# On the Notion of *Assumption* in Logical Systems *

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## 1. The asymmetry between assumptions and assertions

When a logical system is specified and the notion of a derivation or formal proof is explained, we are told (i) which formulas can be used to start a derivation and (ii) which formulas can be derived given that certain other formulas have already been derived. Formulas of the sort (i) are either *assumptions* or *axioms*, formulas of the sort (ii) are *conclusions* of (proper) *inference rules*. Axioms may be viewed as conclusions of (improper) inference rules, viz. inference rules without premisses. In what follows I refer to conclusions of proper or improper inference rules as *assertions*.[1] In natural deduction systems, inference rules deal both with assumptions and assertions, as the assumptions on which the conclusion of an inference rule depends, are not necessarily given by the collection of all assumptions on which the premisses depend, in case the rule permits the discharging of assumptions. For example, the rule of implication introduction

$$\frac{\begin{matrix}[A]\\ B\end{matrix}}{A\to B}\ (\to I)$$

enables us to derive $A\to B$, given a derivation of $B$, where the resulting $A\to B$ no longer depends on $A$, if $B$ depends on $A$.[2]

Axioms and rules of inference are, of course, specified by reference to the *form* of the formulas involved. Normally, not everything can be derived, but only formulas of a certain kind. For example, *modus ponens*

$$\frac{A\to B \quad A}{B}$$

says that a formula $B$ can be derived if formulas of the forms $A\to B$ and $A$, respectively, have been derived. This is different with assumptions. There normally *any* formula whatsoever can serve as an assumption. The rule of assumption introduction merely says: State $A$ as an assumption. I call this the *unspecific* way of introducing an assumption, as the form of $A$ is not specified. At the same time, it is an unspecific way of introducing an assertion, namely by introducing $A$ as depending on itself.

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* I should like to thank the attendees of my talk at the GAP.5 conference in Bielefeld for their helpful comments and suggestions.

[1] More precisely, assertions are conclusions of *applications* of inference rules, as an assertion occurs at a certain place within a derivation.

[2] I use the tree-style variant of natural deduction as proposed by Gentzen (1934) and investigated by Prawitz (1965).

However, the important point is that for assumptions there is *only* this trivial way of introducing them, whereas for assertions this trivial step is one amongst others. Besides unspecifically and trivially asserting $A$ as depending on itself, there are normally several *specific* ways of asserting a formula as an axiom or a conclusion of an inference rule.

This is an issue which not only concerns formal systems, but reasoning in general, as formal systems like natural deduction are intended to capture and codify reasoning. Reasoning is assertion-oriented: Starting with assumptions, which may be chosen arbitrarily, we proceed towards assertions, which are specified by the inference rules given.

This asymmetry is reflected both within the Curry-Howard interpretation of natural deduction and within proof-theoretic semantics in Prawitz's style. According to the Curry-Howard interpretation[3], rather than simply proving $A$, we proceed by proving a more complex judgement $t : A$ for some term $t$ codifying the proof of $A$. The form of $t$ varies with the form of the inference rule applied. The crucial point here is that assumptions are represented by means of variables, i.e. if $A$ is asserted by placing $A$ as an assumption, this is expressed by using

$$x : A$$

as a context. The variable $x$ indicates that *nothing specific* is required of an assumption $A$, in contradistinction to a judgement

$$t : A$$

made on the basis of other (non-trivial) rules, which manipulate the form of the term $t$. The variable $x$ can of course be replaced by any term $t$ representing a proof of $A$, which expresses the view that assumptions are *placeholders* for proofs, whereas assertions are *results* of proofs.

In Prawitz-style proof-theoretic semantics[4], the validity of proofs is defined in terms of the complexity of the end formula (= asserted formula), starting with closed (= assumption free) proofs as a basis. It presupposes the notion of normalizing procedures which allow the reduction of closed proofs to closed *canonical* proofs, where a proof is called canonical if it uses an introduction rule in the last step. Then the definition of validity with respect to an atomic system $S$, whose proofs are considered as valid outright, runs as follows:

*(I) Every closed proof in S is S-valid.*

*(II) A closed canonical proof (proof in I-form) of a complex formula is S-valid, if its immediate subderivations are S-valid.*

[3] For natural deduction as based on the Curry-Howard interpretation see de Groote (1995) and Troelstra & Schwichtenberg (1996).

[4] See Prawitz (1974, 2005) and the detailed discussion with references in Schroeder-Heister (2005). For the general concept of proof-theoretic semantics see Kahle & Schroeder-Heister (2005).

*(III) A closed non-canonical proof is S-valid, if it* reduces *to an S-valid canonical proof.*

*(IV) An open proof* $\begin{array}{c} A_1 \ldots A_n \\ \mathcal{D} \\ B \end{array}$*, where all open assumptions of* $\mathcal{D}$ *are among* $A_1, \ldots, A_n$*,*

*is S-valid, if for every* $S' \geq S$ *and for every list of closed* $S'$*-valid* $\begin{array}{c} \mathcal{D}_i \\ A_i \end{array}$ $(1 \leq i \leq n)$,

$$\begin{array}{c} \mathcal{D}_1 \quad \mathcal{D}_n \\ A_1 \ldots A_n \\ \mathcal{D} \\ B \end{array} \text{ is } S'\text{-valid.}$$

Important in the present context is the last clause, which basically says that assumptions are *placeholders* for *closed* proofs. This is exactly the asymmetry I am claiming to hold: An assumption is an unspecific placeholder, whereas an assertion is specified by its meaning (which is determined by its form).

More generally, this idea underlies the *constructive view* of proofs, which is inherent in the Brouwer-Heyting-Kolmogorov (BHK) interpretation of intuitionistic logic[5], according to which a proof from assumptions is a construction which transforms a proof of the assumptions into a proof of the assertion. Assumptions are then nothing but argument places of constructive functions.

