, that is, of assertions which do not contain any logical constant. The rules are like predicator rules (Kamlah and Lorenzen 1967), rules of an atomic production system (as, e.g., in operative logic (Lorenzen 1955) or in logic programming (Hallnäs and Schroeder-Heister 1990)) or like the rules in an inductive definition (cf. Aczel 1977). Following the terminology of logic programming, such definitional rules for atomic assertions will also be called *clauses*. The second extension concerns an alternative understanding of implications. The notion of implication is closely related to that of a rule (in Lorenzen's 1955 operative logic it is even identified with it). We want to establish dialogues based on the interpretation of implications as rules which can be used to justify assertions. In contradistinction to the

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first extension by definitional clauses for atomic assertions, this second extension is about arbitrarily complex implications, which are understood as rules. These rules constitute a kind of dynamic database, which is generated by the opponent (cf. Piecha 2012 and Piecha and Schroeder-Heister 2012).

In the following, we will first provide the basic notions of dialogical logic, and we will outline the standard dialogues for intuitionistic propositional logic. One feature of these standard dialogues is the fact that atomic assertions cannot be attacked. We will then introduce so-called *definitional dialogues* as a means to treat definitions of atomic assertions dialogically. These definitional dialogues do allow for attacks on atomic assertions. Definitions are also important in certain applications of logic, as for example in logic programming, where they figure as logic programs. Both logic programming and the operative interpretation of implications suggest an alternative understanding of implications as rules. On this understanding, implications are of a kind different from the other logical constants. In order to grasp this difference dialogically, we introduce specific dialogues for implications as rules, which crucially differ from the standard dialogues. Finally, we will discuss these differences. For further aspects of the extensions of dialogical logic considered here we refer the reader to Piecha 2012.

2 Dialogues for intuitionistic propositional logic

We define our language, the argumentation forms for logical constants, and the concepts of dialogue and winning strategy. We follow the presentation of Felscher (1985, 2002) with slight deviations. We focus on dialogues for intuitionistic propositional logic. Intuitionistic logic is of special interest, since in it implication is a distinct logical constant which cannot be defined by, for example, negation and disjunction as in classical logic.

Definition 2.1

(i) The *language* consists of propositional *formulas* $A, B, \dots, A_1, A_2, \dots$ which are constructed from *atomic formulas* (*atoms*) $a, b, \dots, a_1, a_2, \dots$ with the *logical constants* $\wedge$ (conjunction), $\vee$ (disjunction), $\to$ (implication) and $\neg$ (negation).

(ii) *Special symbols* are ?1, ?2 and ?$\vee$.

(iii) The symbols **P** (›proponent‹) and **O** (›opponent‹) are used as *signatures*.

(iv) An *expression e* is either a formula or a special symbol. For each expression *e*, there is a **P**-*signed expression* **P** *e* and an **O**-*signed expression* **O** *e*.

(v) A signed expression is called an *assertion* if the expression is a formula; it is called a *symbolic attack* if the expression is a special symbol. X and Y, where $X \neq Y$, are used as variables for **P** and **O**.

Definition 2.2 For each logical constant an *argumentation form* determines how a complex formula (having this constant in main position) that is asserted by X can be attacked by Y, and how this attack can be defended (if possible) by X. The argumentation forms are as follows:

AF($\wedge$):	assertion:	$X\,A_1 \wedge A_2$	
	attack:	$Y\,?i$	(Y chooses $i = 1$ or $i = 2$)
	defense:	$X\,A_i$	
AF($\vee$):	assertion:	$X\,A_1 \vee A_2$	
	attack:	$Y\,?\vee$	
	defense:	$X\,A_i$	(X chooses $i = 1$ or $i = 2$)
AF($\to$):	assertion:	$X\,A \to B$	
	attack:	$Y\,A$	
	defense:	$X\,B$	
AF($\neg$):	assertion:	$X\,\neg A$	
	attack:	$Y\,A$	
	defense:	*no defense*	

In the literature, the argumentation forms are also called ›particle rules‹ (›Partikelregeln‹) or ›logical rules‹.

Definition 2.3 For each $n \geq 0$, let $\delta(n)$ be a signed expression and $\eta(n)$ a pair $[m, Z]$ where Z is either A (for ›attack‹) or D (for ›defense‹), and where $\eta(0)$ is empty. The pairs $\langle\delta(n), \eta(n)\rangle$ are called *moves*.

A move $\langle\delta(n), \eta(n)\rangle$ where $\eta(n)$ is of the form $[m, A]$ is called an *attack move* (short: *attack*), and a move $\langle\delta(n), \eta(n)\rangle$ where $\eta(n)$ is of the form $[m, D]$ is called a *defense move* (short: *defense*).

Thus $\delta(n)$ is a function mapping natural numbers $n \geq 0$ to signed expressions $X\,e$, and $\eta(n)$ is a function mapping natural numbers $n \geq 0$ to pairs $[m, Z]$. The numbers in the domains of $\delta(n)$ and $\eta(n)$ are called *positions*.

When talking about a move $\langle\delta(n), \eta(n)\rangle$, we write $\langle\delta(n) = X\,e, \eta(n) = [m, Z]\rangle$ to express that $\delta(n)$ has the value $X\,e$ for position n, and that $\eta(n)$ has the value $[m, Z]$ for position n. For example, $\langle\delta(n) = \mathbf{P}\,A, \eta(n) = [m, D]\rangle$ denotes a defense move which is made by the proponent $\mathbf{P}$ at position n by asserting the formula A; this defense move refers to a move made at position m. A concrete move like $\langle\delta(4) = \mathbf{P}\,?1, \eta(4) = [3, A]\rangle$ will also be written as

$$4. \quad \mathbf{P}\,?1 \quad [3, A]$$

This is an attack move with the symbolic attack $\mathbf{P}\,?1$; it is made at position 4 and refers to a move made at position 3.

The notation $\langle \delta(n) = \mathbf{X}\, e, \eta(n) = [m, Z] \rangle$ has the advantage that in speaking about a move $\langle \mathbf{X}\, e, [m, Z] \rangle$, we can include information about the position n at which this move is made.

Although moves are always pairs $\langle \delta(n), \eta(n) \rangle$, we will also refer to moves by giving only their $\delta(n)$-component, as long as it is clear from the context which move is meant, or if it is irrelevant whether the move is an attack or a defense, or if it is irrelevant which position the move refers to. And instead of $\langle \delta(n) = \mathbf{X}\, e, \eta(n) \rangle$ we will also speak of ›the move $\mathbf{X}\, e$ made at position n.‹

Definition 2.4 A *pre-dialogue* is a finite or infinite sequence of moves $\langle \delta(n), \eta(n) \rangle$ (for $n = 0, 1, 2, \dots$) satisfying the following *dialogue conditions*:

(D00) $\delta(n)$ is a **P**-signed expression if n is even and an **O**-signed expression if n is odd. The expression in $\delta(0)$ is a complex formula.

(D01) If $\eta(n) = [m, A]$, then the expression in $\delta(m)$ is a complex formula and $\delta(n)$ is an attack on this formula as determined by the relevant argumentation form.

(D02) If $\eta(p) = [n, D]$, then $\eta(n) = [m, A]$ for $m < n < p$ and $\delta(p)$ is the defense of the attack $\delta(n)$ as determined by the relevant argumentation form.

Definition 2.5 An attack $\langle \delta(n), \eta(n) = [m, A] \rangle$ at position n on an assertion at position m is called *open at position k* for $k > n$ if there is no position n' such that $n < n' \leq k$ and $\eta(n') = [n, D]$, that is, if there is no defense at or before position k to an attack at position n.

Since there is no defense against an attack $\langle \delta(n) = \mathbf{Y}\, A, \eta(n) = [m, A] \rangle$ on $\delta(m) = \mathbf{X} \neg A$ for $m < n$, the attack at position n is open at all positions $k > n$.

We define DI^{P}-dialogues and winning proponent strategies. With regard to the literature on dialogical logic, DI^{P}-dialogues can be considered to be the standard dialogues for intuitionistic propositional logic. The following definition of DI^{P}-dialogues is based on the definition of pre-dialogues.

Definition 2.6 A DI^{P}*-dialogue* (short: *dialogue*) is a pre-dialogue satisfying the following *dialogue conditions* (in addition to (D00), (D01) and (D02)):

(D10) If, for an atomic formula a, $\delta(n) = \mathbf{P}\, a$, then there is an $m < n$ such that $\delta(m) = \mathbf{O}\, a$.
That is, **P** may assert an atomic formula only if it has been asserted by **O** before.

(D11) If $\eta(p) = [n, D]$, $n < n' < p$, $n' - n$ is even and $\eta(n') = [m, A]$, then there is a p' such that $n' < p' < p$ and $\eta(p') = [n', D]$.
That is, if at a position $p - 1$ there is more than one open attack, then only the last of them may be defended at position p.

(D12) For every m there is at most one n such that $\eta(n) = [m, D]$.
That is, an attack may be defended at most once.
(D13) If m is even, then there is at most one n such that $\eta(n) = [m, A]$.
That is, a **P**-signed formula may be attacked at most once.

Dialogue conditions are also called ›structural rules‹ or ›frame rules‹ (›Rahmenregeln‹) in the literature.

A $\mathrm{DI^P}$-dialogue beginning with $\mathbf{P}\,A$ (i.e., where $\delta(0) = \mathbf{P}\,A$ for a complex formula A) is called a $\mathrm{DI^P}$-*dialogue for the formula A*.

Proponent **P** and opponent **O** are not interchangeable due to the asymmetries between **P** and **O** introduced by (D10) and (D13). For atomic formulas a, the proponent move $\langle \delta(n) = \mathbf{P}\,a, \eta(n) = [m, Z]\rangle$ is possible only after an opponent move $\langle \delta(m) = \mathbf{O}\,a, \eta(m) = [k, Z]\rangle$ for $k < m < n$, and **O** can attack a **P**-signed formula only once, whereas **P** can attack **O**-signed formulas repeatedly.

These asymmetries are introduced by dialogue conditions only. The argumentation forms themselves (as given in Definition 2.2) are symmetric with respect to the two players **P** and **O**. That is, they are independent of whether the assertion is made by **P** or by **O**.

Definition 2.7 **P** *wins a dialogue for a formula A* if the dialogue is finite, begins with the move $\mathbf{P}\,A$ and ends with a move of **P** such that **O** cannot make another move.

