Bicomplete calculi for intuitionistic propositional logic

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Abstract. We define a general notion of refutability for logical calculi, similar to finite failure for logic programs, and we call a sequent calculus K for a logic L bicomplete, iff every nonderivable sequent of L is refutable by K. Thus for a bicomplete sequent calculus every sequent is either derivable or refutable. Now we show that from any bicomplete calculus for a logic L we may define a canonical semantics for L. For the semantics obtained in this way the corresponding bicomplete calculus provides a solution to the problem of constructing semantical counterexamples for nonprovable formulae of L. In particular for intuitionistic logic we are going to give three such calculi and we are thus obtaining three different types of semantics for it. One of them is the familiar Kripkean semantics. Thus the bicomplete calculus generating this semantics gives at the same time a very perspicuous solution to the notorious problem of finding Kripkean counterexamples for intuitionistically unprovable formulae.

1. Introduction

Semantical methods have been successfully applied for automated theorem proving in classical logic (cf. [7,8]) This is due to the fact that for classical logic there is a very perspicuous relation between semantical definitions of tautologies and some proof theoretical approaches. In fact for certain calculi such as the well known tableau calculi we immediately obtain for any nonprovable formula a semantical counterexample. In contrast to this the situation for intuitionistic logic is much different. Already A. Heyting himself complained in conversation with W. E. Beth, that there is no straightforward way to extract such semantical counterexamples from failing deduction attempts (cf. [4].) In general nonprovable formulae of intuitionistic propositional logic do not even have failing deduction attempts. Usually information regarding nonprovability of a sequent has therefore to be extracted by a kind of metaargumentation from an unending sequence of inference steps. (For instance by considering loops in this sequence.) This situation also shows in the usual semantical completeness proofs for intuitionistic logic as may be found in monographs such as [6] – such completeness proofs are much more complicated than similar proofs for classical logic. In the sequel we are going to show that these problems may be solved by considering so called bicomplete sequent calculi for intuitionistic propositional logic. By this we mean calculi by which any sequent provable in intuitionistic logic is derivable and moreover any sequent not provable in intuitionistic logic is refutable in a sense to be made precise. We are going to show that failing deduction attempts in such calculi provide a very perspicuous means for obtaining semantical counterexamples for arbitrary nonprovable formulae of intuitionistic logic. Moreover these calculi also yield a thorough analysis of the concept of a semantics itself: To any such calculus there is a canonical semantics based on it, and the relation between any bicomplete calculus and its semantics exactly mirrors the relation between classical semantics and the usual tableaux calculi for classical provability. Therefore in a sense these calculi serve to embed semantical approaches into proof theory and thus provide a means to make semantical approaches fruitful for automated deduction in intuitionistic logic.

2. Logical Calculi

We consider a recursive set of as yet unspecified objects. Moreover we consider so called *sequents*, i.e. finite multisets of such objects. Then a *premiss* consists of a sequent, the so called *side sequent* and a recursive (in general infinite) set of objects, the so called *parameter set*. An *inference* consists of a designated object, the so called *principal* object and a finite set of premisses. An *inference system* is a recursive set of inferences where any object occurs only finitely often as principal object of one of the inferences. An axiom system for an inference system I is a recursive set of sequents consisting entirely of pairwise distinct so called *atomic* objects, i.e. objects which do not occur as principal objects of one of the inferences of I. Finally a sequent calculus consists of an inference system I and an axiom system for I. Derivability of a sequent s by a sequent calculus K is defined in the obvious way: s is derivable in 0 steps, iff s contains one of the axioms of K as a submultiset and s is derivable in $n+1$ steps, iff either s is derivable in 0 steps or s contains an object a which is the principal object of an inference of K with premisses $p_1, \ldots, p_k$ and all sequents $d(I, s, p_1), \ldots, d(I, s, p_k)$ are derivable in n steps, where $d(I, s, p_i)$ results from s by replacing a with the side sequent of p_i and removing all objects of s which do not occur in the parameter set of p_i.