Thus the central observation of this section is the following: *The standard notion of proof is assertion-driven.*

## 2. The sequent calculus

There is no a priori reason for why assumptions should be treated in the same way as assertions. Reasoning might be inherently asymmetric with respect to these two concepts. However, a symmetric model of formal proofs is already available. This is the model of the sequent calculus.[6]

Although formally, the sequent calculus is a system for assumption-free reasoning with sequents as assertions, the rules for asserting sequents are symmetric. There are rules for introducing formulas in the antecedent as well as in the succedent of a sequent. Taking the intuitionistic sequent calculus with a single formula in the succedent as a basis, then, for example, the rules for conjunction run as follows:

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[5] See Troelstra & Schwichtenberg (1996), Ch. 2.5.1, and references therein.

[6] See Gentzen (1934) and Troelstra & Schwichtenberg (1996).

$$\frac{\Gamma \vdash A \quad \Delta \vdash B}{\Gamma, \Delta \vdash A \wedge B}\,(\vdash \wedge) \qquad \frac{\Gamma, A \vdash C}{\Gamma, A \wedge B \vdash C} \quad \frac{\Gamma, B \vdash C}{\Gamma, A \wedge B \vdash C}\,(\wedge \vdash)\,.$$

If natural deduction is presented in sequent style, coding a derivation of $A$ from $\Gamma$ by means of the sequent $\Gamma \vdash A$, then the formulas $\Gamma$ in the antecedent can be viewed as assumptions and the formula $A$ in the succedent as an assertion. The introduction of $\wedge$ in the succedent $(\vdash \wedge)$ is the usual and-introduction rule, but the rule $(\wedge \vdash)$ is different. Whereas in sequent-style natural deduction there should be $\wedge$-elimination rules of the form

$$\frac{\Gamma \vdash A \wedge B}{\Gamma \vdash A} \quad \frac{\Gamma \vdash A \wedge B}{\Gamma \vdash B}\,,$$

the left-introduction rule $(\wedge \vdash)$ governs the introduction of an assumption, if the turnstile $\vdash$ is seen as representing deducibility. $(\wedge \vdash)$ can be paraphrased as follows: Suppose we can already claim $C$ under the assumption $A$ (and other "parametric") assumptions $\Gamma$, then we can claim $C$ under the assumption $A \wedge B$, and the same with $B$ instead of $A$. This reading of the left-introduction rules interprets them as *specific* assumption introduction rules, as the form of the left-introduction rule depends on the form of the formula being introduced (in the example its being a conjunction). There is also the unspecific way of assumption introduction via initial sequents

$$A \vdash A$$

by means of which a derivation can be started. An initial sequent introduces an assumption $A$ and at the same time the assertion $A$ as depending on $A$, irrespective of the form of $A$. According to this reading, the sequent calculus provides both the means of introducing an assumption in an unspecific way (by an initial sequent) as well as in a specific way according to its form (by a left-introduction rule).

My interpretation of the left-introduction rules deviates from the standard way the sequent calculus is understood in relation to natural deduction. Gentzen, who invented the sequent calculus, intended it to be a technical device created in order to have a calculus without proper assumptions as in natural deduction. That is why he called it a "logistic" calculus (and used the designations "**LI**" and "**LK**"), where by "logistic" he means a calculus in the sense of the Hilbert-style tradition of calculi without assumptions.[7] In a similar vein, Prawitz interprets it as a *metacalculus*, which expresses the "vertical" dependency of a formula from assumptions by means of the sequent sign. According to Prawitz, a left-

[7] See Gentzen (1934), pp. 184, 190. At that time, the term "logistic" was still in use as a denotation of modern symbolic logic, as distinguished from pre-Fregean traditional logic. This term was originally proposed by Couturat, Itelson and Lalande in 1904 (see Gabriel 1984) and used by Carnap (1929) and Quine (1934) as a textbook title.

introduction rule corresponds to the idea of an upwards extension of a natural deduction proof, i.e.,

$$\frac{\Gamma, A \vdash C}{\Gamma, A \wedge B \vdash C}$$

is interpreted as

$$\begin{array}{c} \dfrac{A \wedge B}{A} \\ \vdots \\ C \end{array}$$

(see Prawitz 1965, pp. 91). Prawitz did not consider the idea of such an extension as a primitive operation, which might constitute a variant of natural deduction, which would correspond to the idea of specific assumption introduction as propagated here.

The sequent calculus is not a faithful metacalculus of natural deduction in the sense that every inference step in natural deduction is represented by an inference step in the sequent calculus. If the sequent calculus is to be a metacalculus, it is a metacalculus of a different form of natural deduction. Specifying such a natural deduction system means understanding the sequent calculus as a calculus in its own right, thus taking a symmetric stance with respect to assumptions and assertions. I call this system the *natural-deduction-style sequent calculus*, in short: *ND-style sequent calculus*.

### 3. ND-style sequent calculus

In sequent-style natural deduction, a sequent

$$\Gamma \vdash A$$

represents a derivation

$$\begin{array}{c} \Gamma \\ \vdots \\ A \end{array}$$

of $A$ from assumptions $\Gamma$. In the ND-style sequent calculus, derivations are modelled in a natural deduction framework according to the rules of the sequent calculus. I speak of an ND-style sequent calculus, because what is being used are the rules of the sequent calculus, even if they are written down in a natural deduction format with

$$\begin{array}{c}\Gamma\\ \vdots\\ A\end{array}$$

standing for

$\Gamma \vdash A$.[8]

An ND-style system, which faithfully represents the sequent calculus, allows the introduction of assumptions in the course of a derivation and not only at the beginning. For example, the ND-style rule representing the left-introduction-rule

$$\frac{\Gamma,\, A \vdash C}{\Gamma,\, A \wedge B \vdash C}$$

is

$$\frac{\overline{A \wedge B} \quad \begin{array}{c}[A]\\ C\end{array}}{C}\ (\wedge\overline{E})\,,$$

which is to be read as: If we have derived $C$ from the assumption $A$, then we may introduce the assumption $A \wedge B$ and assert $C$ under this assumption. In contradistinction to standard natural deduction, $A \wedge B$, being the major premiss of an $\wedge$-elimination rule[9], *may only occur in top position*. As $A \wedge B$ is introduced as an assumption by means of this rule, it is not allowed to be the conclusion of another rule at the same time. This is indicated by the bar over $A \wedge B$.

