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Atomic Systems in Proof-Theoretic Semantics: Two Approaches

Thomas Piecha and Peter Schroeder-Heister

Abstract Atomic systems are systems of rules containing only atomic formulas. In proof-theoretic semantics for minimal and intuitionistic logic they are used as the base case in an inductive definition of validity. We compare two different approaches to atomic systems. The first approach is compatible with an interpretation of atomic systems as representations of states of knowledge. The second takes atomic systems to be definitions of atomic formulas. The two views lead to different notions of derivability for atomic formulas, and consequently to different notions of proof-theoretic validity. In the first approach, validity is stable in the sense that for atomic formulas logical consequence and derivability coincide for any given atomic system. In the second approach this is not the case. This indicates that atomic systems as definitions, which determine the meaning of atomic sentences, might not be the proper basis for proof-theoretic validity, or conversely, that standard notions of proof-theoretic validity are not appropriate for definitional rule systems.

Key words: Proof-theoretic semantics, atomic systems, higher-level rules, definitions, definitional reflection, minimal logic, intuitionistic logic

1 Introduction

Within proof-theoretic semantics for logical constants the validity of atomic formulas, or atoms, is usually defined in terms of derivability of these formulas in atomic systems. Such systems can be sets of atomic formulas, figuring as atomic axioms, or

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sets of atomic rules, that is, of rules which only contain atomic formulas. Examples of such rules are production rules or definite Horn clauses. One can also allow for atomic rules which can discharge atomic assumptions, or even consider higher-level atomic rules which can discharge assumed atomic rules. Further crucial use of atomic systems is made in explaining the logical constant of implication. An implication $A \to B$ is valid with respect to an atomic system S (in short: S-valid) if and only if *for all extensions S' of S* it holds that whenever A is S'-valid then B is S'-valid. The reference to extensions guarantees that validity is monotone with respect to atomic systems. Otherwise it might happen that a formula, which is valid with respect to S, is invalid with respect to an extension of S.

This monotonicity requirement is motivated by the interpretation of atomic systems as knowledge bases. What is valid should remain valid, if our knowledge as incorporated in an atomic knowledge base is extended. However, there are contexts in which we do not expect monotonicity to hold, as studied, for example, in the various branches of non-monotonic logic. Here we study definitional contexts as a particular case. When we interpret atomic systems as definitions, we cannot postulate monotonicity. If we extend the definition of a term, a valid proposition may lose its validity. Correspondingly, when basing proof-theoretic validity on atomic systems as definitions, for the S-validity of implication and consequence we should not refer to arbitrary extensions S' of S. In his recent publications, Prawitz, who coined the notion of proof-theoretic validity in [14], prefers this definitional reading of atomic systems ('bases') and explicitly refrains from the reference to extensions of atomic systems:

> A base is seen as determining the meanings of the atomic sentences. [18, section 5]
>
> To consider extensions of the given base [...] is natural when a base is seen as representing a state of knowledge, but is in conflict with the view adopted here that a base is to be understood as giving the meanings of the atomic sentences. [18, fn. 12]

This view leads to problems, however. We will show that, if validity is based on atomic systems understood as definitions, then it is not *stable*, that is, logical consequence and derivability diverge already at the atomic level. This negative result is even independent of whether consequence and implication are characterized with respect to arbitrary extensions of atomic systems or not. This shows that the definitional view of atomic systems is not compatible with the concept of proof-theoretic validity in its given form. This result depends, of course, on the theory of definitions used. In this paper we rely on the approach based on the idea of definitional reflection (see [6, 7]; cf. [23]) according to which the definitional reading of atomic rules is implemented by a rule schema which expresses that the clauses given for a certain atom *exhaustively* characterize that atom.

We confine ourselves to propositional logic, as this suffices to make our point. In Section 2 we consider notions of proof-theoretic validity which are monotone with respect to extensions of atomic systems. In Section 3 we compare this approach to Kripke semantics and show that proof-theoretic validity corresponds to considering validity in a specific Kripke model. In Section 4 we describe the idea of atomic systems as definitions and establish that stability is lost under the definitional reading

of atomic systems. Derivability from assumptions and validity of consequence do not even coincide in the atomic case.