(**Remark.** Note that this definition enforces a "context free" manner of evaluation of a given sequent, because the possibility of applying any given inference to a sequent s depends only on the presence of a single object in s which is then replaced by a finite multiset of such objects; thus e.g. the well known sequent calculus rules of the form

$$\frac{M, Ta, Tb}{M, Ta, Ta \to b}$$

(cf. [1,2]) would not be covered by this definition. In contrast to other context free formalisms such as evaluation of logic programs we have here, however, the further option to remove nonprincipal objects during the evaluation process.)

The simplest example of a sequent calculus according to this definition is the well known tableaux style calculus LK for classical propositional logic. If for simplicity we restrict the language to the single connective $\to$, the objects of this calculus are signed formulas Fv or Tv, where v is a formula in the usual sense, built up from atoms from a prespecified set by means of the connective

$\rightarrow$. The axioms of this calculus are all sequents of the form Fv, Tv. Moreover we have a set of one premiss-inferences by which any formula $Fa \rightarrow b$ is replaced by the two formulas Ta and Fb and a set of two premiss-inferences by which any formula $Ta \rightarrow b$ is replaced by Tb in the first premiss and by Fa in the second premiss. Furthermore for all premisses we have that the parameter set is the full set of all signed formulas. (Thus no nonprincipal formulas of a sequent are ever removed during the evaluation process.) Usually a more convenient notation for these inferences is chosen, vic. as logical rules of the form

$$(\mathrm{E}{\rightarrow})\ \frac{M, Tb \qquad\qquad M, Fa}{M, Ta \rightarrow b} \qquad\qquad \text{resp.} \qquad\qquad (\mathrm{I}{\rightarrow})\ \frac{M, Ta, Fb}{M, Fa \rightarrow b}$$

All logical calculi defined in this way are monotonic in the following sense:

Proposition 1. *If a sequent s is derivable in n steps by a calculus K and a sequent t is obtained from s by adding to it a sequent s', then t is derivable in n steps, too.*

Proof. If s is derivable in 0 steps, then s contains some axiom of K as a submultiset, hence t contains the same axiom as a submultiset and thus is also derivable in 0 steps. If s is derivable in $n + 1$ steps and it is not derivable in 0 steps, then there is an inference I of K with premisses $p_1, \ldots, p_k$, having parameter sets $M_1, \ldots, M_k$, and the sequents $d(I, s, p_1), \ldots, d(I, s, p_k)$ are derivable in n steps. Then by the induction hypothesis all sequents t_i obtained from the sequents $d(I, s, p_i)$ by adding all objects in $s' \cap M_i$ are derivable in n steps, and thus according to the definition of derivability the sequent t is derivable in $n+1$ steps. $\square$

While the notion of derivability of a sequent by a calculus is defined in a canonical way, the notion of refutability admits several different definitions for different types of calculi:

3. Strong bicompleteness

Any definition of refutability sequents with respect to a logical calculus starts with a set of sequents which are certainly underivable by the calculus in question and for the recursion step it makes use of a construction for obtaining new underivable sequents from previously known such sequents. The most obvious definition of refutability thus consists in a straightforward adaptation of the concept of finite failure from logic programming:

Strong refutability. A sequent s is refutable in 0 steps, iff all objects of s are atomic and moreover s does not contain an axiom of K as a submultiset and s is refutable in $n+1$ steps, iff it does not contain an axiom of K as a submultiset and either it is refutable in 0 steps or for any object a of s and any inference I with principal object a, there is a premiss p of I such that the sequent $d(I, s, p)$ is refutable in n steps. (Note that the definition of a calculus requires that the number of such inferences is finite.) We call this notion of refutability *strong refutability.*

For any calculus the sets of derivable and strongly refutable sequents are obviously disjoint. For the above calculus for classical propositional logic it holds moreover that the union of these two sets is the full set of all sequents: This is proved by a straightforward induction on the number of connectives of a sequent s. We call calculi having this property *strongly bicomplete.* Obviously the usual calculi for intuitionistic logic such as the familiar calculus LJ, consisting of the same axioms as LK and the two rules

$$(\mathrm{E}\!\to)\ \ \frac{M, Ta \to b, Fa \qquad\qquad M, Ta \to b, Tb}{M, Ta \to b} \qquad \text{and} \qquad (\mathrm{I}\!\to)\ \ \frac{M^T, Ta, Fb}{M, Fa \to b}$$

(where M^T denotes M with all F-signed formulae removed) are by no means bicomplete. In fact going backwards from a nonderivable sequent with a signed formula $Ta \to b$ we will never arrive at a sequent without this formula.

