Improved Decision Procedures for the Modal Logics K, T and S4

Jörg Hudelmaier

WSI, Universität Tübingen Sand 13, D72076 Tübingen
Tel. (49)7071 297361
joerg@logik.informatik.uni-tuebingen.de

Abstract. We propose so called contraction free sequent calculi for the three prominent modal logics K, T, and S4. Deduction search in these calculi is shown to provide more efficient decision procedures than those hitherto known. In particular space requirements for our logics are lowered from the previously established bounds of the form n^2, n^3 and n^4 to $n \log n$, $n \log n$, and $n^2 \log n$ respectively.

1 Introduction

Modal logics are extensions of classical propositional logic by the necessity operator $\Box$. Different properties of necessity give rise to various such logics. Here we consider three logics K, T, and S4. For all these logics basic properties required of the necessity operator are validity of the so called necessitation rule

$$\text{(N)} \quad \frac{a}{\Box a}$$

and of the axiom (K) $\Box(a \to b) \to (\Box a \to \Box b)$. For the logic T we need in addition to (N) and (K) the axiom (T) $\Box a \to a$ and for S4 we need besides (N), (K), and (T) the axiom (4) $\Box a \to \Box\Box a$. Semantics for all these logics are based on so called Kripke models, i.e. sets of possible worlds with binary accessibility relations on them. Validity of a formula is then defined as validity in all suitable Kripke models, where suitability of a model for a given logic is described in terms of properties of the accessibilty relation. So for the logic K any Kripke model is suitable, whereas for T only Kripke models with reflexive accessibilty are suitable and for S4 only models with reflexive and transitive accessibility are suitable. Validity of a formula in a given Kripke model m is defined recursively over the structure of the formula, where the Boolean cases coincide with the definition of classical validity and a formula $\Box a$ is valid in m, iff it is valid in all worlds accessible from m.

It has been shown that validity of formulas for any of these three logics is PSPACE complete, even for their Horn fragments [4] and for formulas with only one propositional variable [6]. In particular Ladner in the 1970s has given decision procedures using Kripke models for the modal logics K, T, and S4 of space complexity respectively n^2, n^3 and n^4 [1]. Now instead of Kripke models most systems designed for actual theorem proving in modal logic (cf. [5], [8]) use so called sequent calculi [3]. These calculi are, however, built on formalized search for counter examples in Kripke models. Thus they can not provide smaller space bounds. Moreover building theorem provers for such calculi is made difficult by the fact that depth first search is not guaranteed to terminate. Thus more sophisticated search procedures have to be applied, e.g. procedures using loop checking. In contrast to this it has recently been shown that for intuitionistic propositional logic so called contraction free sequent calculi, i.e. calculi for which there is a measure such that for each rule of the calculus the measure of all its premisses is smaller than the measure of its conclusion, yield more efficient decision procedures [2]. Thus it was natural to try to extend this approach to modal logic. However it turned out that contraction free calculi could not be obtained directly from these calculi; most notably for the logics T and S4 a new connective had to be added to the language, in order to be able to define a suitable measure function for the contraction free calculus.

2 Basic Clausal Calculi

As our basic language we consider arbitrary modal formulas built up from propositional variables by means of the connectives $\neg$, $\vee$ and $\Box$ and modal clauses, i.e. formulas of the form $\Box \dots \Box(a_1 \vee \dots \vee a_l) \quad (1 \leq l \leq 3)$ or $\Box \dots \Box(a_1 \vee \Box b)$ or $\Box \dots \Box(a_1 \vee \neg\Box b)$, where the a_i are propositional variables or negations of propositional variables and b is a propositional variable. These clauses we write in the form $\Box^s[c_1, \dots, c_n]$. Ordinary sequents are pairs of multisets of arbitrary formulas and clausal sequents are multisets of modal clauses.

We start from an often used family of sequent calculi $\mathrm{F}_0(\mathrm{K})$, $\mathrm{F}_0(\mathrm{T})$ and $\mathrm{F}_0(4)$ for deducing ordinary sequents. All these calculi have axioms of the form $M, v \Rightarrow v, N$, where v is a propositional variable and Boolean rules

$$(\mathrm{EB}\neg_0) \quad \frac{M \Rightarrow N, v}{M, \neg v \Rightarrow N} \qquad\qquad (\mathrm{IB}\neg_0) \quad \frac{M, v \Rightarrow N}{M \Rightarrow N, \neg v}$$

$$(\mathrm{EB}\vee_0) \quad \frac{M, u \Rightarrow N \quad M, v \Rightarrow N}{M, u \vee v \Rightarrow N} \qquad\qquad (\mathrm{IB}\vee_0) \quad \frac{M \Rightarrow N, u, v}{M \Rightarrow N, u \vee v}$$

Moreover $\mathrm{F}_0(\mathrm{K})$ has the modal rule

$$(\mathrm{IK}\Box_0) \quad \frac{M \Rightarrow v}{L, \Box M \Rightarrow N, \Box v}$$

$\mathrm{F}_0(\mathrm{T})$ has the rule $\mathrm{IK}\Box_0$ and the rule

$$(\mathrm{ET}\Box_0) \quad \frac{M, \Box v, v \Rightarrow N}{M, \Box v \Rightarrow N}$$

and $\mathrm{F}_0(4)$ has the rule $\mathrm{ET}\Box_0$ and the rule

$$(\mathrm{I4}\Box_0) \quad \frac{\Box M \Rightarrow v}{L, \Box M \Rightarrow N, \Box v}$$

It is well known that $\mathrm{F}_0(\mathrm{K})$ formalizes the modal logic K, $\mathrm{F}_0(\mathrm{T})$ formalizes T and $\mathrm{F}_0(4)$ formalizes S4 (cf. [3]). Thus in particular for every one of these calculi all the structural rules are admissible, i.e. if a sequent $M \Rightarrow N$ is deducible, then so are the sequents $v, M \Rightarrow N$ and $M \Rightarrow N, v$ (weakening), if a sequent $M, v, v \Rightarrow N$ resp. $M \Rightarrow N, v, v$ is deducible, then so is the sequent $M, v \Rightarrow N$ resp. $M \Rightarrow N, v$ (contraction), and if two sequents $M, v \Rightarrow N$ and $M, \neg v \Rightarrow N$ are deducible, then so is $M \Rightarrow N$ (cut).
We call two sequents s and t equideducible for one of our calculi iff deducibility of s in this calculus implies deducibility of t and vice versa. Then the calculi F_0 allow reduction of sequents to clausal form according to the

