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Intensional Proof-Theoretic Semantics and the Rule of Contraction

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Abstract

The structural rule of contraction, which allows the identification of two occurrences of the same formula, is discussed from the viewpoint of intensional proof-theoretic semantics. It is argued that sometimes such occurrences have different meanings and therefore should not be identified. This suggests a restriction of contraction based on a definitional (and thus intensional) ordering of formulas within proofs, which is related to ramification in type theories such as the one proposed by Marie Duží.

In her impressive scientific work, Marie Duží has put forward and established *Transparent Intensional Logic* as a comprehensive approach to semantics, according to which the meaning of an expression is viewed as a construction that generates its denotation.[1] Here a construction is not understood as a set-theoretical function but as a procedure in a fine-grained sense. This construction-oriented stance makes Duží's approach particularly attractive to proof-theoretic semantics, even though the starting points differ in many respects. In what

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[1] The *Opus Magnum* is Duží, Jespersen and Materna [6], but there is a great number of further papers by Duží propagating this topic. I just mention [3, 5].

follows I would like to make some remarks on what may be called "intensional proof-theoretic semantics" and discuss the structural rule of contraction as a typical example. This rule, which allows one to identify two occurrences of the same formula, gives rise to intensional considerations from a procedural point of view. In the context of paradoxical reasoning I suggest a definitional (and thus intensional) ordering of formula occurrences within proofs corresponding to the idea of ramification in Duží-style type theory.[2]

1 The standard approach to intensional proof-theoretic semantics

What might be called "intensional" proof-theoretic semantics goes back to the development of *general proof theory*, which was proclaimed at the beginning of the 1970s, most notably by Dag Prawitz [12, 13, 14] — he coined this term — and Georg Kreisel [10]. The topic was present also in the writings of Girard, Martin-Löf and others, and in a sense already in Gentzen [9]. Its idea was that in proof theory we should not only be interested in the power of deductive systems and the reduction of deductive systems to others with the aim to establish their consistency (in the spirit of Hilbert's programme), but also in the form and structure of proofs as a topic in its own right. In this sense, proof theory goes way beyond provability theory: beyond our interest in what we can we prove, we should be interested in how something provable is proved. It was only natural that, when proofs as such were put forward as genuine objects of study, the question of proof identity became central. It was explicitly, and under this heading, put on the proof-theoretic agenda by Prawitz [12] and Kreisel [10]. One might relate it to Quine's [15] slogan "no entity without identity": if we want to talk about proofs as objects we must give identity criteria for them. This is definitely an intensional question: if one considers proofs to be extensionally identical when they prove the same theorem, the

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[2] I have mentioned certain aspects of these ideas in [16], [17, in particular Supplement "Definitional Reflection and Paradoxes"] and [18], but never from the intensional perspective.

wider problem of when they are different, though proving the same, is intensional. It is obvious that Euclid's and Euler's proofs establishing that there are infinitely many prime numbers are different, as they are based on different proof ideas (see [1]). Mathematicians often have a very clear intuition of whether two proofs are based on the same proof idea and are essentially identical. However, it is not clear at all how to make this precise.[3]

At the very elementary level of proof theory, and in particular in the neighbouring discipline of categorial proof theory, several proposals have been made. The most prominent one is that two proofs are identical if they can be transformed into each other by certain proof-theoretic reductions along the lines of normalization procedures as given by Prawitz [11]. For example, the proofs

$$\dfrac{\dfrac{\overset{\vdots}{A} \quad \overset{|}{B}}{A \wedge B}}{A} \quad \text{and} \quad \overset{\vdots}{A} \qquad \text{as well as} \qquad \dfrac{\dfrac{\overset{\vdots}{A} \quad \overset{|}{B}}{A \wedge B}}{B} \quad \text{and} \quad \overset{|}{B}$$

are intensionally identical, as they result from removing the redundancy of introduction followed by elimination of conjunction. This very elementary example already shows that we must be very careful in our understanding of proofs and rules. If we replace the proof $\overset{|}{B}$ with an arbitrary proof $\overset{|}{A}$ of A, then the proof

$$\dfrac{\dfrac{\overset{\vdots}{A} \quad \overset{|}{A}}{A \wedge A}}{A} \tag{1}$$

is both identical to $\overset{\vdots}{A}$ and to $\overset{|}{A}$, which implies that any two proofs of A are identical. This *intensional collapse* of all proofs of

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[3] I am using the term "proof" in an ambiguous way to denote both a concrete proof figure (a derivation) and the abstract proof represented by this proof figure. Therefore, when I say that two different proofs are identical I mean that two syntactically different proof figures represent the same abstract proof.