As an example, we consider a dialogue for $(a \vee b) \to \neg\neg(a \vee b)$:

0.	$\mathbf{P}\,(a \vee b) \to \neg\neg(a \vee b)$	
1.	$\mathbf{O}\,a \vee b$	$[0, A]$
2.	$\mathbf{P}\,?\vee$	$[1, A]$
3.	$\mathbf{O}\,a$	$[2, D]$
4.	$\mathbf{P}\,\neg\neg(a \vee b)$	$[1, D]$
5.	$\mathbf{O}\,\neg(a \vee b)$	$[4, A]$
6.	$\mathbf{P}\,a \vee b$	$[5, A]$
7.	$\mathbf{O}\,?\vee$	$[6, A]$
8.	$\mathbf{P}\,a$	$[7, D]$

The dialogue ends with **P**'s move at position 8. The opponent cannot attack a, since it is an atomic formula. Each other **P**-signed formula has been attacked by **O**, thus no more attack moves can be made by **O** due to condition (D13), as these would be repetitions of attacks already made. And since each proponent attack that can be defended according to an argumentation form has already been defended by **O**, no more defense moves are possible either, due to condition (D12). The dialogue is finite, begins with the move $\mathbf{P}\,(a \vee b) \to$

$\neg\neg(a \vee b)$ and ends with a move of **P** such that **O** cannot make another move; the dialogue for $(a \vee b) \to \neg\neg(a \vee b)$ is thus won by **P**.

Definition 2.8 A *winning proponent strategy for a formula A* is a tree S whose nodes are moves and whose branches are dialogues for A won by **P**, such that

(i) S has as root node (with depth 0) the move **P** A;
(ii) if the depth of a node is odd (i. e., if the node is an **O**-move), then it has exactly one immediate successor node (which is a **P**-move);
(iii) if the depth of a node is even (i. e., if the node is a **P**-move), then it has for each possible **O**-move a corresponding immediate successor node.

As we are only interested in winning strategies for the proponent, we will simply refer to them as *winning strategies*.

The following tree is a winning strategy for $(a \vee b) \to \neg\neg(a \vee b)$:

0.	$\mathbf{P}\,(a \vee b) \to \neg\neg(a \vee b)$			
1.	$\mathbf{O}\,a \vee b$			$[0, A]$
2.	$\mathbf{P}\,\neg\neg(a \vee b)$			$[1, D]$
3.	$\mathbf{O}\,\neg(a \vee b)$			$[2, A]$
4.	$\mathbf{P}\,a \vee b$			$[3, A]$
5.	$\mathbf{O}\,?\vee$			$[4, A]$
6.	$\mathbf{P}\,?\vee$			$[1, A]$
7.	$\mathbf{O}\,a$	$[6, D]$	$\mathbf{O}\,b$	$[6, D]$
8.	$\mathbf{P}\,a$	$[5, D]$	$\mathbf{P}\,b$	$[5, D]$

Definition 2.9 A formula A is called *valid* (or DI^{P}-*valid*) if there is a winning strategy for A. Notation: $\vDash_{\mathrm{DI^P}} A$.

Theorem 2.1 (Completeness; see Felscher 1985) *$\vDash_{\mathrm{DI^P}} A$ if and only if A is provable in intuitionistic propositional logic.*

3 Dialogues for definitional reasoning

In standard dialogues, assertions of atomic formulas cannot be attacked, and dialogues won by the proponent always end with the assertion of an *atomic* formula. Compare the two following dialogues:

0.	$\mathbf{P}\,(a \vee b) \to \neg\neg(a \vee b)$		0.	$\mathbf{P}\,(a \vee b) \to \neg\neg(a \vee b)$	
1.	$\mathbf{O}\,a \vee b$	$[0, A]$	1.	$\mathbf{O}\,a \vee b$	$[0, A]$
2.	$\mathbf{P}\,?\vee$	$[1, A]$			
3.	$\mathbf{O}\,a$	$[2, D]$			
4.	$\mathbf{P}\,\neg\neg(a \vee b)$	$[1, D]$	2.	$\mathbf{P}\,\neg\neg(a \vee b)$	$[1, D]$
5.	$\mathbf{O}\,\neg(a \vee b)$	$[4, A]$	3.	$\mathbf{O}\,\neg(a \vee b)$	$[2, A]$
6.	$\mathbf{P}\,a \vee b$	$[5, A]$	4.	$\mathbf{P}\,a \vee b$	$[3, A]$
7.	$\mathbf{O}\,?\vee$	$[6, A]$			
8.	$\mathbf{P}\,a$	$[7, D]$			

The left dialogue is won by **P**; the assertion of the atomic formula a cannot be attacked. The right dialogue is *not* won by **P**, since **O** can attack **P**'s assertion $a \vee b$ with the move $\langle \delta(5) = \mathbf{O}\ ?\vee, \eta(5) = [4, A] \rangle$.

We now consider extensions of logic by a certain kind of definitions for atoms (definienda), where the defining conditions (definientia) are not restricted to conjunctions of atomic formulas, but can be arbitrary (first-order) formulas. These definitions are a generalization of monotone inductive definitions for atoms. They can also be seen as an extension of definite Horn clause programs, which are used in standard logic programming. Predicator rules fall within the scope of our notion of definition, too.

We introduce so-called definitional dialogues, which contain an additional argumentation form of definitional reasoning. This argumentation form allows for attacks on assertions of atomic formulas, which can then be defended by asserting the (atomic or complex) defining conditions of the atomic formula attacked, if a definition for the atomic formula has been given. As we want to reason about definitions whose defining conditions can be complex formulas, we have to make sure that it is possible that dialogues in a strategy can not only end with **P**-moves asserting atomic formulas, but also with **P**-moves asserting complex formulas. We first introduce so-called $\mathrm{EI}^{\mathrm{P}}_{\mathrm{c}}$-dialogues with this property. For this kind of dialogues there is also a completeness result with respect to intuitionistic propositional logic. We then introduce an argumentation form for definitional reasoning, and define definitional dialogues on the basis of $\mathrm{EI}^{\mathrm{P}}_{\mathrm{c}}$-dialogues.

The definitions for atoms need not be well-founded. This leads to paradoxes like Russell's, whose dialogical treatment will be considered as an example of definitional reasoning. The example shows that the structural operation of contraction can be critical in the presence of non-well-founded definitions: without further restrictions, there can be strategies for contradictory assertions in this case.

3.1 EI^p- AND EI^p_c-DIALOGUES

We first define EI^p-dialogues as a restricted form of DI^p-dialogues. They differ from DI^p-dialogues only in that each opponent move must now refer to the immediately preceding proponent move. This restriction yields certain technical advantages without changing the set of valid formulas extensionally.

Definition 3.10 An EI^p-*dialogue* is a DI^p-dialogue with the additional condition

(E) All moves $\langle \delta(n), \eta(n) \rangle$ for n odd are of the form $\langle \delta(n), \eta(n) = [n-1, Z] \rangle$. That is, an opponent move made at position n is either an attack or a defense of the immediately preceding move made by the proponent at position $n-1$.

The notions ›dialogue won by **P**‹ and ›winning strategy‹ as defined for DI^p-dialogues are directly carried over to the corresponding notions for EI^p-dialogues.

The EI^p-dialogues as they are defined here are the E-dialogues of Felscher 1985 and 2002 (references to their original formulation are given therein).

Definition 3.11 A formula A is called EI^p-*valid* if there is an EI^p-winning strategy for A. Notation: $\vDash_{EI^p} A$.

It has been shown by Felscher that there is a recursive function by which every EI^p-winning strategy can be embedded into a DI^p-winning strategy, and that therefore the EI^p-valid formulas are exactly the formulas provable in intuitionistic propositional logic (see Felscher 1985, 221, and Felscher 2002, 119; these results hold not only for the propositional but also for the first-order case). As the DI^p-valid formulas are also exactly the formulas provable in intuitionistic propositional logic, the following holds: $\vDash_{EI^p} A$ *if and only if* $\vDash_{DI^p} A$.

We now define EI^p_c-dialogues as follows:

Definition 3.12 An EI^p_c-*dialogue* is an EI^p-dialogue with the additional condition

(D14) **O** can attack a formula C if and only if either (i) C has not yet been asserted by **O** or (ii) C has already been attacked by **P**.

Again, the notions ›dialogue won by **P**‹ and ›winning strategy‹ as defined for DI^{p}-dialogues are directly carried over to the corresponding notions for $\mathrm{EI}^{\mathrm{p}}_{\mathrm{c}}$-dialogues.

Condition (E) implies condition (D13). Furthermore, condition (E) implies condition (D11) for odd p and condition (D12) for odd n (cf. Definition 2.6). In the presence of condition (E), condition (D13) can therefore be omitted, and conditions (D11) and (D12) can be restricted to conditions (D11′) and (D12′), respectively, as follows:

(D11′) If $\eta(p) = [n, D]$ for odd n, $n < n' < p$, $n' - n$ is even and $\eta(n') = [m, A]$, then there is a p' such that $n' < p' < p$ and $\eta(p') = [n', D]$.
That is, if at a position $p - 1$ there is more than one open attack by **O**, then only the last of them may be defended by **P** at position p.

(D12′) For every odd m there is at most one n such that $\eta(n) = [m, D]$.
That is, an attack by **O** may be defended by **P** at most once.

$\mathrm{EI}^{\mathrm{p}}_{\mathrm{c}}$-dialogues won by **P** need not end with the assertion of an *atomic* formula, but can also end with the assertion of a *complex* formula. Consider the following $\mathrm{EI}^{\mathrm{p}}_{\mathrm{c}}$-dialogue for $(a \vee b) \to \neg\neg(a \vee b)$:

0.	$\mathbf{P}\ (a \vee b) \to \neg\neg(a \vee b)$	
1.	$\mathbf{O}\ a \vee b$	$[0, A]$
2.	$\mathbf{P}\ \neg\neg(a \vee b)$	$[1, D]$
3.	$\mathbf{O}\ \neg(a \vee b)$	$[2, A]$
4.	$\mathbf{P}\ a \vee b$	$[3, A]$

The dialogue is won by **P**, and it is a winning strategy for $(a \vee b) \to \neg\neg(a \vee b)$. The opponent can no longer attack the assertion $a \vee b$ made by **P** in the last move at position 4 with the move $\langle \delta(5) = \mathbf{O}\ ?\vee, \eta(5) = [4, A] \rangle$, due to condition (D14): the formula $a \vee b$ has already been asserted by **O** at position 1, without having been attacked by **P**.

