THE REVIEW OF SYMBOLIC LOGIC
Volume 7, Number 1, March 2014

ON FLATTENING ELIMINATION RULES

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Abstract. In proof-theoretic semantics of intuitionistic logic it is well known that elimination rules can be generated from introduction rules in a uniform way. If introduction rules discharge assumptions, the corresponding elimination rule is a rule of higher level, which allows one to discharge rules occurring as assumptions. In some cases, these uniformly generated elimination rules can be equivalently replaced with elimination rules that only discharge formulas or do not discharge any assumption at all—they can be *flattened* in a terminology proposed by Read. We show by an example from propositional logic that not all introduction rules have flat elimination rules. We translate the general form of flat elimination rules into a formula of second-order propositional logic and demonstrate that our example is not equivalent to any such formula. The proof uses elementary techniques from propositional logic and Kripke semantics.

§1. The flattening problem for elimination rules.

If a sentence φ is defined from conditions $\Delta_1, \ldots, \Delta_n$ by the introduction rules

$$(\varphi \text{ I}) \ \frac{\Delta_1}{\varphi} \ \ldots \ \frac{\Delta_m}{\varphi},$$

then the inversion principle as first proposed by Lorenzen (1955) says that everything that follows from each defining condition Δ_i of φ follows from φ itself. In a natural deduction framework, this idea gives rise to an elimination rule of the form

$$(\varphi \text{ E}) \ \frac{\varphi \quad \begin{matrix}[\Delta_1] \\ r\end{matrix} \quad \ldots \quad \begin{matrix}[\Delta_m] \\ r\end{matrix}}{r},$$

where the square brackets indicate that the Δ_i can be discharged as assumptions in the respective subderivations. This is a general schema which applies to inductive definitions of atoms, (generalized) logic programs and proof-theoretic characterizations of logical constants likewise (see Schroeder-Heister, 2012). Here, we concentrate on logical constants. For the point we want to make we can confine ourselves to propositional logic. Then these rules become rules for an arbitrary n-ary propositional operator c:

Received: December 25, 2012.

doi:10.1017/S1755020313000385

$$(c\,\mathrm{I})\ \frac{\Delta_1(p_1,\ldots,p_n)}{c(p_1,\ldots,p_n)}\ \ldots\ \frac{\Delta_m(p_1,\ldots,p_n)}{c(p_1,\ldots,p_n)}$$

$$(c\,\mathrm{E})\ \frac{c(p_1,\ldots,p_n)\quad \begin{matrix}[\Delta_1(p_1,\ldots,p_n)]\\ r\end{matrix}\ \ldots\ \begin{matrix}[\Delta_m(p_1,\ldots,p_n)]\\ r\end{matrix}}{r}\ .$$

Specializing the general schema $(c\,\mathrm{I})/(c\,\mathrm{E})$ to disjunction, we obtain the common introduction and elimination rules for it:

$$(\vee\mathrm{I})\ \frac{p}{p\vee q}\qquad \frac{q}{p\vee q}\qquad (\vee\mathrm{E})\ \frac{p\vee q\quad \begin{matrix}[p]\\ r\end{matrix}\quad \begin{matrix}[q]\\ r\end{matrix}}{r}\ .$$

Conjunction receives its general elimination rule:

$$(\wedge\mathrm{I})\ \frac{p\quad q}{p\wedge q}\qquad (\wedge\mathrm{E_{GEN}})\ \frac{p\wedge q\quad \begin{matrix}[p,q]\\ r\end{matrix}}{r}\ ,$$

and for absurdity $\bot$ we obtain ex falso quodlibet as an elimination rule without minor premisses

$$[\text{no I rule}]\qquad (\bot\mathrm{E})\ \frac{\bot}{r}\ .$$

In the case of implication, the general schema $(c\,\mathrm{I})/(c\,\mathrm{E})$ forces us to extend our structural means of expression. Since the introduction rule for implication has the form

$$(\to\mathrm{I})\ \frac{\begin{matrix}[p]\\ q\end{matrix}}{p\to q}\ ,$$

the elimination rule according to the schema $(c\,\mathrm{E})$ would have to take a form which might be depicted as follows:

$$\frac{p\to q\quad \begin{matrix}\left[\begin{matrix}[p]\\ q\end{matrix}\right]\\ r\end{matrix}}{r}\ .$$

To capture this idea formally, in Schroeder-Heister (1984) a system was proposed in which not only formulas, but also rules can be introduced as assumptions and discharged at the application of other rules. The elimination rule for implication then takes the form

$$(\to\mathrm{E})\ \frac{p\to q\quad \begin{matrix}[p\Rightarrow q]\\ r\end{matrix}}{r}\ .$$

A rule which may discharge rules as assumptions is called a *higher-level rule*. In our context it is not relevant how exactly the formalism for rules of higher levels is specified. A rule that only discharges formulas as assumptions or no assumption at all is called a standard-level rule. Apparently, if $(c\,\mathrm{I})$ contains at least one standard-level rule discharging an assumption, then $(c\,\mathrm{E})$ becomes a higher-level rule. This procedure can be iterated by assigning different finite levels to higher-level rules. Using higher-level rules gives one a great deal of uniformity, for example in the proof of normalization for the general system based on $(c\,\mathrm{I})/(c\,\mathrm{E})$, or the proof that introduction and elimination rules uniquely determine a connective.