2 Atomic systems and proof-theoretic validity

2.1 First-level atomic systems and validity

Atomic systems have been considered in proof-theoretic approaches to validity by Prawitz [14] and Dummett [5], for example. There atomic systems are sets of production rules for atomic formulas, or atoms, $a, b, \ldots, a_1, a_2, \ldots$, defined as follows:

Definition 1. A *(first-level) atomic system* S is a (possibly empty) set of atomic rules of the form

$$\frac{a_1 \quad \ldots \quad a_n}{b}$$

where the a_i and b are atoms. The set of premisses $\{a_1, \ldots, a_n\}$ in a rule can be empty; in this case the rule is an *atomic axiom* and of *level 0*.

The *derivability* of an atom a from a (possibly empty) set $\{a_1, \ldots, a_n\}$ of atomic assumptions in an atomic system S is written $a_1, \ldots, a_n \vdash_S a$. *Derivations* are defined as usual. For example, for the atomic system S:

$$\frac{}{c} \qquad \frac{a}{b} \qquad \frac{b \quad c}{d}$$

the derivation

$$\dfrac{\dfrac{a}{b} \quad \dfrac{}{c}}{d}$$

shows $a \vdash_S d$.

Extensions S' of atomic systems S are understood in the set-theoretic sense, that is, an atomic system S' is an *extension* of an atomic system S, written $S' \supseteq S$, if S' results from adding a (possibly empty) set of atomic rules to S. For example, $S' = S \cup \{a\}$ is an extension of S by the atomic axiom a. For this extension $\vdash_{S'} d$ holds.

In proof-theoretic notions of validity, the validity of atoms is determined by their derivability in atomic systems, and the validity of complex formulas is defined inductively with respect to such systems. Originally, Prawitz [14–17] gave certain notions of validity for derivations which are constructed from arbitrary inference rules. These notions of validity not only depend on atomic systems but also on reduction procedures ('justifications') which transform such derivations into other derivations (see also [24, 25]).

In what follows, we consider instead notions of validity for formulas (see [13]), which do not depend on reduction procedures. We restrict ourselves to formulas $A, B, \ldots$ in the fragment $\{\rightarrow, \vee, \wedge\}$ of minimal propositional logic; absurdity $\perp$ is just a distinguished atom, not a logical constant.

Definition 2. *S-validity* ($\models_S$) and *validity* ($\models$) are defined as follows:

(S1) $\models_S a \ :\Longleftrightarrow\ \vdash_S a$,

(S2) $\models_S A \to B \ :\Longleftrightarrow\ A \models_S B$,

(S3) $\Gamma \models_S A \ :\Longleftrightarrow\ \forall S' \supseteq S : (\models_{S'} \Gamma \implies \models_{S'} A)$, where Γ is a set of formulas, and where $\models_{S'} \Gamma$ stands for $\{\models_{S'} A_i \mid A_i \in \Gamma\}$,

(S4) $\models_S A \vee B \ :\Longleftrightarrow\ \models_S A$ or $\models_S B$,

(S5) $\models_S A \wedge B \ :\Longleftrightarrow\ \models_S A$ and $\models_S B$,

(S6) $\Gamma \models A \ :\Longleftrightarrow\ \forall S : \Gamma \models_S A$.

By clause (S1), S-validity of atoms is defined in terms of derivability in an atomic system S. Another important use of atomic systems is made in the definition of S-consequence $\Gamma \models_S A$ (S3), and thus of S-validity of implication $\models_S A \to B$ (S2), which is defined by S-consequence $A \models_S B$. In clause (S3), arbitrary extensions of atomic systems are considered. This has the effect that an S-consequence $\Gamma \models_S A$ cannot just hold because some atom on which Γ depends is not valid in S. This would be the case if S-consequence $\Gamma \models_S A$ were, for example, defined by

$$\Gamma \models_S A \ :\Longleftrightarrow\ (\models_S \Gamma \implies \models_S A) \tag{S3$'$}$$

where no extensions of S are considered. In this case, if, for example, $\Gamma = \{a\}$, $A = b$ and $S = \emptyset$, then $\not\models_S a$ and thus trivially $(\models_S a \implies \models_S b)$, and hence $a \models_S b$. Validity with respect to atomic systems would therefore fail to be monotone, since for example for $S' = S \cup \{a\} = \{a\}$ we have $a \not\models_{S'} b$ while $a \models_S b$. This situation is avoided by considering arbitrary extensions in the definition of S-consequence. Indeed, taking extensions into account guarantees monotonicity, as we can easily prove:

$$\Gamma \models_S A \implies \forall S' \supseteq S : \Gamma \models_{S'} A.$$

2.2 Higher-level atomic systems

Atomic systems need not be restricted to systems of first level. Second-level and arbitrary higher-level atomic systems can be considered as well (see [13]; cf. [22] and [20]).