Abstract semantics. We will show that for bicomplete calculi there is a canonical way for defining a semantics for which such calculi are sound and complete: We first define the notion of a K-frame for an arbitrary calculus K: A K-frame is a tree whose nodes are sequents of atomic objects not containing axioms as subsequents and whose edges are labelled with parameter sets from premisses of K in such a way that for any node n and its (single) predecessor n' connected by an edge e which is labelled with M it holds that n contains at least all objects of n' which are contained in M.

Now we say that an atomic object a is *K-valid* in a frame F iff a occurs in the root sequent of F. A nonatomic object a is K-valid in F, iff there are (necessarily finitely many) inferences $I_1, \ldots, I_k$ with principal object a and premisses $p_1^1, \ldots, p_1^{l_1},\ \ldots,\ p_k^1, \ldots, p_k^{l_k}$ having side sequents $s_1^1, \ldots, s_1^{l_1},\ \ldots,\ s_k^1, \ldots, s_k^{l_k}$ and parameter sets $M_1^1, \ldots, M_1^{l_1},\ \ldots,\ M_k^1, \ldots, M_k^{l_k}$ and there are k successors $n_1, \ldots, n_k$ of the root n of F such that for every $1 \leq i \leq k$ there is a j with $1 \leq j \leq l_i$ such that the edge between the root and n_i is labelled with M_i^j and in the subframe of F with root n_i all objects of s_i^j are K-valid.

(Note that both the side sequents s_i^j and the parameter sets M_i^j are uniquely determined by the object a alone; therefore this definition is of the required type, i.e. is a relation between objects and frames.)

Now we have the following abstract completeness theorem for bicomplete calculi:

Theorem 1. *If K is a strongly bicomplete calculus and s is a nonderivable sequent of K, then there is a K-frame in which all objects of s are valid.*

Proof. We recall that the definition of refutability consists of a procedure for constructing for any sequent a tree of sequents witnessing its refutability. Since the calculus K we consider is bicomplete and our sequent s is nonderivable by K, this procedure terminates and gives us a tree T of sequents with root s and leaves consisting of sequents which contain only atomic objects and do not contain an axiom of K as a submultiset. Now we label all edges between any node n of T and its predecessor n', where n was obtained from n' by making use of some premiss p of a K-inference, with the parameter set M of p. Then from the resulting tree we remove all nonatomic objects and we call the new tree T'. It is obvious that T' is a K-frame. Moreover we will show that all objects of all sequents of T are valid in the corresponding subframes of T': The leaves of T are the same as those of T'. So all of their objects are valid in the subframes of T' consisting only of the corresponding leaves. For a node n of T corresponding to a node t of T' with successors $n_1, \ldots, n_k$ corresponding to nodes $t_1, \ldots, t_k$ of T' we consider an arbitrary object a of n: If a is atomic, then it is contained in t and therefore is valid in the subframe of T' with root t. If a is a nonatomic object, then according to the definition of strong refutability for any inference I with principal object a there is a premiss p of I such that the sequent $d(I, n, p)$ is among the n_i. Thus by the induction hypothesis all objects of any sequent $d(I, n, p)$ are valid in the subframe of T' whose root is t_i. In particular all objects belonging to the side sequents of p are valid in this subframe. Moreover, as required, the edge connecting t and t_i by definition is labelled by the parameter set of p. $\Box$

For the above example of classical propositional logic (obviously not a particularly interesting example, but suitable for explaining the procedure we are going to use in more interesting cases) a frame would thus consist of a tree of sequents such that each of these sequents consists entirely of signed propositional variables and none of them contains a propositional variable both F- and T-signed. Moreover this tree would be monotonic in the sense that all variables occurring F- resp. T-signed in a given sequent would occur with the same sign in any successor of this sequent. Validity of a nonatomic signed formula $Fa \to b$ in a frame F would then be declared as validity of both Ta and Fb in some subframe of F having as its root some successor of the root of F. A signed formula $Ta \to b$ on the other hand would be declared valid in F if Fa or Tb were valid in some such subframe of F. (This definition of classical validity is obviously too complicated, but it will be shown below, how the well known Boolean notion of validity may be derived from it.)

