Cut-Elimination in Logics with Definitional Reflection

Peter Schroeder-Heister
Wilhelm-Schickard-Institut, Universität Tübingen
Sand 13, 7400 Tübingen, Germany

Abstract

Definitional Reflection is a principle for introducing atomic assumptions, given a set of definitional rules for atomic formulas. In this paper, proof-theoretic properties of first-order sequent systems with definitional reflection are proved. It is shown that the presence of contraction and the use of implication in the bodies of definitional clauses exclude each other, if cut-elimination is desired.

1. Introduction

"Definitional reflection" denotes an inversion principle for clauses of an inductive definition. For example, suppose an atomic formula A is defined by the inductive clauses

$$\begin{array}{ccc} F_1 & \Rightarrow & A \\ & \vdots & \\ F_n & \Rightarrow & A\,, \end{array}$$

where $F_1, \ldots, F_n$ are formulas of some logic, then this principle says that everything that can be obtained from each definitional condition of A can be obtained from A itself, i.e., for any F, if $F_i \vdash F$ for every i $(1 \leq i \leq n)$, then $A \vdash F$. It is called "definitional reflection", since, when applying this principle, one reflects on the fact that $F_1, \ldots, F_n$ are the *only* conditions defining A, i.e., there is no further condition which allows one to infer A *by means of definition.*

Definitional reflection has been developed and investigated in the context of inductive definitions [12] and in the context of logic programming [13, 22]. It has some (distant) relationship to Clark's "completion" of logic programs [4], to Martin-Löf's elimination rules for predicates in his theory of "iterated inductive defintions" [19] and to Lorenzen's "inversion principle" in his operative interpretation of logical constants [18].

In this paper we focus on systems of first-order logic to which definitional reflection is added. Such systems can be used in the formulation of a declarative semantics for certain programming languages, to be supplemented by an operational semantics guiding the evaluation of queries. This application, however, is not the subject of the present investigation. Rather, we concentrate on the problem of how cut-elimination in Gentzen-style sequent systems is affected by definitional reflection. Actually, the algorithmic questions associated with an operational semantics are much harder to solve.

Our basic results are the following: Cut-elimination holds if the definition of an atom (i.e., the F_i above) does not contain implication (Theorem 3). Furthermore, it holds if the F_i are arbitrary, but the logical system is contraction-free (Theorem 1). If the definition of an atom is allowed to contain implication and the logical system permits contraction, then a counterexample against the cut rule can be given.

The restrictions for cut-elimination are not at all considered a negative result, particularly not for the programming language GCLA based on a subsystem (with implication!) of first-order logic with definitional reflection [3]. Although in many applications of GCLA, such as function evaluation, one works in a contraction-free logic where cut-elimination holds, in others one uses the full system where one may or may not have cut, depending on the specific program (set of inductive clauses) one is considering. We simply do not consider the admissibility of the cut rule to be a matter of principle. A few philosophical remarks may be appropriate to illustrate this point of view.

Normally one considers cut to be a postulate that expresses that the cut-formula F has a well-defined meaning in the sense that the statements which one can infer from F are not stronger than those from which one can infer F, i.e., that one does not gain anything by proceeding via F. According to this approach, eliminability of cut is a necessary condition for the acceptability of a logical system. However, one may also look at a cut with cut-formula F as expressing that F is *totally* defined. The failure of cut-elimination would then express that F is just *partially* defined. That F is totally defined means that one can safely proceed from assertions not containing F, via F, to assertions not containing F without creating anything new; if this is not always possible, F is just partially meaningful.

It is justified in the partial case to say that F has at least *some* meaning, since we are stating fully symmetric conditions for asserting F and for drawing conclusions from F. This holds especially in the case of an atomic F – here this symmetry is due to our principle of definitional reflection. Even the fact that there is no definitional clause for F can be viewed as stating a condition for asserting F (giving rise to the absurdity rule - although this is debatable).

In the case of the definition of a function f, partiality means that for a certain argument a, the function does not return a (unique) value, i.e., that "$f(a)$" cannot be replaced by a value. Analogously, in the case of a partially defined formula F it means that F cannot be fully eliminated from any deductive context. This conceptual relationship between partiality and cut-elimination was first pointed out by Hallnäs [12].

As already mentioned, the principle of definitional reflection refers to a "database" or "program" of clauses, which is handled by our inference system in a certain way. So if we speak of the logical system $\mathcal{D}$ ("$\mathcal{D}$" stands for "definitional reflection"), we mean a system $\mathcal{D}(P)$ over a fixed database P of clauses. For the sake of simplicity, we deal with sequents with a single formula in the succedent. Inspection of proofs will show that methods and results carry over to the case with arbitrarily many formulas in the succedent except in the case of the Lambek-calculus.

We do not say more here about the philosophy of definitional reflection, nor about the theory of definition behind (which gives up monotonicity and does not stick to the least fixpoint interpretation). The reader is referred to the publications mentioned.

In the following, Section 2 describes the first-order system we are dealing with. In Sections 3 and 4 we present the central theorems on the contraction-free and the implication-free system, respectively. Finally, Section 5 gives some hints on how these results carry over to relevance logic, linear logic, and the Lambek calculus. In an appendix we make some remarks concerning recent work by Girard on definitional reflection.

2. First-order logic with definitional reflection

We consider a first-order logic over a certain alphabet with the logical constants $\top$, $\bot$, $\wedge$, $\circ$, $\vee$, $\rightarrow$, $\forall$, $\exists$. Metalinguistic variables for terms are t, for atomic formulas A, B, C, for formulas F, G, H, for finite multisets of formulas X, Y, Z, each with and without primes and indices. Definitional clauses for atoms, in short: clauses, have the form $\top \Rightarrow A$. Thus each clause has a nonempty body, which may be $\top$. Sequents have the form $X \vdash F$. We consider multisets rather than sets as antecedents of sequents since we are dealing in particular with contraction-free systems. Expressions like $X, Y \vdash F$ or $X, A \vdash F$ are understood in the usual way.

A definition P is a finite set of clauses. Let a fixed definition P be given. Let

$$\mathbf{D}(A) := \{F : \textit{there is } \sigma \textit{ such that } F = G\sigma, A = B\sigma \textit{ and } G \Rightarrow B \in P\}.$$

This means, $\mathbf{D}(A)$ is the set of all formulas from which A can be immediately obtained by applying a definitional clause for A (i.e., $\mathbf{D}(A)$ is the set of "definientia" of A). If $F \in \mathbf{D}(A)$, we also say that F is a definitional *condition* of A.

The logical system $\mathcal{D}(P)$ (in short $\mathcal{D}$) we consider is then given by the following inference rules:

$$(I)\ \frac{}{A \vdash A}$$

$$(Thin)\ \frac{X \vdash H}{F, X \vdash H} \qquad (Contr)\ \frac{X, F, F \vdash H}{X, F \vdash H}$$

$$(\vdash\top)\ \frac{}{\vdash \top} \qquad (\top\vdash)\ \frac{X \vdash H}{X, \top \vdash H}$$

$$no\ (\vdash\bot) \qquad (\bot\vdash)\ \frac{}{X, \bot \vdash H}$$

$$(\vdash\wedge)\ \frac{X \vdash F \quad X \vdash G}{X \vdash F \wedge G} \qquad (\wedge\vdash)\ \frac{X, F \vdash H}{X, F \wedge G \vdash H} \qquad \frac{X, G \vdash H}{X, F \wedge G \vdash H}$$

$$(\vdash\circ)\ \frac{X \vdash F \quad Y \vdash G}{X, Y \vdash F \circ G} \qquad (\circ\vdash)\ \frac{X, F, G \vdash H}{X, F \circ G \vdash H}$$

$$(\vdash\vee)\ \frac{X \vdash F}{X \vdash F \vee G} \qquad \frac{X \vdash G}{X \vdash F \vee G} \qquad (\vee\vdash)\ \frac{X, F \vdash H \quad X, G \vdash H}{X, F \vee G \vdash H}$$

$$(\vdash\to)\ \frac{X, F \vdash G}{X \vdash F \to G} \qquad (\to\vdash)\ \frac{X \vdash F \quad Y, G \vdash H}{X, Y, F \to G \vdash H}$$

$$(\vdash\forall)\ \frac{X \vdash F(y)}{X \vdash \forall x F(x)}\ \ y\ new \qquad (\forall\vdash)\ \frac{X, A(t) \vdash H}{X, \forall x F(x) \vdash H}$$

$$(\vdash\exists)\ \frac{X \vdash F(t)}{X \vdash \exists x F(x)} \qquad (\exists\vdash)\ \frac{X, F(y) \vdash H}{X, \exists x F(x) \vdash H}\ \ y\ new$$

$$(\vdash P)\ \frac{X \vdash F}{X \vdash A}\ \ F \in \mathbf{D}(A) \qquad (P\vdash)\ \frac{(X, F \vdash H)_{F \in \mathbf{D}(A)}}{X, A \vdash H}$$

provided $\mathbf{D}(A\sigma) = (\mathbf{D}(A))\sigma$ *for all* σ

$$(Cut)\ \frac{X \vdash F \quad Y, F \vdash G}{X, Y \vdash G}$$

Remark on $\top$ *and* $\bot$*:*

If we admit clauses

$$\Rightarrow A$$

with empty body, we may allow for

$$\overline{\vdash A}$$

to be a limiting case of $(\vdash P)$, where F in $F \Rightarrow B$ is empty and $A = B\sigma$. Then $(\vdash\top)$ and $(\top\vdash)$ are immediate consequences of $(\vdash P)$ and $(P\vdash)$, respectively, if $\top$ is a nullary predicate constant defined by

$$\Rightarrow \top\ .$$

Furthermore, if $\bot$ is a nullary predicate constant not defined by P, i.e., there is no clause with head $\bot$ in P, then $(\bot\vdash)$ is an immediate consequence of $(P\vdash)$ since $\mathbf{D}(\bot)$ is the empty set.