Lemma 1. *Let p be a propositional variable which only occurs at the indicated positions. Then the following holds:*

a) These pairs of sequents are equideducible for all calculi F_0*:*

$M \Rightarrow a, N$	*and*	$M, \neg a \Rightarrow N$
$M, \Box^s[A, a \vee b] \Rightarrow N$	*and*	$M, \Box^s[A, a, b] \Rightarrow N$
$M, \Box^s[A, \neg\neg a] \Rightarrow N$	*and*	$M, \Box^s[A, a] \Rightarrow N$
$M, \Box^s[A, \neg(a \vee b)] \Rightarrow N$	*and*	$M, \Box^s[A, \neg p], \Box^s[p, \neg a], \Box^s[p, \neg b] \Rightarrow N$
$M, \Box^s[A, \neg\Box a] \Rightarrow N$	*and*	$M, \Box^s[A, \neg\Box p], \Box^{s+1}[p, \neg a] \Rightarrow N$
$M, \Box^s[A, \Box a] \Rightarrow N$	*and*	$M, \Box^s[A, \Box p], \Box^{s+1}[\neg p, a] \Rightarrow N$
$M, \Box^s[A, B] \Rightarrow N$	*and*	$M, \Box^s[A, p], \Box^s[\neg p, B] \Rightarrow N$

b) The sequents $M, a \Rightarrow N$ and $M, p, \Box[\neg p, \neg a] \Rightarrow N$ are equideducible for $\mathrm{F}_0(\mathrm{T})$ *and* $\mathrm{F}_0(4)$*.*
c) The sequents $M, \Box\Box a \Rightarrow N$ and $M, \Box a \Rightarrow N$ are equideducible for $\mathrm{F}_0(4)$*.* $\Box$

This lemma is well known (cf. [7]) and using it we may reduce any sequent u to a sequent $\mathrm{CF_K}(u)$ of the form $c_1, \ldots, c_n \Rightarrow$, such that u and $\mathrm{CF_K}(u)$ are equideducible for $\mathrm{F}_0(\mathrm{K})$ and such that the c_i are formulas of the form $\Box^s[a_1, \ldots, a_l]$, $0 \leq s$, $1 \leq l \leq 3$ or $\Box^s[a_1, b]$, where the a_i are propositional variables or negated propositional variables and b is a formula of the form $\Box a$ or $\neg\Box a$, where a is a propositional variable. Moreover we may reduce u to a sequent $\mathrm{CF_T}(u)$ of the form $c_1, \ldots, c_n \Rightarrow$ such that u and $\mathrm{CF_T}(u)$ are equideducible for $\mathrm{F}_0(\mathrm{T})$ and such that the c_i are either propositional variables or formulas of the form $\Box^s[a_1, \ldots, a_l]$ or $\Box^s[a_1, b]$, $1 \leq s$, $1 \leq l \leq 3$. Finally we may reduce u to a sequent $\mathrm{CF}_4(u)$ such that u and $\mathrm{CF}_4(u)$ are equideducible for $\mathrm{F}_0(4)$ and such that the c_i are either propositional variables or formulas of the form $\Box[a_1, \ldots, a_l]$, $1 \leq l \leq 3$ or $\Box[a_1, b]$. In all three cases the number of connectives of the sequents $\mathrm{CF}(u)$ is linearly bounded by the length of u.

The Boolean rules of the calculi F_0 are clearly invertible in the sense that any deduction of a conclusion of such a rule may be converted into a deduction of its premiss(es) of smaller or equal length. Therefore the following family F_1 of calculi is complete for deriving clausal sequents:

Axioms of all calculi F_1 are the sequents of the form $M, a, \neg a \Rightarrow$. In addition $F_1(K)$ has the rule

$$(\mathrm{EK}\vee_1) \quad \frac{M, v_1 \Rightarrow \quad \ldots M, v_n \Rightarrow}{M, [v_1, \ldots, v_n] \Rightarrow}$$

and the rule

$$(\mathrm{IK}\Box_1) \quad \frac{M, \neg v \Rightarrow}{L, \Box M, \neg\Box v \Rightarrow}$$

$F_1(T)$ has this latter rule and the rules

$$(\mathrm{ET}\Box_1) \quad \frac{M, \Box\Box v, \Box v \Rightarrow}{M, \Box\Box v \Rightarrow}$$

and

$$(\mathrm{E4}\Box_1) \quad \frac{M, \Box[v_1, \ldots, v_n], v_1 \Rightarrow \quad \ldots \quad M, \Box[v_1, \ldots, v_n], v_n \Rightarrow}{M, \Box[v_1, \ldots, v_n] \Rightarrow}$$

Finally $F_1(4)$ has the rule $\mathrm{E4}\Box_1$ and the rule

$$(\mathrm{I4}\Box_1) \quad \frac{\Box M, \neg v \Rightarrow}{L, \Box M, \neg\Box v \Rightarrow}$$

Now we consider a further calculus $F_2(T)$ which instead of the two rules $\mathrm{ET}\Box_1$ and $\mathrm{E4}\Box_1$ of F1(T) has only a single rule

$$(\mathrm{ET}\Box_2) \quad \frac{M, \Box^s[v_1, \ldots, v_n], v_1 \Rightarrow \quad \ldots \quad M, \Box^s[v_1, \ldots, v_n], v_n \Rightarrow}{M, \Box^s[v_1, \ldots, v_n] \Rightarrow}$$

The rule $\mathrm{E4}\Box_1$ is just a special case of this rule. Thus in order to show that $F_1(T)$ and $F_2(T)$ are equivalent, it suffices to show that the rule $\mathrm{ET}\Box_1$ is admissible, i.e.