A into a single one is not a contradiction, but something that may be called an *intensional paradox* that we want to avoid if we want to talk about possibly different proofs of a provable sentence A. In the present case it can easily be remedied by distinguishing between the right and left projection of conjunction elimination, which means that, without further annotation, the proof figure (1) is ambiguous. This is not a trivial matter. It implies that the annotation of a proof telling which rule has been applied at which step must be considered a part of the proof itself, something which is immediately relevant for logic teaching and often not mentioned there. When we annotate proofs with terms, this is, of course, automatically satisfied.

In general, the situation is not always that simple. If we have a proof of A which after further steps arrives again at A, that is, $\begin{array}{c}\vdots\\ A\\ |\\ A\end{array}$, we do not want that it reduces to $\begin{array}{c}\vdots\\ A\end{array}$ as long as we do not know what is happening between the upper and the lower A. It is not easy to tell what exactly should be allowed as a reduction generating intensional identity, apart from the negative criterion that we want to avoid an intensional collapse. Ekman's paradox is a very instructive example pointing to these issues (see [20]). It is related to the issue of harmony, that is, the relation between introduction and elimination rules in natural deduction, as harmonious rules give rise to identifications between proofs[4]. I do not want to further elaborate on this approach to intensional proof-theoretic semantics, which puts the identity of proofs in the foreground. Tranchini [21] gives a thorough discussion of it. In the following I would like to draw the reader's attention to another aspect of intensional proof-theoretic semantics.

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[4] For a general discussion see [19].

2 Contraction: a case of intensionality in proof-theoretic semantics

The question of identity of proofs extends the intensional identity problem: from that of objects, functions, concepts and propositions discussed in philosophical semantics since Frege, to another category of entities, namely proofs. However, the identity problem for the common categories still remains present in proof-theoretic semantics and needs to be approached there. I would like to focus on an aspect of propositional identity, that is, of identity between sentences or formulas, that has not yet found adequate consideration. At least four senses of propositional identity can be distinguished (in ascending order from extensional towards intensional):

1. Material equivalence: A is identical to B if A and B have the same truth value. We would nevertheless intensionally distinguish between A and B as they may not be *logically* equivalent.
2. Logical equivalence: A is identical to B if A and B are logically equivalent (with respect to some logical system such as intuitionistic or classical logic). We would nevertheless intensionally distinguish between A and B as they may not be *isomorphic.*
3. Logical isomorphism: A is identical to B if there are proofs between A and B such that the composition of these proofs is the identity proof (with respect to some concept of proof identity). We would neverthelessss intensionally distinguish between A and B as they may not be *syntactically identical.*
4. Syntactic identity: A is identical to A.

The second and third notions are discussed in categorial proof theory (see, e.g., [2]). An example of logical equivalence lacking isomorphism is the relation between $p \wedge q$ and $p \wedge (p \rightarrow q)$, whereas an example of isomorphism is the relation between $p \wedge q$ and $q \wedge p$. The fourth notion is not normally considered explicitly, as it is always taken for granted.

However, I would like to go one step further by arguing that syntactic identity is not the finest-possible semantic identity relation, and that different occurrences of the same sentence or formula may be semantically distinguished. If we have two occurrences of the same sentence A, they point, of course, to the same content, as the content is tied to the sentence itself and not to its occurrences. However, in a non-perfect language, A in one context may mean something different from A in another context, which in the semantic formalism we would disambiguate by different annotations in evaluating A. Therefore I propose to rephrase the fourth identity notion and add a fifth one:

4. Syntactic identity: A is identical to A. We would nevertheless intensionally distinguish between two occurrences of A as they may have a different semantic status.

5. Intensional identity of occurrences: Two occurences of A are identical, if they have the same semantic status.

It turns out that in proof theory this problem — when can two occurrences of formulas be identified and in which context do they denote something different? — manifests itself explicitly in the rule of contraction.