Definition 3.13 A formula A is called $\mathrm{EI}^{\mathrm{p}}_{\mathrm{c}}$-*valid* if there is an $\mathrm{EI}^{\mathrm{p}}_{\mathrm{c}}$-winning strategy for A. Notation: $\vDash_{\mathrm{EI}^{\mathrm{p}}_{\mathrm{c}}} A$.

Theorem 3.2 (Completeness) *The* $\mathrm{EI}^{\mathrm{p}}_{\mathrm{c}}$*-valid formulas are exactly the formulas provable in intuitionistic propositional logic.*

Completeness has been proved constructively in Piecha 2012 by showing that there is an $\mathrm{EI}^{\mathrm{p}}_{\mathrm{c}}$-winning strategy for a formula A if and only if A is provable in the sequent calculus for intuitionistic propositional logic with initial sequents $B \vdash B$, where B can be complex. This result is the theoretical basis for the introduction of definitional dialogues, which will allow us to reason about definitions whose defining conditions can be complex formulas.

3.2 Definitions

We introduce an argumentation form of definitional reasoning for inductive definitions of atoms. Such definitions are collections of definitional clauses which are formulated over a first-order language. We restrict ourselves to the quantifier-free fragment.

Definition 3.14 We extend our language to a (quantifier-free) *first-order language*. Using *variables* $x, y, \ldots$, (*individual*) *constants* $k, l, m, \ldots$ and *function symbols* $f, g, \ldots$, we define *terms* as follows:

(i) Every variable is a term.
(ii) Every individual constant is a term.
(iii) If f is an n-ary function symbol and $t_1, \ldots, t_n$ are terms, then $f(t_1, \ldots, t_n)$ is also a term.

We now use a, b, …, a_1, a_2, … also as *relation symbols*. If a is an n-ary relation symbol and if t_1, …, t_n are terms, then $a(t_1, \ldots, t_n)$ is an *atomic formula* (*atom*). Complex formulas are defined as usual.

Definition 3.15 A *definitional clause* is an expression of the form

$$a \Leftarrow B_1 \wedge \ldots \wedge B_n$$

for $n \geq 0$, where a is atomic and the B_i in the *body* $B_1 \wedge \ldots \wedge B_n$ of the clause are called the *defining conditions* for the *head* a. The defining conditions B_i need not be atomic but can also be complex formulas. Clauses with empty bodies are called *facts*; we indicate empty bodies with the symbol ›⊤‹ (*verum*).

Definition 3.16 A finite set $\mathcal{D}$ of definitional clauses

$$\mathcal{D}\begin{cases} a \Leftarrow \Gamma_1 \\ \quad\vdots \\ a \Leftarrow \Gamma_k \end{cases}$$

is a *definition of the atom a*, where $\Gamma_i = B_1^i \wedge \ldots \wedge B_{n_i}^i$ is the body of the i-th clause (for $1 \leq i \leq k$). These clauses are the *defining clauses of a* with respect to definition $\mathcal{D}$.

We write the bodies Γ_i of definitional clauses as conjunctions $B_1^i \wedge \ldots \wedge B_{n_i}^i$ of the defining conditions $B_{l_i}^i$. They could also be written as a list or set $B_1^i, \ldots, B_{n_i}^i$, where the comma functions as a ›structural conjunction‹. The latter notation is more convenient in a sequent calculus setting. However, for dialogues we would first have to introduce a means to handle such lists or sets, whereas we can handle conjunctions directly via the argumentation form for $\wedge$.

Definition 3.17 A *definition* is any finite set of definitional clauses. Definitions $\mathcal{D}$ have thus the general form

$$\mathcal{D}\begin{cases} a_1 \Leftarrow \Gamma_1^1 & & a_n \Leftarrow \Gamma_1^n \\ \vdots & \cdots & \vdots \\ a_1 \Leftarrow \Gamma_{k_1}^1 & & a_n \Leftarrow \Gamma_{k_n}^n \end{cases}$$

In logic-programming terms, definitions $\mathcal{D}$ are (a generalization of) logic programs, where the bodies of program clauses can be arbitrary formulas.

3.3 Definitional reasoning

We can now define an argumentation form that will allow us to reason about such definitions.

Definition 3.18 For each atom a defined by definitional clauses $a \Leftarrow \Gamma_i$ with defining conditions $\Gamma_i = B_1^i \wedge \ldots \wedge B_{n_i}^i$ (where $1 \leq i \leq k$) the following argumentation form of *definitional reasoning* determines how an atom a that is stated by $\boldsymbol{X}$ can be attacked by $\boldsymbol{Y}$, and how this attack can be defended by $\boldsymbol{X}$. We use ›$?\mathscr{D}$‹ as a special symbol to indicate the attack.

definitional reasoning:	assertion:	$\boldsymbol{X}\, a$
	attack:	$\boldsymbol{Y}\, ?\mathscr{D}$ (only if $a \neq \top$)
	defense:	$\boldsymbol{X}\, \Gamma_i$ ($\boldsymbol{X}$ chooses $i = 1, \ldots, k$)

For the *verum* $\top$ we impose the following restriction: The move $\boldsymbol{X}\, \top$ cannot be attacked with $\boldsymbol{Y}\, ?\mathscr{D}$.

The argumentation form of definitional reasoning is defined in such a way that atoms – with the exception of the *verum* $\top$ – can be attacked independently of whether there are definitional clauses having these atoms in their head or not. In other words, whenever a player asserts an atom, the other player may ask for its definition, regardless of whether a definition has been given or not. If the atom in question is undefined, then there is no defense move. Moreover, we will not give any dialogue conditions which would prohibit attacks on undefined atoms just because they are undefined.

The restriction with respect to the *verum* $\top$ is necessary if $\top$ is treated as an atomic formula. Otherwise it could be attacked as well. This would be in conflict with its intended meaning, suggested by its use as an indicator of empty bodies of definitional clauses, that is, by standing for the empty conjunction. The meaning of the *verum* $\top$ is stipulated by the imposed restriction.

The argumentation form of definitional reasoning comprises the two principles of definitional closure and of definitional reflection, which have been introduced as sequent-style inferences in Hallnäs and Schroeder-Heister 1990 and 1991 (see also Hallnäs 1991 and Schroeder-Heister 1993 and 2012b). In natural deduction they can be formulated as introduction and elimination rules for atoms. Let the atom a be defined by

$$\mathcal{D}\begin{cases} a \Leftarrow \Gamma_1 \\ \quad\vdots \\ a \Leftarrow \Gamma_k \end{cases}$$

Then, for $1 \leq i \leq k$, the *principle of definitional closure* takes the form of an introduction rule for the atom a:

$$\frac{\Gamma_i}{a}\ \text{(def. closure)}$$

And the *principle of definitional reflection* takes the form of a (general) elimination rule for the atom a:

$$\frac{a \quad \begin{matrix}[\Gamma_1] \\ C\end{matrix} \quad \ldots \quad \begin{matrix}[\Gamma_k] \\ C\end{matrix}}{C}\ \text{(def. reflection)}$$

The principle of definitional reflection is related to the *inversion principle* (see Lorenzen 1955, Prawitz 1965; cf. Schroeder-Heister 2007, de Campos Sanz and Piecha 2009) and can also be expressed as follows: *Whatever formula C follows from each of the defining conditions* $\Gamma_1, \ldots, \Gamma_k$ *of the atom a follows from a itself.* It is justified if the given definitions of atoms can be assumed to be complete in the sense that the atoms are defined by the given definitional clauses *and by nothing else*. In mathematical definitions this is sometimes made explicit by giving definitional clauses for something together with the remark that *nothing else* defines it, or by saying that one is defining the *smallest* set for which given definitional clauses hold.

The argumentation form of definitional reasoning is the dialogical equivalent to the principles of definitional closure and definitional reflection. Both principles are incorporated in the single argumentation form of definitional reasoning. For dialogues, the difference between definitional closure and definitional reflection appears at the level of strategies. Here, only a single defense move **P** Γ_i needs to be given for an attack **O** $?\mathscr{D}$, whereas all possible defense moves **O** Γ_i must be given for an attack **P** $?\mathscr{D}$. In other words, in the first case only the defining conditions Γ_i of *one* clause defining the attacked atom are needed, whereas in the second case the defining conditions Γ_i of *each* clause defining the attacked atom have to be taken into account. Thus definitional reasoning in dialogues corresponds to the principles of definitional closure and definitional reflection in natural deduction as follows: Instances of the argumentation form of definitional reasoning in which the attack move is **O** ?$\mathscr{D}$ correspond to applications of definitional closure, and instances of the argumentation form of definitional reasoning in which the attack move is **P** ?$\mathscr{D}$ correspond to applications of definitional reflection.

3.4 Definitional dialogues

Next we introduce definitional dialogues; they are based on $\mathrm{EI}^{\mathrm{P}}_{\mathrm{c}}$-dialogues.

Definition 3.19 *Definitional dialogues* are $\mathrm{EI}^{\mathrm{P}}_{\mathrm{c}}$-dialogues where the following changes are made:

The conditions (D00) and (D01) are replaced by the following conditions (D00′) and (D01′), where the restriction of the expressions in $\delta(0)$ and $\delta(m)$ to complex formulas is discarded; that is, a definitional dialogue can start with the assertion of an atomic formula, and atomic formulas can be attacked:

(D00′) $\delta(n)$ is a **P**-signed expression if n is even, and an **O**-signed expression if n is odd. The expression in $\delta(0)$ is a (complex or atomic) formula.

(D01′) If $\eta(n) = [m, A]$, then the expression in $\delta(m)$ is a formula and $\delta(n)$ is an attack on this formula as determined by the relevant argumentation form.

Condition (D02) remains without change.

Condition (D10) is omitted altogether, so that **P** can now assert atomic formulas without **O** having asserted them before. Conditions (D11′), (D12′) and (E) remain without change. Condition (D14) is replaced by the following condition (D14*), which is (D14) restricted to complex formulas:

(D14*) **O** can attack a complex formula C if and only if either (i) C has not yet been asserted by **O** or (ii) C has already been attacked by **P**.

The following condition is added in order to prohibit attacks by **O** on atoms asserted by **O** before:

(D15) If for an atom a there is a move $\langle \delta(l) = \mathbf{O}\, a, \eta(l) = [k, Z] \rangle$, then there is no attack $\langle \delta(n) = \mathbf{O}\ ?\mathscr{D}, \eta(n) = [m, A] \rangle$ for $\delta(m) = \mathbf{P}\, a$ with $k < l < m < n$.

That is, **O** may attack an atom a by definitional reasoning only if it has not been asserted by **O** before.