For technical reasons it is quite useful to keep the $\top$- and $\bot$-rules separate from P. Due to the presence of $\bot$, we can assume that for any atom A considered, $\mathbf{D}(A)$ is nonempty. (Otherwise we just put $\bot \Rightarrow A$ into the definition P.) It might be noted that $\top$ corresponds to $\mathbf{1}$ and $\bot$ to $\mathbf{0}$ in Girard's linear logic [11].

Remark on $\circ$*:*

The connective $\circ$ is to be distinguished from $\wedge$, if the rule of contraction is absent. It corresponds to "times" in linear logic.

Remark on $(\vdash P)$*:*

This rule decribes the application of a definitional clause. It is contained in various extensions of logic programming (e.g., λ -Prolog, see [20]) and guarantees that the system is closed under definitional clauses. Its operational counterpart is the resolution principle. Adding definitional reflection $(P\vdash)$ can be seen as establishing the symmetry of Right-and Left-rules also for the atomic case ("computational symmetry") by providing means for assuming an atom.

Remarks on the proviso for $(P\vdash)$*:*

1. It ensures that $(P\vdash)$ is only applicable if for all clauses $G\Rightarrow B$ referred to in the definition of $\mathbf{D}(A)$, G contains no free variables beyond those in B ("no extra-variables"). In particular, it guarantees that there are only finitely many premisses for $(P\vdash)$.

2. It ensures that, when A is further substituted to $A\sigma$, no conditions of $A\sigma$ beyond substitution instances of conditions of A have to be taken into account.

Both 1) and 2) guarantee that the rule $(P\vdash)$ is closed under substitution. It would be possible to weaken 1) by admitting extra-variables and treating them like eigenvariables. This would be useful in a logic programming language without quantifiers. However, since here we have existential quantification at our disposal and therefore can express the intended meaning of a clause

$$F(x)\Rightarrow A$$

by

$$(\exists x)F(x)\Rightarrow A,$$

we can actually assume from the beginning that in any clause $G\Rightarrow A$ in P, each free variable of G occurs in A.

The failure of cut-elimination for the full system:

The system $\mathcal{D}(\emptyset)$ with empty database is a standard logical system of first-order intuitionistic logic which admits cut-elimination. However, this does not extend to $\mathcal{D}(P)$ for any P. E.g., for any atom p we can define

$$P := \{p\to\bot\Rightarrow p\}.$$

Then we have the derivation

$$(Cut)\dfrac{(\vdash P)\dfrac{(\vdash\to)\dfrac{(Contr)\dfrac{(P\vdash)\dfrac{(\to\vdash)\dfrac{(I)\dfrac{}{p\vdash p}\qquad (I)\dfrac{}{\bot\vdash\bot}}{p,p\to\bot\vdash\bot}}{p,p\vdash\bot}}{p\vdash\bot}}{\vdash p\to\bot}}{\vdash p}\qquad \dfrac{\dfrac{\dfrac{\dfrac{}{p\vdash p}\quad \dfrac{}{\bot\vdash\bot}}{p,p\to\bot\vdash\bot}}{p,p\vdash\bot}}{p\vdash\bot}}{\vdash\bot}$$

The cut this derivation ends with is not eliminable: Since there is no definitional clause for $\bot$, there is no rule in $\mathcal{D}(P)$ except cut by means of which $\vdash\bot$ can be inferred.

Inspection of the derivation given shows that application of the ordinary reductions used in cut-elimination proofs does not terminate. Actually, such proofs normally proceed by induction on a pair of numbers whose first component is the logical complexity of the cut formula F. This number decreases if F is a logically compound formula and is introduced in the last step of the premiss derivations of the cut by a Right-rule on the left hand side and a Left-rule on the right hand side. In that case the cut is reduced to a cut with a less complex subformula of F. However, if F is atomic, then, depending on P, the cut with F has to be reduced to a cut with a definitional condition of F which may be of higher complexity than F, as in the present case, where $p \rightarrow \perp$ is a definitional condition of p.

Of course, there are non-trivial cases where cut-elimination holds. One example is the case of a well-founded definition P, i.e., a definition whose predicates can be ordered in such a way that, if $p \leq q$ in this ordering, q does not occur in a clause whose head starts with p. In that case, one can attach a degree of complexity to atoms and formulas such that a condition of A is always of lower degree than A. Another example is that of definitions which do not use implications in their bodies. This is treated in Section 4 below. However, as the example above shows, cut-elimination does not hold in general.

The example, which is closely related to Curry's paradox ([5]), uses contraction to derive the premisses of the cut (cf. Curry's [5] explicit statement of contraction as a logical premiss of his paradox). Without contraction, an analogous counterexample cannot be constructed. Rather, standard reductions of the cut-rule terminate. This is shown in detail in the following section.

3. Cut-elimination for the contraction-free system

In rough analogy to a terminology introduced by Girard, we call inference rules with both X and Y in the antecedent of the conclusion *multiplicative rules* and the others *additive rules*. According to this terminology, all single-premiss rules are additive as is $(P\vdash)$ (which may have more than two premisses). Of the two-premiss rules, $(\vdash\circ)$, $(\rightarrow\vdash)$ and (Cut) are multiplicative, all others are additive.[1]

For derivations, we use the following notation: If Π is a derivation, then

$$\begin{array}{c}\Pi\\ X\vdash F\end{array}$$

expresses that Π ends with the sequent $X\vdash F$, whereas

$$\frac{\Pi}{X\vdash F}$$

expresses that $X\vdash F$ results by applying an inference rule to the end sequent of Π, and similarly with notations like

$$\frac{\Pi_1 \quad \Pi_2}{X\vdash F}.$$

[1]In Girard [11], "additive" and "multiplicative" are attributes of connectives. Since we call all single-premiss rules additive, $(\circ\vdash)$ is additive although $\circ$ is a typical multiplicative connective. Our classification of rules is for technical purposes, to be used in the definition of an induction measure in the proof of Theorem 1 (see also remark 6 after the proof of Theorem 1).

Furthermore, we write $(\vdash *)$ and $(* \vdash)$ to denote right- and left-rules in an indefinite or context-dependent way.

We define the **D**-rank $r_{\mathbf{D}}(\Pi)$ of a derivation Π (which may contain cuts) inductively as follows:

$r_{\mathbf{D}}(\Pi) = 0,$	if Π is an application of (I), $(\vdash\top)$, or $(\bot\vdash)$
$r_{\mathbf{D}}(\Pi) = r_{\mathbf{D}}(\Pi_1),$	if Π ends with an application of a single-premiss rule except $(P\vdash)$, whose premiss-derivation is Π_1
$r_{\mathbf{D}}(\Pi) = r_{\mathbf{D}}(\Pi_1) + r_{\mathbf{D}}(\Pi_2),$	if Π ends with an application of a multiplicative rule, whose premiss-derivations are Π_1 and Π_2
$r_{\mathbf{D}}(\Pi) = max(r_{\mathbf{D}}(\Pi_1), r_{\mathbf{D}}(\Pi_2)),$	if Π ends with an application of a two-premiss additive rule except $(P\vdash)$, whose premiss-derivations are Π_1 and Π_2
$r_{\mathbf{D}}(\Pi) = max_{1\leq i\leq n}(r_{\mathbf{D}}(\Pi_i)) + 1,$	if Π ends with an application of $(P\vdash)$, whose premiss-derivations are Π_i $(1 \leq i \leq n)$

The **D**-rank $r_{\mathbf{D}}(\Pi)$ measures the number of applications of $(P\vdash)$ in Π, where these numbers are summed up with multiplicative rules and the maximum is taken with additive rules.

We define the cut-rank $r_{\mathbf{C}}(\Pi)$ to be the logical complexity of the cut-formula of the single cut in Π, if Π ends with a topmost cut, i.e., with a cut whose premiss-derivations do not contain any cut. Otherwise, $r_{\mathbf{C}}(\Pi)$ is not defined. We define $r_{\mathbf{L}}(\Pi)$ to be the length of Π, i.e., the number of rule-applications in Π.

Theorem 1 The rule (Cut) is admissible in the system without $(Contr)$ and without (Cut).[2]

Proof Suppose Π ends with a topmost cut. Then we show by induction on the triple

$$r(\Pi) := \langle r_{\mathbf{D}}(\Pi), r_{\mathbf{C}}(\Pi), r_{\mathbf{L}}(\Pi)\rangle$$

that Π can be transformed into a cut-free derivation Π' such that

$$r_{\mathbf{D}}(\Pi') \leq r_{\mathbf{D}}(\Pi).$$

The proof proceeds similarly to the usual cut-elimination proofs. We present some example steps, particularly those where the assertion $r_{\mathbf{D}}(\Pi') \leq r_{\mathbf{D}}(\Pi)$ comes into play. The following reductions are understood to be performed in the given order, i.e., a certain case is applicable only if no previous case is applicable.