In the case of implication, the use of higher-level rules can be avoided. The general implication elimination rule in the sense of Dyckhoff (1988), Tennant (1992), López-Escobar (1999) and Plato (2001)

$$(\to \mathrm{E}_{\mathrm{GEN}})\ \frac{p \to q \qquad p \qquad \begin{matrix}[q]\\ r\end{matrix}}{r}\ ,$$

which is a natural-deduction analogue of the implication-left rule in the sequent calculus, discharges only the formula q; therefore it is a standard-level elimination rule (for a discussion of this type of rules in relation to higher-level rules, see Schroeder-Heister, 2013). Modus ponens and the projection-style conjunction elimination rules

$$(\to \mathrm{E}_{\mathrm{MP}})\ \frac{p \to q \qquad p}{q} \qquad\qquad (\wedge \mathrm{E}_{\mathrm{P}})\ \frac{p \wedge q}{p} \quad \frac{p \wedge q}{q}$$

even avoid any discharge of assumptions. Using a terminology proposed by Read (2014), we say that an elimination rule is *flat*, if it is a standard-level rule, and that an elimination rule according to the uniform schema (cE) can be *flattened*, if there is a flat elimination rule equivalent to it. Thus the higher-level form of implication elimination ($\to$E) can be flattened both to ($\to \mathrm{E}_{\mathrm{GEN}}$) and to ($\to \mathrm{E}_{\mathrm{MP}}$). The price for flattening is the loss of uniformity, because it means that rules of special forms have to be considered rather than just the general schema (cE), unless there is a uniform schema for flat elimination rules.

If we just have a single introduction rule for c:

$$(c\,\mathrm{I}_{\mathrm{S}})\ \frac{\begin{matrix}[\Delta_1]\\ q_1\end{matrix} \quad \cdots \quad \begin{matrix}[\Delta_m]\\ q_m\end{matrix}}{c(p_1, \ldots, p_n)}\ ,$$

where $q_1, \ldots, q_m$ are propositional variables and $\Delta_1, \ldots, \Delta_m$ are lists of propositional variables, where all these propositional variables are taken from $\{p_1, \ldots, p_n\}$, then a uniform schema for flat elimination rules can be given as follows:

$$(c\,\mathrm{E}_{\mathrm{S}})\ \frac{c(p_1, \ldots, p_n) \qquad \Delta_1}{q_1} \quad \cdots \quad \frac{c(p_1, \ldots, p_n) \qquad \Delta_m}{q_m}\ .$$

In the formalism with higher-level rules it can be easily shown that these rules are together equivalent to the higher-level elimination rule according to schema (cE):

$$\frac{c(p_1, \ldots, p_n) \qquad \begin{matrix}[(\Delta_1 \Rightarrow q_1),\ \ldots\ , (\Delta_m \Rightarrow q_m)]\\ r\end{matrix}}{r}\ .$$

Consider, for example, the 5-place connective c' with the single introduction rule

$$(c'\mathrm{I})\ \frac{\begin{matrix}[p_1, p_2]\\ p_3\end{matrix} \qquad \begin{matrix}[p_4]\\ p_5\end{matrix}}{c'(p_1, p_2, p_3, p_4, p_5)}\ .$$

Its uniform elimination rule

$$\frac{c'(p_1, p_2, p_3, p_4, p_5) \qquad \begin{matrix}[((p_1, p_2) \Rightarrow p_3)\ , (p_4 \Rightarrow p_5)]\\ r\end{matrix}}{r}$$

can be replaced with the following two flat elimination rules

$$(c'\mathrm{E_S})\quad \frac{c'(p_1,p_2,p_3,p_4,p_5)\quad p_1\quad p_2}{p_3}\qquad \frac{c'(p_1,p_2,p_3,p_4,p_5)\qquad p_4}{p_5}\quad .$$

Alternatively, one or both of them could be formulated along the lines of $(\rightarrow \mathrm{E_{GEN}})$:

$$(c'\mathrm{E_{GEN}})\quad \frac{c'(p_1,p_2,p_3,p_4,p_5)\quad p_1\quad p_2\qquad \begin{array}{c}[p_3]\\ r\end{array}}{r}\qquad \frac{c'(p_1,p_2,p_3,p_4,p_5)\quad p_4\qquad \begin{array}{c}[p_5]\\ r\end{array}}{r}\quad .$$

However, this method cannot be applied if we have more than one introduction rule. Consider the ternary connective $\star$ with the following introduction and elimination rules:

$$(\star\,\mathrm{I})\quad \frac{\begin{array}{c}[p_1]\\ p_2\end{array}}{\star(p_1,p_2,p_3)}\qquad \frac{p_3}{\star(p_1,p_2,p_3)}\qquad\qquad (\star\,\mathrm{E})\quad \frac{\star(p_1,p_2,p_3)\qquad \begin{array}{c}[p_1 \Rightarrow p_2]\\ r\end{array}\qquad \begin{array}{c}[p_3]\\ r\end{array}}{r}\quad .$$