Lemma 2. *If a sequent* $M, \Box^{s+1}v, \Box^s v \;\Rightarrow$ *is deducible by* $\mathrm{F}_2(\mathrm{T})$*, then so is* $M, \Box^{s+1}v \Rightarrow$*.*

Proof. If s is 0, then this is rule $\mathrm{ET}\Box_2$. Otherwise suppose our sequent is an axiom $M, \Box^{s+1}v, \Box^s v, \neg\Box^{s+1}v \Rightarrow$. Then $M, \Box^{s+1}v, \neg\Box^{s+1}v \Rightarrow$ is also an axiom. If it is an axiom of the form $M, \Box^{s+1}v, \Box^s v, \neg\Box^s v \Rightarrow$, then we obtain the required sequent from the axiom $v, \neg v \Rightarrow$ by one application of $\mathrm{ET}\Box_2$ leading to $\Box v, \neg v \Rightarrow$ and s consecutive applications of $\mathrm{IK}\Box_1$ leading to $\Box^{s+1}v, \neg\Box v \Rightarrow$ and finally using admissibility of weakening to introduce M.
If our sequent is neither an axiom nor the conclusion of an $\mathrm{IK}\Box_1$- inference nor the conclusion of an $\mathrm{ET}\Box_2$-inference with prinicipal formula $\Box^{s+1}v$ or $\Box^s v$, then these two formulas are present in every premiss and by the induction hypothesis the latter formula may be dropped. Applying the same inference on the transformed premisses therefore results in a deduction of the required sequent. If our sequent is of the form $L, \Box M, \Box^{s+1}v, \Box^s v, \neg\Box w \Rightarrow$ and it is the conclusion of an $\mathrm{IK}\Box_1$-inference, then its premiss is of the form $M, \Box^s v, \Box^{s-1}v, w \Rightarrow$, where $s - 1 \geq 0$. Thus by the induction hypothesis the sequent $M, \Box^s v, w \Rightarrow$ is deducible and by an application of $\mathrm{IK}\Box_1$ we obtain the required sequent. If $M, \Box^{s+1}v, \Box^s v \Rightarrow$ is the conclusion of an application of $\mathrm{ET}\Box_2$ with principal formula $\Box^{s+1}v$ or $\Box^s v$, then its premisses are of the form $M, \Box^{s+1}v, \Box^s v, a_i \Rightarrow$ and by the induction hypothesis we may deduce the sequents $M, \Box^{s+1}v, a_i \Rightarrow$. From these we obtain the sequent $M, \Box^{s+1}v \Rightarrow$ by an application of $\mathrm{ET}\Box_2$. $\Box$

The rule $\mathrm{EK}\vee_1$ of $\mathrm{F}_1(\mathrm{K})$ is also just a special case of $\mathrm{ET}\Box_2$. Therefore it suffices to show parts b) and c) of the

Lemma 3. *a) There is a transformation converting every* $\mathrm{F}_1(\mathrm{K})$*-deduction of a given sequent into another deduction of the same sequent such that in the new deduction every premiss* $M, \neg\Box b \Rightarrow$ *of an* $\mathrm{EK}\vee_1$*-inference with principal formula* $[a, \neg\Box b]$ *is the conclusion of an* $\mathrm{IK}\Box_1$*-inference with principal formula* $\Box b$*.*
b) There is a transformation converting every $\mathrm{F}_2(\mathrm{T})$*-deduction of a given sequent into another deduction of the same sequent such that in the new deduction every premiss* $M, \Box[a, \neg\Box b], \neg\Box b \Rightarrow$ *of an* $\mathrm{ET}\Box_2$*- inference is the conclusion of an* $\mathrm{IK}\Box_1$*- inference with principal formula* $\Box b$*.*
c) There is a transformation converting every $\mathrm{F}_1(4)$*-deduction of a given sequent into another deduction of the same sequent such that in the new deduction every premiss* $M, \Box[a, \neg\Box b], \neg\Box b \Rightarrow$ *of an* $\mathrm{E4}\Box_1$*- inference is the conclusion of an* $\mathrm{I4}\Box_1$*- inference with principal formula* $\Box b$*.*

Proof. b) For every such premiss P of $\mathrm{ET}\Box_2$ we consider the number $n(P)$ which is the maximal number of sequents preceding P in which $\Box b$ appears and for a given deduction d we let $n(d)$ be the sum of all $4^{n(P)} - 1$ for all such premisses P in d. If this number is 0, then all these premisses are conclusions of an application of $\mathrm{IK}\Box_1$. But if the principal formula of such an $\mathrm{IK}\Box_1$-inference I is different from $\Box b$, then we may drop the $\mathrm{ET}\Box_2$-inference and derive its conclusion directly

from the premiss of I by means of IK$\Box_1$. Thus we arrive at a deduction having the required property.
If $n(d)$ is greater than 0, then we consider a maximal application of ET$\Box_2$ with a premiss of this form which is not the conclusion of an IK$\Box_1$- inference, w.l.o.g. a situation of the form

$$\dfrac{K, a \Rightarrow \qquad \dfrac{K, \neg\Box b, c \Rightarrow \qquad \dfrac{M, \Box^{s-1}[a, \neg\Box b], \Box^{t-1}[c, \neg\Box d], e \Rightarrow}{K, \neg\Box b, \neg\Box d \Rightarrow}\,\mathrm{IK}\Box_1}{K, \neg\Box b \Rightarrow}\,\mathrm{ET}\Box_2}{K \Rightarrow}\,\mathrm{ET}\Box_2$$

where K abbreviates $L, \Box M, \Box^s[a, \neg\Box b], \Box^t[c, \neg\Box d]$. If e equals d, then we replace this by

$$\dfrac{\mathrm{ET}\Box_2\,\dfrac{K, c, a \Rightarrow \qquad K, c, \neg\Box b \Rightarrow}{K, c \Rightarrow} \qquad \dfrac{M, \Box^{s-1}[a, \neg\Box b], \Box^{t-1}[c, \neg\Box d], d \Rightarrow}{K, \neg\Box d \Rightarrow}\,\mathrm{IK}\Box_1}{K \Rightarrow}\,\mathrm{ET}\Box_2$$

Here the deduction of the sequent $K, c, a \Rightarrow$ is obtained from the deduction of $K, a \Rightarrow$ by weakening (which obviously does not increase $n(d)$). Thus one premiss P of measure $n(P)$ is replaced by atmost 3 new premisses each of measure atmost $n(P) - 1$ and therefore the measure n of the new deduction has decreased.
If e is different from d, then $\neg\Box e$ is in K and we may replace this series of inferences by a single IK$\Box_1$-inference with principal formula $\neg\Box e$.
c) is proved in the same way, just dropping the superscripts from the $\Box$ and placing a $\Box$ in front of the M at appropriate places. $\Box$