[2]We do not claim originality for this theorem (but, of course, for the proof and its induction measure). The theorem is suggested by the approaches to avoid logical and set-theoretical paradoxes by using contraction-free systems, which date back to Fitch [9] and Ackermann [2] (for an historical overview see [6]).

1. If Π has the form

$$(Cut)\dfrac{\begin{array}{c}\Pi_1\\ X\vdash A\end{array} \quad (I)\dfrac{}{A\vdash A}}{X\vdash A},$$

we replace Π with Π_1, which is cut-free. Furthermore, $r_{\mathbf{D}}(\Pi_1) = r_{\mathbf{D}}(\Pi)$.

2. If Π has the form

$$(Cut)\dfrac{\begin{array}{c}\Pi_1\\ X\vdash F\end{array} \quad (Thin)\dfrac{\begin{array}{c}\Pi_2\\ Y\vdash H\end{array}}{Y, F\vdash H}}{X, Y\vdash H},$$

we replace Π with

$$(Thin)\dfrac{\dfrac{\begin{array}{c}\Pi_2\\ Y\vdash H\end{array}}{\vdots}}{X, Y\vdash H}.$$

The new derivation is cut-free. Furthermore, its **D**-rank is not higher than that of Π.

3. If the right premiss-derivation of the cut Π ends with does *not* introduce the cut-formula by means of a $(*\vdash)$-rule, we perform a permutative reduction according to the pattern of the following examples.

3.1. If Π has the form

$$(Cut)\dfrac{\begin{array}{c}\Pi_1\\ X\vdash F\end{array} \quad (\bot\vdash)\dfrac{}{Y, F, \bot\vdash H}}{X, Y, \bot\vdash H},$$

we replace Π with the following derivation Π':

$$(\bot\vdash)\dfrac{}{X, Y, \bot\vdash H}.$$

Obviously, $r_{\mathbf{D}}(\Pi') = 0 \leq r_{\mathbf{D}}(\Pi)$.

3.2. If Π has the form

$$(Cut)\dfrac{\begin{array}{c}\Pi_1\\ X\vdash F\end{array} \quad (\vdash\forall)\dfrac{\begin{array}{c}\Pi_2\\ Y, F\vdash G(y)\end{array}}{Y, F\vdash \forall x G(x)}}{X, Y\vdash \forall x G(x)},$$

we replace Π with the following derivation Π' after renaming y to an appropriate z, if necessary:

$$(\vdash\forall)\dfrac{\left.(Cut)\dfrac{\begin{array}{c}\Pi_1\\ X\vdash F\end{array} \quad \begin{array}{c}\Pi_2[y/z]\\ Y, F\vdash G(z)\end{array}}{X, Y\vdash G(z)}\right\}\Pi_1'}{X, Y\vdash \forall x G(x)},$$

where $\Pi_2[y/z]$ results from Π_2 by substituting z for y throughout. For the subderivation Π'_1 of Π' we have the following:

$$\begin{aligned}
r_{\mathbf{D}}(\Pi'_1) &= r_{\mathbf{D}}(\Pi_1) + r_{\mathbf{D}}(\Pi_2) \\
&= r_{\mathbf{D}}(\Pi) \\
r_{\mathbf{C}}(\Pi'_1) &= r_{\mathbf{C}}(\Pi) \\
r_{\mathbf{L}}(\Pi'_1) &< r_{\mathbf{L}}(\Pi).
\end{aligned}$$

Thus, by the induction hypothesis, cut is eliminable from Π'_1, yielding a cut-free derivation Π''_1 of $X, Y \vdash G(z)$ with $r_{\mathbf{D}}(\Pi''_1) \leq r_{\mathbf{D}}(\Pi'_1)$. Therefore, replacing Π'_1 by Π''_1 in Π' yields a cut-free derivation Π'' for which $r_{\mathbf{D}}(\Pi'') \leq r_{\mathbf{D}}(\Pi)$.

Note that replacing y with z in Π_2 requires the closure of inference rules under substitution, which in the case of $(P\vdash)$ is guaranteed by the proviso.

3.3. If Π has the form

$$(Cut)\dfrac{\begin{matrix}\Pi_1 \\ X \vdash F\end{matrix} \qquad (\vdash\wedge)\dfrac{\begin{matrix}\Pi_2 \\ Y, F \vdash G\end{matrix} \quad \begin{matrix}\Pi_3 \\ Y, F \vdash H\end{matrix}}{Y, F \vdash G \wedge H}}{X, Y \vdash G \wedge H},$$

we replace Π with the following derivation Π':

$$(\vdash\wedge)\dfrac{\left.(Cut)\dfrac{\begin{matrix}\Pi_1 \\ X \vdash F\end{matrix} \quad \begin{matrix}\Pi_2 \\ Y, F \vdash G\end{matrix}}{X, Y \vdash G}\right\}\Pi'_1 \qquad \left.(Cut)\dfrac{\begin{matrix}\Pi_1 \\ X \vdash F\end{matrix} \quad \begin{matrix}\Pi_3 \\ Y, F \vdash H\end{matrix}}{X, Y \vdash H}\right\}\Pi'_2}{X, Y \vdash G \wedge H}.$$

For the subderivations Π'_1 and Π'_2 of Π' we have the following:

$$\begin{aligned}
r_{\mathbf{D}}(\Pi'_1) &= r_{\mathbf{D}}(\Pi_1) + r_{\mathbf{D}}(\Pi_2) \\
&\leq r_{\mathbf{D}}(\Pi_1) + max(r_{\mathbf{D}}(\Pi_2), r_{\mathbf{D}}(\Pi_3)) \\
&= r_{\mathbf{D}}(\Pi) \\
r_{\mathbf{D}}(\Pi'_2) &\leq r_{\mathbf{D}}(\Pi) \text{ (similarly)} \\
r_{\mathbf{C}}(\Pi'_1) &= r_{\mathbf{C}}(\Pi'_2) = r_{\mathbf{C}}(\Pi) \\
r_{\mathbf{L}}(\Pi'_1) &< r_{\mathbf{L}}(\Pi) \\
r_{\mathbf{L}}(\Pi'_2) &< r_{\mathbf{L}}(\Pi).
\end{aligned}$$

Thus, by the induction hypothesis, cut is eliminable from Π'_1 and Π'_2, yielding cut-free derivations Π''_1 and Π''_2 with

$$\begin{aligned}
r_{\mathbf{D}}(\Pi''_1) &\leq r_{\mathbf{D}}(\Pi'_1), \\
r_{\mathbf{D}}(\Pi''_2) &\leq r_{\mathbf{D}}(\Pi'_2).
\end{aligned}$$

Replacing Π'_1 and Π'_2 in Π' by Π''_1 and Π''_2 yields the following cut-free derivation Π'':

$$(\vdash\wedge)\dfrac{\Pi''_1 \quad \Pi''_2}{X, Y \vdash G \wedge H},$$

for which

$r_{\mathbf{D}}(\Pi'') = max(r_{\mathbf{D}}(\Pi''_1), r_{\mathbf{D}}(\Pi''_2)) \leq r_{\mathbf{D}}(\Pi)$

holds.

3.4. If Π has the form

$$(Cut)\dfrac{\begin{matrix}\Pi_1\\ X\vdash F\end{matrix} \quad (\rightarrow\vdash)\dfrac{\begin{matrix}\Pi_2 \\ Y\vdash G\end{matrix} \quad \begin{matrix}\Pi_3 \\ Z, F, H\vdash H'\end{matrix}}{Y, Z, F, G\rightarrow H\vdash H'}}{X, Y, Z, G\rightarrow H\vdash H'},$$

we replace Π with the following derivation Π':

$$(\rightarrow\vdash)\dfrac{\begin{matrix}\Pi_2\\ Y\vdash G\end{matrix} \quad \left.(Cut)\dfrac{\begin{matrix}\Pi_1 \\ X\vdash F\end{matrix} \quad \begin{matrix}\Pi_3 \\ Z, F, H\vdash H'\end{matrix}}{X, Z, H\vdash H'}\right\}\Pi'_1}{X, Y, Z, G\rightarrow H\vdash H'}.$$

For the subderivation Π'_1 of Π' we have the following:

$$\begin{aligned} r_{\mathbf{D}}(\Pi'_1) &= r_{\mathbf{D}}(\Pi_1) + r_{\mathbf{D}}(\Pi_3) \\ &\leq r_{\mathbf{D}}(\Pi_1) + r_{\mathbf{D}}(\Pi_2) + r_{\mathbf{D}}(\Pi_3) \\ &= r_{\mathbf{D}}(\Pi) \\ r_{\mathbf{C}}(\Pi'_1) &= r_{\mathbf{C}}(\Pi) \\ r_{\mathbf{L}}(\Pi'_1) &< r_{\mathbf{L}}(\Pi). \end{aligned}$$

Thus, by the induction hypothesis, cut is eliminable from Π'_1, yielding a cut-free derivation Π''_1 with

$r_{\mathbf{D}}(\Pi''_1) \leq r_{\mathbf{D}}(\Pi'_1)$.

Replacing Π'_1 in Π' with Π''_1 yields the following cut-free derivation Π'':

$$(\rightarrow\vdash)\dfrac{\Pi_2 \quad \Pi''_1}{X, Y, Z, G\rightarrow H\vdash H'},$$

for which

$$\begin{aligned} r_{\mathbf{D}}(\Pi'') &= r_{\mathbf{D}}(\Pi_2) + r_{\mathbf{D}}(\Pi''_1) \\ &\leq r_{\mathbf{D}}(\Pi_2) + r_{\mathbf{D}}(\Pi_1) + r_{\mathbf{D}}(\Pi_3) \\ &= r_{\mathbf{D}}(\Pi) \end{aligned}$$

holds.