The notions ›dialogue won by **P**‹ and ›winning strategy‹ as defined for $\mathrm{EI}^{\mathrm{P}}_{\mathrm{c}}$-dialogues are directly carried over to the corresponding notions for definitional dialogues.

The omission of condition (D10) is compensated for by the fact that **O** can attack any atom asserted by **P** with a move **O** ?$\mathscr{D}$. The restriction of condition (D14) to complex formulas (yielding condition (D14*)) was not necessary in the treatment of $\mathrm{EI}^{\mathrm{P}}_{\mathrm{c}}$-dialogues because attacks on atomic formulas are not possible there.

3.4.1 Examples of propositional definitional reasoning

We consider the definition

$$\mathcal{D}_1 \begin{cases} a \Leftarrow \top \\ b \Leftarrow \top \\ b \Leftarrow a \\ c \Leftarrow a \wedge b \end{cases}$$

With respect to $\mathcal{D}_1$ there is a winning strategy for the atom c:

0.		$\mathbf{P}\, c$		
1.		$\mathbf{O}\, ?\mathscr{D}$		$[0, A]$
2.		$\mathbf{P}\, a \wedge b$		$[1, D]$
3.	$\mathbf{O}\, ?1$	$[2, A]$	$\mathbf{O}\, ?2$	$[2, A]$
4.	$\mathbf{P}\, a$	$[3, D]$	$\mathbf{P}\, b$	$[3, D]$
5.	$\mathbf{O}\, ?\mathscr{D}$	$[4, A]$	$\mathbf{O}\, ?\mathscr{D}$	$[4, A]$
6.	$\mathbf{P}\, \top$	$[5, D]$	$\mathbf{P}\, a$	$[5, D]$
7.			$\mathbf{O}\, ?\mathscr{D}$	$[6, A]$
8.			$\mathbf{P}\, \top$	$[7, D]$

At position 0 the proponent asserts the atom c. In definitional dialogues this is allowed by condition (D00′), whereas in standard dialogues with condition (D00) only complex formulas can be asserted in initial moves at position 0. At position 1 this assertion is attacked by **O** according to the argumentation form of definitional reasoning. The proponent defends this attack by asserting the defining conditions $a \wedge b$ of the attacked atom c, as given by the last clause of definition $\mathcal{D}_1$. The opponent attacks $a \wedge b$ at position 3, and **P** defends at position 4 by asserting the atoms a and b, respectively. The proponent can assert the atomic formulas a and b – without **O** having asserted them before – as there is no condition (D10) in definitional dialogues which would prohibit these moves. However, **O** can attack any atoms asserted by **P** (if not prohibited by condition (D15)), and does so with the move $\mathbf{O}\, ?\mathscr{D}$ at position 5 in each of the two dialogues.

In the left dialogue, **P** defends **O**'s attack on a by asserting $\top$ at position 6 (there are no defining conditions for the atom a; it is given as a fact by the first clause in $\mathcal{D}_1$). In the right dialogue, **P** chooses to defend by asserting the defining condition a of b, as given in the third clause of $\mathcal{D}_1$. The right dialogue then proceeds like the left one. Alternatively, **P** could have defended **O**'s attack by choosing to use the second clause of $\mathcal{D}_1$. This clause gives b as a fact, and **P**'s defense would thus be the *verum* $\top$. That is, the right dialogue would end with the move $\mathbf{P}\, \top$ already at position 6.

Both dialogues in the above winning strategy end with the assertion of the *verum* $\top$. As no attack on $\top$ is possible, both dialogues are won by **P**. This winning strategy contains only such applications of definitional reasoning in which **O** attacks atomic formulas with moves $\mathbf{O}\ ?\mathscr{D}$; that is, only the principle of definitional closure is employed here.

An example where the principle of definitional reflection is used with respect to the definition $\mathcal{D}_1$ is the following winning strategy for $b \to a$:

<table>
<tr><td>0.</td><td colspan="4" align="center">$\mathbf{P}\, b \to a$</td></tr>
<tr><td>1.</td><td colspan="3" align="center">$\mathbf{O}\, b$</td><td>$[0, A]$</td></tr>
<tr><td>2.</td><td colspan="3" align="center">$\mathbf{P}\ ?\mathscr{D}$</td><td>$[1, A]$</td></tr>
<tr><td>3.</td><td>$\mathbf{O}\, \top$</td><td>$[2, D]$</td><td>$\mathbf{O}\, a$</td><td>$[2, D]$</td></tr>
<tr><td>4.</td><td>$\mathbf{P}\, a$</td><td>$[1, D]$</td><td>$\mathbf{P}\, a$</td><td>$[1, D]$</td></tr>
<tr><td>5.</td><td>$\mathbf{O}\ ?\mathscr{D}$</td><td>$[4, A]$</td><td></td><td></td></tr>
<tr><td>6.</td><td>$\mathbf{P}\, \top$</td><td>$[5, D]$</td><td></td><td></td></tr>
</table>

The first application of definitional reasoning (comprising positions 1–3) is according to the principle of definitional reflection. Here the defining conditions of each of the definitional clauses for the attacked atom b have to be considered. As $\mathcal{D}_1$ contains two clauses for b, there are two defense moves (made at position 3) to be considered. In the left dialogue, **P** can only defend **O**'s attack made at position 1 by asserting the atom a. The following attack by **O**, asking for defining conditions of a, is defended by **P** with $\top$ (using the first clause of $\mathcal{D}_1$, which is the only definitional clause for a). Here, the principle of definitional closure has been employed. In the right dialogue, **P** makes the same defense move at position 4 as in the left dialogue. Due to condition (D15) the opponent cannot attack this assertion of the atom a, since in this dialogue **O** has asserted a before (at position 3).

The proponent could also make the move $\mathbf{P}\ ?\mathscr{D}$ at position 4 in the right dialogue instead. The dialogue would then end thus:

	⋮	
3.	$\mathbf{O}\, a$	$[2, D]$
4.	$\mathbf{P}\ ?\mathscr{D}$	$[3, A]$
5.	$\mathbf{O}\, \top$	$[4, D]$
6.	$\mathbf{P}\, a$	$[1, D]$

This yields a winning strategy in which the principle of definitional reflection has been employed twice.

3.4.2 *Examples of first-order definitional reasoning*

Definition 3.20 A substitution σ is a *unifier* of two atoms a and b if $a\sigma \equiv b\sigma$, that is, if $a\sigma$ and $b\sigma$ are syntactically identical.

A unifier σ of two atoms a and b is a *most general unifier* of a and b if for each unifier τ of a and b there is a substitution ρ such that $\tau = \sigma\rho$.

In the case of first-order clauses one has to consider substitution instances of heads and bodies of clauses. Let the substitution σ be a most general unifier for the atom a and the head a' of at least one first-order clause. Then the body Γ_i of such a clause with head a' can be chosen in a defense $X\,\Gamma_i\sigma$ to an attack $Y\ ?\mathscr{D}$ on $X\,a$ since $a\sigma \equiv a'\sigma$. That is, in order to defend such an attack, we first have to look for a most general unifier σ which unifies a with the head of a clause $a' \Leftarrow \Gamma_i$. If it exists (this is decidable by the unification algorithm), we apply it to Γ_i, and the defense move is $X\,\Gamma_i\sigma$. For example, if the first-order clause $a(t) \Leftarrow b(x)$ is given by definition, then an attack $Y\ ?\mathscr{D}$ on a move $X\,a(x)$ can be defended with the move $b(t)$. That is, the definitional reasoning for the given clause is of the form

$$\begin{array}{l} X\,a(x) \\ Y\ ?\mathscr{D} \\ X\,b(t) \end{array}$$

where the substitution $\sigma = [t/x]$ is here the most general unifier for the atom $a(x)$ and the head $a(t)$ of the definitional clause. Applying σ to the body $b(x)$ of the clause yields $b(t)$, which is asserted in the defense move (for further details see Piecha 2012).

We now consider the following (first-order) definition $\mathcal{D}_2$, in which the atoms $even(x)$ and $odd(x)$ are two unary relation symbols and s is a unary function symbol (interpreted as the successor function on natural numbers):

$$\mathcal{D}_2 \begin{cases} even(0) \Leftarrow \top \\ even(s(x)) \Leftarrow odd(x) \\ odd(x) \Leftarrow \neg\, even(x) \end{cases}$$

Then for the given definition $\mathcal{D}_2$ the following definitional dialogue is a winning strategy for $\neg\, even(s(0))$:

0.	$\mathbf{P}\ \neg\, even(s(0))$		
1.	$\mathbf{O}\ even(s(0))$	$[0, A]$	
2.	$\mathbf{P}\ ?\mathscr{D}$	$[1, A]$	(variable binding: $[0/x]$)
3.	$\mathbf{O}\ odd(0)$	$[2, D]$	
4.	$\mathbf{P}\ ?\mathscr{D}$	$[3, A]$	(variable binding: $[0/x]$)

5.	**O** $\neg$ *even*(0)	[4, *D*]
6.	**P** *even*(0)	[5, *A*]
7.	**O** ?$\mathscr{D}$	[6, *A*]
8.	**P** $\top$	[7, *D*]

The applications of definitional reasoning comprising the moves at positions 1–3 and 3–5, respectively, are according to the principle of definitional reflection. The opponent's first defense move depends on the substitution $[0/x]$, which unifies the attacked atom *even*(*s*(0)) with the head *even*(*s*(*x*)) of clause 2 and yields the corresponding defining condition *odd*(*x*)$[0/x]$ = *odd*(0), asserted by **O** at position 3. The opponent's second defense move depends on the same substitution $[0/x]$; it unifies *odd*(0) with the head *odd*(*x*) of the third clause, allowing **O** to defend with the defining condition $\neg$ *even*(*x*)$[0/x]$ = $\neg$ *even*(0) in the move at position 5. The moves at positions 6–8 are definitional reasoning by the principle of definitional closure. As $\top$ cannot be attacked, the dialogue ends with **P**'s move at position 8. By reasoning about the definition $\mathcal{D}_2$ we have thus shown $\neg$ *even*(*s*(0)).

From a logic-programming perspective this can be described as follows: The initial move expresses in a formal way a query about the given definition (or program) $\mathcal{D}_2$, like »Does $\neg$ *even*(*s*(0)) hold with respect to $\mathcal{D}_2$?« We then try to answer that query by searching for a winning strategy with respect to $\mathcal{D}_2$, that is, by employing definitional reasoning (in addition to purely logical reasoning). Finding a winning strategy means that the query has a positive answer. In addition, one can in general gain further information from the variable bindings which have been computed in the construction of a winning strategy.

