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Failure of Completeness in Proof-Theoretic Semantics

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Abstract Several proof-theoretic notions of validity have been proposed in the literature, for which completeness of intuitionistic logic has been conjectured. We define validity for intuitionistic propositional logic in a way which is common to many of these notions, emphasizing that an appropriate notion of validity must be closed under substitution. In this definition we consider atomic systems whose rules are not only production rules, but may include rules that allow one to discharge assumptions. Our central result shows that Harrop's rule is valid under substitution, which refutes the completeness conjecture for intuitionistic logic.

Keywords Proof-theoretic semantics · Intuitionistic logic · Mints's rule · Harrop's rule · Completeness

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1 Introduction

Within proof-theoretic semantics [25] certain notions of validity have been proposed, notably by Prawitz [16–19] (for a discussion and overview see [24]; cf. also [2]). Prawitz [17, 19] conjectured that intuitionistic first-order logic is complete

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with respect to one such notion. We show that this conjecture is not even true for propositional logic, if certain plausible assumptions are made about validity.

For a language without disjunction, Sandqvist [20] has shown that the laws of classical logic are valid with respect to a variant of proof-theoretic semantics, which corresponds to the one we are using here (for a discussion see [3] and [12]). His result cannot be extended to formulas containing disjunction, unless disjunction is defined classically, e.g., in terms of implication and negation. However, if we want to give a proper counterexample to the completeness of intuitionistic logic, we need to find a formula or rule which is not derivable in a calculus of intuitionistic logic, though *all its substitution instances*, including those containing disjunction, are valid. A formula which is valid, but one of whose substitution instances is not valid, can never be derivable in intuitionistic logic for the trivial reason that derivability in intuitionistic logic is closed under substitution. That a notion of validity is not closed under substitution is, of course, a highly significant result in itself, but a result which rather demonstrates that such a notion is not even a candidate for completeness. Therefore a thorough discussion of completeness or incompleteness of intuitionistic logic should at least consider a concept of validity closed under substitution. This does not infringe Sandqvist's justification of classical logic, where only disjunction-free substitution instances need to be considered, with respect to which validity is indeed closed under substitution.

Like all other notions of validity in the works mentioned above, we rely on atomic systems, with respect to which the validity of atomic formulas is defined. However, we not only consider standard atomic systems whose rules are production rules, but also atomic systems whose rules can discharge assumptions.

2 Validity

We define a notion of validity which is not necessarily closed under substitution. We then define *valid under substitution* as *valid for all substitution instances*, so that validity under substitution is by definition closed under substitution. Our proof-theoretic notion of (intuitionistic) validity for propositional logic is based on atomic deductive systems, which determine the validity of atomic formulas. The validity of complex formulas is defined inductively relative to such systems. In this section we use, for simplicity, atomic systems based on production rules. In Section 5 we consider atomic systems whose rules can discharge assumptions.

We use propositional formulas $A, B, \ldots, A_1, A_2, \ldots$ constructed from proposition letters, called *atoms*, $\bot, a, b, \ldots, a_1, a_2, \ldots$ by means of the logical constants $\rightarrow$, $\vee$ and $\wedge$. We use $\neg A$ as an abbreviation for $A \rightarrow \bot$. It is crucial that $\bot$ is an atom, as this makes it possible to deal with minimal negation independently of *ex falso quodlibet*.

Definition 1 An *atomic system S* is a (possibly empty) set of atomic rules of the form

$$\frac{a_1 \quad \ldots \quad a_n}{b}$$

where the a_i and b are atoms. The set of premisses $\{a_1, \ldots, a_n\}$ in a rule can be empty; in this case the rule is an atomic axiom.

An atomic system S_1 is an *extension* of an atomic system S (written $S_1 \supseteq S$), if S_1 results from adding a (possibly empty) set of atomic rules to S. The *derivability* of an atomic formula a from a (possibly empty) set $\{a_1, \ldots, a_n\}$ of atomic assumptions in an atomic system S is written $a_1, \ldots, a_n \vdash_S a$.

Definition 2 *S-validity* ($\vDash_S$) and *validity* ($\vDash$) are defined as follows:

(S1) $\vDash_S a \;:\Longleftrightarrow\; \vdash_S a$,
(S2) $\vDash_S A \to B \;:\Longleftrightarrow\; A \vDash_S B$,
(S3) $\Gamma \vDash_S A \;:\Longleftrightarrow\; \forall S_1 \supseteq S : (\vDash_{S_1} \Gamma \Longrightarrow \vDash_{S_1} A)$, where Γ is a set of formulas, and where $\vDash_{S_1} \Gamma$ stands for $\{\vDash_{S_1} A_i \mid A_i \in \Gamma\}$,
(S4) $\vDash_S A_1 \vee A_2 \;:\Longleftrightarrow\; \vDash_S A_1$ or $\vDash_S A_2$,
(S5) $\vDash_S A_1 \wedge A_2 \;:\Longleftrightarrow\; \vDash_S A_1$ and $\vDash_S A_2$,
(S5) $\Gamma \vDash_S \Delta \;:\Longleftrightarrow\; \forall A \in \Delta : \Gamma \vDash_S A$,
(S6) $\Gamma \vDash A \;:\Longleftrightarrow\; \forall S : \Gamma \vDash_S A$.

Definition 3 *S-validity under substitution* ($\Vdash_S$) and *validity under substitution* ($\Vdash$) are defined as follows:

(i) $\Gamma \Vdash_S A \;:\Longleftrightarrow\;$ for each substitution instance Γ',A' of Γ,A: $\Gamma' \vDash_S A'$.
(ii) $\Gamma \Vdash A \;:\Longleftrightarrow\;$ for each substitution instance Γ',A' of Γ,A: $\Gamma' \vDash A'$.