Definition 3. A *second-level atomic system* S is a (possibly empty) set of atomic rules of the form

$$\frac{\begin{array}{ccc}[\Gamma_1] & & [\Gamma_n] \\ a_1 & \ldots & a_n\end{array}}{b}$$

where the a_i and b are atoms, and the Γ_i are finite sets of atoms. The sets Γ_i may be empty, in which case the rule is a *first-level rule*. The set of premisses $\{a_1, \ldots, a_n\}$ can be empty as well; in this case the rule is an axiom.

Such a rule can be applied as follows: If the premisses $a_1, \ldots, a_n$ have been derived in S from certain assumptions $\Gamma_1, \ldots, \Gamma_n$, then one may conclude b, where, for each i,

in the branch of the subderivation leading to a_i assumptions belonging to Γ_i may be discharged.

Second-level atomic systems are now further generalized to the higher-level case by allowing for atomic rules which can discharge not only atoms but atomic rules as assumptions (see [22, 26] and [11]; cf. [13]).

Definition 4. We use the following linear notation for atomic *higher-level rules*:
(i) Every atom a is a rule of level 0.
(ii) If $R_1, \ldots, R_n$ are rules ($n \geq 1$), whose maximal level is ℓ, and a is an atom, then $(R_1, \ldots, R_n \rhd a)$ is a rule of level $\ell + 1$.

In tree notation, higher-level rules have the form

$$\frac{\begin{array}{ccc}[\Gamma_1] & & [\Gamma_n] \\ a_1 & \ldots & a_n\end{array}}{b}$$

where the a_i and b are atoms, and the Γ_i are finite sets $\{R_1^i, \ldots, R_k^i\}$ of rules, which may be empty. The set of premisses $\{a_1, \ldots, a_n\}$ of such a rule can again be empty, in which case the rule is an axiom.

Definition 5. A *higher-level atomic system S* is a (possibly empty) set of higher-level rules.

Higher-level rules can be represented by formulas in the fragment $\{\rightarrow, \wedge\}$:

Definition 6. With every rule R in a set of rules S a formula R^* representing R is associated as follows:
(i) $a^* \coloneqq a$, for atoms a.
(ii) $(R_1, \ldots, R_n \rhd a)^* \coloneqq R_1^* \wedge \ldots \wedge R_n^* \rightarrow a$, for a rule $R_1, \ldots, R_n \rhd a$.
Then S^* is defined as the set of formulas representing the rules in S.

In the higher-level case, atomic *rules* can be used as (dischargable) assumptions, whereas in the second-level case only atoms could be used in that way. This difference requires a definition of the notion of *derivation* of atoms from rules:

Definition 7. For a level-0 rule a,

$$\frac{}{a}\, a$$

is a *derivation* of a from $\{a\}$.

Now consider a level-$(\ell+1)$ rule $(\Gamma_1 \rhd a_1), \ldots, (\Gamma_n \rhd a_n) \rhd b$. Suppose that for each i $(1 \leq i \leq n)$ a derivation

$$\begin{array}{c}\Sigma_i \cup \Gamma_i \\ \mathscr{D}_i \\ a_i\end{array}$$

of a_i from $\Sigma_i \cup \Gamma_i$ is given. Then

$$\dfrac{\begin{array}{ccc}\Sigma_1 & & \Sigma_n\\ \mathscr{D}_1 & & \mathscr{D}_n\\ a_1 & \ldots & a_n\end{array}}{b}\,(\Gamma_1 \rhd a_1),\ldots,(\Gamma_n \rhd a_n) \rhd b$$

is a *derivation* of b from $\Sigma_1 \cup \ldots \cup \Sigma_n \cup \{(\Gamma_1 \rhd a_1),\ldots,(\Gamma_n \rhd a_n) \rhd b\}$.

An atom b is *derivable* from Σ in a higher-level atomic system S, symbolically $\Sigma \vdash_S b$, if there is a derivation of b from $\Sigma \cup S$.