Flattening $(\star\,\mathrm{E})$ according to the pattern of the $(\rightarrow \mathrm{E_{GEN}})$ leads to the following elimination rule:

$$\frac{\star(p_1,p_2,p_3)\quad p_1\qquad \begin{array}{c}[p_2]\\ r\end{array}\qquad \begin{array}{c}[p_3]\\ r\end{array}}{r}\quad .$$

However, this elimination rule is not appropriate for $\star$. This can already be seen from the fact that the same elimination rule would be generated from the pair of introduction rules

$$(\star\star\,\mathrm{I})\quad \frac{\begin{array}{c}[p_1]\\ p_3\end{array}}{\star\star\,(p_1,p_2,p_3)}\qquad \frac{p_2}{\star\star\,(p_1,p_2,p_3)}\quad .$$

Whereas $(\star\,\mathrm{I})$ defines a connective with the meaning $(p_1 \rightarrow p_2) \vee p_3$, $(\star\star\,\mathrm{I})$ defines a connective with the meaning $(p_1 \rightarrow p_3) \vee p_2$. These formulas are only classically, but not intuitionistically equivalent. Other schemata for flat elimination rules according to the pattern of $(\rightarrow \mathrm{E_{GEN}})$ that have been proposed (e.g., by Francez & Dyckhoff, 2012) are likewise not appropriate.

This example shows that in the case of multiple introduction rules certain attempts at flattening elimination rules fail. In what follows we formally show that flattening is actually impossible in this case, more precisely, that the connective $\star$ cannot be given a flat elimination rule. From the point of view of Schroeder-Heister (1984), this result can be seen as a defence of the idea of higher-level rules.

§2. Coding introduction and elimination rules by formulas. As shown by von Kutschera (1968), Prawitz (1979) and Schroeder-Heister (1984) (and extended to more advanced systems by Wansing, 1993), the set of standard connectives of intuitionistic propositional logic is complete, that is, every connective, for which introduction and elimination rules according to the schema $(c\,\mathrm{I})/(c\,\mathrm{E})$ are given, can be expressed by means of $\wedge$, $\vee$, $\rightarrow$ and $\bot$. In order to prove this fact, the introduction rules $(c\,\mathrm{I})$ are translated into $\{\wedge,\vee,\rightarrow,\bot\}$-formulas in a straightforward way. For example, the connective c' has the meaning $((p_1 \wedge p_2) \rightarrow p_3) \wedge (p_4 \rightarrow p_5)$, and $\star$ has the meaning $(p_1 \rightarrow p_2) \vee p_3$. More precisely, consider the following introduction rule for c:

$$(c\,\mathrm{I})\quad \frac{\begin{array}{c}[\Gamma_1]\\ s_1\end{array}\quad \cdots\quad \begin{array}{c}[\Gamma_\ell]\\ s_\ell\end{array}}{c(p_1,\ldots,p_n)}\quad ,$$

in which $s_1, \ldots, s_\ell, s$ are propositional variables and $\Gamma_1, \ldots, \Gamma_\ell$ are (possibly empty) lists of propositional variables. As a limiting case we allow for $\ell = 0$ (which covers the case of the truth constant $\top$). All propositional variables must occur among $p_1, \ldots, p_n$. Let $\bigwedge \Gamma$ denote the conjunction of all elements of Γ. Then we express the content of $(c\,\mathrm{I})$ propositionally as follows:

$$\frac{(\bigwedge \Gamma_1 \to s_1) \wedge \ \ldots\ \wedge (\bigwedge \Gamma_\ell \to s_\ell)}{c(p_1, \ldots, p_n)}\,, \quad \text{in short} \quad \frac{\lambda}{c(p_1, \ldots, p_n)}\,.$$

If we have k introduction rules $(c\,\mathrm{I})_1, \ldots, (c\,\mathrm{I})_k$, then we can express their content propositionally by means of the single introduction rule

$$(c\,\mathrm{I}_{\mathrm{PROP}})\, \frac{\lambda_1 \vee \ldots \vee \lambda_k}{c(p_1, \ldots, p_n)}$$

where each λ_i is of the form specified for λ. The formula

$$(c^I)\ \lambda_1 \vee \ldots \vee \lambda_k \quad (k \geq 1)$$

is called the *introduction meaning* of c, denoted by c^I. We can then prove in intuitionistic propositional logic that the rules $(c\,\mathrm{I})_1, \ldots, (c\,\mathrm{I})_k$ and $(c\,\mathrm{I}_{\mathrm{PROP}})$ are equivalent, that is, by using $(c\,\mathrm{I})_1, \ldots, (c\,\mathrm{I})_k$ we can derive $(c\,\mathrm{I}_{\mathrm{PROP}})$, and by using $(c\,\mathrm{I}_{\mathrm{PROP}})$ we can derive $(c\,\mathrm{I})_1, \ldots, (c\,\mathrm{I})_k$.