Definition 4 *Intuitionistic S-validity* ($\vDash^i_S$) is defined as follows. Suppose $(\bot)$ stands for the set of rules $\left\{\frac{\bot}{a}\middle| a \text{ atomic}\right\}$. Then $\Gamma \vDash^i_S A \;:\Longleftrightarrow\; \Gamma \vDash_{S\cup(\bot)} A$. Correspondingly, $\Gamma \vDash^i A$, $\Gamma \Vdash^i_S A$ and $\Gamma \Vdash^i A$ are defined as $\Gamma \vDash_{(\bot)} A$, $\Gamma \Vdash_{S\cup(\bot)} A$ and $\Gamma \Vdash_{(\bot)} A$, respectively.

In Prawitz's original definitions [16–19], validity is defined for derivations rather than for formulas, and is relativized not only to atomic systems, but also to proof reductions ('justifications'). However, the formulation in Definition 2, which avoids the explicit mentioning of justifications, suffices to make our point. More delicate is the question of whether in (S3) it is appropriate at all to use extensions of atomic systems (a point which makes our definition similar to the definition of validity in a specific Kripke structure). This point is not entirely clear in Prawitz's writings and will be discussed in the final section. We consider the reference to extensions of S to be absolutely essential, as it guarantees the monotonicity of $\vDash$ with respect to S.

A crucial point in our dealing with absurdity (and thus negation) is that we do not, as in Kripke semantics, define absurdity to be something that cannot be validated in any atomic system. If we defined $\bot$ to be a non-atomic constant with the semantical clause

$$\text{There is no } S \text{ such that } \vDash_S \bot$$

we could verify $\neg\neg a$ for any atom a, because $\neg a$ is never valid in any $S_1 \supseteq S$, as a will always become valid in some $S_2 \supseteq S_1$. This fact, that any atom a is validated in some extension of any atomic system, might be considered a fault of validity-based proof-theoretic semantics, since it speaks against the intuitionistic idea of negation $\neg A$ as expressing that A can never be verified. We do not deal with this issue here.

The fact that we consider absurdity $\bot$ to be a distinguished atom means that we have defined a notion of minimal validity, where "minimal" is understood in the sense of minimal logic. If we added $\curlywedge$ as a non-atomic constant with the semantic clause

$$\vDash_S \curlywedge \;:\Longleftrightarrow\; \forall a : \;\vDash_S a$$

(which is the clause used by Sandqvist [20] following Dummett [4, Ch. 13]) then, in the presence of *ex falso quodlibet*, the non-atomic $\curlywedge$ and the atomic $\bot$ become equivalent, more precisely, $\bot \vDash^i \curlywedge$ and $\curlywedge \vDash \bot$.

Lemma 1 (**Properties of S-validity**)

(P1) $\vDash_S$ *is a consequence relation, i.e.,*

- (i) $A \vDash_S A,$
- (ii) $\Gamma \vDash_S A \implies \Gamma, \Delta \vDash_S A,$
- (iii) $(\Gamma \vDash_S A$ *and* $\Delta, A \vDash_S B) \implies \Gamma, \Delta \vDash_S B.$

(P2) $\vDash_S$ *is monotone w.r.t. S, i.e.,* $\Gamma \vDash_S A \implies \forall S_1 \supseteq S : \Gamma \vDash_{S_1} A.$
(P3) $\Gamma \vDash_S A \to B \iff \Gamma, A \vDash_S B.$
(P4) $a_1, \ldots, a_n \vDash_S b \iff a_1, \ldots, a_n \vdash_S b.$

These properties also hold for intuitionistic S-validity, i.e., for $\vDash_S$ replaced with $\vDash^i_S$.

Proof Straightforward. □

Note that (P4) is an atomic completeness result: S-valid consequence between atoms coincides with derivability in S.

3 Formulas and Rules

There is an obvious correspondence between rules and formulas of a certain form. Any atomic system S can be represented by a set of formulas S^*, if axioms and rules are translated into formulas as follows:

Definition 5 The atom a *represents* the axiom a, and the formula $a_1 \wedge \ldots \wedge a_n \to b$ *represents* the rule $\frac{a_1 \quad \ldots \quad a_n}{b}$. Then S^* is defined as the set of formulas representing the axioms and rules in S.

Conversely, any disjunction-free formula A without any left-iterated implication as subformula can be translated into a set of rules S° (a left-iterated implication is an implicational formula $A_1 \to A_2$, such that A_1 contains an implication). Obviously, any such A can be transformed into a set of formulas $A_1, \ldots, A_m$ of the form $a_1 \wedge \ldots \wedge a_n \to b$ by (repeatedly) replacing any $B \to (C \to D)$ with $(B \wedge C) \to D$ and any $B \to (C_1 \wedge \ldots \wedge C_k)$ with the list $B \to C_1, \ldots, B \to C_k$. Call the resulting set of formulas S'. Then we proceed as follows:

Definition 6 The axiom *a corresponds to* the atom a, and the rule $\frac{a_1 \quad \ldots \quad a_n}{b}$ *corresponds to* the formula $a_1 \wedge \ldots \wedge a_n \to b$. Then S° is defined as the set of rules corresponding to the formulas in S'.