This result implies that the family C_0 of calculi is complete for clausal sequents where all the calculi C_0 have the usual axioms and moreover C_0(K) has the only rule

$$(\mathrm{CK}\Box_0) \quad \dfrac{L, \Box M, a_1 \Rightarrow \quad \ldots \quad L, \Box M, a_m \Rightarrow \quad M, \neg b \Rightarrow}{L, \Box M, v \Rightarrow}$$

C_0(T) has the only rule

$$(\mathrm{CT}\Box_0) \quad \dfrac{L, \Box M, \Box^s v, a_1 \Rightarrow \quad \ldots \quad L, \Box M, \Box^s v, a_m \Rightarrow \quad M, \Box^{s-1} v, \neg b \Rightarrow}{L, \Box M, \Box^s v \Rightarrow}$$

and C_0(4) has the rule

$$(\mathrm{C4}\Box_0) \quad \dfrac{L, \Box M, \Box v, a_1 \Rightarrow \quad \ldots \quad L, \Box M, \Box v, a_m \Rightarrow \quad \Box M, \Box v, \neg b \Rightarrow}{L, \Box M, \Box v \Rightarrow}$$

where in all rules the rightmost premiss is only present, when v is of the form $[a_1, \neg\Box b]$. Note that in the C_0-calculi any deduction of a clausal sequent consists entirely of clausal sequents.

Now for the rule $\mathrm{CK}\Box_0$ all premisses have less connectives than the conclusion. Therefore the length of every $\mathrm{C}_0(\mathrm{K})$-deduction of a given sequent is bounded by the number of connectives of its endsequent.

3 Extended Clausal Calculi

For $\mathrm{C}_0(\mathrm{T})$, however, the premisses of $\mathrm{CT}\Box_0$ of the form $L, \Box M, \Box^s v, a_i \Rightarrow$ in general have more connectives than the conclusion. Therefore we extend our language by a new connective $\bigcirc$ and we consider the calculus $\mathrm{C}_1(\mathrm{T})$ consisting of the usual axioms, additional axioms of the form $M, \bigcirc a, \neg a \Rightarrow$ and the rule $(\mathrm{CT}\Box_1)$:

$$(\mathrm{CT}\Box_1)\frac{\begin{array}{c}K, \bigcirc L, \Box M, \bigcirc^s u, a_1 \Rightarrow\\ \cdots\\ K, \bigcirc L, \Box M, \bigcirc^s u, a_l \Rightarrow\end{array}}{K, \bigcirc L, \Box M, \Box^s u \Rightarrow} \quad \frac{\begin{array}{c}K, \bigcirc L, \Box M, \bigcirc^s v, a_1 \Rightarrow\\ K, \bigcirc L, \Box M, \bigcirc^s v, \bigcirc b \Rightarrow\end{array}}{K, \bigcirc L, \Box M, \Box^s v \Rightarrow} \quad \frac{\begin{array}{c}K, \bigcirc L, \Box M, \bigcirc^s w, a_1 \Rightarrow\\ L, M, \Box^{s-1} w, \neg c \Rightarrow\end{array}}{K, \bigcirc L, \Box M, \Box^s w \Rightarrow}$$

where $u = [a_1, \ldots, a_l]$, $v = [a_1, \Box b]$, and $w = [a_1, \neg\Box c]$ and $\bigcirc^s v$ stands for $\bigcirc\Box^{s-1} v$.
For $\mathrm{C}_1(\mathrm{T})$ we call the premisses of the form $K, \bigcirc L, \Box M, \Box^s v, a_i \Rightarrow$ α-premisses, those of the form $K, \bigcirc L, \Box M, \bigcirc^s v, \bigcirc b \Rightarrow$ β-premisses and those of the form $L, M, \Box^{s-1} v, \neg c \Rightarrow$ γ-premisses.

Lemma 4. *a) If $L, \bigcirc M \Rightarrow$ is deducible by $\mathrm{C}_1(\mathrm{T})$, then $L, \Box M \Rightarrow$ is deducible by $\mathrm{C}_0(\mathrm{T})$.*
b) If $M, v, v \Rightarrow$ is deducible by $\mathrm{C}_1(\mathrm{T})$, where v is either a propositional variable or $v = \bigcirc w$ or $v = \neg w$ and w is a propositional variable, then $M, v \Rightarrow$ is deducible by $\mathrm{C}_1(\mathrm{T})$.
c) Weakening is an admissible rule for $\mathrm{C}_1(\mathrm{T})$.

Proof. a) If our sequent is an axiom $M, \bigcirc a, \neg a \Rightarrow$, then $M, \Box a, \neg a \Rightarrow$ is deducible by $\mathrm{C}_0(\mathrm{T})$. If it is the conclusion of a $\mathrm{CT}\Box_1$-inference, then the α- and β- premisses contain $\bigcirc^s v$ which by the induction hypothesis may be changed to $\Box v$, whereas the γ-premiss contains $\Box^{s-1} v$. Hence all premisses necessary for an inference leading to $M, \Box v \Rightarrow$ are deducible and therefore this sequent is deducible, too.
b) For axioms this is obvious. But if our sequent $M, v, v \Rightarrow$ is the conclusion of a $\mathrm{CT}\Box_1$- inference, then v cannot be its principal formula, so in the α- and β-premisses both v's are present and in the γ- premiss either both or none of them is present. Thus by the induction hypothesis all these double occurrences may be contracted in the premisses and by the same $\mathrm{CT}\Box_1$-inference the required sequent $M, v \Rightarrow$ is deducible.
c) is trivial. □

This implies:

Lemma 5. *a) If $M, v \Rightarrow$ is deducible, where v is a propositional variable, then so is $M, \bigcirc v \Rightarrow$.*
b) $M, \Box v \Rightarrow$, where v is a propositional variable is deducible iff $M, \bigcirc v \Rightarrow$ is.
c) If $M, \Box^s[A, v], v \Rightarrow$ is deducible, then so is $M, \bigcirc^s[A, v], v \Rightarrow$.