The calculus LS. To obtain a similar strongly bicomplete calculus for intuitionistic logic we start from the well known so called contraction free calculi (c. [1,2]). These calculi themselves, however, are not subsumed under the above definition of a logical calculus, because they usually have a rule which isn't context free in the sense of the above remark. Therefore we have to use a slightly modified variant LS of these calculi: The sequents we consider are again built up from signed propositional variables and signed implications. The axioms are the same as those of classical propositional logic, and the inferences are given by the three rules:

$$(\mathrm{E}{\to})\ \ \frac{M^T, Fa \qquad M, Tb}{M, Ta \to b} \qquad\qquad (\mathrm{I}{\to})\ \ \frac{M, Ta, Fb}{M, Fa \to b}$$

$$(\mathrm{E}{\to}{\to})\ \frac{M^T, Ta, Ta \to b, Fc \qquad M, Tb}{M, T(a \to b) \to c}$$

We have to show

Theorem 2. *The calculus LS is complete for intuitionistic propositional logic.*

Proof. Consider one of the previously mentioned contraction free calculi LS′: It consists of the usual axioms and the three rules

$$(\mathrm{E}{\to})\frac{M, Ta, Tb}{M, Ta, Ta \to b} \quad (\mathrm{E}{\to}{\to})\frac{M^T, Ta, Ta \to b, Fc \qquad M, Tb}{M, T(a \to b) \to c} \quad (\mathrm{I}{\to})\frac{M, Ta, Fb}{M, Fa \to b}$$

Completeness of LS′ has been proved in [1]. To show completeness of LS we have just to prove admissibility of the rule E→ of LS′ for LS: Any sequent M^T, Ta, Fa is derivable by LS in 0 steps. Thus if a sequent M, Ta, Tb is derivable by LS, then from these two sequents by an application of the rule E→ of LS we arrive at the required sequent $M, Ta, Ta \to b$. □

Now we are able to prove

Theorem 3. *The calculus LS is bicomplete.*

Proof. By the previous theorem we just need to show that any nonderivable sequent s is refutable by LS: this is done by induction on the measure $\kappa(s)$ defined as follows:

$\kappa(a) = 2$, iff a is a propositional variable,
$\kappa(a) = 1 + \kappa(b) \cdot \kappa(c)$, iff a is $b \to c$,
$\kappa(v) = \kappa(a)$, iff v is Fa or Ta and
$\kappa(s) = \kappa(a_1) + \cdots + \kappa(a_n)$, iff s is the sequent $a_1, \ldots, a_n$.

For any premiss p of any LS-inference I it holds that $\kappa(s) > \kappa(d(I, s, p))$. So consider the finitely many inferences $I_1, \ldots, I_k$ which have principal objects occurring in s. Since s is not derivable, there is for every such inference I_j a premiss p_j such that $d(I, s, p)$ is not derivable by LS. Since these sequents have smaller κ-measure, they are refutable by the induction hypothesis, thus by the definition of refutability s is refutable, too. □

The semantics for LS. The semantics that results from the calculus LS is based on frames whose edges are labelled either with the full set of all signed formulas or with the set of all T-signed formulas. (We call a successor n of a node n' of such a frame a 1-successor iff the edge between n and n' is labelled with the set of T-signed formulas and we call it a 0-successor iff this edge is labelled with the set of all formulas. Moreover we call a subframe F' of a frame F with root r a 0-subframe, iff the root r' of F' is a 0-successor of r, and we call it a 1-subframe, iff r' is a 1-successor of r.) Thus these frames are T-monotone in the sense that any T-signed propositional variable which occurs on a given node also occurs with sign T on all successors of this node.

Validity of a signed formula $Fa \to b$ in a frame with root n is then declared as validity of both Ta and Fb in some 0-subframe of F. Validity of a signed formula $Ta \to b$, where a is a propositional variable, is declared as either validity of Fa in some 1-subframe of F or validity of Tb in some 0-subframe of F. Finally validity of $T(a \to b) \to c$ is declared as either validity of Ta and $Tb \to c$ and Fb in some 1-subframe of F or validity of Tc in some 0-subframe of F.