Lemma 2 (**Properties of the formula-rule correspondence**)

(C1) $S^{*\circ} = S, \quad \Delta^{\circ *} \dashv\models \Delta.$
(C2) $\models_S S^*, \quad \models_{\Delta^\circ} \Delta.$
(C3) $(\models_S S_1^* \textit{ and } \Gamma \models_{S \cup S_1} A) \implies \Gamma \models_S A, (\models_S \Delta \textit{ and } \Gamma \models_{S \cup \Delta^\circ} A) \implies \Gamma \models_S A.$
(C4) $\Gamma \models_S A \iff \Gamma, S^* \models A, \quad \Gamma \models_{\Delta^\circ} A \iff \Gamma, \Delta \models A.$

These properties also hold for intuitionistic S*-validity, i.e., for* $\models_S$ *replaced with* $\models^i_S$.

Proof We show the first claim of each pair of propositions.
(C1) and (C2): Straightforward.
(C3): By induction. $\models_{S \cup S_1} a \iff \vdash_{S \cup S_1} a$. Lemma 1, (P4) implies for $\models_S S_1^*$ that all rules of S_1 are derivable in S. One therefore obtains $\vdash_S a$, and thus $\models_S a$.

$$
\begin{aligned}
\models_{S \cup S_1} A \to B &\iff A \models_{S \cup S_1} B && \text{by (P3)} \\
&\implies A \models_S B && \text{by } \models_S S_1^* \text{ and i.h.} \\
&\implies \models_S A \to B && \text{by (P3).}
\end{aligned}
$$

(The cases for the remaining connectives are also trivial.)

$$
\begin{aligned}
\Gamma \models_{S \cup S_1} A &\iff \forall S_2 \supseteq (S \cup S_1) : (\models_{S_2} \Gamma \implies \models_{S_2} A) \\
&\iff \forall S_3 : (\models_{S_3 \cup S \cup S_1} \Gamma \implies \models_{S_3 \cup S \cup S_1} A) \\
&\iff \forall S_3 : (\models_{S_3 \cup S} \Gamma \implies \models_{S_3 \cup S} A) && \text{by } \models_{S_3 \cup S} S_1^* \text{ and i.h.} \\
&\iff \Gamma \models_S A && \text{by Def.}
\end{aligned}
$$

(C4): "$\Longleftarrow$" follows from (C2) and Lemma 1, (P1), (iii).
"$\Longrightarrow$": $\Gamma \models_S A \iff \forall S_1 \supseteq S : (\models_{S_1} \Gamma \implies \models_{S_1} A) \iff \forall S_1 : (\models_{S \cup S_1} \Gamma \implies \models_{S \cup S_1} A)$. Suppose $\models_{S_2} S^*$ and $\models_{S_2} \Gamma$. Then $\models_{S \cup S_2} \Gamma$ by Lemma 1, (P2), and therefore $\models_{S \cup S_2} A$. From $\models_{S_2} S^*$ and (C3) one gets $\models_{S_2} A$. □

4 The Failure of Strong Completeness

We now consider natural deduction for intuitionistic logic and show that it is not complete for validity.

Definition 7 *Natural deduction for intuitionistic logic NI* is given by the following rules:

$$\dfrac{\begin{matrix}[A]\\ B\end{matrix}}{A \to B}(\to\mathrm{I}) \qquad \dfrac{A \quad A \to B}{B}(\to\mathrm{E})$$

$$\dfrac{A_i}{A_1 \vee A_2}(\vee\mathrm{I})\ (i = 1 \text{ or } 2) \qquad \dfrac{A_1 \vee A_2 \quad \begin{matrix}[A_1]\\ C\end{matrix} \quad \begin{matrix}[A_2]\\ C\end{matrix}}{C}(\vee\mathrm{E})$$

$$\dfrac{A_1 \quad A_2}{A_1 \wedge A_2}(\wedge\mathrm{I}) \qquad \dfrac{A_1 \wedge A_2}{A_i}(\wedge\mathrm{E})\ (i = 1 \text{ or } 2)$$

$$\dfrac{\bot}{a}(\bot)$$

Note that the rule $(\bot)$ can be assumed to have only atomic conclusions.

The *derivability* of a formula A from a (possibly empty) set of assumptions Γ over an atomic system S is written $\Gamma \vdash_S A$, and derivability in *NI* is written $\Gamma \vdash A$.

Definition 8 (i) *Soundness* of *NI* means: $\Gamma \vdash A \implies \Gamma \vDash^i A$.
(ii) *Strong completeness* of *NI* means: $\Gamma \vDash^i A \implies \Gamma \vdash A$.
(iii) *Completeness* (simpliciter) of *NI* means: $\Gamma \Vdash^i A \implies \Gamma \vdash A$.

Since derivability in *NI* is closed under substitution, soundness implies $\Gamma \Vdash^i A$. As remarked in the introduction, we are mainly interested in completeness rather than strong completeness. Strong completeness parallels a concept of validity, which is not necessarily closed under substitution, with derivability in intuitionistic logic. Therefore, if we do not have strong completeness, this may simply be due to the fact that validity is not closed under substitution, whereas derivability in intuitionistic logic is closed under substitution. In this sense, intuitionistic validity under substitution (and not intuitionistic validity) is the proper concept to be compared to intuitionistic derivability. We shall nevertheless present some results on the failure of strong completeness before we proceed to our main result, which is the failure of completeness (simpliciter).

Lemma 3 *NI is sound.*

Proof By induction on the structure of derivations. □

Theorem 1 *NI is not strongly complete.*

We present and discuss three proofs of this theorem.

Proof 1 In his justification of classical logic, Sandqvist [20] proved $\neg\neg a \vDash^i a$ for any atom a, and even showed that this holds when a is replaced with any formula A not containing disjunction. (We are, of course, using the terminology developed in the present paper.) This was part of his soundness theorem for classical logic. Although he did not explicitly deal with the incompleteness of intuitionistic logic, his result obviously demonstrates that *NI* is not strongly complete, as $\neg\neg a \vdash a$ is false. □

This result is essentially due to the fact that validity is not closed under substitution. For example, $\neg\neg(a \vee \neg a) \vDash^i a \vee \neg a$ is not true, since, using standard principles for $\vDash^i$, we can easily conclude $\vDash^i a \vee \neg a$, which is obviously false. We interpret this as showing that the notion of validity is not properly framed, if it is to be compared with derivability.