Proof. a) For axioms this is obvious. If $M, v \Rightarrow$ is the conclusion of a $\mathrm{CT}\Box_1$-inference, then in the α- and β-premisses v occurs and by the induction hypothesis it may be replaced by $\bigcirc v$, but in the γ-premiss v disappears and may be reintroduced by weakening. Thus all premisses of a $\mathrm{CT}\Box_1$-inference leading to $M, \bigcirc v \Rightarrow$ are deducible and so is this latter sequent.
b) If $M, \Box v \Rightarrow$ is an axiom or the conclusion of an inference with principal formula different from $\Box v$, then this is obvious. Otherwise the only premiss of the final $\mathrm{CT}\Box_1$-inference is $M, \bigcirc v, v \Rightarrow$. By a) this may be changed to $M, \bigcirc v, \bigcirc v \Rightarrow$ and from this by part b) of the preceding lemma we obtain $M, \bigcirc v \Rightarrow$.
c) Case 1: v is a propositional variable or of the form $\neg w$, where w is a propositional variable:
If $\Box^s[A, v]$ is the principal formula of the final inference of the given deduction, then one of its premisses is $M, \bigcirc^s[A, v], v, v \Rightarrow$ and from this sequent we obtain $M, \bigcirc^s[A, v], v \Rightarrow$ by an application of the preceding lemma. If $M, \Box^s[A, v], v \Rightarrow$ is the conclusion of an application of $\mathrm{CT}\Box_1$ with principal formula different from $\Box^s[A, v]$, then in the α- and β-premisses both $\Box^s[A, v]$ and v occur, whence $\Box^s[A, v]$ may be changed to $\bigcirc^s[A, v]$ by the induction hypothesis and in the γ-premiss only $\Box^{s-1}[A, v]$ occurs. Thus all the premisses for an application of $\mathrm{CT}\Box_1$ leading to $M, \bigcirc^s[A, v], v \Rightarrow$ are deducible.
Case 2: v is of the form $\Box w$, where w is a propositional variable:
If $\Box^s[A, v]$ is the principal formula of the final inference of the given deduction, then one of its premisses is $M, \bigcirc^s[A, v], \bigcirc w, \Box w \Rightarrow$ and from this sequent by b) we obtain $M, \bigcirc^s[A, v], \bigcirc w, \bigcirc w \Rightarrow$ and by the preceding lemma we obtain $M, \bigcirc^s[A, v], \bigcirc w \Rightarrow$ and again by b) $M, \bigcirc^s[A, v], v \Rightarrow$. If $\Box^s[A, v]$ is not the principal formula of the last inference, then as in case 1 the induction hypothesis applies. $\Box$

This immediately implies:

Lemma 6. *The rule* $\mathrm{CT}\Box_0$ *is admissible for* $\mathrm{C}_1(\mathrm{T})$. $\Box$

Thus $\mathrm{C}_0(\mathrm{T})$ and $\mathrm{C}_1(\mathrm{T})$ coincide on sequents without $\bigcirc$. Now for $\mathrm{C}_1(\mathrm{T})$ the parameter

> $d(s) :=$ the number of connectives of a sequent s plus twice the number of $\Box$ of s

decreases under backwards application of the rule $\mathrm{CT}\Box_1$. Thus every deduction of a sequent s has length at most $d(s)$.

For $C_0(4)$ we also use the connective $\bigcirc$, and we start from the calculus $C_1(4)$ which has the usual axioms and a rule

$$(\mathrm{C4}\Box_1)\ \frac{K, \bigcirc L, \Box M, \Box v, a_1 \Rightarrow\ \dots\ K, \bigcirc L, \Box M, \Box v, a_m \Rightarrow\quad \bigcirc L, \Box M, \Box v, \neg b \Rightarrow}{K, \bigcirc L, \Box M, \Box v \Rightarrow}$$

Thus $C_4\Box_1$ results from $C_4\Box_0$ by simply adding parameters $\bigcirc L$ in premisses and conclusion. Therefore the following holds trivially:

Lemma 7. *$M \Rightarrow$ is deducible by $C_0(4)$ iff $\bigcirc L, M \Rightarrow$ is deducible by $C_1(4)$.* □

Thus the cut rule in the form

$$(\mathrm{Cut})\quad \frac{M, v \Rightarrow \qquad M, \neg v \Rightarrow}{M \Rightarrow}$$

is admissible for $C_1(4)$, where $\neg v$ is an abbreviation for the set of clauses resulting from the negation of v.

Next we consider the calculus $C_2(4)$ with the usual axioms and two rules

$$(\mathrm{C4}\Box_2\sigma)\ \frac{\begin{array}{c} K, \bigcirc L, \Box M, \bigcirc u, a_1 \Rightarrow \\ \dots \\ K, \bigcirc L, \Box M, \bigcirc u, a_l \Rightarrow \end{array}}{K, \bigcirc L, \Box M, \Box u \Rightarrow} \qquad \frac{\begin{array}{c} K, \bigcirc L, \Box M, \bigcirc v, a_1 \Rightarrow \\ K, \bigcirc L, \Box M, \Box b \Rightarrow \end{array}}{K, \bigcirc L, \Box M, \Box v \Rightarrow}$$

where u and v are *shallow* formulas, i.e. either the form $[a_1, \dots, a_l]$ or of the form $[a_1, \Box b]$, and

$$(\mathrm{C4}\Box_2\delta)\quad \frac{K, \Box L, \Box M, \Box a_1, a_1 \Rightarrow \qquad \Box L, \Box M, \Box v, \neg b \Rightarrow}{K, \bigcirc L, \Box M, \Box v \Rightarrow}$$

where v is a *deep* formula, i.e. of the form $[a_1, \neg\Box b]$. We call the two types of premisses of the $\mathrm{C4}\Box_2\sigma$-rule $\alpha\sigma$- and β-premisses respectively and the two premisses of the $\mathrm{C4}\Box_2\delta$ -rule we call $\alpha\delta$- and γ-premisses. The following lemma shows that the rule $\mathrm{C4}\Box_1$ is admissible for $C_2(4)$:

Lemma 8. *a) If $M, v, v \Rightarrow$ is deducible by* $\mathrm{C}_2(4)$*, where v is a propositional variable or v is of the form $\neg w$ or $\bigcirc w$, where w is a propositional variable, then $M, v \Rightarrow$ is deducible, too.*
b) If $M, \Box v, a_i \Rightarrow$ is deducible, where v is shallow, then so is $M, \bigcirc v, a_i \Rightarrow$.
c) If $M, \Box[A, v] \Rightarrow$ or $M, \bigcirc[A, v] \Rightarrow$ is deducible, then so is $M, \Box A \Rightarrow$.
d) If $M, \Box v, \Box v \Rightarrow$ is deducible, where v is a propositional variable, then $M, \Box v \Rightarrow$ is deducible, too.
e) If $M, \Box[A, \Box v] \Rightarrow$ is deducible, then so is $M, \Box v \Rightarrow$.