In the sequent calculus the rule of contraction allows one to identify two occurrences of a formula A:

$$\frac{\ldots, A, A, \ldots \vdash \ldots}{\ldots, A, \ldots \vdash \ldots}$$

In the classical variant of the sequent calculus — here I only consider the intuitionistic case —, we have an analogous rule also for the right side of the turnstile. The rule of contraction is often concealed by the fact that the antecedents and succedents of sequents are taken as sets, in which the multiplicity of occurrences of the same formula does not count. In natural deduction contraction occurs in the form that more than one occurrence of A may be discharged as assumption. For example, when introducing $A \to B$ from a derivation of B which uses the assumption A at more than one place

$$\begin{array}{c} \overset{(1)}{[A]} \quad \ldots \quad \overset{(1)}{[A]} \\ | \\ \dfrac{B}{A \to B}\,{}^{(1)} \end{array}$$

then these occurrences become a single occurrence in the implication introduced.[5]

Contraction is normally seen as absolutely unproblematic. The validity of an argument is not affected by using an assumption twice rather than once. However, in substructural logics, in particular in the context of relevance logic and entailment as well as in linear logic, the rule of contraction has become an issue of discussion. Putting the taxonomy of possible formal systems aside, the arguments for taking contraction-free logics seriously are not very convincing, at least not to me. Mathematical and other argumentation is full of contractions.

If we want to argue against the identification of two occurrences of A, we must argue that these two occurrences are semantically different. Semantical difference here means that they belong to different semantical contexts, so that they can be disambiguated by distinguishing these contexts, perhaps by attaching corresponding context-labels to them. This is definitely a topic of intensional proof-theoretic semantics, as we would argue that the formula A presented by the two occurrences of A is given to us *in two different ways*. In other words, the *way of being given* or *mode of presentation* of the formula A differs between the two occurrences of A. “Way of being given” or “mode of presentation” (“Art des Gegebenseins”) is actually Frege’s term for ‘sense’ in contradistinction to ‘reference’. The planet Venus is given to us in two different ways when described as ‘the morning/evening star’. In the deductive context, if we wanted to abolish contraction, we would have first to argue for the difference in meaning (sense) of

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[5] As a side remark, I should mention that the notion of occurrence, though appearing in every logic textbook, is still an underexposed concept that deserves further considerations. A detailed elaboration has been provided by Gazzari [8].

the two occurrences of A and then argue that this difference counts in the given surrounding. That there is such an argument for *any* contraction candidate A can be doubted.

Actually, there is one area where the absence of contraction would solve big problems: that of logical and semantical paradoxes. Rather than criticizing their internal construction (such as negative self-referentiality etc.), one may attack the logical derivation of a contradiction from these constructions. As has been pointed out already by Fitch [7], this derivation uses contraction at an essential place. Certainly, it also uses other logical means (at essential places). However, one can at least say that abolishing contraction blocks the derivation of inconsistency.

I do not want to go into the construction details of any paradoxes here. For the point to be made it suffices to assume that there is a sentence R, which by definition has the same meaning as its opposite $\neg R$. Thus I assume that in the sequent calculus there are definitional rules of the following form, which govern the meaning of R ("R" should remind one of "**R**ussell").

$$\frac{\ldots, \neg R, \ldots \vdash \ldots}{\ldots, R, \ldots \vdash \ldots}\,\text{DEF} \qquad\qquad \frac{\ldots \vdash \neg R}{\ldots \vdash R}\,\text{DEF}$$

In a natural deduction system, we would, as definitional rules, postulate introduction and elimination rules for R, which allow one to pass from $\neg R$ to R and from R to $\neg R$.

Then the following derivations generate $\neg R$ and R, respectively, and thus a contradiction. The right derivation is nothing more than the left derivation extended with an additional step. Thus, when discussing the issue of contraction, we can focus on the left one.

$$\dfrac{\dfrac{\dfrac{\dfrac{R \vdash R}{R, \neg R \vdash}}{R, R \vdash}\,\text{DEF}}{R \vdash}\,\text{Contr}}{\vdash \neg R} \qquad\qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{R \vdash R}{R, \neg R \vdash}}{R, R \vdash}\,\text{DEF}}{R \vdash}\,\text{Contr}}{\vdash \neg R}}{\vdash R}\,\text{DEF}$$

The natural deduction formulation would be as follows.

$$\dfrac{\dfrac{\overset{(1)}{[R]}}{\neg R}\,\text{DEF} \qquad \overset{(1)}{[R]}}{\neg R}\,(1) \qquad\qquad \dfrac{\dfrac{\dfrac{\overset{(1)}{[R]}}{\neg R}\,\text{DEF} \qquad \overset{(1)}{[R]}}{\neg R}\,(1)}{R}\,\text{DEF}$$

The critical use of contraction is obvious: in the sequent calculus formulation as the application of an explicit rule, in the natural deduction formulation as the joint cancellation of two occurrences of R when applying the rule of constructive reductio (negation introduction).