We give an example derivation for the atomic system

$$S\begin{cases} (b \rhd e) \rhd f \\ ((a \rhd b) \rhd c) \rhd e \end{cases}$$

and the set of assumptions $\Sigma = \{((a \rhd b) \rhd d), ((d,b) \rhd c)\}$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\overline{a}\,[a]^1}{b}\,[a \rhd b]^2}{d}\,(a \rhd b) \rhd d \qquad \dfrac{}{b}\,[b]^3}{(d,b) \rhd c}}{c}\,2\\ \dfrac{c}{e}\,\langle((a \rhd b) \rhd c) \rhd e\rangle\,3\\ \dfrac{e}{f}\,\langle(b \rhd e) \rhd f\rangle\,1$$

The derivation shows $\Sigma \vdash_S f$. (Angle brackets $\langle\,\rangle$ are used to indicate the rules of S, and square brackets $[\,]$ with numerals indicate the discharge of assumptions.)

The definition of validity for second-level or higher-level atomic systems is exactly the same as that for first-level atomic systems (Definition 2). The generalization from first- to higher-level atomic systems does not affect the monotonicity of validity: S-validity, and hence validity, for higher-level atomic systems is monotone with respect to extensions $S' \supseteq S$.

2.3 Completeness issues

It can be shown that minimal logic is not complete with respect to validity. A counterexample is the consequence

$$a \to (b \vee c) \vDash (a \to b) \vee (a \to c)$$

which holds independently of the level of atomic systems. Here it is important that a, b and c are individual atoms, not propositional variables. This counterexample ceases to hold for arbitrary substitutions of complex formulas for atoms. If, for example, $b \vee c$ is substituted for a, then the resulting consequence is no longer valid. This shows that validity is not closed under substitution. Since derivability in minimal logic is closed under substitution, one could demand that a notion of validity proposed for

minimal logic should be closed under substitution as well. This can be done by definition:

Definition 8. *S-validity under substitution* ($\Vdash\!\models_S$) and *validity under substitution* ($\Vdash\!\models$) are defined as follows:
(i) $\Gamma \Vdash\!\models_S A \iff$ for each substitution instance Γ', A' of Γ, A: $\Gamma' \models_S A'$.
(ii) $\Gamma \Vdash\!\models A \iff$ for each substitution instance Γ', A' of Γ, A: $\Gamma' \models A'$.

These strengthened notions of validity can be extended to intuitionistic logic. There one considers the following notion of validity:

Definition 9. Let $(\bot)$ stand for the set of rules $\left\{ \dfrac{\bot}{a} \;\middle|\; a \text{ atomic} \right\}$. Then *intuitionistic S-validity* is defined as follows: $\Gamma \models^i_S A \iff \Gamma \models_{S\cup(\bot)} A$.

Intuitionistic validity $\Gamma \models^i A$ is defined as $\Gamma \models_{(\bot)} A$, and the corresponding notions closed under substitution, $\Gamma \Vdash\!\models^i_S A$ and $\Gamma \Vdash\!\models^i A$, are defined as $\Gamma \Vdash\!\models_{S\cup(\bot)} A$ and $\Gamma \Vdash\!\models_{(\bot)} A$, respectively.

For the case of higher-level atomic systems S it could be shown (see [13]) that intuitionistic propositional logic is not complete for intuitionistic validity under substitution ($\Vdash\!\models^i$). A counterexample is the intuitionistically non-derivable but valid Harrop formula (where $\neg A := A \to \bot$):

$$(\neg A \to (B \vee C)) \to ((\neg A \to B) \vee (\neg A \to C)).$$

If we restrict ourselves to first-level atomic systems, the question of completeness is still open. However, in view of the fact that proof-theoretic validity amounts to considering a single Kripke model rather than the totality of all Kripke models (see Section 3 below), we would conjecture that, as in the higher-level case, we lose the completeness of intuitionistic logic. Proof-theoretic validity characterizes at best (that is, if validity is closed under substitution) some intermediate logic between the intuitionistic and classical systems.

For details concerning completeness we refer to [13] and [12]. Here we just remark that completeness (or failure of completeness) of logical systems for the proposed notions of validity depends essentially on the kind of atomic systems on which these notions are based.