Proof 2 It is clear that $a \rightarrow (b \vee c) \vdash (a \rightarrow b) \vee (a \rightarrow c)$ is false. We show that $a \rightarrow (b \vee c) \vDash (a \rightarrow b) \vee (a \rightarrow c)$ holds. Suppose that $\vDash_S a \rightarrow (b \vee c)$ for some atomic system S. We have to show that $\vDash_S (a \rightarrow b) \vee (a \rightarrow c)$. We know that for every $S_1 \supseteq S$ for which $\vDash_{S_1} a$, we also have that $\vDash_{S_1} b \vee c$, which means that either $\vDash_{S_1} b$ or $\vDash_{S_1} c$. Now choose S_1 to be S extended with a as an axiom. Then by (C4) either $a \vDash_S b$ or $a \vDash_S c$, which implies $\vDash_S (a \rightarrow b) \vee (a \rightarrow c)$. □

This is a counterexample against strong completeness of minimal logic and thus of *NI*, which has also been presented by Goldfarb [5] in his discussion of Dummett's boundary rules. Like Sandqvist's double-negation example, it shows that validity is not closed under substitution. If $b \vee c$ were substituted for a, then it would have to be shown that either $\vDash (b \vee c) \rightarrow b$ or $\vDash (b \vee c) \rightarrow c$, which cannot be achieved. The advantage of this counterexample over Sandqvist's is that it is not tied to the particular format of atomic systems. It just expects that we can extend an atomic system by adding an atom as an axiom, whereas Sandqvist's example ceases to be valid if we consider atomic systems with assumption-discharging rules, which we shall consider below in Section 5.

Proof 3 As is well-known, *Mints's rule*

$$\frac{(A \rightarrow B) \rightarrow (A \vee C)}{((A \rightarrow B) \rightarrow A) \vee ((A \rightarrow B) \rightarrow C)}$$

is not derivable in *NI*, i.e., $(A \rightarrow B) \rightarrow (A \vee C) \nvdash ((A \rightarrow B) \rightarrow A) \vee ((A \rightarrow B) \rightarrow C)$ (cf. Mints [14]). However, assuming strong completeness, we show $(A \rightarrow B) \rightarrow (A \vee C) \vDash^i ((A \rightarrow B) \rightarrow A) \vee ((A \rightarrow B) \rightarrow C)$, which contradicts completeness. Therefore *NI* is not complete. Suppose $\vDash^i_S (A \rightarrow B) \rightarrow (A \vee C)$. By (C4), $S^* \vDash^i (A \rightarrow B) \rightarrow (A \vee C)$. Assuming strong completeness, $S^* \vdash (A \rightarrow B) \rightarrow (A \vee C)$, i.e., there is an open derivation in *NI* of the premiss of Mints's rule from assumptions S^*. This derivation can be transformed into normal form. Since S^* does not contain disjunctions, this normal form must be of the following form, having an introduction rule in the last step:

$$\frac{\begin{array}{c}[A \rightarrow B]^n, S^* \\ \mathscr{D} \\ A \vee C\end{array}}{(A \rightarrow B) \rightarrow (A \vee C)}\,(\rightarrow\mathrm{I})^n$$

The subderivation $\mathscr{D}$ either ends with ($\vee$I) or with an elimination rule. (It cannot end with the rule ($\bot$), which has only atomic conclusions.) If $\mathscr{D}$ ends with ($\vee$I), then either $A \to B, S^* \vdash A$ or $A \to B, S^* \vdash C$. If $\mathscr{D}$ ends with an elimination rule, then there is a path through formulas $F_1, \ldots, F_n, F_{n+1}$, where each $F_1, \ldots, F_n$ is the major premiss of an elimination rule, and F_{n+1} is either the major premiss of an elimination rule or the endformula. The path starts with F_1, which is the open assumption $A \to B$ and major premiss of an application of ($\to$E). Hence, there is a derivation of the minor premiss A of this application of ($\to$E), i.e., $A \to B, S^* \vdash A$ or even $S^* \vdash A$. If $A \to B, S^* \vdash A$ or $S^* \vdash A$, then $S^* \vdash (A \to B) \to A$, and if $A \to B, S^* \vdash C$, then $S^* \vdash (A \to B) \to C$, each by ($\to$I). In both cases $S^* \vdash ((A \to B) \to A) \vee ((A \to B) \to C)$, by ($\vee$I). By soundness $S^* \vDash^i ((A \to B) \to A) \vee ((A \to B) \to C)$, and $\vDash^i_S ((A \to B) \to A) \vee ((A \to B) \to C)$ by (C4). □

This proof is independent of the form of the formulas A, B and C, i.e., it holds for A, B and C used as schematic letters for arbitrary formulas. However, our proof is indirect. Assuming strong completeness, it shows that Mints's rule is valid under substitution, thus providing a counterexample to completeness (hence *a fortiori* to strong completeness). What makes this proof interesting is that it does not rely in an obvious way on the fact that validity is not closed under substitution. The assumption $S^* \vDash^i (A \to B) \to (A \vee C)$, to which the hypothesis of strong completeness was applied, is not in any clearcut manner non-closed under substitution. It might even be that it is closed under substitution after all, so that we have a proper refutation of completeness. As the previous one, this proof is not dependent on the format of atomic systems. If we consider assumption-discharging rules, everything stays as it is.