Although for this semantics validity of any signed formula with respect to any frame may be determined, this definition is not an homomorphic definition over the construction of formulae of our language. This obviously means that it does not provide a semantical definition of the meaning of propositional connectives in the usual sense.

Therefore we will now establish two other semantics based on a new notion of refutability.

4. Weak bicompleteness

The defintion of refutability need not start from sequents s to which no inference is applicable. It suffices that for any possible inference I one of the sequents $d(I, s, p)$ is contained in s. Then derivability of s in $n+1$ would imply derivability of $d(I, s, p)$ in n steps and therefore by proposition 1 the sequent s itself would be derivable in n steps, leading to a contradiction. Moreover in the recursion step of the definition of refutability of a sequent s we may also restrict ourselves to inferences I for which no sequent $d(I, s, p)$ is contained in s. Thus we obtain the following notion of *weak refutability* for a logical calculus K:

Weak refutability. Let us call an inference I redundant for a sequent s, iff there is a premiss p of I such that the sequent $d(I, s, p)$ is a submultiset of s. Then a sequent s is weakly refutable in 0 steps, iff all inferences with principal formulas from s are redundant for s and moreover s does not contain an axiom of K as a submultiset. The sequent s is refutable in $n + 1$ steps, iff s does not contain an axiom of K as a submultiset and either it is refutable in 0 steps or for any nonredundant inference I with principal formula from s one of the sequents $d(I, s, p)$ is refutable in n steps.

All calculi K we encounter here admit the so called contraction rule: *If a sequent M, a, a is derivable by K in n steps, then the sequent M, a is derivable in n steps, too.* For such calculi the definition of redundance may be weakened somewhat: An inference I is then called redundant for a sequent s, iff after removing multiple occurrences of objects from some sequent $d(I, s, p)$ this sequent becomes a submultiset of s. The definition of refutability for such calculi may then be based on this weakened notion of redundance and otherwise be kept the same.

The calculus LT. Again we call a calculus for which any sequent is either derivable or weakly refutable a weakly bicomplete calculus. Then theorem 1 readily extends to the case of weakly bicomplete calculi. But again the usual calculi for intuitionistic propositional are not even weakly bicomplete. For instance in the calculus LJ the sequent $T(c \to a) \to a, T(a \to b) \to a, Fa$ is neither derivable nor refutable. In fact the construction of an LJ-refutation for this sequent does not terminate, but yields an infinite tree which starts as follows:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\vdots}{M, Tc, Fa} \qquad M, Tc, Ta, Fb}{M, Tc, Fa, Fc \to a, Fa \to b}}{M, Tc, Fa, Fc \to a}}{M, Tc, Fa} \qquad M, Ta, Fb}{M, Fa, Fc \to a, Fa \to b}}{\dfrac{M, Fa, Fc \to a}{M, Fa}}$$

(where M consists of the two formulas $T(c \to a) \to a$ and $T(a \to b) \to a$.) Here the branches ending in M, Ta, Fb and M, Tc, Ta, Fb are finite because the right premisses of both possible E$\to$-inferences equal the conclusion; the leftmost branch, however, contains infinitely many copies of the sequent M, Tc, Fa. This shows that it is in general impossible to read off a semantical counterexample from a failing LJ-deduction. Thus the calculus LJ is not a solution to the problem of Heyting's mentioned above. There is, however, a surprisingly simple variant LT of LJ which turns out to be bicomplete: LT differs from LJ by adding to the rule I$\to$ a second premiss obtained from the first one by omitting the superscript ^{T}, i.e. LT is the calculus consisting of the usual axioms and the two rules

$$(\mathrm{E}{\to})\,\frac{M, Ta \to b, Fa \qquad M, Ta \to b, Tb}{M, Ta \to b} \qquad (\mathrm{I}{\to})\,\frac{M, Ta, Fb \qquad M^T, Ta, Fb}{M, Fa \to b}$$

We easily show:

Proposition 2 *The calculi LJ and LT are equivalent.*

Proof. The new two-premiss I$\to$-rule is a fortiori admissible for the calculus LJ; on the other hand if a sequent M^T, Ta, Fb is derivable by the new calculus LT, then by proposition 1 the sequent M, Ta, Fb is also derivable. Thus the one-premiss I$\to$-rule is admissible for LT and the two calculi are equivalent. □

Now we have to show:

Theorem 4. *LT is weakly bicomplete.*

Proof. We need three lemmas:

Lemma 1 *If a sequent M, Tb is refutable, then so is the sequent $M, Ta \to b$.*

Proof. If the E$\to$-inference with principal formula $Ta \to b$ is an irredundant inference for the sequent $M, Ta \to b$, then this inference is the only such inference which is not necessarily irredundant for the sequent M, Tb. Therefore refutability of this latter sequent implies refutability of $M, Ta \to b$. □

Lemma 2 *If a sequent M, Ta, Fb is refutable, then so is the sequent M, Ta, Fb, $Fa \to b$.*

Proof. The I→-inference with principal formula $Fa \to b$ is a redundant inference for $M, Ta, Fb, Fa \to b$ because of the left premiss; therefore refutability of M, Ta, Fb implies refutability of $M, Ta, Fb, Fa \to b$. □

Lemma 3 *If a sequent $M, Ta, Tb \to c$ is refutable, then so is the sequent $M, Ta, T(a \to b) \to c$.*

Proof. If the E→-inference with principal formula $Tb \to c$ is a redundant inference for our sequent $M, Ta, Tb \to c$, then we have to consider two cases: Either M contains Tc and thus the E→-inference with principal formula $T(a \to b) \to c$ is a redundant inference for the sequent $M, Ta, T(a \to b) \to c$ and refutability of the former sequent implies refutability of the latter sequent. Or M contains Fb and thus according to lemma 2 the sequent $M, Ta, Tb \to c, Fa \to b$ is refutable if the given sequent is refutable and moreover the sequent $M, Ta, T(a \to b) \to c, Fa \to b$ is also refutable, if the given sequent is refutable, because the only inference for $M, Ta, Tb \to c, Fa \to b$, which is not an inference for $M, Ta, Tb \to c, Fa \to b$, is the redundant inference with principal formula $T(a \to b) \to c$. Now the sequent $M, Ta, T(a \to b) \to c, Fa \to b$ belongs to the left premiss of the only inference which may be irredundant for our sequent $M, Ta, (a \to b) \to c$ without being irredundant for the given sequent $M, Ta, Tb \to c$. Consequently refutability of $M, Ta, Tb \to c$ implies refutability of $M, Ta, T(a \to b) \to c, Fa \to b$ and these two together imply refutability of $M, Ta, (a \to b) \to c$.

If the E→-inference with principal formula $Tb \to c$ is an irredundant inference for our sequent $M, Ta, Tb \to c$ and this sequent is refutable in $n + 1$ steps, then there are again two cases: Either the sequent $M, Ta, Tb \to c, Fb$ is refutable in n steps, then by the induction hypothesis the sequent $M, Ta, T(a \to b) \to c, Fb$ is refutable and by lemma 2 the sequent $M, Ta, T(a \to b) \to c, Fb, Fa \to b$ is also refutable. Now the only irredundant inference for the sequent $M, Ta, T(a \to b) \to c, Fa \to b$ which is not an irredundant inference for $M, Ta, Tb \to c, Fb$ is the $I \to$-inference with principal formula $Fa \to b$. Therefore refutability of the sequent $M, Ta, T(a \to b) \to c, Fb, Fa \to b$ implies refutability of $M, Ta, T(a \to b) \to c, Fa \to b$ and since the E→-inference with principal formula $T(a \to b) \to c$ is the only irredundant inference for the sequent $M, Ta, T(a \to b) \to c$ which is not an irredundant inference of $M, Ta, T(a \to b) \to c, Fa \to b$, the sequent $M, Ta, T(a \to b) \to c$ is also refutable. In the second case the sequent $M, Ta, Tb \to c, Tc$ is refutable in n steps and thus by the induction hypothesis the sequent $M, Ta, T(a \to b) \to c, Tc$ is refutable. Then the E→-inference with principal formula $T(a \to b) \to c$ is the only irredundant inference for the sequent $M, Ta, T(a \to b) \to c$ which is not an irredundant inference for $M, Ta, T(a \to b) \to c, Tc$ and therefore refutability of this latter sequent implies refutability of $M, Ta, T(a \to b) \to c$. □