Analogously, we can define the *elimination meaning* of c by coding the elimination rule for c. For that we have first to fix what we consider to be the general form of an elimination rule. We have discussed the uniformly generated elimination rule $(c\,\mathrm{E})$, but also alternative elimination rules, in particular flat ones, for example $(\to \mathrm{E}_{\mathrm{GEN}})$ or $(\wedge\,\mathrm{E}_{\mathrm{GEN}})$. Here we are interested in flat elimination rules. As the most general schema of a flat elimination rule for $c(p_1, \ldots, p_n)$ we consider the following:

$$(c\,\mathrm{E}_{\mathrm{FLAT}})\ \frac{c(p_1, \ldots, p_n) \qquad \begin{matrix}[\Gamma_1] \\ s_1\end{matrix} \quad \cdots \quad \begin{matrix}[\Gamma_\ell] \\ s_\ell\end{matrix}}{s}\,,$$

where $s_1, \ldots, s_\ell, s$ are propositional variables and $\Gamma_1, \ldots, \Gamma_\ell$ are (possibly empty) lists of propositional variables. As a limiting case we allow for $\ell = 0$ (which covers rules such as $(\wedge\,\mathrm{E}_{\mathrm{P}})$ or $(\bot\mathrm{E})$). We do not impose any restrictions on the propositional variables occurring in $(c\,\mathrm{E}_{\mathrm{FLAT}})$. They may (and will normally) comprise $p_1, \ldots, p_n$, but any number of propositional variables beyond $p_1, \ldots, p_n$ may be present. This generalizes the fact that in many elimination rules considered, we have used r as such an additional propositional variable. $(c\,\mathrm{E}_{\mathrm{FLAT}})$ is a plausible candidate for elimination rules, as it has $c(p_1, \ldots, p_n)$ as its major premiss which is eliminated by the rule, and arbitrarily many side premisses which may or may not discharge assumptions. As the Γ_i can only contain propositional variables, the rule is flat.

We suppose that for a constant c finitely many elimination rules of this form are given. We express the content of $(c\,\mathrm{E}_{\mathrm{FLAT}})$ propositionally as follows:

$$\frac{c(p_1, \ldots, p_n)}{((\bigwedge \Gamma_1 \to s_1) \wedge \ \ldots\ \wedge (\bigwedge \Gamma_\ell \to s_\ell)) \to s}\,, \quad \text{in short} \quad \frac{c(p_1, \ldots, p_n)}{\kappa}\,.$$

If we have k elimination rules $(c\,\mathrm{E}_{\mathrm{FLAT}})_1, \ldots, (c\,\mathrm{E}_{\mathrm{FLAT}})_k$, then we can express their total content by means of

$$\frac{c(p_1, \ldots, p_n)}{\kappa_1 \wedge \ldots \wedge \kappa_k}$$

where each κ_i is of the form specified for κ. When applying an elimination inference, the propositional variables beyond $p_1, \ldots, p_n$ can be instantiated with any formula without affecting the meaning of the eliminated premiss $c(p_1, \ldots, p_n)$, so they are understood universally quantified. To express this fact, we use second-order propositional logic, that is intuitionistic propositional logic with propositional quantifiers. This system is called **PL2**. Let

$$(c^E_{\mathrm{FLAT}}) \quad \overline{\forall}(\kappa_1 \wedge \ldots \wedge \kappa_k) \quad (k \geq 1)$$

be the prenex formula starting with a prefix of universal quantifiers which bind all propositional variables except $p_1, \ldots, p_n$. That is, $\overline{\forall}$ stands for $\forall q_1 \ldots \forall q_j$, where $\{q_1, \ldots, q_j\}$ is the set of those variables occurring in $\kappa_1 \wedge \ldots \wedge \kappa_k$, which are different from any variable in $\{p_1, \ldots, p_n\}$. We denote this formula by c^E_{FLAT} and call it the elimination meaning of c with respect to the elimination rules $(c\,\mathrm{E}_{\mathrm{FLAT}})_1, \ldots, (c\,\mathrm{E}_{\mathrm{FLAT}})_k$. Then we can prove in **PL2** that the rules $(c\,\mathrm{E}_{\mathrm{FLAT}})_1, \ldots, (c\,\mathrm{E}_{\mathrm{FLAT}})_k$ and

$$(c\,\mathrm{E}_{\mathrm{PROP}}) \quad \frac{c(p_1, \ldots, p_n)}{c^E_{\mathrm{FLAT}}}$$

are equivalent, that is, using $(c\,\mathrm{E}_{\mathrm{FLAT}})_1, \ldots, (c\,\mathrm{E}_{\mathrm{FLAT}})_k$ we can derive $(c\,\mathrm{E}_{\mathrm{PROP}})$, and using $(c\,\mathrm{E}_{\mathrm{PROP}})$ we can derive$(c\,\mathrm{E}_{\mathrm{FLAT}})_1, \ldots, (c\,\mathrm{E}_{\mathrm{FLAT}})_k$. In an analogous way we can define the elimination meaning c^E for elimination rules for c, which are not necessarily flat, that is, in which Γ_i may contain rules rather than only propositional variables. This can be achieved by translating any rule $q_1, \ldots, q_r \Rightarrow q$ by the formula $(q_1 \wedge \ldots \wedge q_r) \to q$. Then c^E_{FLAT} becomes a special case of c^E. For the purpose of this paper, we need not further elaborate on the handling of rules as assumptions.