5 Atomic Higher-Level Rules

In order to give a direct counterexample to completeness, we extend the notion of an atomic system by allowing for rules that discharge assumptions. Atomic systems in the sense of Definition 1 are now called *first-level atomic systems*.

Definition 9 A *second-level atomic system* S is a (possibly empty) set of atomic rules of the form

$$\frac{\begin{matrix}[\Gamma_1] \\ a_1\end{matrix} \quad \ldots \quad \begin{matrix}[\Gamma_n] \\ a_n\end{matrix}}{b}$$

where the a_i and b are atoms, and the Γ_i are finite sets of atoms. The sets Γ_i may be empty, in which case the rule is a first-level rule. The set of premisses of this rule can be empty as well, in which case the rule is also called an axiom.

The intended meaning of such a rule as suggested by the notation is as follows: If in S we have derived $a_1, \ldots, a_n$ from certain assumptions, then we may pass over to b, where, for each i, in the branch of the subderivation leading to a_i assumptions belonging to Γ_i may be discharged. Rules which discharge assumptions are present in logical calculi, for example in the implication introduction or disjunction elimination rules in *NI*. Here the idea of having rules discharging assumptions is carried over to the atomic case. As before, the *derivability* of an atomic formula a from a (possibly empty) set $\{a_1, \ldots, a_n\}$ of atomic assumptions in an atomic system S is written $a_1, \ldots, a_n \vdash_S a$.

This idea of atomic discharging rules can be extended to the higher-level case where not only atoms but atomic rules can be introduced and discharged as assumptions, an idea first proposed in [22] (for the more general case of arbitrary, non-atomic rules). We cannot spell out the full formalism of this approach here, but sketch it in sufficient detail. A recent exposition can be found in [26] and [15]. First we need a linear notation for rules. A basic rule of a first-level atomic system (Definition 1) is linearly written as $a_1, \ldots, a_n \rhd b$, a basic rule of a second-level atomic system (Definition 9) as $(\Gamma_1 \rhd a_1), \ldots, (\Gamma_n \rhd a_n) \rhd b$. The precise definition of atomic higher-level rules runs as follows:

Definition 10 (i) Every atom a is a rule of level 0.
(ii) If $R_1, \ldots, R_n$ are rules ($n \geq 1$), whose maximal level is ℓ, and a is an atom, then $(R_1, \ldots, R_n \rhd a)$ is a rule of level $\ell + 1$.

The intended meaning of a rule $(\Gamma_1 \rhd a_1), \ldots, (\Gamma_n \rhd a_n) \rhd b$ is nothing but a generalization of the second-level case: Suppose, for each i $(1 \leq i \leq n)$, we have derived a_i from Γ_i; then we may pass over to b. This gives rise to the notion of a higher-level atomic system.

Definition 11 A *higher-level atomic system* S is a (possibly empty) set of atomic rules of the form

$$\frac{\begin{matrix}[\Gamma_1] \\ a_1\end{matrix} \quad \ldots \quad \begin{matrix}[\Gamma_n] \\ a_n\end{matrix}}{b}$$

where the a_i and b are atoms, and the Γ_i are now finite sets of rules, which may be empty. The set of premisses of such a rule can be empty as well, in which case the rule is also called an axiom.

The fundamental difference to the second-level case is that now *rules* and not only atoms can function as assumptions, which can be discharged. This has to be taken into account to define the notion of a *derivation* of an atom a from rules $R_1, \ldots, R_n$.

Definition 12 For a level-0 rule a,

$$\frac{}{a}\,a$$

is a *derivation* of a from $\{a\}$.

Now consider a level-$(\ell + 1)$ rule $(\Gamma_1 \rhd a_1), \ldots, (\Gamma_n \rhd a_n) \rhd b$. Suppose that, for each i $(1 \leq i \leq n)$ a derivation

$$\begin{matrix}\Sigma_i \cup \Gamma_i \\ \mathscr{D}_i \\ a_i\end{matrix}$$

of a_i from $\Sigma_i \cup \Gamma_i$ is given. Then

$$\frac{\begin{array}{ccc}\Sigma_1 & & \Sigma_n\\ \mathscr{D}_1 & & \mathscr{D}_n \\ a_1 & \ldots & a_n\end{array}}{b}\,(\Gamma_1 \rhd a_1), \ldots, (\Gamma_n \rhd a_n) \rhd b$$

is a *derivation* of b from $\Sigma_1 \cup \ldots \cup \Sigma_n \cup \{(\Gamma_1 \rhd a_1), \ldots, (\Gamma_n \rhd a_n) \rhd b\}$.

We say that b is *derivable* from Σ in a higher-level atomic system S, symbolically $\Sigma \vdash_S b$, if there is a derivation of b from $\Sigma \cup S$.

An example may illustrate what a particular derivation looks like. Suppose the atomic system S comprises the rules $(b \rhd e) \rhd f$ and $((a \rhd b) \rhd c) \rhd e$. Then the following derivation demonstrates that $((a \rhd b) \rhd d), ((b, d) \rhd c) \vdash_S f$:

$$3\,\dfrac{2\,\dfrac{\dfrac{\dfrac{}{b}\,[b]^3 \qquad 1\,\dfrac{\dfrac{\dfrac{}{a}\,[a]^1}{b}\,[a \rhd b]^2}{d}\,(a \rhd b) \rhd d}{c}\,b, d \rhd c}{e}\,\langle((a \rhd b) \rhd c) \rhd e\rangle}{f}\,\langle(b \rhd e) \rhd f\rangle$$

Here, the rules enclosed in angle brackets $\langle \ldots \rangle$ are primitive rules of S. As usual, square brackets $[\ldots]$ with numerals indicate the discharge of assumptions. The definitions of S-validity (Definitions 2, 3 and 4) remain unchanged, with the reference to derivability in S now understood in the higher-level sense.