3.5. If Π has the form

$$(Cut)\dfrac{\begin{matrix}\Pi_1\\ X\vdash F\end{matrix} \quad (P\vdash)\dfrac{\left(\begin{matrix}\Pi_G \\ Y, F, G\vdash H\end{matrix}\right)_{G\in\mathbf{D}(A)}}{Y, F, A\vdash H}}{X, Y, A\vdash H},$$

we replace Π with the following derivation Π':

$$(P\vdash)\frac{\left(\left.\begin{array}{c}\Pi_1 \qquad \Pi_G \\ (Cut)\dfrac{X\vdash F \quad Y, F, G\vdash H}{X, Y, G\vdash H}\end{array}\right\}\Pi'_G\right)_{G\in\mathbf{D}(A)}}{X, Y, A\vdash H}.$$

For each subderivation Π'_G of Π' we have the following:

$$\begin{aligned} r_{\mathbf{D}}(\Pi'_G) &= r_{\mathbf{D}}(\Pi_1) + r_{\mathbf{D}}(\Pi_G) \\ &< r_{\mathbf{D}}(\Pi_1) + max_{G\in\mathbf{D}(A)}(r_{\mathbf{D}}(\Pi_G)) + 1 \\ &= r_{\mathbf{D}}(\Pi). \end{aligned}$$

Thus, by the induction hypothesis, cut is eliminable from Π'_G for each G, yielding cut-free derivations Π''_G with $r_{\mathbf{D}}(\Pi''_G) \leq r_{\mathbf{D}}(\Pi'_G)$. Replacing each Π'_G in Π' by Π''_G yields the following cut-free derivation Π'':

$$(P\vdash)\frac{(\Pi''_G)_{G\in\mathbf{D}(A)}}{X, Y, A\vdash H},$$

for which

$$\begin{aligned} r_{\mathbf{D}}(\Pi'') &= max_{G\in\mathbf{D}(A)}(r_{\mathbf{D}}(\Pi''_G)) + 1 \\ &\leq max_{G\in\mathbf{D}(A)}(r_{\mathbf{D}}(\Pi'_G)) + 1 \\ &< r_{\mathbf{D}}(\Pi) + 1 \end{aligned}$$

and thus

$$r_{\mathbf{D}}(\Pi'') \leq r_{\mathbf{D}}(\Pi)$$

holds.

4. If the right premiss-derivation of the cut Π ends with introduces the cut-formula by means of a $(*\vdash)$-rule, but the left premiss-derivation does *not* introduce the cut-formula by means of a $(\vdash*)$-rule, we perform a permutative reduction according to the pattern of the following examples:

4.1. If Π has the form

$$(Cut)\frac{(\bot\vdash)\dfrac{}{X, \bot\vdash F} \qquad (*\vdash)\dfrac{\Pi_1}{Y, F\vdash H}}{X, Y, \bot\vdash H},$$

we replace Π with the following derivation Π':

$$(\bot\vdash)\frac{}{X, Y, \bot\vdash H}.$$

Obviously, $r_{\mathbf{D}}(\Pi') = 0 \leq r_{\mathbf{D}}(\Pi)$.

4.2. If Π has the form

$$
(Cut)\dfrac{(\vee\vdash)\dfrac{\begin{array}{cc}\Pi_1 & \Pi_2\\ X,G\vdash F & X,H\vdash F\end{array}}{X,G\vee H\vdash F} \qquad (*\vdash)\dfrac{\Pi_3}{Y,F\vdash H'}}{X,Y,G\vee H\vdash H'},
$$

we replace Π with the following derivation Π':

$$
(\vee\vdash)\dfrac{(Cut)\dfrac{\begin{array}{c}\Pi_1\\ X,G\vdash F\end{array} \quad (*\vdash)\dfrac{\Pi_3}{Y,F\vdash H'}}{X,Y,G\vdash H'} \qquad (Cut)\dfrac{\begin{array}{c}\Pi_2\\ X,H\vdash F\end{array} \quad (*\vdash)\dfrac{\Pi_3}{Y,F\vdash H'}}{X,Y,H\vdash H'}}{X,Y,G\vee H\vdash H'}.
$$

The argument proceeds analogously to case 3.3 above.

4.3. If Π has the form

$$
(Cut)\dfrac{(\to\vdash)\dfrac{\begin{array}{cc}\Pi_1 & \Pi_2\\ X\vdash G & Y,H\vdash F\end{array}}{X,Y,G\to H\vdash F} \qquad (*\vdash)\dfrac{\Pi_3}{Z,F\vdash H'}}{X,Y,Z,G\to H\vdash H'},
$$

we replace Π with the following derivation Π':

$$
(\to\vdash)\dfrac{\begin{array}{c}\Pi_1\\ X\vdash G\end{array} \qquad (Cut)\dfrac{\begin{array}{c}\Pi_2\\ Y,H\vdash F\end{array} \quad (*\vdash)\dfrac{\Pi_3}{Z,F\vdash H'}}{Y,Z,H\vdash H'}}{X,Y,Z,G\to H\vdash H'}.
$$

The argument proceeds analogously to case 3.4 above.

4.4. If Π has the form

$$
(Cut)\dfrac{(P\vdash)\dfrac{\left(\begin{array}{c}\Pi_G\\ X,G\vdash F\end{array}\right)_{G\in\mathbf{D}(A)}}{X,A\vdash F} \qquad (*\vdash)\dfrac{\Pi_1}{Y,F\vdash H}}{X,Y,A\vdash H},
$$

we replace Π with the following derivation Π':

$$
(P\vdash)\dfrac{\left((Cut)\dfrac{\begin{array}{c}\Pi_G\\ X,G\vdash F\end{array} \quad (*\vdash)\dfrac{\Pi_1}{Y,F\vdash H}}{X,Y,G\vdash H}\right)_{G\in\mathbf{D}(A)}}{X,Y,A\vdash H}.
$$

The argument proceeds analogously to case 3.5 above.

5. If the right premiss-derivation of the cut Π ends with introduces the cut-formula by means of a $(*\vdash)$-rule, and the left premiss-derivation introduces it by means of a $(\vdash *)$-rule, we perform a logical reduction according to the pattern of the following examples:

5.1. If Π has the form

$$(Cut)\dfrac{(\vdash\top)\dfrac{}{\vdash\top}\qquad (\top\vdash)\dfrac{\begin{matrix}\Pi_1\\ X\vdash H\end{matrix}}{X,\top\vdash H}}{X\vdash H},$$

we replace Π with Π_1. Obviously, $r_{\mathbf{D}}(\Pi_1) = r_{\mathbf{D}}(\Pi)$.

5.2. If Π has the form

$$(Cut)\dfrac{(\vdash\circ)\dfrac{\begin{matrix}\Pi_1\\ X\vdash F\end{matrix}\quad\begin{matrix}\Pi_2\\ Y\vdash G\end{matrix}}{X,Y\vdash F\circ G}\qquad (\circ\vdash)\dfrac{\begin{matrix}\Pi_3\\ Z,F,G\vdash H\end{matrix}}{Z,F\circ G\vdash H}}{X,Y,Z\vdash H},$$

we replace Π with the following derivation Π':

$$(Cut)\dfrac{\begin{matrix}\Pi_1\\ X\vdash F\end{matrix}\qquad \left.(Cut)\dfrac{\begin{matrix}\Pi_2\\ Y\vdash G\end{matrix}\quad\begin{matrix}\Pi_3\\ Z,F,G\vdash H\end{matrix}}{Y,Z,F\vdash H}\right\}\Pi_2'}{X,Y,Z\vdash H}.$$

For the subderivation Π_2' of Π' we have the following:

$$\begin{aligned}
r_{\mathbf{D}}(\Pi_2') &= r_{\mathbf{D}}(\Pi_2) + r_{\mathbf{D}}(\Pi_3)\\
&\le r_{\mathbf{D}}(\Pi_1) + r_{\mathbf{D}}(\Pi_2) + r_{\mathbf{D}}(\Pi_3)\\
&= r_{\mathbf{D}}(\Pi)\\
r_{\mathbf{C}}(\Pi_2') &< r_{\mathbf{C}}(\Pi)\ .
\end{aligned}$$

Thus, by the induction hypothesis, cut is eliminable from Π_2', yielding a cut-free derivation Π_2'' with

$$r_{\mathbf{D}}(\Pi_2'') \le r_{\mathbf{D}}(\Pi_2').$$

Replacing Π_2' in Π' with Π_2'' yields the following derivation Π'':

$$(Cut)\dfrac{\Pi_1\qquad \Pi_2''}{X,Y,Z\vdash H},$$

for which

$$\begin{aligned}
r_{\mathbf{D}}(\Pi'') &= r_{\mathbf{D}}(\Pi_1) + r_{\mathbf{D}}(\Pi_2'')\\
&\le r_{\mathbf{D}}(\Pi_1) + r_{\mathbf{D}}(\Pi_2) + r_{\mathbf{D}}(\Pi_3)\\
&= r_{\mathbf{D}}(\Pi)\\
r_{\mathbf{C}}(\Pi'') &< r_{\mathbf{C}}(\Pi)
\end{aligned}$$

holds. Thus, again by the induction hypothesis, cut is eliminable from Π'', yielding a cut-free derivation Π''' of $X, Y, Z \vdash H$, for which

$$r_{\mathbf{D}}(\Pi''') \leq r_{\mathbf{D}}(\Pi'') \leq r_{\mathbf{D}}(\Pi)$$

holds (see remarks 2 and 4 below).