We say that introduction and elimination rules proposed for c are in *harmony* with one another, if in **PL2** we can prove that introduction meaning and elimination meaning (with respect to these rules) are equivalent: $c^I \dashv\vdash c^E$. If certain introduction rules for c are given, we call a set of elimination rules *appropriate* for c, if they are in harmony with the introduction rules. Note that, in order to establish this harmony, the equivalence between c^I and c^E needs to be demonstrated in **PL2** alone, without introduction and elimination rules for the c in question. For example, the general rules $(\wedge\,\mathrm{E}_{\mathrm{GEN}})$ are appropriate for conjunction, since in **PL2** we can show that

$$p_1 \wedge p_2 \dashv\vdash \forall r(((p_1 \wedge p_2) \to r) \to r)\ .$$

Similarly, the general elimination rule $(\to\mathrm{E}_{\mathrm{GEN}})$ is appropriate for implication, since we can show in **PL2** that

$$p_1 \to p_2 \dashv\vdash \forall r((p_1 \wedge (p_2 \to r)) \to r)\ .$$

For our connective c' with the introduction rule $(c'\mathrm{I})$, the elimination rules $(c'\mathrm{E}_{\mathrm{GEN}})$ are appropriate, since in **PL2** we have

$$((p_1 \wedge p_2) \to p_3) \wedge (p_4 \to p_5) \dashv\vdash \forall r(((p_1 \wedge p_2 \wedge (p_3 \to r)) \to r) \wedge ((p_4 \wedge (p_5 \to r)) \to r))\ .$$

For the same connective with the same introduction rule, the elimination rules $(c'\mathrm{E}_{\mathrm{S}})$ are appropriate as well, since in this case both the introduction and elimination meaning are identical and have the form $((p_1 \wedge p_2) \to p_3) \wedge (p_4 \to p_5)$, without any propositional quantification.

Our uniform elimination rules (c E) are appropriate for c, if c is given by our introduction rules (c I). This is not surprising, as they have been designed to achieve harmony. Therefore, the nonflat elimination rule ($\star$E) is appropriate for $\star$, which can be directly seen from the following equivalence which is provable in **PL2**:

$$(p_1 \to p_2) \vee p_3 \dashv\vdash \forall r((((p_1 \to p_2) \to r) \wedge (p_3 \to r)) \to r) .$$

The fact that ($\star$E) is not flat is reflected by the fact that on the right-hand side of this equivalence, implication is iterated (nested) twice to the left. For flat elimination rules, implication can be iterated to the left maximally once.

In the third part of this paper we show that there are no flat elimination rules which are appropriate for $\star$. In other words: no formula of the form c^E_{FLAT} is equivalent in **PL2** to $(p_1 \to p_2) \vee p_3$.

It should be emphasized that we have not presented a *foundational* approach to harmony based on some basic equilibrium between introduction and elimination rules independent of any given connectives. We are rather adopting a *reductive* approach by expressing harmony in terms of **PL2**, a system we here take for granted (see Schroeder-Heister, 2014b). This way of proceeding is fully sufficient to establish our non-flattening result; in fact, **PL2** provides the technical framework to prove it[1]. A more foundational perspective on harmony corresponding to the view normally taken in the literature (see, e.g., Tennant, 1978; Dummett, 1991; Read, 2010, 2014; Francez & Dyckhoff, 2012) is developed in Schroeder-Heister (2014a).

§3. There are no flat elimination rules for $\star$.

In this part we are no longer dealing with n-ary propositional connectives but with the special case of the ternary connective $\star$. Therefore it is more convenient to use the variables p, q, r rather than p_1, p_2, p_3. The introduction meaning of $\star$ is then given by the formula $(p \to q) \vee r$. We define so-called *γ-formulas*, which represent the elimination meaning of ternary connectives with respect to flat elimination rules. That is, γ-formulas correspond for the ternary case to what in the previous section was called c^E_{FLAT}. Our main theorem then says that no γ-formula is intuitionistically equivalent to $(p \to q) \vee r$.

We work in a language with a countable set Var of propositional variables which includes p, q and r, connectives $\to$ and $\wedge$ and the universal propositional quantifier $\forall$. A formula is called *elementary* if it is of the form $K \to s$ where s is a propositional variable and K is a conjunction of propositional variables (we admit the case when K is empty and equivalent to $\top$). A formula is called *simple* if it is of the form $K \to s$ where s is a propositional variable and K is a conjunction of elementary formulas (again, we admit the case when K is empty and equivalent to $\top$). A formula is called a *quantified simple formula* if it is of the form $\forall s_1 \dots s_n B$, where B is a simple formula and $s_1 \dots s_n$ is the list of all variables of B except for possibly p, q and r. We call a formula a *γ-formula* if it is of the form $\forall s_1 \dots s_n A$, where A is a conjunction of simple formulas and $s_1 \dots s_n$ is the list of all variables of A except for possibly p, q and r. It is obvious that every γ-formula is intuitionistically equivalent to a conjunction of quantified simple formulas.

[1] **PL2** has the advantage that rules have a faithful translation into formulas of **PL2**, as the definiton of c^I and c^E shows, which is lost, when we use the translation of **PL2** into **PL** discovered by Pitts (1992).