However, should one regard this observation as a proper argument against contraction? The argument is of a consequentialist nature. Assuming contraction, we get into trouble: thus abolish contraction. This looks like it would fit well into Popperian falsificationist methodology. However, many other assumptions may be responsible for the trouble of paradoxicality, such as the above-mentioned internal construction of the paradoxical sentence, something we have deliberately ignored. There should be additional arguments that speak for the selection of contraction as the critical feature (rather than any other feature) to be rejected. These arguments are not very strong. One does not get very far when trying to do mathematics (or other argumentation) in a logic without contraction. Rejecting a logical principle which has proved almost indispensible in the context of mathematics should be our last resort after testing other, more specific principles, which are crucial for the paradoxes but do not affect mathematical reasoning. Actually, formulating principles as specific as possible and testing those would be really in line with Popperian methodology.

Can such specific principles be found in our framework? What if we can find a plausible restriction of contraction, which disallows contraction in the case of paradoxical reasoning, but would otherwise, in standard mathematical contexts, allow for the use of contraction? I will argue that such a restriction can indeed be formulated by intensional considerations. In the case of the paradoxes, the two occurrences of R can be claimed to have different semantical contents. This is not arguing ad hoc, saying that contraction should be disallowed when it has a paradoxical outcome, but arguing by general semantical considerations for a difference in meaning of the two occurrences of R, which, when ignored, has paradoxical consequences, and that this difference plays no role in standard mathematical reasoning.

3 Levels of definitional evaluation (*order of meaning*)

Consider the upper part of the the derivation of inconsistency down to the application of contraction.

$$\dfrac{\dfrac{\dfrac{\boxed{R} \vdash R}{\boxed{R}, \neg R \vdash}}{\boxed{R}, \textcircled{R} \vdash}\;\text{DEF}}{R \vdash}\;\text{Contr}$$

Certain occurrences of R are annotated by boxing or encircling them. Contraction is applied to a boxed and an encircled occurrence of R. It is obvious that these two occurrences have a different 'derivational history'. The boxed occurrence of R is the repetition of an R occurring in an initial identity sequent. It does not result from a definitional rule, that is, a meaning rule. Contrary to that, the encircled occurrence of R is the result of such a rule, leading from $\neg R$ to R on the left side of the turnstile, thus using the definition that defines R in terms of $\neg R$. In this sense the encircled R is a *specific* or evaluated occurrence of R, whereas the boxed R is completely *unspecific*, as it has just been laid down as an assumption within an initial sequent.

In the corresponding part of the natural deduction formulation we have the analogous situation.

$$\dfrac{\dfrac{\overset{(1)}{[\textcircled{R}]}}{\neg R}\;\text{DEF} \qquad \overset{(1)}{[\boxed{R}]}}{\neg R}\;(1)$$

The encircled R is the major (actually the only) premiss of a meaning inference and thus understood according to its definition, while the boxed R, which is discharged simultaneously and thus identified with the encircled R, is just laid down as an assumption, without presupposing any meaning rule associated with it.

Now we argue that it makes a semantical difference of whether, in the sequent calculus, an occurrence of a formula A is specific, that is, results from a meaning rule, or whether it is unspecific, that is, does not result from such a rule. In the first case, establishing it (either in the antecedent or in the succedent of a sequent) requires knowledge of its definition, whereas in the second case no definitional knowledge is presupposed, as the formula just remains as it has been stated, without any reference to its content. In the case of natural deduction, an occurrence A of a formula is specific, if it is involved in a definitional rule, either as the conclusion of an introduction of A, or as the major premiss of an elimination of A, and unspecific, if it is not of this form. If this difference is considered intensionally significant, which we argue it is, contraction should be disallowed in these cases, even though the two occurrences have the same shape. Understanding A as a sentence *defined* in a certain way, that is, having a specific meaning, is different from regarding A just as an arbitrary sentence whose semantic content does not play any role. These are two different ways in which A is given to us, that is, two senses in Frege's terminology (the reference here being the sentence A). Using another terminology, we can understand the meaning of A as the *construction* embodied by its meaning rules. A specific use of A is one in which A has actually been construed according to this meaning, whereas in the unspecific use such a construction is not taken into account. This might be viewed as a *procedural distinction* between A as the dynamic result/output of a specific operation (or, in natural deduction, as its argument/input), and A as being static.