2.4 Stability of S-validity

Let Δ^* be the set of formulas representing a finite set Δ of atomic rules (in the sense of Definition 6). One can show that S-validity is *stable* in the sense that

$$\Delta^* \models_S b \iff \Delta^* \vdash_S b$$

holds for any atomic systems S. This includes *atomic completeness*

$$a_1,\ldots,a_n \models_S b \implies a_1,\ldots,a_n \vdash_S b$$

and *atomic soundness*

$$a_1,\ldots,a_n \vdash_S b \implies a_1,\ldots,a_n \models_S b$$

as special cases. Intuitionistic S-validity ($\models^i_S$) is stable as well.

Stability is an important feature of S-validity, since it guarantees that S-validity is not creative in the sense that atomic completeness fails, and that it is not destructive in the sense that atomic soundness fails (see also the discussion in [21] on conservativeness as a desideratum). If we consider notions of S-validity which lack stability, we must take into account that atomic derivability from assumptions $a_1,\ldots,a_n \vdash_S a$ can be different from the corresponding S-consequence, even though atomic derivability $\vdash_S a$ is (by definition) equivalent with the S-validity of a. In this case, atomic systems would be used merely as a device to generate valid atoms, where the induced relation of derivability from assumptions can be totally disregarded. Technically, this is no problem. However, conceptually, this would not be much different from looking at atomic systems as sets of atoms which are valid by definition.

3 Proof-theoretic validity and Kripke semantics

The formulation of Definition 2 has a striking resemblance to the definition of validity in Kripke semantics (see e.g. [2, 10, 28]). It can actually be viewed as a definition of validity in a special Kripke model.

In Kripke semantics for propositional intuitionistic logic a Kripke model $\mathscr{K}$ consists of a partial order $\leq$ between objects called *nodes* (or *reference points* or *worlds*) together with a valuation function v which tells which atoms are true at which node. Thus $v(a,k) = 1$ means that the atom a is true at node k. This valuation function must satisfy the monotonicity condition that, if $k' \geq k$ and $v(a,k) = 1$, then $v(a,k') = 1$. Intuitively, this means that what is true at some stage, must remain true. Then the validity $\models^{\mathscr{K}}_k A$ of a formula A in $\mathscr{K}$ at a node k, the validity $\Gamma \models^{\mathscr{K}}_k A$ of a consequence of A from Γ in $\mathscr{K}$ at k, the validity $\Gamma \models^{\mathscr{K}} A$ of a consequence of A from Γ in $\mathscr{K}$, and the validity (simpliciter) $\Gamma \models A$ of a consequence of A from Γ (i.e., logical validity) are defined as follows:

Definition 10.

(K1) $\models^{\mathscr{K}}_k a \;:\Longleftrightarrow\; v(a,k) = 1$,

(K2) $\models^{\mathscr{K}}_k A \to B \;:\Longleftrightarrow\; A \models^{\mathscr{K}}_k B$,

(K3) $\Gamma \models^{\mathscr{K}}_k A \;:\Longleftrightarrow\; \forall k' \geq k : (\models^{\mathscr{K}}_{k'} \Gamma \implies \models^{\mathscr{K}}_{k'} A)$, where Γ is a set of formulas, and where $\models^{\mathscr{K}}_{k'} \Gamma$ stands for $\{\models^{\mathscr{K}}_{k'} A_i \mid A_i \in \Gamma\}$,

(K4) $\models^{\mathscr{K}}_k A \vee B \;:\Longleftrightarrow\; \models^{\mathscr{K}}_k A$ or $\models^{\mathscr{K}}_k B$,

(K5) $\models^{\mathscr{K}}_k A \wedge B \;:\Longleftrightarrow\; \models^{\mathscr{K}}_k A$ and $\models^{\mathscr{K}}_k B$,

(K6) $\Gamma \models^{\mathscr{K}} A \;:\Longleftrightarrow\; \forall k : \Gamma \models^{\mathscr{K}}_k A$,

(K7) $\Gamma \models A \ :\Longleftrightarrow\ \forall \mathscr{K} : \Gamma \models^{\mathscr{K}} A$.