5.3. If Π has the form

$$(Cut)\frac{(\vdash\rightarrow)\dfrac{\begin{matrix}\Pi_1\\ X, F \vdash G\end{matrix}}{X \vdash F \rightarrow G} \quad (\rightarrow\vdash)\dfrac{\begin{matrix}\Pi_2 & \Pi_3\\ Y \vdash F & Z, G \vdash H\end{matrix}}{Y, Z, F \rightarrow G \vdash H}}{X, Y, Z \vdash H},$$

we replace Π with the following derivation Π':

$$(Cut)\frac{\left.(Cut)\dfrac{\begin{matrix}\Pi_2 & \Pi_1\\ Y \vdash F & X, F \vdash G\end{matrix}}{X, Y \vdash G}\right\}\Pi_1' \qquad \begin{matrix}\Pi_3\\ Z, G \vdash H\end{matrix}}{X, Y, Z \vdash H}.$$

For the subderivation Π_1' of Π' we have:

$$\begin{aligned} r_{\mathbf{D}}(\Pi_1') &= r_{\mathbf{D}}(\Pi_2) + r_{\mathbf{D}}(\Pi_1)\\ &\leq r_{\mathbf{D}}(\Pi_1) + r_{\mathbf{D}}(\Pi_2) + r_{\mathbf{D}}(\Pi_3)\\ &= r_{\mathbf{D}}(\Pi)\\ r_{\mathbf{C}}(\Pi_1') &< r_{\mathbf{C}}(\Pi)\ . \end{aligned}$$

We argue analogously to the previous case 5.2 (see remarks 2, 3 and 4 below).

5.4. If Π has the form

$$(Cut)\frac{(\vdash\exists)\dfrac{\begin{matrix}\Pi_1\\ X \vdash F(t)\end{matrix}}{X \vdash \exists x F(x)} \quad (\exists\vdash)\dfrac{\begin{matrix}\Pi_2\\ Y, F(y) \vdash H\end{matrix}}{Y, \exists x F(x) \vdash H}}{X, Y \vdash H},$$

we replace Π with the following derivation Π':

$$(Cut)\frac{\begin{matrix}\Pi_1 & \Pi_2[y/t]\\ X \vdash F(t) & Y, F(t) \vdash H\end{matrix}}{X, Y \vdash H},$$

where $\Pi_2[y/t]$ results from Π_2 by substituting t for y throughout. (Here, again, closure of rules under substitution is used.) Obviously, for Π' we have that

$$\begin{aligned} r_{\mathbf{D}}(\Pi') &= r_{\mathbf{D}}(\Pi)\\ r_{\mathbf{C}}(\Pi') &< r_{\mathbf{C}}(\Pi), \end{aligned}$$

so that the induction hypothesis is applicable to Π'.

5.5. If Π has the form

$$
(Cut)\dfrac{(\vdash P)\dfrac{\begin{matrix}\Pi_1\\ X\vdash F\end{matrix}}{X\vdash A} \quad (P\vdash)\dfrac{\left(\begin{matrix}\Pi_G\\ Y, G\vdash H\end{matrix}\right)_{G\in \mathbf{D}(A)}}{Y, A\vdash H}}{X, Y\vdash H},
$$

we replace Π with the following derivation Π':

$$
(Cut)\dfrac{\begin{matrix}\Pi_1\\ X\vdash F\end{matrix} \quad \begin{matrix}\Pi_F\\ Y, F\vdash H\end{matrix}}{X, Y\vdash H}.
$$

Since $F \in \mathbf{D}(A)$, Π_F must be among the Π_G. Obviously, $r_{\mathbf{D}}(\Pi') < r_{\mathbf{D}}(\Pi)$, so that the induction hypothesis can be applied to Π' (see remark 1 below).

□

Remarks on the proof of Theorem 1

1. The introduction of the **D**-rank as the first component of the induction value is essential for the logical reduction of the P-rules (case 5.5), since in that case the formula F may be of higher complexity than the atom A.

2. That part of the induction hypothesis which says that the **D**-rank is not increased by cut-elimination, becomes crucial when subsequent cuts have to be eliminated. This happens in the logical reductions of $\circ$ and $\rightarrow$ (cases 5.2 and 5.3).

3. It is essential that we use the (normal) multiplicative version of $(\rightarrow\vdash)$. With an additive version like

$$
\dfrac{X\vdash F \quad X, G\vdash H}{X, F\rightarrow G\vdash H},
$$

the reduction in case 5.3 would not be valid, save we also used an additive version of cut like

$$
\dfrac{X\vdash F \quad X, F\vdash G}{X\vdash G}.
$$

4. There is a crucial difference between cases 5.2 ($\circ$) and 5.3 ($\rightarrow$) above. In the reduced derivation Π' of case 5.2 ($\circ$), the cuts with F and G as cut-formulas can just be interchanged by interchanging Π_1 and Π_2 as subderivations of Π' without changing the structure of Π'. Here Π_1 and Π_2 can be viewed as two "independent" input-sequents for a (generalized version of) cut (see Section 4). In case 5.3 ($\rightarrow$), interchanging the cuts with F and G in Π' would change Π' to the following derivation:

$$
(Cut)\dfrac{\begin{matrix}\Pi_2\\ Y\vdash F\end{matrix} \quad (Cut)\dfrac{\begin{matrix}\Pi_1\\ X, F\vdash G\end{matrix} \quad \begin{matrix}\Pi_3\\ Z, G\vdash H\end{matrix}}{X, Z, F\vdash H}}{X, Y, Z\vdash H},
$$

whose structure differs from that of Π', although the order of Π_1 and Π_2 remains the same. The derivations Π_1 and Π_2 are not "independent", but members of a "chain" of cuts. This peculiarity of $\to$ prevents reducing the two cuts in Π' in a single step with an appropriate induction measure which would allow for the reduction of contraction.

5. In order to reduce contraction, one would have to replace the derivation

$$(Cut)\dfrac{\begin{array}{c}\Pi_1\\ X\vdash F\end{array} \quad (Contr)\dfrac{\begin{array}{c}\Pi_2\\ Y,F,F\vdash H\end{array}}{Y,F\vdash H}}{X,Y\vdash H}$$

with the derivation

$$(Contr)\dfrac{(Cut)\dfrac{\begin{array}{c}\Pi_1\\ X\vdash F\end{array} \quad (Cut)\dfrac{\begin{array}{c}\Pi_1\\ X\vdash F\end{array} \quad \begin{array}{c}\Pi_2\\ Y,F,F\vdash H\end{array}}{X,Y,F\vdash H}}{X,X,Y\vdash H}}{X,Y\vdash H}.$$

Due to the duplication of Π_1, the **D**-rank may increase in an uncontrolled way. If $\to$ is not present, this problem can be avoided (see Section 4).

6. The distinction between additive and multiplicative rules in the definition of the **D**-rank of a derivation has the following reason. If one treated all rules in the additive way, taking the maximum of the **D**-ranks of the subderivations, in the right permutative reduction with $(P\vdash)$ (case 3.5) the **D**-rank may increase if Π_1 is of highest **D**-rank. If one treated all rules in the multiplicative way, summing up the **D**-ranks of subderivations, it may increase in cases like 3.3, where a subderivation is duplicated.

4. Cut-elimination for implication-free definitions

Now we admit contraction, but consider only definitions P which do not contain implication. We first show that a generalized version of cut, the "multicut" rule

$$(mc)\dfrac{X_1\vdash F_1 \ \ldots \ X_n\vdash F_n \quad F_1,\ldots,F_n,Y\vdash H}{X_1,\ldots,X_n,Y\vdash H}$$

is admissible provided $F_1,\ldots,F_n$ do not contain implication (Theorem 2).[3] From that we obtain as a corollary that for implication-free definitions, the atomic cut-rule

$$(Cut_0)\dfrac{X\vdash A \quad A,Y\vdash G}{X,Y\vdash G}$$

is admissible (Corollary 1). By combining this result with the ordinary cut-elimination procedure for intuitionistic logic, we then obtain the admissibility of (Cut) for implication-free definitions (Theorem 3). In the last case, implication is allowed to occur in cut-formulas, but not in bodies of definitional clauses.

[3] The multicut rule and the idea to prove cut-elimination via multicut-elimination is due to Slaney [23]. In his system with implication, multicut is actually not admissible.

In a multicut (mc), the $F_1, \dots, F_n$ are called "cut-formulas". The rightmost premiss of a multicut is called its *major premiss*, the other ones its *minor premisses*. We also write

$$(mc)\frac{(X_i \vdash F_i)_{i \leq n} \quad F_1, \dots, F_n, Y \vdash H}{X_1, \dots, X_n, Y \vdash H}.$$

We only consider derivations ending with a topmost multicut, i.e., with an application of (mc), above which there is no further application of (mc). Define for such a derivation Π the **D**-rank $r_{\mathbf{D}}(\Pi)$ to be the number of applications of $(Contr)$ and of $(P\vdash)$ above the major premiss of the multicut Π ends with, $r_{\mathbf{C}}(\Pi)$ to be the sum of the logical complexities of the cut formulas $F_1, \dots, F_n$ of that multicut, and $r_{\mathbf{L}}(\Pi)$ to be the length of Π.