The translation of atomic rules into formulas and vice versa (Definitions 5 and 6) can easily be carried over to the higher-level case as follows.

Definition 13 With every rule R in a set of rules S a formula R^* representing R is associated as follows:

(i) $a^* := a$, for atoms a.
(ii) $(R_1, \ldots, R_n \rhd a)^* := R_1^* \wedge \ldots \wedge R_n^* \to a$, for a rule $R_1, \ldots, R_n \rhd a$.

Then S^* is defined as the set of formulas representing the rules in S.

Conversely, with a formula A not containing disjunction a rule or finite set of rules S° is associated as follows. Carry out the following transformations on subformulas until an irreducible formula A' is reached:

(i) Replace any subformula of the form $C \to D_1 \wedge \ldots \wedge D_n$ with $(C \to D_1) \wedge \ldots \wedge (C \to D_n)$.
(ii) Replace any subformula of the form $C \to (D \to E)$ with $(C \wedge D) \to E$.

Then the operation $^\#$ associating a rule or set of rules with A' is defined as follows:

(i) $a^\# := a$, for atoms a,
(ii) $((B_1 \wedge \ldots \wedge B_n) \to a)^\# := B_1^\#, \ldots, B_n^\# \rhd a$,
(iii) $(B_1 \wedge \ldots \wedge B_n)^\# := \{B_1^\#, \ldots, B_n^\#\}$.

Finally set $S^\circ := A'^{\#}$.

As an example, if A is the formula $(a \wedge b) \to (c \wedge ((d \to \bot) \to \bot))$, then A' is the formula $((a \wedge b) \to c) \wedge ((a \wedge b \wedge (d \to \bot)) \to \bot)$, and S° is the set consisting of the two rules $a, b \rhd c$ and $a, b, (d \rhd \bot) \rhd \bot$.

The properties (P1)-(P3) in Lemma 1 continue to hold. Note that (P1)-(P3) also hold for the intuitionistic case, which we need now. Lemma (1), (P4) now takes the form

$$\Delta^* \vDash_S b \iff \Delta^* \vdash_S b$$

where Δ^* is the set of formulas representing a finite set Δ of atomic rules.

Lemma 2 continues to hold with * and ° understood in the new way.

6 The Failure of Completeness

First we note as lemmas two interesting completeness results, which show that in the current framework of higher-level atomic systems strong completeness holds for disjunction-free formulas as well as for arbitrary negativeformulas.

Lemma 4 (**Strong completeness for disjunction-free formulas**) *Suppose Γ and A do not contain disjunction. Then $\Gamma \vDash^i A \iff \Gamma \vdash A$.*

Proof Follows immediately from Lemma 1, (P4), together with the translation between formulas and rules. □

Remark 1 For disjunction-free Γ and A we also have strong minimal completeness $\Gamma \vDash A \iff \Gamma \vdash^m A$, where $\vdash^m$ denotes derivability in minimal logic, i.e., without using the rule $(\bot)$.

Lemma 4 depends on the availability of higher-level rules, which makes it possible to represent any nested implication as a rule. It can be extended to arbitrary negative formulas, as from negative formulas disjunctions can be eliminated.

Lemma 5 (i) *Any formula $\neg A$ is intuitionistically equivalent to a formula A', which does not contain disjunction.*
(ii) $\vDash^i_S \neg A \iff \vDash^i_S A'$ *for any S.*

Proof (i) The following equivalences hold for *NI* (see [9, §§26-27]):
$\neg(A \vee B) \dashv\vdash \neg A \wedge \neg B$, $\neg(A \wedge B) \dashv\vdash \neg(\neg\neg A \wedge \neg\neg B)$, $\neg(A \to B) \dashv\vdash \neg\neg A \wedge \neg B$.
(ii) Soundness of intuitionistic logic. □

Remark 2 This result does not hold in the framework of minimal logic, as we need *ex falso quodlibet* (i.e., the rule $(\bot)$) in the translation of a negated implication.

Lemma 6 (**Strong completeness for negative formulas**) *For any formula of the form* $\neg A$ *it holds that* $\vDash^i \neg A \iff \vdash \neg A$.

Proof Suppose $\vDash^i \neg A$. By Lemma 5, we we have $\vDash^i A'$ for some disjunction-free A' which is intuitionistically equivalent to $\neg A$. By Lemma 4 we have that $\vdash A'$, which again is equivalent to $\vdash \neg A$. □

Now we can present our counterexample to the completeness of intuitionistic logic.

Theorem 2 *Intuitionistic logic is not complete with respect to the semantics based on higher-level atomic systems.*

Proof Harrop's rule[1]

$$\frac{\neg a \to (b \vee c)}{(\neg a \to b) \vee (\neg a \to c)}$$

is not derivable in intuitionistic logic, i.e., $\neg a \to (b \vee c) \nvdash (\neg a \to b) \vee (\neg a \to c)$.[2] We show that $\neg A \to (B \vee C) \vDash^i (\neg A \to B) \vee (\neg A \to C)$ holds for any formulas A, B, C, which means that Harrop's formula is intuitionistically valid under substitution. Suppose that $\vDash^i_S \neg A \to (B \vee C)$. We have to show that $\vDash^i_S (\neg A \to B) \vee (\neg A \to C)$. We know that for every $S_1 \supseteq S$ for which $\vDash^i_{S_1} \neg A$, we also have that $\vDash^i_{S_1} B \vee C$, which means that either $\vDash^i_{S_1} B$ or $\vDash^i_{S_1} C$. By Lemma 5, $\vDash^i_{S_1} \neg A$ is equivalent to $\vDash^i_{S_1} A'$ for some disjunction-free A' which is intuitionistically equivalent to $\neg A$. Now choose S_1 to be $S \cup (A')^\circ$. Then by (C2) we know that $\vDash^i_{S_1} A'$, and therefore $\vDash^i_{S_1} \neg A$. Thus either $\vDash^i_{S_1} B$ or $\vDash^i_{S_1} C$. Thus, by (C4), either $A' \vDash^i_S B$ or $A' \vDash^i_S C$, i.e., either $\neg A \vDash^i_S B$ or $\neg A \vDash^i_S C$. □