Essential for the ND-style sequent calculus is the usage of *generalized elimination rules*, which bring all elimination rules in line with "indirect" rules such as $\vee$-elimination. The generalized $\wedge$-elimination rules are

8 My terminology is different from that of Negri & von Plato (2001a; 2001b, Ch. 5.2). By "sequent calculus in natural deduction style" they denote a system having the format of a *sequent calculus* in which certain features of natural deduction systems concerning the discharging of assumptions are incorporated. By "ND-style sequent calculus" I mean a system which has the format of a *natural deduction system* (two-dimensional notation etc.), but uses inference rules corresponding to those of the sequent calculus (right and left introduction rules). I chose this terminology in analogy with the common "sequent-style natural deduction". Derivations in sequent-style natural deduction are derivations in a natural deduction framework (with introduction and elimination rules), but represented in the format of a sequent calculus. Correspondingly, derivations in the "ND-style sequent calculus" are derivations in a sequent calculus framework, but represented in natural deduction format.

9 More precisely, of an $\wedge$-elimination rule in generalized form, see the next paragraph.

$$\frac{A\wedge B \quad \begin{matrix}[A]\\ C\end{matrix}}{C} \quad \frac{A\wedge B \quad \begin{matrix}[B]\\ C\end{matrix}}{C} \quad \text{or} \quad \frac{A\wedge B \quad \begin{matrix}[A \quad B]\\ C\end{matrix}}{C} \quad ,$$

depending on the structural features of the system considered, while for implication the generalized elimination rule is

$$\frac{A\to B \quad A \quad \begin{matrix}[B]\\ C\end{matrix}}{C} \quad .$$

They can be shown to be equivalent to the standard "direct" elimination rules in natural deduction which are

$$\frac{A\wedge B}{A} \quad \frac{A\wedge B}{B} \quad \text{and} \quad \frac{A\to B \quad A}{B} \, ,$$

respectively. If we allow for the major premisses of generalized elimination rules to occur only in top position, we obtain the ND-style sequent calculus.[10] Prawitz's idea of upwards extensions of natural deduction proofs should lead to a system of equal strength, but deviates from the usual framework of presenting rules.

I do not want to go into details of how the ND-style sequent calculus can be framed. For that, the reader is referred to Tennant (1992) and von Plato (2001). What is crucial for me here is that the sequent calculus, apart from being a useful technical device, represents a novel idea of deduction, which has not yet been fully appreciated philosophically.

It might be mentioned that, when presented in natural deduction style, *cut* and *cut elimination* receive a new reading. The rule of cut

$$\frac{\Gamma \vdash A \quad \Delta, A \vdash B}{\Gamma, \Delta \vdash B}$$

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[10] The history of this system is the following. The idea of generalized elimination, which is a crucial ingredient of the system, was developed and investigated in Schroeder-Heister (1984a,b), following earlier work by Prawitz (1978) and von Kutschera (1968). For $\to$-elmination, a more general version with higher-level assumptions was proposed, rather than the variant above which directly corresponds to the left-introduction rule for $\to$ in the sequent calculus. The generalized elimination rules became part of logical frameworks and related systems (see Harper, Honsell & Plotkin 1993, Basin & Matthews 2002). The idea of a natural deduction system with generalized elimination rules and major premisses of elimination rules occurring only in top position was considered by the author shortly after 1984, but never published. It was independently discovered by Tennant (1992) and von Plato (2001) (see also Tennant 2002). Tennant (1992) quotes Paulson (1989) for the generalized ("parallel") $\wedge$-elimination rules (who himself refers to Schroeder-Heister 1984a,b). Von Plato (2001) was not aware of Tennant (1992). Neither Tennant (1992) nor von Plato (2001) mention Prawitz's (1978) presentation of generalized elimination rules.

now means that derivations $\begin{array}{c}\mathcal{D}\\A\end{array}$ and $\begin{array}{c}A\\\mathcal{D}'\\B\end{array}$ can be combined to a derivation $\begin{array}{c}\mathcal{D}\\A\\\mathcal{D}'\\B\end{array}$. That such a combination is possible, i.e., that *transitivity of deduction* holds, is no longer a trivial matter, but something in need of a proof, corresponding to cut elimination. This issue will be pursued in Section 5 below.

## 4. The challenge from logic programming: definitional reflection

The discussion so far focused on logical constants. However, the issues of right and left introduction rules in the sequent calculus and the notion of assumption apply to a more general context. Let us consider definite clauses as in logic programming, i.e., expressions of the form

(1) $A \Leftarrow B_1, \ldots, B_n$ .

Whereas in the standard theory of logic programming such clauses are read as disjunctions of the form $A \lor \neg B_1 \lor \ldots \lor \neg B_n$, they may as well be read as *inference rules*, which actually would sound quite natural for a PROLOG programmer. This suggests a proof-theoretic treatment, according to which a clause of the form (1) generates an introduction rule

$$\frac{B_1 \quad \ldots \quad B_n}{A}$$

in a natural deduction framework, and a right-introduction rule

$$\frac{\Gamma_1 \vdash B_1 \quad \ldots \quad \Gamma_n \vdash B_n}{\Gamma_1, \ldots, \Gamma_n \vdash A}$$

in a sequent framework. Clauses of the form (1) can be generalized to the case where $A$ is still an atom, but the $B_i$ are arbitrary first-order formulas, in particular formulas which contain implications.[11] Now it is only natural to extend this proof-theoretic procedure and equip the sequent system with left-introduction rules. This leads to the theory of *definitional reflection*, which was first proposed by Hallnäs und has been further developed and investigated by Lars Hallnäs and the author (Hallnäs 1991, Schroeder-Heister 1993). Sup-

[11] More precisely, such generalized clauses can be given a *declarative* sense. For *computational* purposes they would have to be restricted to a language based on conjunction, implication and universal quantification. See Hallnäs & Schroeder-Heister (1990, 1991).