To formally deal with this situation, I propose to introduce an indexing discipline. With every formula occurrence in a sequent calculus derivation, a natural number is associated as a *meaning index*, which is increased if a meaning rule (left- or right-introduction rule in the sequent calculus) is applied. Contraction is then prohibited, if the meaning indices of the sentences involved differ. In the above example, the encircled R undergoing contraction receives a higher meaning index under this discipline than the boxed one, as it results from an application of a definitional rule, whereas the boxed R is available without any such application. This procedure can be iterated in that any application of a meaning rule adds to the index. For example, using an upper index to denote the meaning level, the following derivation ends with two occurrences of R in the antecedent (or as assumptions in natural deduction), which due to the different meaning index must not be contracted[6].

$$\dfrac{\begin{array}{c}\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{R^0 \vdash R^0}{R^0, \neg R^0 \vdash}}{\neg R^0 \vdash \neg R^0}}{\neg R^0 \vdash R^1}\,\textsc{def}}{R^1 \vdash R^1}\,\textsc{def}}{R^1, \neg R^1 \vdash}}{R^1, R^2 \vdash}\,\textsc{def} \\ \hdashline R^? \vdash \end{array}\;\textsc{not allowed}}{\vdash \neg R^?}$$

In natural deduction:

$$\begin{array}{c} \dfrac{\overset{(2)}{R^2}}{\neg R^1}\,\textsc{def} \qquad \dfrac{\dfrac{\dfrac{\overset{(2)}{R^1}}{\neg R^0}\,\textsc{def} \qquad \overset{(1)}{R^0}}{\neg R^0}\,(1)}{R^1}\,\textsc{def} \\ \hdashline \neg R^? \end{array}\;(2)\ \textsc{not allowed}$$

Due to the non-local features of natural deduction the indexing regime is not so straightforward as in the sequent calculus, something that speaks for the sequent calculus in this situation (which is anyway the system making the contraction rule fully explicit).

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[6] For simplicity, we do not apply the indexing regime to the negation rules. In a properly worked-out setting, indexing would be applied to all logical constants, as the rules governing their meaning have a definitional status.

The general result is a kind of higher-order theory with respect to meaning, according to which the application of a meaning rule increases the order. If in the underlying theory a sort of typing is already under way, this yields a ramified type theory, which represents another link to Duží's work (see, e.g., [4]). Ramification essentially means classification by definitional order, which is exactly what we are doing. We distinguish items according to their definition. The definiendum is of higher order than its definiens. In the current situation, R is defined by $\neg R$ and thus occurs in its definiens, which in the context of contraction has the effect that the defined R is of higher order than the R from which it is defined. In this way one may look at the proposal made here as a way of evading impredicativity by ramification. The crucial issue, however, is that we carry out this ramification at the deduction level and not at the definitional level. As far as definitions are concerned, we do not impose any restrictions and thus allow R to be defined by $\neg R$ (or being obtained from $\neg R$ by other means).

Our claim is that in ordinary mathematical reasoning, this situation does not show up. Our proposal to disallow contraction, but only when the occurrences to be contracted differ in their order of meaning, can be seen as a well-founded semantical approach which at the same time is precisely targeted at the paradoxes. This shows that a definitional hierarchy according to the order of meaning is not only a conceptually well-founded, but also a useful device. At least there is no need for a Russellian reducibility postulate to enable proper mathematical reasoning.

Note that in such a system we do not obtain full admissibility of cut, but have a restriction for the cut formula analogous to that for contraction. The cut rule

$$\frac{\Gamma \vdash A \quad A, \Delta \vdash C}{\Gamma, \Delta \vdash C}$$

can be shown to be admissible, but only if the occurrence of A in the left premiss and the occurrence of A in the right premiss, that is, the two occurrences of the cut formula, are of the same order of meaning. With an unrestricted cut rule we could easily override our restriction for contraction, as we could generate sequents such as $R^1 \vdash R^0$ and thus transform a sequent of the form $\ldots, R^1, R^0, \ldots \vdash \ldots$ into the sequent $\ldots, R^1, R^1, \ldots \vdash \ldots$. However, this outcome is only natural. When performing a cut, we are identifying two occurrences of the cut formula, and we should expect that they have the same meaning apart from representing the same formula.

These considerations, which still need to be spelled out in detail, show that intensionality in proof-theoretic semantics goes way beyond the question of identity of proofs. They support a ramified approach, which may be seen as an inferentialist extension of the constructive approach found in Duží's procedural semantics.

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