For a conjunction K of elementary formulas we define $S(K) \subseteq Var$ as follows:

$S^0(K) = \{s \in Var \mid s \text{ or } \top \rightarrow s \text{ is a conjunct in } K\}$;

$S^{k+1}(K) = S^k(K) \cup \{t \mid s_1 \ldots s_n \in S^k(K), (s_1 \wedge \ldots \wedge s_n) \rightarrow t \text{ is a conjunct in } K\}$;

$S(K) = \bigcup_k S^k(K)$.

We write $\models A$ to express that formula A is intuitionistically valid, and $B \models A$ to express that formula A follows intuitionistically from formula B.

Now we can state a simple syntactic criterion for the intuitionistic validity of a simple formula:

LEMMA 3.1. *A simple formula $K \rightarrow s$ is intuitionistically valid iff $s \in S(K)$. This means that $S(K) = \{s \in Var \mid K \models s\}$.*

Proof. The right-to-left direction of the lemma is obvious. For the other direction, if $s \notin S(K)$, then consider a Kripke model with a single world w accessible from itself, such that $w \Vdash p$ iff $p \in S(K)$. Then of course $w \nVdash s$. On the other hand, if $(s_1 \wedge \ldots \wedge s_n) \rightarrow t$ is a conjunct in K, then either $t \in S(K)$ or t plus at least one of $s_1 \ldots s_n$ are not in $S(K)$. In both cases, we will have $w \Vdash (s_1 \wedge \ldots \wedge s_n) \rightarrow t$ and therefore $w \nVdash K \rightarrow s$. □

In a Kripke model for intuitionistic logic, we call w an end-world, iff w is the only world accessible from w in this model. The following three lemmas are necessary stepping stones to the main result.

LEMMA 3.2. *If w is an end-world of a Kripke model M, $K \rightarrow s$ is a simple formula, then $M, w \Vdash K \rightarrow s$ iff either $M, w \nVdash K$ or $M, w \Vdash s$.*

Proof. Immediate from semantic definitions. □

LEMMA 3.3. *If w is a world of a Kripke model M, $K \rightarrow s$ is a simple formula, $M, w \nVdash S(K)$, and*

$$\forall v((wRv \wedge w \neq v) \Rightarrow M, v \Vdash K \rightarrow s), \tag{1}$$

then $M, w \Vdash K \rightarrow s$.

Proof. Since $M, w \nVdash S(K)$, then, by Lemma 3.1, we have that $M, w \nVdash K$. The rest follows from semantic definitions, given our assumption (1). □

LEMMA 3.4. *Let t, u be propositional variables and let $(K \wedge (t \rightarrow u)) \rightarrow s$ be a simple formula. If $\models (K \wedge (t \rightarrow u)) \rightarrow s$ and $t \notin S(K)$, then $\models K \rightarrow s$.*

Proof. By Lemma 3.1, we know that $s \in S(K \wedge (t \rightarrow u))$, and we need to show that $s \in S(K)$. It will suffice to show that for all natural i, $S^i(K \wedge (t \rightarrow u)) \subseteq S^i(K)$ by induction on i.

Basis. It is clear that

$$\{s \in Var \mid s \text{ is a conjunct in } K\} = \{s \in Var \mid s \text{ is a conjunct in } K \wedge (t \rightarrow u)\}.$$

Induction step. Let $i = k + 1$. If $s' \in S^{k+1}(K \wedge (t \rightarrow u)) \cap S^k(K \wedge (t \rightarrow u))$, then use induction hypothesis. If $s \in S^{k+1}(K \wedge (t \rightarrow u)) \smallsetminus S^k(K \wedge (t \rightarrow u))$, then there is a conjunct $(s_1 \wedge \ldots \wedge s_n) \rightarrow s'$ in $K \wedge (t \rightarrow u)$ such that $s_1 \ldots s_n \in S^k(K \wedge (t \rightarrow u))$. We know that $t \notin S(K)$, therefore $t \notin S^k(K)$ whence, by induction hypothesis, $t \notin S^k(K \wedge (t \rightarrow u))$.

Therefore, $(s_1 \wedge \ldots \wedge s_n) \to s' \neq t \to u$ and so $(s_1 \wedge \ldots \wedge s_n) \to s'$ is a conjunct in K. Also by induction hypothesis, $s_1 \ldots s_n \in S^k(K)$ and finally $s' \in S^{k+1}(K)$. □

Now we can prove our main theorem:

THEOREM 3.5. *No γ-formula is intuitionistically equivalent to $(p \to q) \vee r$.*

Proof. Assume that a γ-formula F intuitionistically follows from $(p \to q) \vee r$. We will show that in this case $\not\models F \to ((p \to q) \vee r)$.

If F intuitionistically follows from $(p \to q) \vee r$, then both

$$\models (p \to q) \to F \tag{2}$$

and

$$\models r \to F. \tag{3}$$

Assume that F is intuitionistically equivalent to the following conjunction of quantified simple formulas:

$$F \;=\!\!|\!\!\models\; \overline{\forall}(K_1 \to s_1) \wedge \ldots \wedge \overline{\forall}(K_n \to s_n)^2\ . \tag{4}$$

We may assume that F is not intuitionistically valid, for otherwise our conclusion would follow immediately. This means that at least one conjunct in (4) is not intuitionistically valid. Moreover, we may even assume that none of the above conjuncts are intuitionistically valid, for, as we already know that some of them are not valid, we can simply omit the other conjuncts from the formula.