3.5 Definitional dialogues and contraction

In dialogical logic the structural operation of contraction, which allows one to treat several assumptions of the same form as just one assumption, is only implicitly present in the dialogue conditions. This is comparable to the calculus of natural deduction, where contraction is also only implicitly present, namely in the way how assumptions are discharged. In dialogues for intuitionistic logic, the twofold use made by **P** of a formula A asserted by **O** corresponds to the structural operation of contraction, contracting A, A to A. The twofold use can consist either (1) in the twofold attack of a formula by **P**, (2) in the twofold assertion by **P** of a formula asserted by **O** before or (3) in an attack of a formula A by **P** together with the assertion of A by **P**. That is, the twofold use can be of the following forms:

$$
\begin{array}{llll}
(1) & k. & \mathbf{O}A & [k-1,Z] \\
 & & \vdots & \\
 & l. & \mathbf{P}e & [k,A] \\
 & & \vdots & \\
 & m. & \mathbf{P}e & [k,A]
\end{array}
\qquad
\begin{array}{llll}
(2) & k. & \mathbf{O}A & [k-1,Z] \\
 & & \vdots & \\
 & l. & \mathbf{P}A & [i<l,Z] \\
 & & \vdots & \\
 & m. & \mathbf{P}A & [j<m,Z]
\end{array}
$$

$$
\begin{array}{llll}
(3) & k. & \mathbf{O}A & [k-1,Z] \\
 & & \vdots & \\
 & l. & \mathbf{P}e & [k,A] \\
 & & \vdots & \\
 & m. & \mathbf{P}A & [i<m,Z]
\end{array}
\quad \text{or} \quad
\begin{array}{lll}
k. & \mathbf{O}A & [k-1,Z] \\
 & \vdots & \\
l. & \mathbf{P}A & [i<l,Z] \\
 & \vdots & \\
m. & \mathbf{P}e & [k,A]
\end{array}
$$

In the following example, the twofold use made by **P** of an assertion made by **O** is of the form (1); for comparison we also show a corresponding derivation in the calculus of natural deduction (wherein $\neg a := a \to \bot$, where $\bot$ is the *falsum*):

$$
\begin{array}{lll}
0. & \mathbf{P}\,\neg(a \wedge \neg a) & \\
1. & \mathbf{O}\,a \wedge \neg a & [0,A] \\
2. & \mathbf{P}\,?1 & [1,A] \\
3. & \mathbf{O}\,a & [2,D] \\
4. & \mathbf{P}\,?2 & [1,A] \\
5. & \mathbf{O}\,\neg a & [4,D] \\
6. & \mathbf{P}\,a & [5,A]
\end{array}
\qquad
\dfrac{\dfrac{\dfrac{[a \wedge \neg a]^1}{a}\,(\wedge\,\text{elim.}) \qquad \dfrac{[a \wedge \neg a]^1}{\neg a}\,(\wedge\,\text{elim.})}{\bot}\,(\to \text{elim.})}{\neg(a \wedge \neg a)}\,(\to \text{intro.})^1
$$

The twofold attack at positions 2 and 4 corresponds to the contraction of $a \wedge \neg a$, $a \wedge \neg a$ to $a \wedge \neg a$. Without a twofold attack by **P** on $a \wedge \neg a$ there is no winning strategy for $\neg(a \wedge \neg a)$, just as in the calculus of natural deduction there is no corresponding derivation without discharging two occurrences of the same assumption.

We now consider the paradoxical definitional clause $a \Leftarrow \neg a$, to which in our context many antinomies can be reduced. For example, in the case of Russell's antinomy we have for $t \in \{x \mid A\} \Leftarrow A[t/x]$ with $t = \{x \mid \neg(x \in x)\}$ and $A = \neg(x \in x)$ that $t \in t \Leftarrow \neg(t \in t)$. The latter clause is of the form $a \Leftarrow \neg a$. If such a clause is given as definition for a, then there are winning strategies for a as well as for $\neg a$:

$$
\begin{array}{lll}
0. & \mathbf{P}\,a & \\
1. & \mathbf{O}\,?\mathscr{D} & [0,A] \\
2. & \mathbf{P}\,\neg a & [1,D] \\
3. & \mathbf{O}\,a & [2,A]
\end{array}
\qquad\qquad
\begin{array}{lll}
0. & \mathbf{P}\,\neg a & \\
1. & \mathbf{O}\,a & [0,A] \\
2. & \mathbf{P}\,?\mathscr{D} & [1,A] \\
3. & \mathbf{O}\,\neg a & [2,D]
\end{array}
$$

4.	$\mathbf{P}\ ?\mathscr{D}$	$[3, A]$		4.	$\mathbf{P}\,a$	$[3, A]$
5.	$\mathbf{O}\,\neg a$	$[4, D]$				
6.	$\mathbf{P}\,a$	$[5, A]$				

These two winning strategies correspond to the following two natural-deduction derivations for the given definitional clause $a \Leftarrow \neg a$, respectively (where again $\neg a := a \to \bot$):

$$\dfrac{\dfrac{\dfrac{[a]^2 \qquad \dfrac{[a]^2 \quad [\neg a]^1}{\bot}\,(\to \text{elim.})}{\bot}\,(\text{def. reflection})^1}{\neg a}\,(\to \text{intro.})^2}{a}\,(\text{def. closure}) \qquad\qquad \dfrac{\dfrac{[a]^2 \qquad \dfrac{[a]^2 \quad [\neg a]^1}{\bot}\,(\to \text{elim.})}{\bot}\,(\text{def. reflection})^1}{\neg a}\,(\to \text{intro.})^2$$

The existence of winning strategies for a as well as for $\neg a$ depends on the fact that, in the last move, **P** can state the formula a (in the moves $\langle \delta(6) = \mathbf{P}\,a$, $\eta(6) = [5, A]\rangle$ and $\langle \delta(4) = \mathbf{P}\,a, \eta(4) = [3, A]\rangle$, respectively), which has been attacked by **P** with definitional reasoning before (in the moves $\langle \delta(4) = \mathbf{P}\ ?\mathscr{D}$, $\eta(4) = [3, A]\rangle$ and $\langle \delta(2) = \mathbf{P}\ ?\mathscr{D}, \eta(2) = [1, A]\rangle$, respectively).

That a is stated in the last move of a dialogue in a winning strategy means that a is used without reference to its definition, just as the assumption a, which is used as minor premiss in the inference ($\to$ elim.) of the corresponding natural-deduction derivations. However, here this move is possible only after having reflected on the definition of a by definitional reasoning; this corresponds to the use of the assumption a as the major premiss (i. e., as the left premiss) in the inference of definitional reflection in the natural-deduction derivations. Hence, the formula a has been used both with and without referring to its definition. This means that the occurrences of the formula a which are used in different ways have been *contracted* implicitly. In other words, **P** has not only made twofold use of the formula a (asserted by **O** at position 3) in the moves at positions 4 and 6 of the left dialogue and correspondingly in the moves at positions 2 and 4 of the right dialogue (i. e., contractions of the form (3)), but the formula a has also been used in two different senses: once as an arbitrary assumption and once according to its given definition.

One way to avoid paradoxes of the above kind lies thus in restricting the structural operation of contraction in a suitable way (cf. Schroeder-Heister 2012a). Disallowing contraction altogether would be too strong, since there would then, for example, no longer be a winning strategy for $\neg(a \land \neg a)$, an instance of the principle of noncontradiction. What would be needed is a restriction of contraction to only such occurrences of formulas which are not used in different senses (see Ekman 2014). Several dialogical approaches to this problem have been considered in Piecha 2012.

4 Dialogues for implications as rules

We now want to reconsider the meaning of the logical constant of implication ›→‹ by interpreting implications $A \to B$ as rules. For the sequent calculus, an alternative left introduction rule for implication has been introduced (see Schroeder-Heister 2011), which is motivated by the interpretation of implications as rules. Here, we will look at its dialogical counterpart by giving a dialogical framework for implications as rules (see also Piecha and Schroeder-Heister 2012).

Usually, constructive interpretations of implication are more or less directly given by the Brouwer–Heyting–Kolmogorov (BHK) interpretation, according to which a proof of an implication $A \to B$ consists of a construction transforming any given proof of A into a proof of B; in the formulation of Heyting:

> The *implication* $\mathfrak{p} \to \mathfrak{q}$ can be asserted, if and only if we possess a construction $\mathfrak{r}$, which, joined to any construction proving $\mathfrak{p}$ (supposing that the latter be effected), would automatically effect a construction proving $\mathfrak{q}$. In other words, a proof of $\mathfrak{p}$, together with $\mathfrak{r}$, would form a proof of $\mathfrak{q}$. (Heyting 1971, 102 f.)

The standard dialogical interpretation of implication is based on the same idea: An implication $A \to B$ is attacked by claiming A and defended by claiming B. In order to have a winning strategy for $A \to B$, the proponent must be able to produce a sub-winning strategy (cf. Definition 4.28 below) for B from what the opponent uses in defending A. A difference to standard constructive interpretations is that the opponent need not give a full proof of A which is then transformed into a proof of B. Instead, the proponent may force the opponent to produce certain fragments of a proof of A that are sufficient to produce a sub-winning strategy for B.

A more elementary view of implication is based on the conception that an implication $A \to B$ is a rule which allows one to pass over from A to B. This view is in particular supported by the treatment of implication in the calculus of natural deduction. There *modus ponens* (i. e., implication elimination ($\to$ elim.))

$$\frac{A \quad A \to B}{B}$$

can be read as the application of $A \to B$ as a rule which is used to infer B from A; that is, *modus ponens* can be read as a schema of rule application:

$$\frac{A}{B}\,(A \to B)$$

The introduction of an implication $A \to B$ by ($\to$ intro.)

$$\frac{\begin{matrix}[A]\\ B\end{matrix}}{A \to B}$$

(where assumptions A may be discharged) can be read as establishing a rule, namely, by deriving its conclusion B from its premiss A. Applications of logic such as logic programming or definitional reasoning support this approach. When implications are read as rules, an elementary meaning is given to implication which is conceptually prior to the meaning of the other logical constants (see Schroeder-Heister 2011).

In the following, we explain how the implications-as-rules approach can be carried over to dialogues. This is done in two steps: We first introduce preliminary EI°-dialogues, which implement the implications-as-rules approach. These preliminary dialogues will be found lacking, since they are not sufficient for intuitionistic logic. In the second step, we correct this by making an addition to preliminary EI°-dialogues, yielding EI°-dialogues for intuitionistic logic. We will only treat the propositional case; the results can be generalized to the first-order case.