Theorem 2 Suppose P is implication-free. Then in the system without (Cut), the multicut rule (mc) is admissible for implication-free cut-formulas.

Proof We proceed, for a given Π ending with a topmost multicut, by induction on the triple

$$r(\Pi) := \langle r_{\mathbf{D}}(\Pi), r_{\mathbf{C}}(\Pi), r_{\mathbf{L}}(\Pi) \rangle.$$

1. If Π has the form

$$(mc)\frac{\begin{array}{c}\Pi_1\\ X \vdash A\end{array} \quad A \vdash A}{X \vdash A},$$

we replace Π with Π_1, which does not contain any multicut (see remark 1 below).

2. If Π has the form

$$(mc)\frac{\left(\begin{array}{c}\Pi_i\\ X_i \vdash F_i\end{array}\right)_{i \leq n} \quad (Thin)\dfrac{\begin{array}{c}\Pi_{n+1}\\ F_1, \dots, F_{n-1}, Y \vdash H\end{array}}{F_1, \dots, F_n, Y \vdash H}}{X_1, \dots, X_n, Y \vdash H},$$

we replace Π with the following derivation Π':

$$(Thin)\frac{\left.(mc)\dfrac{\left(\begin{array}{c}\Pi_i\\ X_i \vdash F_i\end{array}\right)_{i \leq n-1} \quad \begin{array}{c}\Pi_{n+1}\\ F_1, \dots, F_{n-1}, Y \vdash H\end{array}}{X_1, \dots, X_{n-1}, Y \vdash H}\right\} \Pi_1'}{\dfrac{\vdots}{X_1, \dots, X_n, Y \vdash H}}.$$

For the subderivation Π_1' of Π' we have the following:

$r_{\mathbf{D}}(\Pi_1') = r_{\mathbf{D}}(\Pi)$
$r_{\mathbf{C}}(\Pi_1') \leq r_{\mathbf{C}}(\Pi)$
$r_{\mathbf{L}}(\Pi_1') < r_{\mathbf{L}}(\Pi)$.

Therefore we can apply the induction hypothesis.

3. If Π has the form

$$
(mc)\frac{\left(\begin{array}{c}\Pi_i\\ X_i\vdash F_i\end{array}\right)_{i\le n}\qquad (Contr)\dfrac{\begin{array}{c}\Pi_{n+1}\\ F_1,\dots,F_n,F_n,Y\vdash H\end{array}}{F_1,\dots,F_n,Y\vdash H}}{X_1,\dots,X_n,Y\vdash H},
$$

we replace Π with the following derivation Π':

$$
(Contr)\frac{\left.(mc)\dfrac{\left(\begin{array}{c}\Pi_i\\ X_i\vdash F_i\end{array}\right)_{i\le n}\quad \begin{array}{c}\Pi_n\\ X_n\vdash F_n\end{array}\quad \begin{array}{c}\Pi_{n+1}\\ F_1,\dots,F_n,F_n,Y\vdash H\end{array}}{X_1,\dots,X_n,X_n,Y\vdash H}\right\}\Pi_1'}{\dfrac{\vdots}{X_1,\dots,X_n,Y\vdash H}}.
$$

For the subderivation Π_1' of Π' we have $r_{\mathbf{D}}(\Pi_1') < r_{\mathbf{D}}(\Pi)$, since the number of contractions above the major premiss of the multicut is reduced by 1. Therefore we can apply the induction hypothesis.

4. If the derivation of the major premiss of the multicut Π ends with does *not* introduce *any* cut-formula $F_1,\dots,F_n$ by means of a $(*\vdash)$-rule, we perform a permutative reduction according to the pattern of the following example:

If Π has the form

$$
(mc)\frac{\left(\begin{array}{c}\Pi_i\\ X_i\vdash F_i\end{array}\right)_{i\le n}\qquad (\wedge\vdash)\dfrac{\begin{array}{c}\Pi_{n+1}\\ F_1,\dots,F_n,Y,F\vdash H\end{array}}{F_1,\dots,F_n,Y,F\wedge G\vdash H}}{X_1,\dots,X_n,Y,F\wedge G\vdash H},
$$

we replace Π with the following derivation Π':

$$
(\wedge\vdash)\frac{\left.(mc)\dfrac{\left(\begin{array}{c}\Pi_i\\ X_i\vdash F_i\end{array}\right)_{i\le n}\quad \begin{array}{c}\Pi_{n+1}\\ F_1,\dots,F_n,Y,F\vdash H\end{array}}{X_1,\dots,X_n,Y,F\vdash H}\right\}\Pi_1'}{X_1,\dots,X_n,Y,F\wedge G\vdash H}.
$$

For the subderivation Π_1' of Π' we have the following:

$r_{\mathbf{D}}(\Pi_1') = r_{\mathbf{D}}(\Pi)$
$r_{\mathbf{C}}(\Pi_1') = r_{\mathbf{C}}(\Pi)$
$r_{\mathbf{L}}(\Pi_1') < r_{\mathbf{L}}(\Pi)$.

Therefore we can apply the induction hypothesis.

5. If the derivation of the major premiss of the multicut Π ends with introduces a cut-formula by means of a $(*\vdash)$-rule, but the derivation of the corresponding minor premiss

does *not* introduce this cut-formula by means of a ($\vdash *$)-rule, we perform a permutative reduction according to the pattern of the following example:

If Π has the form

$$(mc)\frac{\left(\begin{array}{c}\Pi_i\\ X_i\vdash F_i\end{array}\right)_{i\leq n-1} \quad (\to\vdash)\dfrac{\begin{array}{cc}\Pi'_n & \Pi''_n\\ X\vdash F & Z,G\vdash F_n\end{array}}{X,Z,F\to G\vdash F_n} \quad (*\vdash)\dfrac{\Pi_{n+1}}{F_1,\ldots,F_n,Y\vdash H}}{X_1,\ldots,X_{n-1},X,Y,Z,F\to G\vdash H},$$

we replace Π with the following derivation Π':

$$(\to\vdash)\frac{\begin{array}{c}\Pi'_n\\ X\vdash F\end{array} \quad \left.(mc)\dfrac{\left(\begin{array}{c}\Pi_i\\ X_i\vdash F_i\end{array}\right)_{i\leq n-1} \quad \begin{array}{c}\Pi''_n\\ Z,G\vdash F_n\end{array} \quad (*\vdash)\dfrac{\Pi_{n+1}}{F_1,\ldots,F_n,Y\vdash H}}{X_1,\ldots,X_{n-1},Y,Z,G\vdash H}\right\}\Pi'_1}{X_1,\ldots,X_{n-1},X,Y,Z,F\to G\vdash H}.$$

For the subderivation Π'_1 of Π' we have the following:

$r_{\mathbf{D}}(\Pi'_1) = r_{\mathbf{D}}(\Pi)$
$r_{\mathbf{C}}(\Pi'_1) = r_{\mathbf{C}}(\Pi)$
$r_{\mathbf{L}}(\Pi'_1) < r_{\mathbf{L}}(\Pi)$.

Therefore we can apply the induction hypothesis.

6. If the derivation of the major premiss of the multicut Π ends with introduces a cut-formula by means of a ($*\vdash$)-rule, and the derivation of the corresponding minor premiss introduces the same formula by means of a ($\vdash *$)-rule, we perform a logical reduction according to the pattern of the following examples:

6.1. If Π has the form

$$(mc)\frac{\left(\begin{array}{c}\Pi_i\\ X_i\vdash F_i\end{array}\right)_{i\leq n-1} \quad (\vdash\vee)\dfrac{\begin{array}{c}\Pi_n\\ X_n\vdash F\end{array}}{X_n\vdash F\vee G} \quad (\vee\vdash)\dfrac{\begin{array}{cc}\Pi'_{n+1} & \Pi''_{n+1}\\ F_1,\ldots,F_{n-1},F,Y\vdash H & F_1,\ldots,F_{n-1},G,Y\vdash H\end{array}}{F_1,\ldots,F_{n-1},F\vee G,Y\vdash H}}{X_1,\ldots,X_n,Y\vdash H},$$

we replace Π with the following derivation Π':

$$(mc)\frac{\left(\begin{array}{c}\Pi_i\\ X_i\vdash F_i\end{array}\right)_{i\leq n-1} \quad \begin{array}{c}\Pi_n\\ X_n\vdash F\end{array} \quad \begin{array}{c}\Pi'_{n+1}\\ F_1,\ldots,F_{n-1},F,Y\vdash H\end{array}}{X_1,\ldots,X_n,Y\vdash H}.$$

For Π' we have the following:

$r_{\mathbf{D}}(\Pi') \leq r_{\mathbf{D}}(\Pi)$
$r_{\mathbf{C}}(\Pi') < r_{\mathbf{C}}(\Pi)$.

Since $F\vee G$ is implication-free, F is implication-free, so that the induction hypothesis can be applied.