7 Critical Discussion

By means of a counterexample, we have shown that intuitionistic logic is incomplete for a semantics based on higher-level atomic systems. By appropriate coding, the usage of higher-level rules can be reduced to the usage of second-level rules (see [21]). Thus, in effect, we have shown the incompleteness of intuitionistic logic for a semantics based on second-level atomic systems. However, there is no way in sight how to carry over this result to a semantics based on standard first-level rules in the sense of Definition 1 (in the following called *standard semantics*). The arguments in Section 4 show that intuitionistic logic is not strongly complete for standard semantics. This means that the question of whether intuitionistic logic is complete (simpliciter) with respect to standard semantics is still open.

[1] Also known as *Kreisel-Putnam rule* (cf. [10]) or *independence of premiss rule*.

[2] Harrop's rule was proposed as an example of a formula, which is admissible, but not derivable in intuitionistic logic (see [8]). It should be pointed out that admissibility is different from validity, although there are some similarities between these concepts (see [3]).

The significance of this point is reinforced by the fact that serious objections can be raised against second-level and higher-level atomic systems. By admitting atomic rules that discharge assumptions, a great deal of logic is already put into the atomic system, namely fundamental ideas underlying the framing of implication in natural deduction. With higher-level atomic systems, everything that is independent of disjunction is already present at the atomic level. This is reflected by the fact that the strong completeness of implication-conjunction logic is nearly trivially proved (Lemmas 4 and 6). In fact, once we start to include implication-specific features such as assumption discharge in the atomic system, there is no genuine reason why we should exclude further means of expression. If we included propositional quantification, which is very useful in the framing of logical rules (see [26]), in the atomic system, we would gain means to express disjunction-like features at the atomic level, giving us the completeness of intuitionistic logic in a relatively simple way.[3] Overall this means that there are good reasons to argue that the 'real' validity-based proof-theoretic semantics is standard semantics, in which atomic systems consist just of production rules.

What we have shown speaks neither for nor against completeness with respect to standard semantics. If the latter could be established, we would have the interesting fact that, in view of Sandqvist's result for classical logic, there would be a justification for classical as well as for intuitionistic propositional logic, where intuitionistic logic, being based on a wider range of connectives, demands stronger requirements concerning closure under substitution.

However, there are further points that affect the validity concept as a whole, as it is used here and in related works. One crucial point already mentioned in Section 2 is the handling of negation. If we consider negation to be a proper logical constant as dealt with in Kripke semantics, namely as expressing that something can never turn out to be true, then most of our techniques fail. Our way of proceeding depends on the fact that, by means of adding rules to a given atomic system, we can force a negated statement to be true. By adding the rule $a \triangleright \bot$ to S we can generate an extension of S, in which a is false. Theoretically, we could even make an atomic system inconsistent by adding absurdity $\bot$ as an axiom to it. This is not possible if absurdity is a logical constant which by definition can never be established. If the semantics is restricted in such a way that only consistent extensions of atomic systems are allowed, i.e., extensions in which absurdity $\bot$ cannot be derived, then completeness can be achieved (Goldfarb [5]; see also Litland [11]).

Another crucial point is that our framework and results rest on the assumption that in the interpretation of hypothetical consequence in (S3) (and therefore implicitly in the interpretation of implication in (S2)) we are considering arbitrary extensions of atomic systems. Prawitz used the idea of extensions of atomic systems in [16], but from 1973 [17] on never refers to them, without making it explicit that he does not need them. There might be arguments against extensions of atomic systems as describing evolving knowledge; an atomic system might instead be considered to

[3] See also Sandqvist [21], who proposed some sort of semantics for disjunction corresponding to the use of propositional quantification in atomic rules, for which completeness follows almost immediately.

be an (inductive) definition that delineates the meaning of atomic expressions. With respect to definitions one would not expect monotonicity (in the sense of (P2)), as an extension of a definition changes what is being defined. A system for definitional reasoning is definitely worth developing. For that purpose one might use, for example, Martin-Löf's theory of iterated inductive definitions [13] or Hallnäs's idea of definitional reflection (see [6, 7, 23]). It would then not be enough to just drop the reference to extensions in (S2). It would rather be necessary to add a full-fledged definitional theory.

In any case we have shown that if extensions are considered, which is a common case considered by many authors, and absurdity and negation are dealt with in the way indicated, then we do *not* have intuitionistic completeness, at least when assumption-discharging atomic rules are considered. This is a significant result, which goes against certain intuitions concerned with the harmonious relationship between introduction and elimination rules (cf. [1]) as put forward by Prawitz [16–18], and also by Dummett [4]. However, even if, given the standard introduction rules, there are no stronger elimination rules than the intuitionistic ones, this does not preclude that stronger rules, which do *not* have the form of elimination rules, can be validated. Harmony between syntactically specified introduction and elimination rules is one matter, the validity of arbitrary rules is a different matter.