Proof. a) is trivial. b) If $M, \Box v, a_i \Rightarrow$ is an axiom, then so is $M, \bigcirc v, a_i \Rightarrow$. If it is the conclusion of an inference with principal formula $\Box v$, then one of the premisses is $M, \bigcirc v, a_i, a_i \Rightarrow$. From this we obtain the required sequent by a). If it is the conclusion of a $\mathrm{C4}\Box_2\sigma$-inference with principal formula different from $\Box v$, then both $\Box v$ and a_i occur in all premisses and $\Box v$ may thus be changed to $\bigcirc v$ by the induction hypothesis. If it is the conclusion of a $\mathrm{C4}\Box_2\delta$-inference, then the $\alpha\delta$- and γ- premisses leading to a conclusion with $\bigcirc v$ are the same as those for $\Box v$. Therefore in this case, too, $M, \bigcirc v, a_i \Rightarrow$ may be derived.
c) If $M, \Box[A, v] \Rightarrow$ or $M, \bigcirc[A, v] \Rightarrow$ is an axiom, then so is $M, \Box A \Rightarrow$. If it is the conclusion of a $\mathrm{C4}\Box_2\sigma$-inference with principal formula $\Box[A, v]$, then by the induction hypothesis $\bigcirc[A, v]$ in the $\alpha\sigma$- and β-premisses may be changed to $\bigcirc A$ and with these new premisses the given inference yields a deduction of $M, \Box A \Rightarrow$. If it is the conclusion of a $\mathrm{C4}\Box_2\delta$-inference with principal formula $\Box[A, v]$, then either A is a propositional variable or a negated propositional variable and v is of the form $\neg\Box b$ with b a propositional variable or vice versa. In the first case the formula $\Box A$ is shallow and by b) we may change the $\alpha\delta$-premiss of this inference to $M, \bigcirc A, A \Rightarrow$ and from this sequent by an application of $\mathrm{C4}\Box_2\sigma$ we arrive at the desired sequent $M, \Box A \Rightarrow$. In the second case the formula $\Box[A, v]$ occurs in the γ-premiss and by the induction hypothesis may be changed to $\Box A$ and from this new γ-premiss by an application of $\mathrm{C4}\Box_2\delta$ we arrive at the desired sequent. Finally if our sequent is the conclusion of an inference with principal formula different from $\Box[A, v]$, then this formula occurs in every premiss and by the induction hypothesis it may be replaced by $\Box A$.
d) If $M, \Box v, \Box v \Rightarrow$ is not the conclusion of an inference with principal formula $\Box v$, then this is trivial. Otherwise this sequent has a single premiss $M, \bigcirc v, v, \Box \mathrm{v} \Rightarrow$. Thus by b) we obtain from this the sequent $M, \bigcirc v, v, \bigcirc v \Rightarrow$ and by a) we obtain $M, \bigcirc v, v \Rightarrow$. From this we obtain $M, \Box v \Rightarrow$ by an application of $\mathrm{C4}\Box_2\sigma$.
e) If $M, \Box[A, \Box v] \Rightarrow$ is not the conclusion of an inference with principal formula $\Box[A, v]$, then by the induction hypothesis this formula may be changed to $\Box v$ in all premisses. Otherwise the required sequent is the β-premiss. $\Box$

From this follows:

Lemma 9. *The rule* $\mathrm{C4}\Box_1$ *is admissible for* $\mathrm{C}_2(4)$*.*

Proof. For suppose we are given sequents $\ldots\ M, \Box v, a_i \Rightarrow, \ldots,\quad M, \Box v, \Box b \Rightarrow$, where v is shallow, then by b), e) and d) of the preceding lemma we obtain the

sequents $\ldots\ M, \bigcirc v, a_i \Rightarrow\ \ldots,\quad M, \Box b \Rightarrow$, i.e. the premisses of an application of C4$\Box_2\sigma$ leading to the sequent $M, \Box v \Rightarrow$.
If we have sequents $\ldots\ L, \Box M, \Box v, a_i \Rightarrow, \ldots,\quad L, \Box M, \Box v, \neg b \Rightarrow$, where v is deep, then by c), e), and d) of the preceding lemma we obtain $\ldots\ L, \Box M, \Box a_i, a_i \Rightarrow, \ldots,\quad \Box M, \Box v, \neg b \Rightarrow$ and from these by an application of C4$\Box_2\delta$ we obtain $L, \Box M, \Box v \Rightarrow$. □

On the other hand the rules C4$\Box_2$ are also admissible for $\mathrm{C}_1(4)$. This is seen by successive cuts with some $\mathrm{C}_1(4)$-deducible sequent. For the C4$\Box_2\sigma$-rule this is obvious and for the C4$\Box_2\delta$-rule we cut the first premiss and the sequent $\Box L, \Box M, \Box v, \neg\Box b \Rightarrow$ obtained from the second premiss by an application of C4$\Box_1$ with the $\mathrm{C}_1(4)$- deducible sequent

$$\Box[a, \neg\Box b], \neg(\Box a \wedge a), \neg(\Box[a, \neg\Box b] \wedge \neg\Box b) \Rightarrow$$

and obtain the required conclusion. Thus the calculi $\mathrm{C}_1(4)$ and $\mathrm{C}_2(4)$ are equivalent.