To complete the proof of theorem 4 we consider the following calculus CRIP due to Dyckhoff and Pinto (cf. [1]) and we show that any sequent s deducible by CRIP is refutable by LT — since all nonderivable sequents of intuitionistic propositional logic are deducible by CRIP, this proves the theorem: The calculus CRIP has axioms of the form $Tp_1 \to b_1, \ldots, Tp_n \to b_n, M$, where all p_i are propositional variables, M does not contain a complementary pair of signed propositional variables and none of the Tp_i occurs in M and rules

$$(1)\ \frac{M, Tp, Tb}{M, Tp, Tp \to b} \qquad\qquad (2)\frac{M, Tb}{M, T(c \to d) \to b}$$

$$(3)\frac{M_1, P(b_1, c_1, d_1) \quad \ldots \quad M_m, P(b_m, c_m, d_m) \quad M, Te_1, Ff_1 \quad \ldots \quad M, Te_n, Ff_n}{M, Fe_1 \to f_1, \ldots, Fe_n \to f_n}$$

where p in rule (1) is a propositional variable and where in rule (3) M is of the form $T(c_1 \to d_1) \to b_1, \ldots, T(c_m \to d_m) \to b_m, M'$, M_i is M with $T(c_i \to d_i) \to b_i$ deleted, $P(b, c, d)$ is $Tc, Td \to b, Fd$ and all formulas of M' are either propositional variables or formulas of the form $Tp \to b$, where p is a propositional variable not occurring in M' and moreover M' does not contain a complementary pair of signed propositional variables.

If our sequent s is an axiom of CRIP of the form $Tp_1 \to b_1, \ldots, Tp_n \to b_n, M$, where M consists entirely of atomic formulas, then $Tp_1 \to b_1, \ldots, Tp_n \to b_n, M, Fp_1, \ldots, Fp_n$ is refutable in 0 steps and thus the sequent s is refutable in n steps. If s is the conclusion of an application of rule (1) or (2), then by the induction hypothesis the premiss of this application is refutable, and thus by lemma 1 the sequent s itself is refutable. Finally if s is the conclusion of an application of rule (3), then by the induction hypothesis all sequents $Tc_i, Td_i \to b_i, M_i, Fd_i$ and all sequents M', Te_j, Ff_j are refutable, where $p_1, \ldots, p_t$ are all propositional variables for which M contains a formula $Tp_k \to g$. Then by lemma 3 the sequents $Tc_i, T(c_i \to d_i) \to b_i, M_i, Fd_i$ are refutable. Now the sequent $M', Fc_1 \to d_1, \ldots, Fc_m \to d_m, Fe_1 \to f_1, \ldots, Fe_n \to f_n, Fp_1, \ldots, Fp_t$ does not contain an axiom of LT as a submultiset and moreover the inferences with principal formulas $c_i \to d_i$ or $e_j \to f_j$ are the only irredundant inferences for this sequent. Therefore refutability of all sequents $Tc_i, Td_i \to b_i, M_i, Fd_i$ and all sequents M', Te_j, Ff_j implies refutability of this latter sequent. □

The semantics of LT. The semantics resulting from the calculus LT is based on the same type of frames as the semantics for LS. The present semantics, however, admits an homomorphic interpretation for the connective $\to$:

A signed formula $Ta \to b$ is declared valid by this semantics in a frame F, iff either the formulae $Ta \to b$ and Fa are valid in a 0-subframe of F or the formulae $Ta \to b$ and Tb are valid in a 0-subframe of F.

A signed formula $Fa \to b$ is declared valid in F, iff the formulae Ta and Fb are valid in a 0-subframe or a 1-subframe of F.