6.2. If Π has the form

$$(mc)\frac{\left(\begin{array}{c}\Pi_i\\ X_i\vdash F_i\end{array}\right)_{i\leq n-1} \quad (\vdash P)\dfrac{\begin{array}{c}\Pi_n\\ X_n\vdash F\end{array}}{X_n\vdash A} \quad (P\vdash)\dfrac{\left(\begin{array}{c}\Pi_G\\ F_1,\ldots,F_{n-1},G,Y\vdash H\end{array}\right)_{G\in\mathbf{D}(A)}}{F_1,\ldots,F_{n-1},A,Y\vdash H}}{X_1,\ldots,X_n,Y\vdash H},$$

we replace Π with the following derivation Π':

$$(mc)\frac{\left(\begin{array}{c}\Pi_i\\ X_i\vdash F_i\end{array}\right)_{i\leq n-1} \quad \begin{array}{c}\Pi_n\\ X_n\vdash F\end{array} \quad \begin{array}{c}\Pi_F\\ F_1,\ldots,F_{n-1},F,Y\vdash H\end{array}}{X_1,\ldots,X_n,Y\vdash H}.$$

Since $F \in \mathbf{D}(A)$, Π_F must be among the Π_G. Obviously, $r_{\mathbf{D}}(\Pi') < r_{\mathbf{D}}(\Pi)$, since one application of $(P\vdash)$ disappears. Furthermore, due to the restriction on P, F is implication-free. So the induction hypothesis can be applied to Π' (see remark 2 below).

□

Remarks on the proof of Theorem 2

1. Since by means of a multicut we reduce several cuts in a single step and do not have the problem of subsequent applications of cuts, we do not require in the induction hypothesis that the **D**-rank be not increased when a cut is eliminated. For example, in case 1 it is always increased if the **D**-rank of Π_1 is not 0.

2. It is instructive to see what would happen in the presence of implication: Consider just the case with a single minor premiss, where a Π of the form

$$(mc)\frac{(\vdash\to)\dfrac{\begin{array}{c}\Pi_1\\ X,F\vdash G\end{array}}{X\vdash F\to G} \quad (\to\vdash)\dfrac{\begin{array}{c}\Pi_2\\ Y\vdash F\end{array} \quad \begin{array}{c}\Pi_3\\ Z,G\vdash H\end{array}}{Y,Z,F\to G\vdash H}}{X,Y,Z\vdash H}$$

would have to be replaced with the following derivation Π':

$$(mc)\frac{\left.(mc)\dfrac{\begin{array}{c}\Pi_2\\ Y\vdash F\end{array} \quad \begin{array}{c}\Pi_1\\ X,F\vdash G\end{array}}{X,Y\vdash G}\right\}\Pi_1' \quad \begin{array}{c}\Pi_3\\ Z,G\vdash H\end{array}}{X,Y,Z\vdash H}.$$

The fact that here we have two subsequent cuts causes no problem, since for the **D**-rank only the major premiss is relevant. However, the **D**-rank of Π_1' may now be greater than that of Π, since for the **D**-rank of Π the subderivation Π_1 of Π , as occurring in a derivation of a minor premiss, is irrelevant, whereas for the **D**-rank of Π_1' the subderivation Π_1 of Π_1', as being a derivation of a major premiss, has to be taken into account. In the reduction of an implicational cut-formula, minor and major premisses change place. The presence of implication forbids an asymmetric induction measure for which only one of the cut premisses would be counted in the computation of the **D**-rank.

Since (Cut_0) is a special case of (mc), and since by definition its cut-formula is atomic, we immediately have the following corollary:

Corollary 1 Suppose P is implication-free. Then in the system without (Cut), the atomic cut rule (Cut_0) is admissible.

□

Theorem 3 Suppose P is implication-free. Then in the system without (Cut), (Cut) is admissible.

Proof We perform the usual cut-elimination procedure for intuitionistic first-order logic with the difference that we apply Corollary 1 when we encounter an atomic cut.

□

Theorems 1 and 3 show that with definitional reflection contraction and implication exclude each other, if one wants to have cut elimination. The elimination of implicational cuts requires a symmetric induction measure which would be blown up by contraction. On the other hand, the elimination of contraction works with an asymmetric induction measure which is ruled out by implication.[4]

5. Remarks on other logical systems

We give some hints on how our results carry over to logics with other structural postulates.

Relevance logic

There is a great variety of systems of relevance logic. We only consider those which immediately fit into our framework. The easiest one is obtained by taking away the thinning rule $(Thin)$. An inspection of proofs shows that all our results remain valid: Thinning is not involved in the negative result (failure of cut-elimination for the system with contraction and with $\to$ in definitions). In the reductions of the cut-elimination proofs (Theorems 1 and 2), thinning is only used in cases where the original derivations apply $(Thin)$. Besides $(Thin)$, one might also drop $(\bot\vdash)$ in relevance logic. This, again, does not change our results. Here it proves useful that we have kept the $\top$- and $\bot$-rules separate from the P-rules (see the remarks on $\top$ and $\bot$ in Section 2). – The investigation of relevance logics in which the distribution law $A\&(B\vee C)\vdash(A\&B)\vee C$ holds, which in Gentzen-style proof theory would require a second structural operation besides the comma ([7], for references see [21]), is beyond the scope of the present investigation.

[4]In [13] (Part II, Proposition 1) we proved cut elimination for definite Horn clause programming with definitional reflection in a much simpler way than here in Theorem 3, which relies on the multicut elimination of Theorem 2. This was possible because there, for the purpose of logic programming, we considered a fragment in which the underlying logic just contains the implication rules (besides definitional reflection). Since in our treatment of Horn clauses we did not even use explicit conjunction in the bodies of clauses, each premiss in the body of a clause is of the same logical complexity as its head (namely atomic). - It might be added that in the proof given there the notion of length of derivations has to be understood as the cut rank in Gentzen's [10] sense (i.e., the number of occurrences of the cut-formula when proceeding upwards from a topmost cut), not just as the number of rule applications.

Linear logic

In linear logic we have neither thinning nor contraction. Therefore, according to the previous remarks on relevance logic, Theorem 2 holds for linear logic, too. That is, for linear logic with definitional reflection we have cut-elimination. However, this only holds if modal operators (exponentials) are not present. With exponentials, intuitionistic logic can be translated faithfully into linear logic ([11], pp. 78-82) , so that the failure of cut-elimination for intuitionistic logic with definitional reflection carries over to linear logic with exponentials. More precisely, if one defines in linear logic

$$!p \to \bot \Rightarrow p,$$

where $\to$ is now linear implication, then a derivation analogous to the example of Section 2 which demonstrates the invalidity of cut can be given: Both $\vdash !p$ and $!p \vdash \bot$ are then derivable, but not $\vdash \bot$. Note that in linear logic contraction is available for !-formulas (on the left side of the turnstile).

These remarks only apply to linear logic with a single formula on the right side of the turnstile, but can be carried over to the symmetric case (see below).

The Lambek calculus

In the Lambek calculus we have sequences rather than multisets as antecedents of sequents. We do not have a postulate of exchange (nor do we have thinning or contraction).[5] In this system we can distinguish between two implications with the $(\vdash *)$-rules

$$(\vdash /)\frac{X, F \vdash G}{X \vdash G/F} \qquad\qquad (\vdash \backslash)\frac{F, X \vdash G}{X \vdash F\backslash G}.$$

The $(*\vdash)$-rules for these and all other connectives have to be (re-)formulated carefully with attention being paid to the order of formulas and the way they are embedded in a context. For example, the cut-rule would now have to be formulated as

$$\frac{X \vdash F \quad Y, F, Z \vdash G}{Y, X, Z \vdash G}.$$

It can be seen that Theorem 2 holds for this case as well. This reflects the fact that for the Lambek-calculus without definitional reflection cut-elimination holds ([14, 16]), and that our induction measure is independent of the structural postulates assumed.

Logics with multiple formulas in the succedent

In a system whose sequents are of the form $X \vdash Y$ rather than $X \vdash F$, one would have to introduce a connective dual to $\circ$ ("par" in Girard's [11] terminology, here written "+"), as well as to consider rules for thinning and contraction on the right side of the turnstile. Nevertheless, as long as we work with multisets, all the results mentioned carry over to this case (including relevance and linear logic). One has to check all steps of the

[5] We are considering here only the associative case [14], not the even weaker version with a non-associative binary structural connective [15]

proofs in detail. For Theorem 1 this result is plausible, since our induction measure is symmetric and only reductions dual to those presented are added. For Theorem 2 we used an asymmetric induction measure, which in the definition of "**D**-rank" only refers to the major premiss of a multicut. However, the additional logical reduction for + works well with that measure. For simplicity, we present the case of a multicut with a single minor premiss, which is the same as a cut:

If Π has the form

$$(mc)\dfrac{(\vdash +)\dfrac{\begin{matrix}\Pi_1\\ X\vdash F, G, Y\end{matrix}}{X\vdash F+G, Y} \qquad (+\vdash)\dfrac{\begin{matrix}\Pi_2'\\ X', F\vdash Y'\end{matrix} \quad \begin{matrix}\Pi_2''\\ X'', G\vdash Y''\end{matrix}}{X', X'', F+G\vdash Y', Y''}}{X, X', X''\vdash Y, Y', Y''},$$

we replace Π with the following derivation Π':

$$(mc)\dfrac{\left.(mc)\dfrac{\begin{matrix}\Pi_1\\ X\vdash F, G, Y\end{matrix} \quad \begin{matrix}\Pi_2'\\ X', F\vdash Y'\end{matrix}}{X, X'\vdash G, Y, Y'}\right\}\Pi_1' \qquad \begin{matrix}\Pi_2''\\ X'', G\vdash Y''\end{matrix}}{X, X', X''\vdash Y, Y', Y''}.$$

Contrary to all other reductions, here a single multicut is replaced with two subsequent multicuts. However, due to the fact that only the major premiss of a multicut is relevant for the **D**-rank, we have $r_{\mathbf{D}}(\Pi_1') \leq r_{\mathbf{D}}(\Pi)$ and $r_{\mathbf{D}}(\Pi'') \leq r_{\mathbf{D}}(\Pi)$, where Π'' results from Π' by replacing Π_1' with a multicut-free derivation.[6]

For the Lambek calculus with multiple formulas in the succedent, there are problems with cut-elimination anyway, quite independent of definitional reflection, even if certain conditions are posed on the application of cut (see [1, 17]). However, it can be shown that cut is eliminable in the system with definitional reflection if the implication-rules are dropped (and no negation-rules are introduced, which would lead to the same problems).