Now we call a formula $\Box v$ distant in a $\mathrm{C}_2(4)$-deduction d iff below any conclusion of an inference with principal formula $\Box v$ there is an $\alpha\delta$-, β- or γ-premiss. Using this definition one observes

Lemma 10. *If there is a $\mathrm{C}_2(4)$-deduction of a sequent $M, \Box v \Rightarrow$ in which $\Box v$ is distant, then the sequent $M, \bigcirc v \Rightarrow$ is deducible by $\mathrm{C}_2(4)$.*

Proof. Since $\Box v$ has to be distant, this sequent can neither be an axiom nor the conclusion of an inference with principal formula $\Box v$. So if the last inference of this deduction is by an application of C4$\Box_2\sigma$, then $\Box v$ is distant in all deductions of the $\alpha\sigma$-premisses and by the induction hypothesis it may be changed to $\bigcirc v$. From the transformed $\alpha\sigma$-premisses together with the β-premiss we therefore obtain the sequent $M, \bigcirc v \Rightarrow$. But if the last inference is by an application of C4$\Box_2\delta$, then the premisses of $M, \bigcirc v \Rightarrow$ are the same as those of $M, \Box v \Rightarrow$. □

Lemma 11. *There is a transformation sending every deduction of a sequent into another deduction of the same sequent such that in the new deduction for every deep formula $\Box v$ and every $\alpha\sigma$-premiss $M, \Box v \Rightarrow$ the formula $\Box v$ is distant in the deduction of $M, \Box v \Rightarrow$.*

Proof. To every $\alpha\sigma$-premiss P in a deduction d we assign a number $n(P)$, i.e. the maximal number of conclusions of C4$\Box_2\delta$ on a branch of d ending in P which only contains $\alpha\sigma$- and γ-premisses. Then we let $n(d)$ be the sum of all $4^{n(P)} - 1$ for all such premisses P in d.
If this number is 0, then all the $n(P)$ are 0 and for every $\alpha\sigma$-premiss Q of the form $M, \Box v \Rightarrow$ and every path ending in Q which contains the conclusion R of a C4$\Box_2\delta$- inference with principal formula $\Box v$, there is some sequent S between Q and R which is neither an $\alpha\sigma$-premiss nor a γ-premiss. Therefore $\Box v$ is distant in the deduction of $M, \Box v \Rightarrow$.

If this number is greater than 0, then w.l.o.g. there is in d some maximal pair of inferences of the form

$$\dfrac{\dfrac{L, \Box M, \Box v, a, \Box c, c \Rightarrow \qquad \Box M, \Box v, \Box w, \neg d \Rightarrow}{L, \Box M, \bigcirc v, a, \Box w \Rightarrow} \qquad L, \Box M, \Box b, \Box w \Rightarrow}{L, \Box M, \Box v, \Box w \Rightarrow}$$

where $v = [a, \Box b]$ and $w = [c, \neg\Box d]$. We may change this pair of inferences to

$$\dfrac{\dfrac{L, \Box M, \bigcirc v, a, \Box c, c \Rightarrow \qquad L, \Box M, \Box b, \Box c, c \Rightarrow}{L, \Box M, \Box v, \Box c, c \Rightarrow} \qquad \Box M, \Box v, \Box w, \neg d \Rightarrow}{L, \Box M, \Box v, \Box w \Rightarrow}$$

where the deduction of the sequent $L, \Box M, \bigcirc v, a, \Box c, c \Rightarrow$ results from the deduction of $L, \Box M, \Box v, a, \Box c, c \Rightarrow$ by lemma 8 and the deduction of $L, \Box M, \Box b, \Box c, c \Rightarrow$ results from the deduction of $L, \Box M, \Box b, \Box w \Rightarrow$ by the same lemma and by weakening. Thus one $\alpha\sigma$-premiss P of measure $n(P)$ is replaced by atmost 3 new $\alpha\sigma$-premisses each of measure at most $n(P) - 1$ and therefore the induction parameter has decreased. $\Box$

The preceding two lemmas show that the following calculus $\mathrm{C}_3(4)$ is complete which consists of the usual axioms, the rule $\mathrm{C4}\Box_3\delta = \mathrm{C4}\Box_2\delta$ and the rule

$$(\mathrm{C4}\Box_3\sigma) \quad \dfrac{\begin{matrix} J, \bigcirc K, \bigcirc L, \Box M, \bigcirc u, a_1 \Rightarrow \\ \cdots \\ J, \bigcirc K, \bigcirc L, \Box M, \bigcirc u, a_l \Rightarrow \end{matrix}}{J, \bigcirc K, \Box L, \Box M, \Box u \Rightarrow} \qquad \dfrac{\begin{matrix} J, \bigcirc K, \bigcirc L, \Box M, \bigcirc v, a_1 \Rightarrow \\ J, \Box K, \Box L, \Box M, \Box b \Rightarrow \end{matrix}}{J, \bigcirc K, \Box L, \Box M, \Box v \Rightarrow}$$

where u is $[a_1, \ldots, a_l]$ and v is $[a_1, \Box b]$ and all formulas of L are deep.

Now we observe:

Lemma 12. *In a* $\mathrm{C}_3(4)$*-deduction* d *of a sequent without* $\bigcirc$ *any subdeduction of* d *ending in a* $\mathrm{C4}\Box_3\delta$*-inference has an endsequent without* $\bigcirc$*.* $\Box$

This means that the calculus $\mathrm{C}_4(4)$ is complete which has the usual axioms, the rule $\mathrm{C4}\Box_4\sigma = \mathrm{C4}\Box_3\sigma$ and the rule

$$(\mathrm{C4}\Box_4\delta) \quad \dfrac{K, \Box L, \Box M, \Box a, a \Rightarrow \qquad \bigcirc L, \Box M, \Box v, \neg b \Rightarrow}{K, \bigcirc L, \Box M, \Box v \Rightarrow}$$

where v is $[a, \neg\Box b]$. (Note that the sequence L is always empty!)