pose a program of the form

$$(2)\quad \left\{ \begin{array}{lcl} a_1 & \Leftarrow & B_{11} \\ & \vdots & \\ a_1 & \Leftarrow & B_{1k_1} \\ & \vdots & \\ a_n & \Leftarrow & B_{n1} \\ & \vdots & \\ a_n & \Leftarrow & B_{nk_n} \end{array} \right.$$

is given, where the lower case letters represent atoms and the $B_{ij}$ represent formulas of propositional logic. Then this program not only gives rise to the right-introduction rule

$$\frac{\Gamma \vdash B_{ij}}{\Gamma \vdash a_i}\ (\vdash a_i)$$

for every $i$, but also to the left-introduction rule

$$\frac{\Gamma, B_{i1} \vdash C \quad \dots \quad \Gamma, B_{ik_i} \vdash C}{\Gamma, a_i \vdash C}\ (a_i \vdash)\ .$$

This rule allows the introduction of $a_i$ into the antecedent of a sequent at a place, which before was occupied by the premisses of $a_i$ according to the program (2). It represents a generalized rule of assumption introduction, which works for atomic formulas in a way similar to the common left-introduction rules for logical constants. This principle is called *definitional reflection*, as it expresses the definitional reading of the program (2): Suppose the clauses for $a_i$ in (2) are considered as *defining* $a_i$, and suppose *no other* definitional clauses for $a_i$ are available. Then assuming $a_i$ means assuming that one of the definientia of $a_i$, i.e., one of the $B_{ij}$ holds. Thus everything derivable from each of those $B_{ij}$ must be derivable from $a_i$ itself. Definitional reflection is a way of interpreting the *extremal clause* in (normally inductive) definitions, which is often expressed by saying that "nothing else" defines the definiens.

In natural deduction style, the rule of definitional reflection is

$$\frac{\overline{a_i} \quad \begin{array}{c}[B_{i1}]\\ C\end{array} \ \dots\ \begin{array}{c}[B_{ik_i}]\\ C\end{array}}{C}\ (\overline{a_i E})\ ,$$

where, as in the previous section, the bar indicates that the major premiss $a_i$ must occur in top position. For more details on the theory of definitional reflection I refer to Hallnäs (1991), and Schroeder-Heister (1992, 1993, 1994b). Here, I should only mention that one of the advantages of dealing with arbitrary atoms rather than logically compound formulas is that we can now treat non-wellfounded systems of clauses (programs). These are cases where a backwards chain of definitions, starting with a defined atom $a$ and proceeding to the clauses defining the definientia of this atom, does not terminate after finitely many steps. Circular reasoning, as found with the paradoxes, is of that kind.

## 5. Cut and transitivity of deduction

The rule of cut

$$\frac{\Gamma \vdash A \quad \Delta, A \vdash B}{\Gamma, \Delta \vdash B}$$

is normally something that is expected to hold in the sequent calculus in the sense that it is admissible. For the ND-style sequent calculus cut would be read as a statement about deducibility, expressing transitivity of deduction. That this is not *eo ipso* true as in ordinary natural deduction, where it is built into the system, is due to the fact that major premisses of elimination rules can only occur in assumption position. The fact that in ordinary natural deduction, transitivity of deduction is a trivial matter, whereas in the ND-style sequent calculus it is not, implies that it is not trivial that every ordinary natural deduction derivation can be turned into a derivation in the ND-style sequent calculus. In fact, it can be shown that natural deduction derivations *in normal form* correspond to ND-style sequent derivations.[12] This corresponds to the fact that cut elimination is equivalent to normalization, or that normal derivations correspond to cut-free ones. Philosophically, my plea for dealing with assumptions in the manner of the sequent calculus corresponds to a plea for a *cut-free* system as preferable formalism. Taking the sequent calculus as a philosophically significant model of deduction is not as innocent as it may seem at first glance.

However, this makes good sense if the more general case with of definitions of atoms is considered. If such a definition is not well-founded, transitivity of deduction or cut cannot hold without further restriction. A standard example is the definition of an atom $a$ by its own negation $\neg a$:

$$a \Leftarrow \neg a \ ,$$

which may be considered as an abbreviation of, for example, the definition of $R \in R$ by means of $R \notin R$, if $R$ is the Russell set in naïve set theory. I assume that for negation the following standard rules of the sequent calculus are available[13]

$$\frac{\Gamma, A \vdash}{\Gamma \vdash \neg A}\ (\vdash \neg) \qquad \frac{\Gamma \vdash A}{\Gamma, \neg A \vdash}\ (\neg \vdash)$$

[12] More precisely, if *generalized* elimination rules are used for natural deduction, then derivations in the ND-style sequent calculus are normal forms of natural deduction derivations.

[13] Remember that only the intuitionistic case is considered here, with a single formula in the succedent.

or, in the ND-style sequent calculus,

$$\frac{\begin{array}{c}[A]\\ \bot\end{array}}{\neg A}\,(\neg I) \qquad \frac{\overline{\neg A} \quad A}{\bot}\,(\neg \overline{E})$$

for some absurdity constant $\bot$. Then, for $a$ the sequent rules are

$$\frac{\Gamma \vdash \neg a}{\Gamma \vdash a}\,(\vdash a) \qquad \frac{\Gamma, \neg a \vdash C}{\Gamma, a \vdash C}\,(a \vdash)$$

or, in the ND-style sequent calculus,

$$\frac{\neg a}{a}\,(aI) \qquad \frac{\overline{a} \quad \begin{array}{c}[\neg a]\\ C\end{array}}{C}\,(a\overline{E})\,.$$

This yields the following derivations in the sequent calculus

$$\cfrac{\cfrac{\cfrac{\cfrac{a \vdash a}{a, \neg a \vdash}}{a \vdash}\,(a \vdash)}{\vdash \neg a}}{\vdash a}\,(\vdash a) \qquad \cfrac{\cfrac{a \vdash a}{a, \neg a \vdash}}{a \vdash}\,(a \vdash)$$

or, in the ND-style sequent calculus,

$$\cfrac{\cfrac{\cfrac{\overset{(2)}{[a]} \quad \cfrac{\overset{(1)}{[\neg a]} \quad \overset{(2)}{[a]}}{\bot}}{\bot}\,(a\overline{E})\ (1)}{\neg a}\,(2)}{a}\,(aI) \qquad \cfrac{a \quad \cfrac{\overset{(1)}{[\neg a]} \quad a}{\bot}}{\bot}\,(a\overline{E})\ (1)\,.$$