The calculus LU. The calculi LS and LT do certainly not exhaust the possibilities for bicomplete calculi for intuitionistic propositional logic. In fact there are other such calculi giving rise to new types of semantics. We mention just one other bicomplete calculus LU giving a new homomorphic semantics: Besides the usual axioms it has two rules

$$(\mathrm{E}\!\to)\,\frac{M^T, Ta \to b, Fa \qquad M, Ta \to b, Fa \qquad M, Ta \to b, Tb}{M, Ta \to b} \qquad (\mathrm{I}\!\to)\,\frac{M, Ta, Fb}{M, Fa \to b}$$

We omit the proof of bicompleteness for this calculus. We do, however, give the semantics entailed by this calculus: It is based on the usual frames for intuitionistic propositional logic defined before. Then a signed formula $Ta \to b$ is valid in a frame F iff either $Ta \to b$ and Fa are valid in a 1-subframe of F or $Ta \to b$ and Fa are valid in a 0-subframe of F or $Ta \to b$ and Tb are valid in a 0-subframe of F.

A signed formula $Fa \to b$ is valid in F iff both Ta and Fb are valid in a 0-subframe of F.

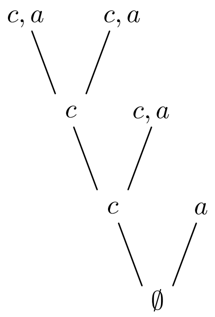
5. Collapsing 0-successors

The various semantics obtained so far do not yet closely resemble the well known Boolean or Kripkean semantics. This is due to the presence of 0-successor nodes in our frames. In usual semantics such 0-successors are identified with their predecessors. In our present setting we can obtain such a collapsing as follows: We first define the notion of *addition* of an object a to a frame F: an object a is added to a frame F with root r, iff it is added to r and it is added to any subframe of F whose root r' is a successor of r and for which a is contained in the parameter set labelling the edge between r and r'. Obviously, if during this process there aren't created any axioms from the sequents of F, then the resulting structure is again a frame as defined above. Moreover any object b which is valid in F is still valid in the frame obtained from F by adding a. Collapsing of a 0-successor node n now works as follows: Let n' be the predecessor of n in F. Then first the objects occurring in n which do not occur in n' are added to the frame with root n'. Since the edge between n and n' is labelled with the set of all objects, this implies that n and n' now contain the same objects. Now the node n is removed from the frame and the edges emanating from n are redirected to n'. Then in the definitions of our semantics we just have to replace all references to 0-subframes of our frame F by references to F itself.

Simplified semantics. By this procedure for our first example of classical propositional logic all nodes are collapsed and we obtain the following clauses for the semantics of complex signed formulae: $Fa \to b$ is valid in F iff both Ta and Fb are valid in F and $Ta \to b$ is valid in F iff Fa or Tb is valid in F. Thus obviously the reference to the frame F may be dropped and we obtain the usual Boolean semantics for classical logic.

For the calculus LS we obtain the definition: A formula $Fa \to b$ is valid in a frame with root n iff both Ta and Fb are valid in F. A formula $Ta \to b$, where a is a propositional variable is valid in F iff Fa is valid in some 1-subframe of F or Tb is valid in F. Finally a formula $T(a \to b) \to c$ is valid in F iff either Ta and $Tb \to c$ and Fb are valid in some 1-subframe of F or Tc is valid in F.

For the calculus LT we obtain the following semantics: A formula $Ta \to b$ is valid in a frame F, iff either the formulae $Ta \to b$ and Fa are valid in F or the formulae $Ta \to b$ and Tb are valid in F. A signed formula $Fa \to b$ is declared valid in F, iff the formulae Ta and Fb are valid in F or a 1-subframe of F. Thus from the calculus LT we obtain the usual Kripkean semantics for intuitionistic propositional logic. E. g. for the previously considered sequent $T(c \to a) \to a, T(a \to b) \to a, Fa$ this gives us the following Kripkean counterexample (among infinitely many others):



Finally for the calculus LU we obtain the semantics defined by the following clauses: A formula $Ta \to b$ is valid in a frame F iff either $Ta \to b$ and Fa are valid in a 1-subframe of F or $Ta \to b$ and Fa are valid in F or $Ta \to b$ and Tb are valid in F. Thus we obtain a new alternative homomorphic semantics for intuitionistic propositional logic.

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