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References

1. de Campos Sanz, W., & Piecha, T. (2009). Inversion by definitional reflection and the admissibility of logical rules. *Review of Symbolic Logic*, *2*(3), 550–569.
2. de Campos Sanz, W., & Piecha, T. (2014). A critical remark on the BHK interpretation of implication. In P.E. Bour, G. Heinzmann, W. Hodges, P. Schroeder-Heister (Eds.), *14th CLMPS 2011 Proceedings, Philosophia Scientiae*, Vol. 18(3). To appear.
3. de Campos Sanz, W., Piecha, T., Schroeder-Heister, P. (2014). Constructive semantics, admissibility of rules and the validity of Peirce's law. *Logic Journal of the IGPL*, *22*(2), 297–308. First published online August 6, 2013.
4. Dummett, M. (1991). *The Logical Basis of Metaphysics*. London: Duckworth.
5. Goldfarb, W. (2014). On Dummett's "Proof-theoretic Justifications of Logical Laws". In T. Piecha & P. Schroeder-Heister (Eds.), *Advances in Proof-Theoretic Semantics*. Trends in Logic. Dordrecht: Springer. Circulated manuscript, 1998.
6. Hallnäs, L. (1991). Partial inductive definitions. *Theoretical Computer Science*, *87*, 115–142.
7. Hallnäs, L. (2006). On the proof-theoretic foundation of general definition theory. In R. Kahle & P. Schroeder-Heister (Eds.), *Proof-Theoretic Semantics. Special issue of Synthese* (Vol. 148, pp. 589–602). Berlin: Springer.
8. Harrop, R. (1960). Concerning formulas of the types $A \to B \vee C$, $A \to (Ex)B(x)$ in intuitionistic formal systems. *Journal of Symbolic Logic*, *25*, 27–32.
9. Kleene, S.C. (1971). *Introduction to Metamathematics*. Wolters-Noordhoff Publishing, Groningen and North-Holland Publishing Company: Amsterdam and London.
10. Kreisel, G., & Putnam, H. (1957). Eine Unableitbarkeitsbeweismethode für den intuitionistischen Aussagenkalkül. *Archiv für Mathematische Logik und Grundlagenforschung*, *3*, 74–78.

11. Litland, J. (2012). *Topics in Philosophical Logic*. Ph.D. thesis, Cambridge: Department of Philosophy, Harvard University.
12. Makinson, D. (2014). On an inferential semantics for classical logic. *Logic Journal of the IGPL*, *22*(1), 147–154.
13. Martin-Löf, P. (1971). Hauptsatz for the intuitionistic theory of iterated inductive definitions. In J.E. Fenstad (Ed.), *Proceedings of the Second Scandinavian Logic Symposium*, Studies in Logic and the Foundations of Mathematics (Vol. 63, pp. 179–216). Amsterdam: North-Holland.
14. Mints, G.E. (1976). Derivability of admissible rules. *Journal of Mathematical Sciences*, *6*, 417–421.
15. Olkhovikov, G.K., & Schroeder-Heister, P. (2014). Proof-theoretic harmony and the levels of rules: Generalised non-flattening results. In E. Moriconi & L. Tesconi (Eds.), *Second Pisa Colloquium in Logic, Language and Epistemology*. Pisa: ETS. To appear.
16. Prawitz, D. (1971). Ideas and results in proof theory. In J.E. Fenstad (Ed.), *Proceedings of the Second Scandinavian Logic Symposium*, Studies in Logic and the Foundations of Mathematics (Vol. 63, pp. 235–307). Amsterdam: North-Holland.
17. Prawitz, D. (1973). Towards a foundation of a general proof theory. In P. Suppes, et al. (Eds.), *Logic, Methodology and Philosophy of Science IV* (pp. 225–250). Amsterdam: North-Holland.
18. Prawitz, D. (1974). On the idea of a general proof theory. *Synthese*, *27*, 63–77.
19. Prawitz, D. (2014). An approach to general proof theory and a conjecture of a kind of completeness of intuitionistic logic revisited. In L.C. Pereira, E.H. Haeusler, V. de Paiva (Eds.), *Advances in Natural Deduction*, Trends in Logic (Vol. 39, pp. 269–279). Berlin: Springer.
20. Sandqvist, T. (2009). Classical logic without bivalence. *Analysis*, *69*, 211–217.
21. Sandqvist, T. (2014). Basis-extension semantics for intuitionistic sentential logic. Submitted manuscript.
22. Schroeder-Heister, P. (1984). A natural extension of natural deduction. *Journal of Symbolic Logic*, *49*, 1284–1300.
23. Schroeder-Heister, P. (1993). Rules of definitional reflection. In *Proceedings of the Eighth Annual IEEE Symposium on Logic in Computer Science (Montreal 1993)* (pp. 222–232). Los Alamitos: IEEE Computer Society.
24. Schroeder-Heister, P. (2006). Validity concepts in proof-theoretic semantics. In R. Kahle & P. Schroeder-Heister (Eds.), *Proof-Theoretic Semantics. Special issue of Synthese* (Vol. 148, pp. 525–571). Berlin: Springer.
25. Schroeder-Heister, P. (2012). Proof-theoretic semantics. In E. Zalta (Ed.), *The Stanford Encyclopedia of Philosophy (Winter 2012 Edition)*. http://plato.stanford.edu/archives/win2012/entries/proof-theoretic-semantics/.
26. Schroeder-Heister, P. (2014). The calculus of higher-level rules, propositional quantification, and the foundational approach to proof-theoretic harmony. In A. Indrzejczak (Ed.), *Special issue, commemorating the 80th anniversary of Gentzens and Jaśkowskis groundbreaking works on assumption based calculi, Studia Logica* (Vol. 103). Berlin: Springer. To appear.