As in Definition 2 we restrict ourselves to minimal logic. Normally, in Kripke semantics, the consequence relations $\Gamma \models^{\mathscr{K}}_k A$ and $\Gamma \models^{\mathscr{K}} A$ are not defined; instead, the validity of implication in $\mathscr{K}$ at node k is defined as:

$$\models^{\mathscr{K}}_k A \to B \ :\Longleftrightarrow\ \forall k' \geq k : (\models^{\mathscr{K}}_{k'} A \implies \models^{\mathscr{K}}_{k'} B).$$

However, it can easily be seen that our Definition 10 comes to the same, as far as the relations validity $\models^{\mathscr{K}}_k A$ of a formula and logical validity $\Gamma \models A$ are concerned.

From the parallelism between (S1)-(S6) and (K1)-(K6) it is obvious that the definition of validity in Definition 2 is the definition of validity for a *specific* Kripke model $\mathscr{S}$, the nodes of which are the atomic systems S, the accessibility relation $\leq$ between nodes is the inclusion relation $\subseteq$ between atomic systems, and the valuation function v is defined by the derivability in S, that is, $v(a,S) = 1 \ :\Longleftrightarrow\ \vdash_S a$. From this definition of v and the fact that $\leq$ is set inclusion $\subseteq$ it is clear that the monotonicity condition required for v is satisfied. $\Gamma \models A$ in the sense of Definition 2 means the same as $\Gamma \models^{\mathscr{S}} A$ for this Kripke model $\mathscr{S}$.

From this point of view the counterexamples to completeness mentioned in Section 2.3 and established in [19] and [4, 12, 13] are not really surprising. If the definition of validity is merely based on validity in a *specific* Kripke model, we cannot expect completeness for intuitionistic (here: minimal) logic, of which we know that it holds with respect to logical validity, that is, to validity in *all* Kripke models. There is no obvious reason why the model $\mathscr{S}$ should be 'canonical' in that it represents the totality of all Kripke models.

4 Atomic systems as definitions

If atomic systems are understood as knowledge bases, then monotonicity of validity with respect to extensions is certainly a desired property, since increased knowledge should at least account for what is already known. If a consequence $\Gamma \models_S A$ has been established on the basis of some knowledge given by the atomic system S, and an atomic system S' extends that knowledge, then $\Gamma \models_{S'} A$ should hold as well.

There is, however, an alternative view of atomic systems, in which one would not expect monotonicity of consequence with respect to extensions. Atomic systems can be understood as definitions of atoms. As an extension of a definition changes in general what is being defined, it is to be expected that there are consequences which hold with respect to the initial definition but do no longer hold with respect to an extension of that definition.

4.1 Definitional closure

Consider an atomic system

$$S\begin{cases}\Gamma_1 \rhd a\\ \quad\vdots\\ \Gamma_k \rhd a\end{cases}$$

of k higher-level atomic rules. This can be read as a *definition* of the atom a by *defining conditions* Γ_i, for $1 \leq i \leq k$. In this definitional reading the atomic rules $\Gamma_i \rhd a$ are also called *definitional clauses*. The defining conditions in such clauses can be empty. In the terminology of inductive definitions (see [1]) one can thus distinguish basis clauses of the form $\emptyset \rhd a$ (or just a) and inductive clauses of the form $\Gamma_i \rhd a$ (for non-empty Γ_i).

A direct application of such a definition consists in passing from some defining condition Γ_i of a to the defined atom a:

$$\frac{\Gamma_i}{a}$$

Inferences of this kind are also called steps of *definitional closure*. They correspond to the individual steps in a derivation of an atom in a higher-level atomic system.

4.2 Definitional reflection

In the reading of atomic systems as definitions a difference is introduced by the fact that in the case of a definition of an atom a it is assumed that nothing else defines a. This assumption, the extremality condition, is usually made only implicitly (for example in mathematical definitions), just by saying that something is a definition. Sometimes it is stated explicitly by saying that the clauses for a in a definition define *the smallest set* of objects for which the given clauses hold, or by adding a clause, the extremal clause, saying that *nothing else defines a*.

When this assumption is taken into consideration, an additional reasoning principle becomes available for definitions. For an atom a defined by

$$S\begin{cases}\Gamma_1 \rhd a\\ \quad\vdots\\ \Gamma_k \rhd a\end{cases}$$

one can, in addition to definitional closure, also reason by *definitional reflection* (see [6, 7]; cf. [23]):

$$\frac{a \qquad \begin{array}{c}[\Gamma_1]\\ C\end{array} \;\ldots\; \begin{array}{c}[\Gamma_k]\\ C\end{array}}{C}$$

This rule says that whenever a formula C follows from each of the defining conditions Γ_i of an atom a, then C follows from the defined atom a alone.