Now we call a formula $\Box v$ distant in a $\mathrm{C}_4(4)$-deduction d iff below any inference with principal formula $\Box v$ in d there is an $\alpha\delta$-, $\alpha\sigma$-, or β-premiss. Then from lemma 10 follows

Lemma 13. *If there is a* $\mathrm{C}_4(4)$*-deduction of a sequent* $M, \Box v \Rightarrow$*, where* v *is deep and* $\Box v$ *is distant, then the sequent* $M, \bigcirc v \Rightarrow$ *is deducible by* $\mathrm{C}_4(4)$*.* □

From this follows:

Lemma 14. *A sequent* $\Box M, \Box[a, \neg\Box b], \neg b \Rightarrow$ *is deducible by* $\mathrm{C}_4(4)$ *iff the sequent* $\Box M, \bigcirc[a, \neg\Box b], \neg b \Rightarrow$ *is.*

Proof. This is proved by induction on the maximal number of successive γ-premisses preceding $\Box M, \Box[a, \neg\Box b], \neg b \Rightarrow$. If it is 0, then our sequent is either an axiom or it is the conclusion of an application of $\mathrm{C4}\Box_4\sigma$. The first case is trivial, and in the second case the α-premisses already have $\bigcirc[a, \neg\Box b]$ and the β- premiss is the same for both $\Box[a, \neg\Box b]$ and $\bigcirc[a, \neg\Box b]$. Thus the same inference applied to the transformed premisses yields the required deduction. If this number is greater than 0, then we distinguish cases according to whether one of these successive $\mathrm{C4}\Box_4\delta$- inferences has principal formula $\Box[a, \neg\Box b]$ or not. In the latter case $\Box[a, \neg\Box b]$ is distant, and we may replace it by $\bigcirc[a, \neg\Box b]$. In the former case the induction hypothesis applies to the γ-premisses of this inference, and by repeating the following inferences with the transformed formula we obtain the required deduction of $\Box M, \bigcirc[a, \neg\Box b], \neg b \Rightarrow$. □

This finally shows that the calculus $\mathrm{C}_5(4)$ is complete which consists of the usual axioms, the rule $\mathrm{C4}\Box_5\sigma = \mathrm{C4}\Box_4\sigma$ and the rule

$$(\mathrm{C4}\Box_5\delta) \quad \frac{K, \Box L, \Box M, \Box a, a \Rightarrow \qquad \bigcirc L, \Box M, \bigcirc v, \neg b \Rightarrow}{K, \bigcirc L, \Box M, \Box v \Rightarrow}$$

Now for $\mathrm{C}_5(4)$ we observe that for the measure

> $d(s) =$(the total number of connectives of the sequent s times (the number of □ plus the number of ○)) minus the number of ○

and for both rules the measure of the conclusions is greater than the measure of all premisses. Therefore every deduction of a given sequent is bounded in length by some quadratically growing function of the number of connectives of its endsequent.

4 Space Bounds for the Calculi

In order to obtain space bounds for the decision procedures resulting from backwards application of the rules of our calculi $\mathrm{C}_0(\mathrm{K})$, $\mathrm{C}_1(\mathrm{T})$, and $\mathrm{C}_5(4)$ we consider the set of subclauses $\mathrm{sc}(s)$ of a given sequent s:
For the logic K we let $\mathrm{sc}(\Box^s v)$ be $\{v, a_1, \ldots, a_m, \neg b_1\}$, where v is $[a_1, \ldots, a_m, \neg\Box b]$ and for T we let $\mathrm{sc}(\Box^s v)$ be $\{v, a_1, \ldots, a_l, b, \neg c\}$, where v is $[a_1, \ldots, a_l, \Box b, \neg\Box c]$ and for S4 we let $\mathrm{sc}(\Box v)$ be $\{v, [a_1, \ldots, a_l, \Box b], a_1, \ldots, a_l, b, \neg c\}$ for the same v. Then it is obvious that in any deduction of a given sequent s all the occurring

formulas are of the form $\Box^s v$ or $\bigcirc^s v$, where v is a subclause of one of the formulas of s. Therefore we may denote any sequent in such a deduction by a string of numbers shorter than twice the greatest exponent in s which has one entry for every subclause of s: If the entry corresponding to a certain subclause v is 0, then this subclause does not occur in the respective sequent; if it is $2(n+1)$, then the formula $\Box^n v$ occurs, and if it is $2(n+1)+1$, then the formula $\bigcirc^n v$ occurs. Thus given any string of this form which denotes a premiss of one of our inferences together with the principal formula of this inference we may obtain both its conclusion and the next premiss. Therefore we need not store whole branches of our deductions but only one sequent string at a time together with a list of the principal formulas on the current branch. Since the number of subclauses of a sequent s is linear in the number of connectives of s we need space $n \log n$ to store this one sequent string. Moreover to store the list of principal formulas we need space $\log n$ times the maximal length of a deduction of s. Thus for K and T we arrive at a decision procedure which requires space $n \log n$ and for S4 we need space $n^2 \log n$, where n is the number of connectives of the endsequent which is in clausal form. But in reducing any given sequent to clausal form we add linearly many connectives; therefore we arrive at the

Theorem 15. *Provability in the modal logics K and T is in the complexity class SPACE*$(n \log n)$ *and provability in the modal logic S4 is in SPACE*$(n^2 \log n)$.

5 Conclusion

We have presented so called contraction free sequent calculi for the three prominent PSPACE complete modal logics K, T, and S4. Using these calculi we have demonstrated how to define decision procedures for these logics which both admit simpler implementation, relying entirely on depth first search, and require less space than conventional decision procedures. Thus we could lower space bounds for the logic K from the previously known bound of n^2 to $\log n$ and for the logic T from n^3 to $n \log n$ and for S4 from n^4 to $n^2 \log n$.

References

1. Ladner, R. E.: The computational complexity of provability in systems of modal propositional logic. Siam Journal of Computing **6** (1977) 467–480
2. Hudelmaier, J.: An $n \log n$-SPACE decision procedure for intuitionistic propositional logic. Journal of Logic and Computatation **3** (1993) 63–75
3. Fitting, M.: Proof Methods for Modal and Intuitionistic Logics. Reidel Publ. Co 1983
4. Chen, C. C. & I. P. Lin: The computational complexity of satisfiability of modal Horn clauses for modal propositional logics. Theoretical Computer Science **129** (1994) 95–121
5. Ognjanovic, Z.: A tableau-like proof procedure for normal modal logics. Theoretical Computer Science **129** (1994) 167–186

6. Halpern, J. Y.: The effect of bounding the number of primitive propositions and the depth of nesting on the complexity of modal logic. Artificial Intelligence **75** (1995) 361–372
7. Mints, G. E.: Gentzen-type systems and resolution rules. In: P. Martin-Löf, G. Mints (eds.): COLOG-88 (Springer LNCS 417) 198–231
8. Wallen, L.: Automated proof search in nonclassical logics. MIT Press 1990