If we were entitled to apply cut (within the sequent calculus) or transitivity of deduction (within the ND-style sequent calculus), we would obtain a contradiction in the form of the empty sequent $\vdash$ or absurdity $\bot$, respectively. However, since there is no derivation of a contradiction (as there is no rule of inference available by means of which $\vdash$ or $\bot$ could be proved), cut and transitivity of deduction fail to hold in this case.[14]

[14] Likewise, we cannot extend the right derivation (of $\bot$ from $a$) to

As cut corresponds to normalization in standard natural deduction, this result corresponds to the fact that derivations of absurdity generated by the paradoxes are not normalizable, which was already observed by Prawitz (1965, Appendix B). From a philosophical point of view, the loss of normalizability is not counterintuitive. That paradoxes generate no direct or straightforward derivations appears even natural, as the paradoxes are based on“artificial” and “peculiar” constructions. However, transitivity of deduction would usually be seen as a much more fundamental principle than normalizability, which has a rather technical stance. From that point of view, it is worthwhile to consider possibilities of regaining transitivity within our model of deduction. This is possible if we return to the distinction between specific and unspecific assumptions, which was the starting point of this paper.

## 6. Treating specific and unspecific assumptions differently

If the distinction between specific and unspecific assumptions is to provide additional expressive power, specific and unspecific assumptions must function differently. Specific assumptions are assumptions introduced *according to their meaning*, whereas unspecific assumptions are nothing but placeholders. In the following I sketch three approaches which take this difference into account. With respect to each proposal, cut and transitivity of deduction turn out to be valid. Therefore, if one of these proposals is accepted, there is no need to question cut or transitivity.

### 6.1 Kreuger's restriction

In the context of definitional reflection, Kreuger (1994) proposed a restriction which can be adapted to the present context as follows:

> *The unspecific introduction of an assumption is only allowed if no specific way of introducing this assumption is defined.*

Philosophically or, more precisely, semantically, this may be paraphrased as

> *If a meaning rule is specified for a statement, then this statement can be introduced as an assumption only according to this meaning rule.*

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$$\dfrac{\dfrac{\overset{(2)}{[a]} \qquad \dfrac{\overset{(1)}{[\neg a]} \quad \overset{(2)}{[a]}}{\bot}}{\bot}\,(a\overline{E})\;(1)}{\neg a}\,(2)$$

(which is the same as the left derivation without its last step), and then, together with the left derivation of $a$, apply negation elimination to obtain $\bot$. Such a derivation would violate the requirement that, in order to apply $(\neg\overline{E})$ to $\neg a$, $\neg a$ must stand in top position.

In the sequent calculus this means that an initial sequent

$$A \vdash A$$

can only be used if there is no specific (i.e., left-introduction) rule for $A$. In the logical case (without definitional reflection) this corresponds to the requirement that $A$ be atomic. For example, if $A$ has the form $A_1 \wedge A_2$, then instead of using $A \vdash A$, we can introduce $A$ by specific rules, reducing initial sequents to the case of $A_1$ and $A_2$:

$$\dfrac{\dfrac{A_1 \vdash A_1}{A_1 \wedge A_2 \vdash A_1} \quad \dfrac{A_2 \vdash A_2}{A_1 \wedge A_2 \vdash A_2}}{A_1 \wedge A_2 \vdash A_1 \wedge A_2} \quad .$$

Actually, in presentations of the sequent calculus, such a restriction is often made. In the logical case, this is a matter of technical convenience, as nonatomic initial sequents can always be reduced to atomic ones. Kreuger's restriction develops its power in non-wellfounded cases of definitional reflection, where the chain of clausal definitions starting with a defined atom does not terminate. Considering the circular definition

$$a \Leftarrow \neg a \ ,$$

then, by Kreuger's restriction, we cannot start a derivation with

$$a \vdash a \ ,$$

as $a$ is defined by $\neg a$. However, we cannot begin with

$$\neg a \vdash \neg a$$

either, as for negation, being a logical constant, a specific left-introduction rule is available. This means that a derivation, which, in the presence of cut, leads to absurdity (see Section 5 above), cannot be started at all.

Kreuger's restriction reconstitutes cut and transitivity of deduction. It can be shown that for a system with definitional reflection, which allows for paradoxical constructions such as the definition of $a$ above, cut elimination (in the sequent calculus) and transitivity of deduction (in the ND-style sequent calculus) are valid in the presence of Kreuger's restriction (see Schroeder-Heister 1994a).[15]

[15] Actually, in three-valued or four-valued semantics of the sequent calculus, which are, for example, used for dealing with negation as failure in logic programming or in reflexive theories of truth, we find ideas similar to Kreuger's restriction: There, the fact that an initial sequent $A \vdash A$ is defined, corresponds to the fact that $A$ has a definite truth value (true or false), which is not always the case. See Jäger & Stärk (1998) and Halbach & Horsten (2004).

In ordinary natural deduction, Kreuger's restriction corresponds to the following requirement: Suppose $A$ occurs as a *premiss* of an application of an introduction rule. Suppose furthermore that an introduction rule is *specified* for $A$. Then $A$ must be *derived* by means of this introduction rule, i.e. $A$ is neither an assumption nor a conclusion of an elimination rule (more precisely: it is not the last formula of a segment, which starts with the conclusion of an elimination rule).[16] In other words: Formulas must be broken down as far as possible before they can be reassembled according to their meaning. Normalization for natural deduction with full definitional reflection can then be demonstrated. This means that, given Kreuger's restriction, every natural deduction proof corresponds to a proof in the sequent calculus, i.e. there is a full parallelism between natural deduction and ND-style sequent calculus.

Kreuger's restriction says that everything which can in principle be assumed according to its meaning, must be assumed that way. It might be argued that this is too strong of a restriction, as there should be the freedom to use or not to use an assumption according to its meaning. For that reason, I also consider other possibilities in the presence of which standard natural deduction and ND-style sequent calculus are equivalent.

### 6.2 Contraction-free logic

Contraction-free logics are logics without the contraction rule

$$\frac{\Gamma, A, A \vdash B}{\Gamma, A \vdash B} .$$

The antecedent of a sequent can then no longer be considered a set, but e.g. as a multiset. In a contraction-free natural deduction system or ND-style sequent calculus, we would instead disallow the discharging of several occurrences of the same formula by the same inference step. Philosophically, this means that every occurrence of an assumption is treated as a separate statement, which in particular excludes the possibility that assumptions with different semantic functions are identified.