If no additional logical rules are available, or if no additional rules are available for the decomposition or construction of higher-level rules, then C will in general be an atomic formula. An exception is the case where a is an undefined atom, say $\bot$, that is, where S does not contain any clauses of the form $\Gamma \rhd \bot$. Then any formula C can be inferred from $\bot$ by definitional reflection, since the set of defining conditions of the undefined atom $\bot$ is empty. This means that for atomic systems S as definitions a principle of *ex falso quodlibet*

$$\bot \vdash_S C$$

is available as long as at least one atom $\bot$ is undefined in S.

Definitional reflection is only justified for atomic systems as definitions, that is, when an extremality condition is assumed. Without this assumption only definitional closure can be used.

4.3 Properties of derivability

In general, a *definition* is any finite atomic system

$$S\begin{cases} \Gamma_1^1 \rhd a_1 & & \Gamma_1^n \rhd a_n \\ \quad\vdots & \cdots & \quad\vdots \\ \Gamma_{k_1}^1 \rhd a_1 & & \Gamma_{k_n}^n \rhd a_n \end{cases}$$

Definitions in this sense need not have basis clauses $\emptyset \rhd a_i$. They are thus similar to logic programs, where such a restriction is not made either.

We here consider only atomic systems of higher-level atomic rules, which could be represented by formulas in the fragment $\{\to, \wedge\}$ (see Definition 6). When atomic systems are used as definitions one could also allow the defining conditions $\Gamma_{j_i}^i$ in definitional clauses $\Gamma_{j_i}^i \rhd a_i$ to be arbitrary formulas (see [8, 9]). However, this is not permitted in our setting here.

As an example, consider the following definition:

$$S\begin{cases} \Gamma \rhd a & & \Gamma \rhd b \\ \Delta \rhd a & & \Delta \rhd b \\ & & \Sigma \rhd b \end{cases}$$

Using definitional closure and definitional reflection we can show that $a \vdash_S b$ (but not $b \vdash_S a$) holds:

$$\dfrac{a \qquad \dfrac{[\Gamma]^1}{b}\,(\text{def. closure}),\ \langle \Gamma \rhd b\rangle \qquad \dfrac{[\Delta]^1}{b}\,(\text{def. closure}),\ \langle \Delta \rhd b\rangle}{b}\,(\text{def. reflection on } S)\ 1$$

The set of subderivations $\left\{\frac{\Gamma}{b},\frac{\Delta}{b}\right\}$ shows that b can be derived from each of the defining conditions of a, namely Γ and Δ. Thus definitional reflection can be applied to a, discharging the assumptions Γ and Δ. Without definitional reflection, $a \vdash_S b$ cannot be shown.

For the extension $S' = S \cup \{\Theta \rhd a\}$ we do not have $a \vdash_{S'} b$, if $\Theta \vdash_{S'} b$ does not hold. In other words, since b cannot be derived from each of the defining conditions of a (the exception being Θ), we cannot apply definitional reflection here, and it thus cannot be shown that b is derivable in S' from a as the only assumption. This example shows that atomic systems behave quite differently when they are treated as definitions. It shows in particular that derivability fails to be monotone with respect to extensions of atomic systems: For the given $S' \supseteq S$ we have $a \vdash_S b$ but $a \nvdash_{S'} b$. Monotonicity is already lost in the case of first-level atomic systems, as can be seen by letting the defining conditions $\Gamma, \Delta, \Sigma, \Theta$ be sets of atoms. By the same argument we can see that for the extension $S'' = S \cup \{a \rhd a\}$ we do not have $a \vdash_{S''} b$, because for that to hold we would already need $a \vdash_{S''} b$, which is exactly what we want to prove. In effect, the addition of the clause $a \rhd a$ to a definition blocks the application of definitional reflection with respect to a, as one of the premisses of definitional reflection would already require as proven what one intends to prove.

4.4 Validity based on definitions

We now consider S-validity and validity in the context of atomic systems as definitions. That is, we now consider the situation where derivability $\vdash_S$ is defined with respect to atomic systems S understood as definitions.