Now, consider the Kripke model $M = \langle W, R, V \rangle$ (see Figure 1), where

$$\begin{aligned} &W = \{w, v\},\\ &R = \{\langle w, v\rangle, \langle w, w\rangle, \langle v, v\rangle\},\\ &V(q) = \emptyset,\\ &V(p) = V(r) = \{v\}. \end{aligned}$$

We define M only for p, q, and r because these are the only free variables in the formulas under consideration.

We will show that $M, w \Vdash F$, but $M, w \nVdash (p \to q) \vee r$. The latter claim is immediate. To show the former claim, assume that $t_1 \ldots t_m$ is the list of variables occurring in F which are distinct from p, q, and r. Fix arbitrary valuations for these variables in M and choose any natural i such that $1 \leq i \leq n$. It will suffice to show that $M, w \Vdash K_i \to s_i$ under this valuation of $t_1 \ldots t_m$.

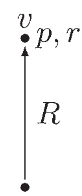

Fig. 1. Structure of M.

[2] We assume that $\overline{\forall}$ binds all variables except for possibly p, q and r.

It follows from (2)–(4) that

$$\models (K_i \wedge r) \rightarrow s_i \tag{5}$$

$$\models (K_i \wedge (p \rightarrow q)) \rightarrow s_i . \tag{6}$$

Since both r and $p \rightarrow q$ are elementary, this means that our criterion of validity from Lemma 3.1 applies, and we have $s_i \in S(K_i \wedge r) \cap S(K_i \wedge (p \rightarrow q))$ for every $1 \leq i \leq n$. Also, since we know that $\not\models K_i \rightarrow s_i$, it follows from (6) and Lemma 3.4 that

$$p \in S(K_i) . \tag{7}$$

We will show the following

Claim. $M, \upsilon \Vdash K_i \rightarrow s_i$.

Once this claim is established, the rest is easy; we observe that by (7) and $M, w \nVdash p$ we have $M, w \nVdash S(K_i)$, and then apply Lemma 3.3 to this fact and our claim to obtain $M, w \Vdash K_i \rightarrow s_i$.

In order to prove the claim, we must consider two cases.

Case 1. Under the fixed valuations of $t_1 \dots t_m$ we have $M, \upsilon \Vdash S(K_i \wedge r)$. Then, by (5) and Lemma 3.1, we know that s_i is in $S(K_i \wedge r)$ and so is true in M, υ, and since υ is an end-world, we also have $M, \upsilon \Vdash K_i \rightarrow s_i$ by Lemma 3.2.

Case 2. Under the fixed valuations of $t_1 \dots t_m$ we have $M, \upsilon \nVdash S(K_i \wedge r)$. We take the least j such that there is a variable u_j such that $M, \upsilon \nVdash u_j$ and $u_j \in S^j(K_i \wedge r)$ and proceed by induction on j.

Basis. If $j = 0$, then u_j must be a conjunct in $K_i \wedge r$, but since $M, \upsilon \nVdash u_j$ but $M, \upsilon \Vdash r$, we have $u_j \neq r$, and so u_j must be a conjunct in K. Therefore, we have $M, \upsilon \nVdash K_i$, and since υ is an end-world, we get $M, \upsilon \Vdash K_i \rightarrow s_i$ by Lemma 3.2.

Induction step. If $j = k + 1$, then for some $m > 0$ there must be a conjunct $(s'_1 \wedge \dots \wedge s'_m) \rightarrow u_j$ in K_i, such that (due to the minimality of j) $M, \upsilon \Vdash s'_1 \dots s'_m$, but not in $M, \upsilon \nVdash u_j$. This means that $M, \upsilon \nVdash (s'_1 \wedge \dots \wedge s'_m) \rightarrow u_j$, and therefore $M, \upsilon \nVdash K_i$. So, since υ is an end-world, we, again by Lemma 3.2, get $M, \upsilon \Vdash K_i \rightarrow s_i$. □

§4. On flattening introduction rules. Using the example $\star$ we have shown that not every connective characterized by flat introduction rules has harmonious flat elimination rules. As a dual question one might ask whether every connective characterized by flat elimination rules has harmonious flat introduction rules. The answer is again negative. Consider the ternary connective $\circ$ with the single elimination rule

$$(\circ\, \mathrm{E}) \quad \frac{\circ(p, q, r) \qquad \begin{matrix}[p] \\ q\end{matrix}}{r} \quad .$$

This connective has

$$(\circ\, \mathrm{I}) \quad \frac{\begin{matrix}[p \Rightarrow q] \\ r\end{matrix}}{\circ(p, q, r)}$$

as a higher-level introduction rule. However, there is no *flat* introduction rule for $\circ$, which is in harmony with ($\circ$ E). In order to establish this result, we show that the formula $(p \rightarrow q) \rightarrow r$, which is the elimination meaning of $\circ$, is not equivalent to any formula of the form $K_1 \vee \dots \vee K_n$, which would be the introduction meaning of $\circ$ for n flat introduction rules for $\circ$.

THEOREM 4.1. *Formula $(p \to q) \to r$ is not intuitionistically equivalent to any formula of the form*

$$K_1 \vee \ldots \vee K_n,$$

where for every $1 \leq i \leq n$, K_i is a conjunction of elementary formulas.