It can be shown that by prohibiting contraction, we regain cut and transitivity of deduction, even in the presence of definitional reflection and non-wellfounded definitions. Cut elimination holds for the contraction-free sequent calculus with definitional reflection (Schroeder-Heister 1992). Correspondingly, we have normalization in contraction-free natural deduction with definitional reflection. Thus in the context of contraction-free logics, natural deduction and the ND-style sequent calculus correspond to each other. Definitions such as

$$a \Leftarrow \neg a$$

[16] Segments are maximal sequences of consecutive occurrences of minor premisses of elimination rules, which end with the conclusion of an elimination rule. In our generalized case, any elimination rules gives rise to segments. For the notion of "segment" see Prawitz (1965).

are harmless. Actually, the idea of associating the paradoxes of logic and set theory with the rule of contraction goes back to Fitch (1936) and Curry (1942) and is well investigated.[17]

However, from the point of view proposed here, fully blocking contraction goes too far. We can only argue that assumptions with different semantic functions should not be identified if they have the same shape. This leads to a modified restriction on contraction which goes as far as it should go.

### 6.3 Restricting specific/unspecific contraction

If we want to treat specific and unspecific assumptions differently, it is natural to prohibit only the contraction of specific with unspecific assumptions. This means that the contraction rule

$$\frac{\Gamma, A, A \vdash B}{\Gamma, A \vdash B}$$

is not applicable, if the first $A$ is an unspecific assumption and the second $A$ is a specific assumption, or vice versa. The distinction between specific and unspecific assumptions is the basic distinction drawn, so we must be careful to keep these two sorts of assumptions separate. The resulting systems are called *weakly contraction-free*.

This can be illustrated with the derivation of $a \vdash$ with respect to the non-wellfounded definition

$$a \Leftarrow \neg a \ ,$$

which has the form

$$\mathcal{D} \left\{ \dfrac{\dfrac{a \vdash a}{a, \neg a \vdash} (\neg \vdash)}{a \vdash} (a \vdash) \right.$$

and which would generate a contradiction in the form of the empty sequent, if cut is available:

$$\dfrac{\dfrac{\dfrac{\mathcal{D}}{\vdash \neg a} (\neg I)}{\vdash a} (\vdash a) \qquad \mathcal{D}}{\vdash} Cut \ .$$

[17] Petersen even considers contraction-free logic as the basis of dialectics, see Petersen (2002), I 304, III 1568–71, 1921–44.

(see the derivations in section 5). Obviously, $\mathcal{D}$ implicitly uses contraction in the second step. Making this contraction explicit, $\mathcal{D}$ reads as follows:

$$\dfrac{\dfrac{\dfrac{a \vdash a}{a, \neg a \vdash}(\neg\vdash)}{a, a \vdash}(a\vdash)}{a \vdash}\,Contr \quad .$$

This contraction is of the incriminated sort, i.e. between an $a$ which stems from the initial sequent at the top and is therefore an unspecific assumption, and an $a$ which stems from the application of the left-introduction rule for $a$ and is therefore a specific assumption. If this sort of contraction is disallowed, the contradictory derivation is no longer available. Something similar holds for the ND-style sequent calculus. Consider the derivation $\mathcal{D}'$ of $\neg a$

$$\mathcal{D}' \left\{ \dfrac{\dfrac{\overset{(2)}{[a]} \quad \dfrac{\overset{(1)}{[\neg a]} \quad \overset{(2)}{[a]}}{\bot}}{\bot}(a\overline{E})\ (1)}{\neg a}\,(2) \right.$$

from which, by using transitivity of deduction and implication elimination, a contradiction could be obtained as follows:

$$\dfrac{\mathcal{D}' \quad \dfrac{\mathcal{D}'}{a}(aI)}{\bot}\,Transitivity + (\rightarrow\overline{E}) \quad .$$

Obviously, $\mathcal{D}'$ uses a critical contraction, as two occurrences of $a$ are discharged (indicated with the number 2), where the left occurrence is the major premiss of $a$-elimination, whereas the right one is an unspecific assumption. Blocking this contraction means blocking the proof of inconsistency.

Thus by prohibiting *contraction of specific with unspecific assumptions*, cut and transitivity of deduction can be upheld even in the generalized framework with arbitrary definitions of atoms. Under this specific restriction, ordinary natural deduction and ND-style sequent calculus correspond to each other.

When carrying out this approach in detail, it turns out that the distinction between specific and unspecific assumptions is not precise enough as it stands. An unspecific assumption can easily be turned into a specific one by using intermediate derivation steps. For example, if $\mathcal{D}$ is replaced with

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{a \vdash a}{a, \neg a \vdash}\,(\neg\vdash)}{\neg a \vdash \neg a}\,(\vdash\neg)}{\neg a \vdash a}\,(\vdash a)}{a \vdash a}\,(a\vdash)\,(*)}{a, \neg a \vdash}\,(\neg\vdash)}{a, a \vdash}\,(a\vdash)\,(*)}{a \vdash}\,Contr \qquad ,
$$

the contraction step is now a contraction between two *specific* occurrences of $a$ which are both introduced by an application of $(a \vdash)$ (indicated by an asterisk).

To deal with such cases, I propose to introduce an indexing discipline. With every formula occurrence in a derivation, a natural number is associated as a *meaning index*, which increases if a meaning rule (left- or right-introduction rule) is applied. Contraction is then prohibited, if the meaning indices of the formulas involved differ. In the above example, the right $a$ undergoing contraction receives a higher meaning index under this discipline than the left one, as it results from a single application of $(\vdash a)$ plus a single application of $(a \vdash)$, whereas the left $a$ results from a single application of $(a \vdash)$ alone (if negation rules are disregarded). The technicalities of this approach will be spelled out elsewhere.