4.6 Preliminary EI°-dialogues

The guiding idea for implications-as-rules dialogues is the following: Once an implication $C \to A$ has been claimed by **O**, it is considered to be a rule in a kind of database, which can later be used by **P** to reduce the justification of its conclusion A to the justification of its premiss C. This is achieved by allowing **P** to defend an attack on A by asserting C whenever $C \to A$ has been claimed by **O** before. In case no such claim has been made before (i.e., if no applicable rule is available in the database), the argument for A continues as usual with an opponent attack on A (which must eventually be defended by **P**), depending on the respective form of A. When making an assertion A, the proponent must be prepared to either defend A in the standard way against an attack by **O**, or else make the assertion C for some C for which **O** has already claimed $C \to A$, that is, for which the implication-as-rule $C \to A$ is sufficient to generate A. This is implemented by saying that every assertion made by **P** is symbolically questioned by **O**, following which

P chooses which of the two ways described **P** is prepared to take. Contrary to the proponent, the opponent is not given a choice. The opponent's non-implicational assertions are attacked and defended as usual, whereas **O**'s implicational assertions are considered as providing rules which **P** can use, but not question; so there are no attacks and defenses defined for them.

Definition 4.21 For each logical constant we first define *argumentation forms* which determine how a complex formula (having the respective constant in main position) that has been asserted by **O** can be attacked (if possible) and how this attack can be defended (if possible):

AF(**O**$\wedge$):	assertion:	**O** $A_1 \wedge A_2$	
	attack:	**P** ?i	(**P** chooses $i = 1$ or $i = 2$)
	defense:	**O** A_i	
AF(**O**$\vee$):	assertion:	**O** $A_1 \vee A_2$	
	attack:	**P** ?$\vee$	
	defense:	**O** A_i	(**O** chooses $i = 1$ or $i = 2$)
AF(**O**$\to$):	assertion:	**O** $A \to B$	
	attack:	*no attack*	
	defense:	*no defense*	
AF(**O**$\neg$):	assertion:	**O** $\neg A$	
	attack:	**P** A	
	defense:	*no defense*	

Except for AF(**O**$\to$), these argumentation forms coincide with the standard ones (cf. Definition 2.2) in case of assertions made by **O**. The argumentation form AF(**O**$\to$) could also be omitted, to the same effect; we present it to make explicit that implications $A \to B$ asserted by **O** cannot be attacked.

We now extend our language by the two *special symbols* ? and $|\cdot|$. For assertions made by **P** there is a pair of argumentation forms for each logical constant (depicted below as trees having two branches which are separated by $|$). An assertion A made by **P** can be questioned by **O** with the move **O** ? (such a move is only possible if the expression stated in the **P**-move is an assertion, that is, a formula; if it is not an assertion but a symbolic attack, then it cannot be questioned by means of the move **O** ?).

The proponent can then answer this question either by allowing an attack on the assertion (this is indicated by the special symbol $|\cdot|$; see the argumentation forms on the left side of $|$ below) or by asserting any formula C for which **O** has asserted the implication $C \to A$ at an earlier position. We call this latter part the *rule condition* (R):

(R) **P** may answer a question **O** ? on a formula A by choosing C provided **O** has asserted the formula $C \to A$ before.

The argumentation forms for assertions made by **P** are then defined as follows:

AF(**P**∧): assertion: **P** $A_1 \wedge A_2$
question: **O** ?
choice: **P** $|A_1 \wedge A_2|$ | **P** C (R)
attack: **O** ?i ($i = 1$ or 2)
defense: **P** A_i

AF(**P**∨): assertion: **P** $A_1 \vee A_2$
question: **O** ?
choice: **P** $|A_1 \vee A_2|$ | **P** C (R)
attack: **O** ?∨
defense: **P** A_i ($i = 1$ or 2)

AF(**P**→): assertion: **P** $A \to B$
question: **O** ?
choice: **P** $|A \to B|$ | **P** C (R)
attack: **O** A
defense: **P** B

AF(**P**¬): assertion: **P** $\neg A$
question: **O** ?
choice: **P** $|\neg A|$ | **P** C (R)
attack: **O** A
defense: *no defense*

In the case of an attack **O** ?i according to the argumentation form AF(**P**∧) for conjunctive formulas asserted by **P**, the opponent chooses $i = 1$ or $i = 2$, and in the case of a defense **P** A_i to an attack **O** ?∨ according to the argumentation form AF(**P**∨) for disjunctive formulas asserted by **P**, the proponent chooses $i = 1$ or $i = 2$. The argumentation forms on the left (i.e., the respective left-hand branches) correspond to the argumentation forms given in Definition 2.2 for standard dialogues (where the device of question and choice moves is not needed). The argumentation forms on the right (i.e., the respective right-hand branches) reflect the implications-as-rules view.

For assertions of atomic formulas a made by **P**, an argumentation form is given by the rule condition (R) itself:

AF(R): assertion: $\mathbf{P}\,a$
question: $\mathbf{O}$?
choice: $\mathbf{P}\,C$ only if $\mathbf{O}$ has asserted $C \to a$ before

In Definition 2.2, the argumentation forms for standard dialogues were defined independently of whether the assertion is made by **P** or by **O**. This symmetry is not preserved here.

Definition 4.22 We extend the definition of moves (see Definition 2.3) as follows: As before, *moves* are pairs $\langle \delta(n), \eta(n) \rangle$, where $\delta(n)$, for $n \geq 0$, is again a signed expression, and $\eta(n)$ is again a pair $[m, Z]$ with $0 \leq m < n$, but where Z is now either A (for ›attack‹), D (for ›defense‹), Q (for ›question‹) or C (for ›choice‹). As before, $\eta(n)$ is empty for $n = 0$, that is, $\eta(0) = \emptyset$. We have thus the following types of moves:

$$\begin{array}{rl} \textit{attack move} & \langle \delta(n) = \mathbf{X}\,e, \eta(n) = [m, A] \rangle, \\ \textit{defense move} & \langle \delta(n) = \mathbf{X}\,A, \eta(n) = [m, D] \rangle, \\ \textit{question move} & \langle \delta(n) = \mathbf{O}\,?, \eta(n) = [m, Q] \rangle, \\ \textit{choice move} & \left\{ \begin{array}{l} \langle \delta(n) = \mathbf{P}\,|A|, \eta(n) = [m, C] \rangle, \\ \langle \delta(n) = \mathbf{P}\,A, \eta(n) = [m, C] \rangle. \end{array} \right. \end{array}$$

A question move can only be made by **O**, and a choice move can only be made by **P**. The other types of moves are available for both **P** and **O**. In a choice move, $\delta(n)$ can have the form $\mathbf{P}\,|A|$ or $\mathbf{P}\,A$. In the first case, **P** allows an attack on the formula A. In the second case, **P** asserts the formula A in accordance with the rule condition (R), that is, A is the antecedent of an implication asserted by **O** before.

Dialogues for the implications-as-rules approach can now be defined as follows:

Definition 4.23 A *preliminary* EI°-*dialogue* is a sequence of moves $\langle \delta(n), \eta(n) \rangle$ ($n = 0, 1, 2, \ldots$) satisfying the following conditions:

(D00′) $\delta(n)$ is a **P**-signed expression if n is even and an **O**-signed expression if n is odd. The expression in $\delta(0)$ is a (complex or atomic) formula.

(D01°) If $\eta(n) = [m, A]$ for even n, then the expression in $\delta(m)$ is a complex formula. If $\eta(n) = [n-1, A]$ for odd n, then the expression in $\delta(n-1)$ is of the form $|B|$ for a complex formula B. In both cases $\delta(n)$ is an attack on this formula as determined by the relevant argumentation form.

(D02) If $\eta(p) = [n, D]$, then $\eta(n) = [m, A]$ for $m < n < p$ and $\delta(p)$ is the defense of the attack $\delta(n)$ as determined by the relevant argumentation form.

(D03°) If $\eta(n) = [m, Q]$ (for odd n), then for $m < n$ the expression in $\delta(m)$ is a (complex or atomic) formula, $\eta(m) = [l, Z]$ for $l < m$, $Z = A$, D or C, and the expression in $\delta(n)$ is the question mark ›?‹.

(D04°) If $\eta(n) = [m, C]$ (for even n), then $\eta(m) = [l, Q]$ for $l < m < n$ and $\delta(n)$ is the choice answering the question $\delta(m)$ as determined by the relevant argumentation form.

(D11′) If $\eta(p) = [n, D]$ for odd n, $n < n' < p$, $n' - n$ is even and $\eta(n') = [m, A]$, then there is a p' such that $n' < p' < p$ and $\eta(p') = [n', D]$.
That is, if at a position $p - 1$ there is more than one open attack by **O**, then only the last of them may be defended by **P** at position p.

(D12′) For every odd n there is at most one m such that $\eta(m) = [n, D]$.
That is, an attack by **O** may be defended by **P** at most once.

(D14′) **O** can question a formula C if and only if either (i) C has not yet been asserted by **O** or (ii) C has already been attacked by **P**.

(E) All moves $\langle \delta(n), \eta(n) \rangle$ for n odd are of the form $\langle \delta(n), \eta(n) = [n-1, Z] \rangle$, for $Z = Q$, A or D.
That is, an **O**-move made at position n is either a question, an attack or a defense of the immediately preceding **P**-move made at position $n - 1$.

The notions ›dialogue won by **P**‹ and ›winning strategy‹ as defined for DI^p-dialogues are directly carried over to the corresponding notions for (preliminary) EI°-dialogues.

Preliminary EI°-dialogues are similar to EI^p_c-dialogues without condition (D10) for the argumentation forms given in Definition 4.21 and satisfying the condition (D14′) instead of (D14), where (D14′) differs from (D14) only in that the latter is a condition for **O** *attacking* a formula C, whereas the former is a condition for **O** *questioning* a formula C. Condition (D00′) is the same as for definitional dialogues (cf. Definition 3.19). Thus (preliminary) EI°-dialogues can also begin with the assertion of an atomic formula. Condition (D01°) differs from condition (D01) in EI^p_c-dialogues in that it allows for attacks by **O** on expressions of the form $|A|$ for complex formulas A. Condition (D02) is as given in Definition 2.4 for dialogues. Conditions (D03°) and (D04°) have been added for the question and choice moves, respectively.