Proof. Assuming that

$$\models (K_1 \vee \ldots \vee K_n) \to ((p \to q) \to r), \tag{8}$$

we will show that

$$\not\models ((p \to q) \to r) \to (K_1 \vee \ldots \vee K_n).$$

By (8) we must have

$$\models (K_i \wedge (p \to q)) \to r$$

for every $1 \leq i \leq n$. Since $K_i \wedge (p \to q)$ is a conjunction of elementary formulas, $(K_i \wedge (p \to q)) \to r$ is a simple formula. Now suppose $p \notin S(K_i)$. Then by Lemma 3.4 we have that $\models K_i \to r$, and therefore, by Lemma 3.1, that $r \in S(K_i)$. Thus for every $1 \leq i \leq n$ we have either $p \in S(K_i)$ or $r \in S(K_i)$.

Then consider the model M consisting of two worlds w and v, where v is accessible from w, but not vice versa. As for the evaluation of variables, assume that $V(w) = \emptyset$ and $V(v) = \{p\}$. It is straightforward that $M, w \models (p \to q) \to r$, but we have neither $M, w \models p$, nor $M, w \models r$, which means that we must have $M, w \not\models S(K_i)$ for every $1 \leq i \leq n$. Therefore, by Lemma 3.1 we have $M, w \not\models K_i$ for every $1 \leq i \leq n$ and so we are done. □

§5. Concluding remarks. The fact that elimination rules cannot always be flattened given certain introduction rules, and, conversely, that introduction rules cannot always be flattened given certain elimination rules, can be generalized to higher levels. Consider an $(n+2)$-place connective $\star_n$ with a higher-level introduction rule giving it the introduction meaning $p_1 \vee (((\ldots(p_2 \to p_3) \ldots \to p_n) \to p_{n+1}) \to p_{n+2})$. Then any appropriate set of (harmonious) elimination rules must exceed the level of the introduction rules. That is, the elimination meaning of $\star_n$ cannot be described by a formula using lower or equal nesting of implications to the left. The proof of this fact, which is more involved than the one given here for $\star$, is presented in Olkhovikov & Schroeder-Heister (2014). This paper also contains a proof of the dual result that an $(n+2)$-place connective $\circ_n$ with a higher-level elimination rule giving it the elimination meaning $((\ldots(p_1 \to p_2) \ldots \to p_n) \to p_{n+1}) \to p_{n+2}$ cannot be given harmonious introduction rules of equal or lower level than the elimination rules. That is, its introduction meaning cannot be described by using equal or lower nesting of implications to the left. Thus, as a general result, we have that, when moving either from introductions to eliminations or from eliminations to introductions in a general setting, one needs to increase the level of rules. This supports the idea of rules of higher levels as proposed in Schroeder-Heister (1984), at least as long as a *general* schema for introductions and eliminations is concerned. It speaks against general eliminations rules following the pattern of $(\to \mathrm{E}_{\mathrm{GEN}})$ as the basis of a general schema.

It is clear that our results depend on the syntactic forms which introduction and elimination rules are allowed to take. Our schemata for introductions and eliminations are well motivated as long as we are dealing with connectives, which are characterized in the context of natural deduction with the standard structural assumptions presupposed and without additional means of expression. When this background is changed, for example by admitting a classically behaved denial operator or by switching to a multiple-conclusion framework we will obtain different results. In a classical context, $\star$ would have flat elimination rules, as pointed out by Read (2014) (he uses the connective with the meaning $(p_1 \rightarrow p_2) \vee (p_2 \rightarrow p_1)$ as discussed by Dyckhoff—see below). A modal context would change the situation as well. Thus our result applies first and foremost to intuitionistically inspired proof-theoretic semantics based on standard natural deduction consequence, even though it can be extended to other fields such as relevant logic.

Historically, a connective similar to $\star$ was already discussed as early as 1968 by von Kutschera (1968, p. 15) as a counterexample to flat elimination rules (von Kutschera used the ternary connective with the meaning $(p_1 \rightarrow p_2) \vee (p_3 \rightarrow p_2)$), and by Zucker & Tragesser (1978). More recently, similar connectives have been investigated by Dyckhoff (2009) (he uses the binary connective with the meaning $(p_1 \rightarrow p_2) \vee (p_2 \rightarrow p_1)$), Read (2014) and Schroeder-Heister (2013). However, no attempt has been made in the literature to make the strong intuition that $\star$-like connectives cannot be captured by flat elimination rules formally precise and to prove that this intuition is correct.

In this paper we have focussed on the nonflattening result. The approach to consider independent schemata for introduction and elimination rules and to use second-order propositional logic to describe the introduction and elimination meaning of connectives is significant beyond the results presented here, in particular, as it provides a formal account of proof-theoretic harmony and of notions such as conservativeness and uniqueness. This issue is discussed in Schroeder-Heister (2014b).

§6. Acknowledgements. This work was carried out within the French-German ANR-DFG project "Hypothetical Reasoning—Its Proof-Theoretic Analysis" (HYPOTHESES) (DFG Schr 275/16-2). We would like to thank two anonymous reviewers for their helpful comments and suggestions.

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