I conjecture that among the three restrictions dealt with in this section, the last one is optimal in that it is the weakest with the desired consequences (validity of cut and transitivity of deduction). It is clear that neither general contraction nor Kreuger's restriction are weaker. For general contraction this is obvious. For Kreuger's restriction this is clear from the fact that no unspecific assumption can be generated if a specific assumption of the same shape is available. However, what still must be established is that Kreuger's restriction and general prohibition of contraction are *properly* stronger than the third restriction. By "properly stronger" I mean that they block derivations, which are not blocked in the weakly contraction-free system, but which are "significant" in some sense. Otherwise, Kreuger's restriction or strongly contraction-free systems should be preferred as they are easier to handle than our weakly contraction-free system with its indexing discipline.

## 7. Validity and invalidity of cut reconsidered

I have argued as follows: Taking the idea of a specific introduction of assumptions seriously, leads to the cut-free sequent calculus or, equivalently, to the ND-style sequent calculus, as the preferable model of reasoning. If this approach is generalized by including assumption rules for atoms which are clausally defined, then the universal validity of cut (and transitivity of deduction) is lost. However, by imposing certain restrictions motivated by the difference between specific and unspecific assumptions, cut and transitivity of deduction can be reestablished. The wish to ensure the validity of cut and transitivity of deduction was a premiss of this line of argument.

I should like to finish with two remarks which (1) further illuminate the conditions under which cut might fail to hold, und (2) challenge the presumption that losing cut must be avoided under any circumstances.

### 7.1 The validity of cut in implicaton-free logics

To show the failure of cut or transitivity of deduction I chose the circular definition

(3) $a \Leftarrow \neg a$

as an example or prototype of a non-wellfounded definition. In the context of clausal definitions, being wellfounded means that iterating the step from the head of a clause to an atom in its body terminates at some stage. However, in this sense, the definition

(4) $a \Leftarrow a$

is non-wellfounded as well, doing no harm even when the rule of cut is explicitly added. The fact that cut is lost does not rest on non-wellfoundedness alone. Actually, it can be shown that cut is valid if the definitions of atoms are definite Horn clauses only, i.e., if they are of the form

(5) $a \Leftarrow b_1 \wedge \ldots \wedge b_n$

for atoms $b_1, \ldots b_n$ (see Hallnäs & Schroeder-Heister 1991). Clause (4) is of that form, but clause (3) is not. Clause (3) uses negation, or more precisely, implication in its body, when we write $\neg a$ as $a \to \bot$. It is actually the presence of implication which destroys the monotonicity of inductive definitions. This somehow reflects the philosophical claim often made that it is *negative* self-reference which makes the paradoxes paradoxical. It also indicates that the field of *non-monotonic* inductive definitions, whose clauses are not necessarily of the form (5), is of outstanding philosophical relevance for the treatment of the paradoxes and problems of restricting cut and/or the introduction of assumptions.

### 7.2 The invalidity of global cut as a significant feature

Not everything is lost if cut or transitivity of deduction is not universally valid. That cut or transitivity fail to hold means philosophically that if we want to pass from $A$ to $C$, we *must proceed via a cut formula $B$*. So a derivation from $A$ to $C$

$$\begin{array}{c} A \\ \vdots \\ C \end{array}$$

could never be a “direct” derivation, but a derivation ”passing along” $B$ as an intermediate step, which might be symbolized as

$$\begin{array}{c} A \\ \vdots \\ B \\ \hdashline B \\ \vdots \\ C \end{array} \quad .$$

Here $B$ would be a *lemma* on the path from $A$ to $C$. In certain cases the fact that we must move through $B$ may even be considered an advantage. There might be contexts in which we are particularly interested in the lemmas used in proving a theorem, so the failure of cut or transitivity may give us expressive power to deal with such a situation. Actually, mathematicians would consider which lemmas it uses to be a crucial ingredient of the “content” of a proof, and would not always consider a proof equally informative if it is reduced so as to proceed directly without passing through certain intermediate lemmas.[18].

It should also be mentioned that such an invalidity of cut and transitivity would be a *global* feature of deduction. It would be independent of the fact that introduction and elimination rules, or right-introduction and left-introduction rules are in harmony with each other, which in philosophy is often considered to be the justification of cut or normalization. If cut cannot be eliminated globally, it still holds that asserting a formula $A$ (by means of an introduction rule or right-introduction rule for $A$) is *in harmony* with assuming that formula by means of an elimination rule or left-introduction rule for $A$. The conditions for asserting $A$ are at the same time the consequences of assuming $A$. However, this harmony is a *local* feature which does not extend to cut elimination or normalization as a global property, and it would be worth discussing whether it should do so from a meaning-theoretic point of view. The global validity of cut, normalization or transitivity of deduction is not as firm a philosophical principle as it is often considered to be.

Concluding, the proper treatment of assumptions in logical systems is closely connected with structural rules such as cut and contraction, and with fundamental philosophical and semantical topics concerning the principles of logical reasoning. The notion of assumption is an issue which deserves more attention within philosophical logic than it it is receiving currently.

## References


Basin, D. & Matthews, S. (2002): “Logical frameworks”, in: D. Gabbay & F. Guenthner (eds.), *Handbook of Philosophical Logic, Second Edition*, Vol. 9, Dordrecht: Kluwer, 89–163.


[18] This was pointed out by Contu. See Schroeder-Heister & Contu (2004), and Contu (2005).

Carnap, R. (1929): *Abriß der Logistik. Mit besonderer Berücksichtigung der Relationstheorie und ihrer Anwendungen.* Wien: Springer.

Contu, P. (2005): "The justification of the logical laws revisited", in: R. Kahle & P. Schroeder-Heister (eds.), *Proof-Theoretic Semantics*. Special issue of *Synthese*.

Curry, H.B. (1942): "The inconsistency of certain formal logics", *Journal of Symbolic Logic* 7, 115–117.

Fitch, F.B. (1936): "A system of formal logic without an analogue to the Curry *W* operator", *Journal of Symbolic Logic* 1, 92–100.

Gabriel, G. (1984): "Logistik", in: J. Mittelstraß (ed.), *Enzyklopädie Philosophie und Wissenschaftstheorie*, Vol. 2, Mannheim: Bibliographisches Institut, 703.

Gentzen, G. (1934): "Untersuchungen über das logische Schließen", *Mathematische Zeitschrift* 39 (1934/35), 176–210, 405–431, English translation ("Investigations into Logical Deduction") in: M.E. Szabo (ed.), *The Collected Papers of Gerhard Gentzen*, Amsterdam: North Holland 1969, 68–131.