Condition (D10) is not needed in the definition of preliminary EI°-dialogues because **O** can question assertions of atomic formulas made by **P**. In dialogues with (D10) there is no winning strategy, say, for $a \to b$, since the dialogue

0.	**P** $a \to b$	
1.	**O** a	$[0, A]$

cannot be continued with the move $\langle \delta(2) = \mathbf{P}\, b, \eta(2) = [1, D] \rangle$; this would only be possible if b had been asserted by **O** before. In (preliminary) EI°-dialogues (where (D10) is absent) there is no winning strategy for $a \to b$ either. The (preliminary) EI°-dialogue begins with the moves

0.	$\mathbf{P}\, a \to b$	
1.	$\mathbf{O}\, ?$	$[0, Q]$
2.	$\mathbf{P}\, \lvert a \to b \rvert$	$[1, C]$
3.	$\mathbf{O}\, a$	$[2, A]$
4.	$\mathbf{P}\, b$	$[3, D]$
5.	$\mathbf{O}\, ?$	$[4, Q]$

where **P** can now assert b at position 4 without **O** having asserted it before. However, **O** can make a question move at position 5, in accordance with the argumentation form AF(R). The proponent cannot make the choice move $\langle \delta(6) = \mathbf{P}\, \lvert b \rvert, \eta(6) = [5, C] \rangle$ here, since there is no such argumentation form for atomic formulas. The only possible choice move would be one according to the argumentation form AF(R), that is, a move of the form $\langle \delta(6) = \mathbf{P}\, C$, $\eta(6) = [5, C] \rangle$ for a formula $C \to b$ asserted by **O** before. But such a formula has not been asserted by **O** in this dialogue.

Due to condition (D14′), (preliminary) EI°-dialogues won by **P** need not end with the assertion of an atomic formula but can also end with the assertion of a complex formula. For example, the following dialogue is a (preliminary) EI°-winning strategy for $(a \vee b) \to \neg\neg(a \vee b)$:

0.	$\mathbf{P}\, (a \vee b) \to \neg\neg(a \vee b)$	
1.	$\mathbf{O}\, ?$	$[0, Q]$
2.	$\mathbf{P}\, \lvert (a \vee b) \to \neg\neg(a \vee b) \rvert$	$[1, C]$
3.	$\mathbf{O}\, a \vee b$	$[2, A]$
4.	$\mathbf{P}\, \neg\neg(a \vee b)$	$[3, D]$
5.	$\mathbf{O}\, ?$	$[4, Q]$
6.	$\mathbf{P}\, \lvert \neg\neg(a \vee b) \rvert$	$[5, C]$
7.	$\mathbf{O}\, \neg(a \vee b)$	$[6, A]$
8.	$\mathbf{P}\, a \vee b$	$[7, A]$

The opponent cannot question $a \vee b$, since neither of the two conditions (i) and (ii) of (D14′) is satisfied: $a \vee b$ has already been asserted by **O** at position 3, and $a \vee b$ has not been attacked by **P**.

In order to clarify the interpretation of implications as rules, we consider the following dialogue, which is also a (preliminary) EI°-winning strategy for $(a \to b) \to ((b \to c) \to (a \to c))$:

0.	$\mathbf{P}\,(a \to b) \to ((b \to c) \to (a \to c))$		
1.	$\mathbf{O}\,?$	$[0, Q]$	
2.	$\mathbf{P}\,\lvert(a \to b) \to ((b \to c) \to (a \to c))\rvert$	$[1, C]$	
3.	$\mathbf{O}\,a \to b$	$[2, A]$	($a \to b$ in database)
4.	$\mathbf{P}\,(b \to c) \to (a \to c)$	$[3, D]$	
5.	$\mathbf{O}\,?$	$[4, Q]$	
6.	$\mathbf{P}\,\lvert(b \to c) \to (a \to c)\rvert$	$[5, C]$	
7.	$\mathbf{O}\,b \to c$	$[6, A]$	($b \to c$ in database)
8.	$\mathbf{P}\,a \to c$	$[7, D]$	
9.	$\mathbf{O}\,?$	$[8, Q]$	
10.	$\mathbf{P}\,\lvert a \to c\rvert$	$[9, C]$	
11.	$\mathbf{O}\,a$	$[10, A]$	
12.	$\mathbf{P}\,c$	$[11, D]$	
13.	$\mathbf{O}\,?$	$[12, Q]$	
14.	$\mathbf{P}\,b$	$[13, C]$	($b \to c$ used as rule)
15.	$\mathbf{O}\,?$	$[14, Q]$	
16.	$\mathbf{P}\,a$	$[15, C]$	($a \to b$ used as rule)

At position 3, the opponent asserts the implication $a \to b$. The formula b, which occurs also as the consequent of this implication, is questioned at position 15. In accordance with the rule condition (R), the proponent asserts a, the antecedent of the implication, in the last move; the opponent cannot question this move due to condition (D14′). The implication $b \to c$ is asserted by **O** in the move at position 7. The opponent questions c at position 13, which enables **P** to answer according to the rule condition (R) with the choice move $\mathbf{P}\,b$ at position 14. The implications $a \to b$ and $b \to c$ have thus been used as rules: the latter implication-as-rule allowed **P** to answer the question on c with b, and the former allowed **P** to answer the question on b with a.

4.7 EI°-DIALOGUES WITH CUT

For the preliminary EI°-dialogues considered so far, there is no winning strategy for $a \to ((a \to (b \land c)) \to b)$. Consider the following dialogue:

0.	$\mathbf{P}\,a \to ((a \to (b \land c)) \to b)$	
1.	$\mathbf{O}\,?$	$[0, Q]$
2.	$\mathbf{P}\,\lvert a \to ((a \to (b \land c)) \to b)\rvert$	$[1, C]$
3.	$\mathbf{O}\,a$	$[2, A]$
4.	$\mathbf{P}\,(a \to (b \land c)) \to b$	$[3, D]$

(*continued on next page*)

5.	$\mathbf{O}$?	[4, Q]
6.	$\mathbf{P}\,\|(a \to (b \wedge c)) \to b\|$	[5, C]
7.	$\mathbf{O}\,a \to (b \wedge c)$	[6, A]
8.	$\mathbf{P}\,b$	[7, D]
9.	$\mathbf{O}$?	[8, Q]

The moves at positions 0–4 and at positions 4–7 are made according to the argumentation form AF($\mathbf{P}\to$). In the choice moves at positions 2 and 6 the proponent can only choose $|a \to ((a \to (b \wedge c)) \to b)|$ and $|(a \to (b \wedge c)) \to b|$, respectively, since $\mathbf{O}$ has not asserted any implications so far which could be used as rules by choosing their antecedents. At position 7 the opponent asserts the implication $a \to (b \wedge c)$. At position 8 the proponent responds to the attack $\mathbf{O}\,a \to (b \wedge c)$ by asserting b; assertions by $\mathbf{P}$ of atomic formulas not asserted by $\mathbf{O}$ before are not prohibited in (preliminary) EI°-dialogues (they would be prohibited by condition (D10), for example in $\mathrm{EI}_{\mathrm{c}}^{\mathrm{P}}$-dialogues). This move can be questioned by $\mathbf{O}$ at position 9, and $\mathbf{P}$ loses this dialogue, since $\mathbf{P}$ cannot make another move at position 10: $\mathbf{P}$ can choose neither $|b|$ nor C. Since no move $\mathbf{O}\,C \to b$ has been made for such a formula C, there is no attack for $\mathbf{O}\,a \to (b \wedge c)$ (by definition of AF($\mathbf{O}\to$)), and because a is atomic there is no attack for the move $\mathbf{O}\,a$ made at position 3.

Although there is no preliminary EI°-winning strategy, there is an $\mathrm{EI}_{\mathrm{c}}^{\mathrm{P}}$-winning strategy for $a \to ((a \to (b \wedge c)) \to b)$:

0.	$\mathbf{P}\,a \to ((a \to (b \wedge c)) \to b)$	
1.	$\mathbf{O}\,a$	[0, A]
2.	$\mathbf{P}\,(a \to (b \wedge c)) \to b$	[1, D]
3.	$\mathbf{O}\,a \to (b \wedge c)$	[2, A]
4.	$\mathbf{P}\,a$	[3, A]
5.	$\mathbf{O}\,b \wedge c$	[4, D]
6.	$\mathbf{P}$?1	[5, A]
7.	$\mathbf{O}\,b$	[6, D]
8.	$\mathbf{P}\,b$	[3, D]

Therefore, in contradistinction to $\mathrm{EI}_{\mathrm{c}}^{\mathrm{P}}$-dialogues (for which we have Theorem 3.2), preliminary EI°-dialogues cannot be complete for intuitionistic propositional logic. There are, however, EI°-winning strategies for both $a \to ((a \to (b \wedge c)) \to (b \wedge c))$ and $(b \wedge c) \to b$. What would be needed is a means to concatenate these two winning strategies. Since such a concatenation corresponds to the cut rule in the sequent calculus, we will also speak of ›cut‹. In order to achieve completeness of the dialogical implications-as-rules framework for intuitionistic propositional logic, we have to add a form of cut to our preliminary EI°-dialogues. We first define an argumentation form for cut, extend our definition of moves by adding cut moves, and adjust our definition of preliminary EI°-dialogues accordingly, yielding the final definition of EI°-dialogues. The implications-as-rules approach as such is independent of the presence of cut. However, cut moves have to be allowed if more than a fragment of intuitionistic (propositional) logic is to be captured.

Definition 4.24 We define an *argumentation form* AF(Cut) such that any expression *e* (i.e., question, symbolic attack or formula) stated by **O** can be followed by a move **P** *A*, which in turn can be followed by the move **O** *A*:

AF(Cut):	statement:	**O** *e*
	cut:	**P** *A*
	cut:	**O** *A*

The formula *A* is arbitrary and is called *cut formula*.

The argumentation form AF(Cut) differs from the other argumentation forms in that the move **O** *e* need not be an assertion (i.e., the statement of a formula) but can be the statement of any expression *e* (i.e., question, symbolic attack or formula). Another difference is that the cut formula is completely independent of the expression *e*. Calling the **P**-move an attack and the subsequent **O**-move a defense, as in the other argumentation forms, would thus be inadequate. We therefore simply speak of *cut moves* in both cases. The idea behind AF(Cut) is that at any (even) position the proponent can introduce an arbitrary formula *A* as a lemma. The proponent must then later be prepared both to defend the lemma *A* as an assertion, and to defend the original claim (i.e., the assertion made in the initial move at position 0) given this lemma, that is, given the opponent's claim of *A*.