We distinguish two cases. In the first case S-validity and validity are exactly as given by Definition 2, where S-consequence $\Gamma \models_S A$ is defined using extensions $S' \supseteq S$:

$$\Gamma \models_S A \;:\Longleftrightarrow\; \forall S' \supseteq S : (\models_{S'} \Gamma \implies \models_{S'} A) \tag{S3}$$

In the second case we consider validity without extensions, that is, we define S-consequence $\Gamma \models_S A$ as follows:

$$\Gamma \models_S A \;:\Longleftrightarrow\; (\models_S \Gamma \implies \models_S A) \tag{S3$'$}$$

We show that atomic soundness fails for validity using extensions, and that atomic completeness fails for validity without extensions. In each case we give a very simple counterexample which only uses the framework of first-level rules.

Case 1: Validity with extensions. Atomic soundness does not hold. For the empty definition $S = \emptyset$ we have $a \vdash_S b$ by definitional reflection, since a is not defined. Now consider the extension $S' = S \cup \{a\} = \{a\}$ in which a is defined. Then $\vdash_{S'} a$ and thus $\models_{S'} a$, while $\nvdash_{S'} b$ and therefore $\not\models_{S'} b$. Hence $\forall S' \supseteq S : (\models_{S'} a \implies \models_{S'} b)$ fails to hold, which means $a \not\models_S b$.

Case 2: Validity without extensions. Atomic completeness does not hold. The definition $S = \{a \rhd a\}$ yields a counterexample. We have $\nvdash_S a$ and thus $\nvDash_S a$; hence $a \vDash_S b$ by clause (S3′). But $a \nvdash_S b$, since in S only a can be derived from a.

Summing up, we have:

Proposition 1. *S-validity (with or without extensions) is not stable.*

As this result is independent of whether extensions are considered or not, it hints at a deeper issue in the relation between definitional bases and proof-theoretic validity. In definitional reasoning, consequence $\Gamma \vdash_S a$ is based on specific definitional rules, in particular on rules, which allow one to assume an atom in a specific way by means of definitional reflection. This has the effect that the biconditional

$$\Gamma \vdash_S a \iff (\vdash_S \Gamma \implies \vdash_S a)$$

is no longer guaranteed. On the other hand, proof-theoretic validity is fundamentally based on the biconditional

$$\Gamma \vDash_S a \iff (\vDash_S \Gamma \implies \vDash_S a)$$

(we disregard extensions). This suggests that definitional reasoning and proof-theoretic validity aim at different notions of consequence and therefore implication. It is possible indeed to build a notion of validity on top of definitional bases. However, this would not proceed according to a validity definition as set out in Definition 2, but by considering introduction and elimination rules for logical constants as instances of definitional rules and thus by incorporating logic into the realm of definitional reasoning (cf. [3] and [27]). Definitional reflection would then be considered to be a general reasoning principle which applies to the atomic and logical cases likewise.

5 Conclusion

We considered two approaches to atomic systems. They show that within proof-theoretic semantics widely differing notions of validity can be formulated, depending on how atomic systems are understood. The first approach dealt with atomic systems of production rules (first-level), of assumption-discharging rules (second-level) and of arbitrary higher-level rules, which allow for the discharge of assumed atomic rules. Such atomic systems can be understood as knowledge bases. Notions of proof-theoretic validity based on these kinds of atomic systems are monotone with respect to extensions of atomic systems. The choice of the kind of atomic systems can make a difference with respect to completeness (see [12, 13]).

In the second approach, where atomic systems are understood as definitions, the situation is quite different. The additional principle of definitional reflection induces a derivability relation which is not monotone with respect to extensions of such systems. It is doubtful whether notions of proof-theoretic validity in the sense of Definition 2

should be based on atomic systems understood as definitions: Atomic soundness does not hold for validity using extensions, and atomic completeness fails for validity not using extensions. This means that S-validity is not stable. Definitional reflection is a principle leading to a different notion of validity. Besides the points mentioned in the last paragraph of Section 4, definitional reflection goes beyond the scope of atomic systems, since in principle it allows one to derive not only atoms from atoms but also complex formulas from atoms. Although the underlying definitions are atomic systems, they might then no longer be foundational for the meaning explanations for the logical constants given in standard notions of proof-theoretic validity.

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