Groote, P. de (ed.) (1995): *The Curry-Howard Isomorphism.* Louvain-la-Neuve: Academia.

Halbach, V. & Horsten, L. (2004): "Partial Kripke-Feferman", *Notre Dame Journal of Formal Logic*.

Hallnäs, L. (1991): "Partial inductive definitions", *Theoretical Computer Science* 87, 115–142.

Hallnäs, L. & Schroeder-Heister, P. (1990): "A proof-theoretic approach to logic programming. I. Clauses as rules", *Journal of Logic and Computation* 1 (1990/91), 261–283.

Hallnäs, L. & Schroeder-Heister, P. (1991): "A proof-theoretic approach to logic programming. II. Programs as definitions", *Journal of Logic and Computation* 1 (1990/91), 635–660.

Harper, R., Honsell, F. & Plotkin, G. (1993): "A framework for defining logics", *Journal of the Association for Computing Machinery* 40, 143–184.

Jäger, G. & Stärk, R.F. (1998): "A proof-theoretic framework for logic programming", in: S. Buss (ed.), *Handbook of Proof Theory*. Amsterdam: Elsevier, 639–682.

Kahle, R. & Schroeder-Heister, P. (2005): "Introduction: Proof-theoretic semantics", in: R. Kahle & P. Schroeder-Heister (eds.), *Proof-Theoretic Semantics*. Special issue of *Synthese*.

Kreuger, P. (1994): "Axioms in definitional calculi", in: R. Dyckhoff (ed.), *Extensions of Logic Programming. Proceedings of the 4th International Workshop, ELP '93, St. Andrews, March/April 1993*. Berlin: Springer LNCS, Vol. 798, 196–205.

Kutschera, F. von (1968): "Die Vollständigkeit des Operatorensystems $\{\neg, \wedge, \vee, \supset\}$ für die intuitionistische Aussagenlogik im Rahmen der Gentzensemantik", in: *Archiv für mathematische Logik und Grundlagenforschung* 11, 3–16.

Negri, S. & Plato, J. von (2001a): "Sequent calculus in natural deduction style", *Journal of Symbolic Logic* 66, 1803–1816.

Negri, S. & Plato, J. von (2001b): *Structural Proof Theory*. Cambridge University Press.

Paulson, L.C. (1989): "The foundation of a generic theorem prover", *Journal of Automated Reasoning* 5, 363–397.

Petersen, U. (2002): *Diagonal Method and Dialectical Logic. Tools, Materials, and Groundworks for a Logical Foundation of Dialectic and Speculative Philosophy*, 3 Vols., Osnabrück: Der Andere Verlag.

Plato, J. von (2001): "Natural deduction with general elimination rules", *Archive for Mathematical Logic* 40, 541–567.

Prawitz, D. (1965): *Natural Deduction: A Proof-Theoretical Study*. Stockholm: Almqvist & Wiksell.

Prawitz, D. (1974): "On the idea of a general proof theory", *Synthese* 27, 63–77.

Prawitz, D. (1978): "Proofs and the meaning and completeness of the logical constants", in: J. Hintikka et al. (eds.), *Essays on Mathematical and Philosophical Logic*, Dordrecht: Reidel, 25–40 (revised German translation in *Conceptus* 16 (1982).

Prawitz. D. (2005): "Meaning approached via proofs", in: R. Kahle & P. Schroeder-Heister (eds.), *Proof-Theoretic Semantics*. Special issue of *Synthese*.

Quine, W.V.O. (1934): *A System of Logistic*. Cambridge Mass.: Harvard University Press.

Schroeder-Heister, P. (1984a) "A natural extension of natural deduction", *Journal of Symbolic Logic* 49, 1284–1300.

Schroeder-Heister, P. (1984b): "Generalized rules for quantifiers and the completeness of the intuitionistic operators &, $\vee$, $\supset$, $\curlywedge$, $\forall$, $\exists$". in: M.M. Richter et al. (eds.), *Computation and Proof Theory. Proceedings of the Logic Colloquium held in Aachen, July 1983, Part II.* Berlin: Springer LNM, Vol. 1104, 399–426.

Schroeder-Heister, P. (1992): "Cut-elimination in logics with definitional reflection", in: D. Pearce & H. Wansing (eds.), *Nonclassical Logics and Information Processing. International Workshop, Berlin 1990, Proceedings.* Berlin: Springer LNCS, Vol. 619, 146–171.

Schroeder-Heister, P. (1993): Rules of definitional reflection. In: *8th Annual IEEE Symposium on Logic in Computer Science (Montreal 1993).* Los Alamitos: IEEE Computer Society Press, 222–232.

Schroeder-Heister, P. (1994a): "Cut elimination for logics with definitional reflection and restricted initial sequents", in: *Proceedings of the Post-Conference Workshop of ICLP 1994 on Proof-Theoretic Extensions of Logic Programming (Washington, December 1994).* Can be downloaded from the author's homepage.

Schroeder-Heister, P. (1994b): "Definitional reflection and the completion", in: R. Dyckhoff (ed.), *Extensions of Logic Programming. Proceedings of the 4th International Workshop, ELP '93, St. Andrews, March/April 1993*. Berlin: Springer LNCS, Vol. 798, 333–347.

Schroeder-Heister, P. (2005): "Validity concepts in proof-theoretic semantics.", in: R. Kahle & P. Schroeder-Heister (eds.), *Proof-Theoretic Semantics*. Special issue of *Synthese*.

Schroeder-Heister, P. & Contu, P. (2004): “Folgerung”, in: W. Spohn, P. Schroeder-Heister & E.J. Olsson (eds.), *Logik in der Philosophie*, Heidelberg: Synchron Wissenschaftsverlag.

Tennant, N. (1992): *Autologic*. Edinburgh: University Press.

Tennant, N. (2002): “Ultimate normal forms for parallelized natural deductions”, *Logic Journal of the IGPL* 10, 299–337.

Troelstra, A.S. & Schwichtenberg, H. (1996): *Basic Proof Theory.* Cambridge University Press, 2nd edition 2000.