Appendix

In an electronically distributed message of 5 February 1992, entitled "A Fixpoint Theorem for Linear Logic"[7], Girard proposes a logical system with an elimination rule for atoms similar to our rule of definitional reflection and states a theorem similar to our Theorem 1 above. It seems that his first presentention of these ideas was at the German Workshop on Artificial Intelligence (GWAI), Bonn, September 1991. This suggests some remarks on how Girard's system relates to ours.

1. Girard's idea of inverting introduction rules for atoms ("definitional reflection" in our terminology) is basically the same as ours (see our [Hallnäs' and my] reply to Girard of 19 February 1992[8]).

[6]In the presence of thinning and contraction, it is sufficient just to consider additive connectives. However, since we are also considering the relevant case where thinning is absent, we have to take into account the multiplicatives even for Theorem 2.

[7]Linear logic mailing list (linear@cs.stanford.edu, set up by Patrick Lincoln)

[8]Same mailing list

2. In our notation, for a system with a single formula in the succedent and a definition P, Girard's "elimination rule" would be written as

$$(ER)\frac{(X\sigma, F\sigma \vdash G\sigma)_{F \Rightarrow B \in P \text{ and } \sigma = mgu(B,A)}}{X, A \vdash G}.$$

The rule (ER) is equivalent to our rule in the propositional case (i.e., without variables). In the general case, it is stronger than $(P\vdash)$ in the sense that $(P\vdash)$ can be derived (without cut) if (ER) is present. However, when it comes to the computation of bindings, where rules have to be read backwards in some reasonable sense, (ER) is less useful than $(P\vdash)$. For the computation of bindings it is essential that a single substitution is computed at each step. This means that only in the one-premiss case (with a single σ), (ER) can be used to generate a binding. With $(P\vdash)$, there is no such restriction. To obtain a binding, we compute a minimal substitution θ such that for $A\theta$ the proviso for the application of $(P\vdash)$ is satisfied (for details see [13]). Therefore, *computationally,* $(P\vdash)$ is more powerful than (ER). This is why we rejected a rule like (ER) in 1987, when we considered "definitional reflection" as the basis for an extended logic programming language and chose $(P\vdash)$ instead.

Deductively, of course, (ER) is a very interesting rule which has to be further investigated. So far, it has been considered by Eriksson ([8], p. 102). Philosophically speaking, (ER) and $(P\vdash)$ differ in the interpretation of variables in the atom A.

3. For Girard, the cut-elimination theorem for contraction-free systems with definitional reflection (Theorem 1 above) is fundamental, since he considers the admissibility of cut a matter of principle. Therefore he works throughout with a contraction-free logic (in fact, linear logic), whereas for us the failure of cut is a sign of partiality (see Section 1 above). Concerning his proof, Girard gives some hints for the treatment of exponentials and furthermore refers to the small normalization theorem of [11] (p. 71). It is not immediately obvious how to carry over this proof to our logical system. Our proof is different from that.

Acknowledgements

Besides the Berlin conference, the results reported here were presented in part at the Second and at the Third International Workshop on Extensions of Logic Programming (January 1991, Stockholm, and February 1992, Bologna). The conceptual framework including the exact formulation of the $(P\vdash)$-rule is due to both Lars Hallnäs and myself. The basic idea of definitional reflection (assuming atomic formulas in a nontrivial way) and also the term "definitional reflection" has to be credited to Lars alone. I would like to thank Kosta Došen, Lars Hallnäs, David Pearce and Hedwig Schmucker for helpful remarks and suggestions and Muriel Quenzer for assistance with typing.

References

[1] Abrusci, V. M. Phase semantics and sequent calculus for pure noncommutative classical linear propositional logic. *Journal of Symbolic Logic*, **56** (1991), 1403–1451.

[2] Ackermann, W. Widerspruchsfreier Aufbau der Logik.I. Typenfreies System ohne tertium non datur. *Journal of Symbolic Logic*, **15** (1950), 33–57.

[3] Aronsson, M., Eriksson, L.-H., Gäredal, A., Hallnäs, L. & Olin, P. The programming language GCLA: A definitional appraoch to logic programming. *New Generation Computing*, **4** (1990), 381–404.

[4] Clark, K. L. Negation as failure. In: Gallaire, H. & Minker, J. (Eds.), *Logic and Data Bases*, New York 1978, 293–322.

[5] Curry, H. B. The inconsistency of certain formal logics. *Journal of Symbolic Logic*, **7** (1942), 115–117.

[6] Došen, K. A historical introduction to substructural logics. In: Došen, K. & Schroeder-Heister, P. (Eds.) *Substructural Logics: Proceedings of the Conference held in Tübingen, October 1990.* Oxford University Press, 1992.

[7] Dunn, M. Consecution formulation of positive R with co-tenability and t. In: Anderson, A.R. & Belnap, N.D., *Entailment: The Logic of Relevance and Necessity*, Vol. I, Princeton University Press, 1975, 381–391.

[8] Eriksson, L.-H. A finitary version of the calculus of partial inductive definitions. In: Eriksson, L.-H., Hallnäs, L. & Schroeder-Heister, P. (Eds.), *Extensions of Logic Programming. Second International Workshop, ELP-91, Stockholm, January 1991, Proceedings.* Springer LNCS, Vol. 596, Berlin 1992, 89–134.

[9] Fitch. F. B. A system of formal logic without an analogue to the Curry W operator. *Journal of Symbolic Logic*, **1** (1936), 92–100.

[10] Gentzen, G. Untersuchungen über das logische Schließen. *Mathematische Zeitschrift*, **39** (1935), 176–210, 405–431, English translation in: M.E. Szabo (ed.), *The Collected Papers of Gerhard Gentzen*, Amsterdam: North Holland 1969, 68–131.

[11] Girard, J.-Y. Linear logic. *Theoretical Computer Science*, **50** (1987), 1–102.

[12] Hallnäs, L. Partial inductive definitions. *Theoretical Computer Science*, **87** (1991), 115-142. Previous versions published as SICS Research Reports 86005 (1986) and 86005C (1988), and in: Avron, A. et al. (Eds.), Workshop on General Logic, Dept. of Computer Science, University of Edinburgh, Report ECS-LFCS-88-52 (1988).

[13] Hallnäs, L. & Schroeder-Heister, P. A proof-theoretic approach to logic programming. I. Clauses as rules. *Journal of Logic and Computation*, **1** (1990), 261–283; II. Programs as definitions, *ibid.* **1** (1991), 635–660. Previous version published at SICS Research Report 88005, 1988.

[14] Lambek, J. The mathematics of sentence structure. *American Mathematical Monthly*, **65** (1958), 154–170.

[15] Lambek, J. On the calculus of syntactic types. In: R. Jacobson (Ed.), *Proceedings of the 12th Symposium on Applied Mathematics*, AMS, Providence 1961, 166–178, 264–265.

[16] Lambek, J. Logic without structural rules (Another look at cut elimination). In: Došen, K. & Schroeder-Heister, P. (Eds.) *Substructural Logics: Proceedings of the Conference held in Tübingen, October 1990.* Oxford University Press, 1992.

[17] Lambek, J. From categorial grammar to bilinear logic. In: Došen, K. & Schroeder-Heister, P. (Eds.) *Substructural Logics: Proceedings of the Conference held in Tübingen, October 1990.* Oxford University Press, 1992.

[18] Lorenzen, P. *Einführung in die operative Logik und Mathematik.* Springer: Berlin 1955, 2nd ed. Berlin 1969.

[19] Martin-Löf, P. Hauptsatz for the intuitionistic theory of iterated inductive definitions. In: Fenstad, J. E. (Ed.), *Proceedings of the Second Scandinavian Logic Symposium*, North-Holland: Amsterdam 1971, 179–216.

[20] Nadathur, G. & Miller, D. An overview of λ-Prolog. In: Kowalski, R. & Bowen, K. (Eds.), *Fifth International Conference on Logic Programming*, MIT Press 1988, 810–827.

[21] Read, S. *Relevant Logic: A Philosophical Examination of Inference.* Oxford: Basil Blackwell, 1988.

[22] Schroeder-Heister, P. Hypothetical reasoning and definitional reflection in logic programming. In: P. Schroeder-Heister (Ed.), *Extensions of Logic Programming. International Workshop, Tübingen, FRG, December 1989, Proceedings.* Springer LNCS, Vol. 475, Berlin 1991, 327–340.

[23] Slaney, J. Solution to a problem of Ono and Komori. *Journal of Philosophical Logic*, **18** (1989), 103–111.