Definition 4.25 We extend the definition of *moves* (see Definition 4.22) further by adding *cut moves* $\langle \delta(n) = \mathbf{X}\, A, \eta(n) = [Cut] \rangle$. (Note that in the pair $\eta(n) = [m, Z]$ position m is empty and $Z = Cut$.)

Definition 4.26 EI°-*dialogues* are preliminary EI°-dialogues with the following additional dialogue condition (D05°) and two small adjustments in conditions (D03°) and (E) for cut moves:

(D03°) If $\eta(n) = [m, Q]$ (for odd n), then for $m < n$ the expression in $\delta(m)$ is a (complex or atomic) formula, $\eta(m) = [l, Z]$ for $l < m$, $Z = A$, D, C or Cut (where l is empty if $Z = Cut$), and the expression in $\delta(n)$ is the question mark ›?‹.

(D05°) If $\eta(n) = [Cut]$ for even n, then $\eta(m) = [l, Z]$ (where l is empty if $Z = Cut$) for $l < m < n$ and $\delta(n)$ is a formula (namely, the cut formula). If $\eta(n) = [Cut]$ for odd n, then $\eta(m) = [Cut]$ and $\delta(n) = \mathbf{O}\,A$ for $\delta(m) = \mathbf{P}\,A$ (where $m < n$) for some formula A.

(E) All moves $\langle \delta(n), \eta(n) \rangle$ for n odd are of the form $\langle \delta(n), \eta(n) = [n-1, Z] \rangle$, for $Z = Q, A, D$ or Cut (where $n-1$ is empty if $Z = Cut$). That is, an **O**-move made at position n is either a question, an attack or a defense of the immediately preceding **P**-move made at position $n-1$, or it is a cut move with $\delta(n) = \mathbf{O}\,A$ for $\delta(n-1) = \mathbf{P}\,A$.

Definition 4.27 A formula A is called EI°-*valid* if there is an EI°-winning strategy for A. Notation: $\vDash_{\mathrm{EI}^\circ} A$.

In the presence of cut, there is an EI°-winning strategy for $a \to ((a \to (b \wedge c)) \to b)$:

0.	$\mathbf{P}\, a \to ((a \to (b \wedge c)) \to b)$			
1.	$\mathbf{O}\,?$			$[0, Q]$
2.	$\mathbf{P}\, \lvert a \to ((a \to (b \wedge c)) \to b) \rvert$			$[1, C]$
3.	$\mathbf{O}\, a$			$[2, A]$
4.	$\mathbf{P}\, (a \to (b \wedge c)) \to b$			$[3, D]$
5.	$\mathbf{O}\,?$			$[4, Q]$
6.	$\mathbf{P}\, \lvert (a \to (b \wedge c)) \to b \rvert$			$[5, C]$
7.	$\mathbf{O}\, a \to (b \wedge c)$			$[6, A]$
8.	$\mathbf{P}\, b \wedge c$			$[Cut]$
9.	$\mathbf{O}\,?$	$[8, Q]$	$\mathbf{O}\, b \wedge c$	$[Cut]$
10.	$\mathbf{P}\, a$	$[9, C]$	$\mathbf{P}\, ?1$	$[9, A]$
11.			$\mathbf{O}\, b$	$[10, D]$
12.			$\mathbf{P}\, b$	$[7, D]$

Instead of responding with a defense move to **O**'s attack $a \to (b \wedge c)$ made at position 7, the proponent continues by asserting the consequent $b \wedge c$ of that implication in the cut move at position 8. It is questioned at position 9 (in the left dialogue). In accordance with the rule condition (R), the proponent can now answer this question move by asserting in the choice move at position 10 (in the left dialogue) the antecedent a of the implication whose consequent has been questioned. The implication $a \to (b \wedge c)$ asserted by **O** at position 7 was thus used as a rule. The opponent cannot question the formula a due to condition (D14′): **O** has already asserted a (in the attack move at position 3), and **P** has not attacked a (such an attack is not even possible, since a is atomic). In the right dialogue, **O** makes the corresponding cut move at position 9, which is attacked by **P** and defended by **O** with the assertion of b. Now **P** can respond to **O**'s attack from position 7 by asserting b; as

O has already asserted b without its having been attacked by **P**, the opponent cannot question b due to condition (D14′), and **P** also wins the right dialogue.

4.8 Completeness

Completeness for EI°-dialogues and intuitionistic propositional logic can be proved (see Piecha 2012) by showing that there is an EI°-winning strategy for a formula A if and only if there is an $\mathrm{EI}^{\mathrm{P}}_{\mathrm{c}}$-winning strategy for A (see Theorem 4.6 below). Completeness (see Corollary 4.7 below) then follows from our completeness result for $\mathrm{EI}^{\mathrm{P}}_{\mathrm{c}}$-dialogues (see Theorem 3.2).

Definition 4.28 A *sub-winning strategy* is a subtree s of a winning strategy S, comprising as root node a node at an even position in S together with all its descendants given in S.

Lemma 4.3 *(i) The weak cut elimination property holds for* EI°*-winning strategies. That is, every* EI°*-winning strategy containing cut moves made according to the argumentation form* AF(Cut) *can be transformed into an* EI°*-winning strategy of the form*

$$
\begin{array}{rll|l}
 & & \vdots & \\
m. & & \mathbf{O}\,A \to B\ [m-1, Z] & \\
 & & \vdots & \\
n. & & \mathbf{P}\,B\ [Cut] & \\
n+1. & \mathbf{O}\,? & [n, Q] & \mathbf{O}\,B\ \ [Cut] \\
n+2. & \mathbf{P}\,A & [n+1, C] & \quad s_2 \\
n+3. & \mathbf{O}\,? & [n+2, Q] & \\
 & & \quad s_1 &
\end{array}
$$

where the **O***-move at position m is either an attack or a defense (i. e., either $Z = A$ or $Z = D$), and the move $\langle \delta(n+1) = \mathbf{O}\,B, \eta(n+1) = [Cut] \rangle$ is the uppermost cut move made by* **O** *(i. e., there is no cut move at positions $k < n-1$). The* **O***-move at position $n+3$ might not be possible due to* (D14′)*. In this case the left dialogue ends with the* **P***-move at position $n+2$.*
(Note that the cut formula B is a subformula of $A \to B$, asserted by **O** *at position m.)*
(ii) Furthermore, the sub-winning strategy s_2 is either of the same form as the above EI°*-winning strategy, or it depends on a sequence of moves made according to* AF(**O**∧), AF(**O**∨), AF(**O**→) *or* AF(**O**¬).

Corollary 4.4 *As a consequence of the weak cut elimination property,* EI°*-winning strategies have the subformula property. (This is in full analogy to the results in Schroeder-Heister 2011 for the sequent calculus.)*

Lemma 4.5 *(i)* EI°*-winning strategies for formulas of the form*

$$A \to ((A \to (B \wedge C)) \to B)$$

containing a cut move where the cut formula is of the form $B \wedge C$ *cannot be transformed into* EI°*-winning strategies (for the respective formula) containing no cut move. However, they can be transformed into* $\mathrm{EI}^{\mathrm{p}}_{\mathrm{c}}$*-winning strategies (for the respective formula).*
(ii) Every other EI°*-winning strategy (for a given formula) containing a cut move can be transformed into an* $\mathrm{EI}^{\mathrm{p}}_{\mathrm{c}}$*-winning strategy (for the given formula) as well.*

Theorem 4.6 $\vDash_{\mathrm{EI}^{\circ}} A$ *if and only if* $\vDash_{\mathrm{EI}^{\mathrm{p}}_{\mathrm{c}}} A$.

Corollary 4.7 (Completeness) *With Theorem 3.2 we have that the* EI°*-valid formulas are exactly the formulas provable in intuitionistic propositional logic.*

4.9 Discussion

One of the main differences between standard dialogues (like DI^{p}- or EI^{p}-dialogues) and EI°-dialogues is that the argumentation forms in the latter are no longer symmetric with respect to proponent and opponent. In other words, the player-independence of the argumentation forms that obtains in the standard dialogues is given up in EI°-dialogues: different argumentation forms have to be given for proponent and opponent. Although in standard dialogues proponent and opponent are also not interchangeable due to the dialogue conditions, there is a perfect symmetry with respect to the argumentation forms. If in the dialogical paradigm the idea of player-independent argumentation forms is considered essential, then giving it up may seem to amount to giving up the dialogical setting itself as a foundational approach. However, from the implications-as-rules point of view it could be argued that implication is different from the other logical constants, and that this difference requires an asymmetric treatment with respect to the argumentation forms.

As a consequence of this asymmetry in the treatment of implication there is another asymmetry: in EI°-dialogues the proponent can defend an assertion by means of the rule condition (R) independently of its logical form. This is not possible in standard dialogues, where a defense of an assertion always depends on its logical form, and where formulas are always decomposed into subformulas, i. e., according to their logical form. Nonetheless, we have shown that the subformula property holds at least for EI°-winning strategies.

But certain tenets within the dialogical tradition – such as the player-independence of argumentation forms or the decomposition of formulas according to their logical form – might not be essential; particularly not if implications are understood as rules. Rules are *not* logical constants but belong to the general structural framework that underlies definitions or meaning-explanations of logical constants. Given that the proponent has the dialogical role of claiming something to hold, and the opponent the role of providing the assumptions under which something is supposed to hold, the implication-as-rule $A \to B$ means for the proponent that B must be defended on the background of A, whereas the opponent grants with $A \to B$ only the right to *use* this implication as a rule, without any propositional claim. This is exactly what is captured in the EI°-dialogues for implications-as-rules.

An important aspect here is the significance which is given to *modus ponens*. For the implications-as-rules view, *modus ponens* is essential for the meaning of implication as it expresses the idea of *application*, which is the characteristic feature of a rule. In the calculus of natural deduction, *modus ponens* can be understood as the application of the implication $A \to B$ as a rule which allows us to infer B from A. In EI°-dialogues this means that a (partial) dialogue on B can be reduced to a (partial) dialogue on A, if an implication-as-rule $A \to B$ is given. We have thus obtained a dialogical interpretation for implications as rules.

A further complication is introduced by the need for (a restricted form of) cut in order to achieve full intuitionistic (propositional) logic. Although this need is present in both the proof-theoretic setting (e.g., using the sequent calculus) and the dialogical setting for implications-as-rules, the addition of an argumentation form for cut might be conceived as being alien to the dialogical approach as such, as this approach has always been considered as being cut-free *per se*. However, from the perspective of implications-as-rules such a view turns out to be too narrow.

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