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WIRTSCHAFTS- UND
SOZIALWISSENSCHAFTLICHE
FAKULTÄT

Chair of Statistics, Econometrics and Empirical Economics

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S414
Advanced Mathematical Methods
Exercises

WS 2023/24

LINEAR ALGEBRA

EXERCISE 1 **Vector product**

Calculate for $\mathbf{v}' = (-1 \ 0 \ 3 \ -2)$ and $\mathbf{w} = (2 \ 3 \ -1 \ -3)'$ the following expressions:

a) $\mathbf{v}'\mathbf{w}$ b) $\mathbf{v}'\mathbf{v}$

c) $\mathbf{w}' \cdot \mathbf{w} \cdot \mathbf{v}$ d) $\mathbf{w} \cdot \mathbf{v}' \cdot \mathbf{w}$

In subtasks c) and d) consider the dimensions of the vectors in order to figure out which product has to be calculated first.

EXERCISE 2 **Orthogonality**

Determine the components x , y and z in a way, such that the vectors $\mathbf{v}_1 = (1 \ 2 \ -1)'$, $\mathbf{v}_2 = (4 \ 2 \ x)'$ and $\mathbf{v}_3 = (y \ z \ 1)'$ are pairwise orthogonal to each other.

EXERCISE 3 **Linear Combination**

The vectors $\mathbf{v}_1 = (1 \ 1 \ 1)'$, $\mathbf{v}_2 = (1 \ 2 \ 3)'$ and $\mathbf{v}_3 = (2 \ -1 \ 1)'$ are given. Show that the vector $\mathbf{w} = (1 \ -2 \ 5)'$ can be described as a linear combination of the vectors \mathbf{v}_1 , \mathbf{v}_2 and \mathbf{v}_3 .

EXERCISE 4 Matrix Multiplication

Decide whether the matrix multiplications $\mathbf{A} \cdot \mathbf{B}$ and $\mathbf{B} \cdot \mathbf{A}$ are possible and if so, carry them out.

$$\text{a) } \mathbf{A} = \begin{pmatrix} 2 & 1 & -1 \\ 0 & 4 & 2 \end{pmatrix} \quad \mathbf{B} = \begin{pmatrix} -1 & 1 & 0 & 3 \\ 1 & 1 & 1 & 1 \\ 0 & 2 & -3 & 4 \end{pmatrix}$$

$$\text{b) } \mathbf{A} = \begin{pmatrix} -2 & 1 & 2 \\ 1 & 2 & -3 \end{pmatrix} \quad \mathbf{B} = \begin{pmatrix} 3 & 2 \\ 1 & 2 \end{pmatrix}$$

$$\text{c) } \mathbf{A} = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \quad \mathbf{B} = \begin{pmatrix} -2 & 0 & 1 \\ 4 & -1 & 2 \end{pmatrix}$$

$$\text{d) } \mathbf{A} = \begin{pmatrix} 1 & -3 & 4 \\ 0 & 1 & -2 \\ 2 & 2 & -1 \end{pmatrix} \quad \mathbf{B} = \begin{pmatrix} 3 & 5 & 2 \\ -4 & -9 & 2 \\ -2 & -8 & 1 \end{pmatrix}$$

$$\text{e) } \mathbf{A} = \begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix} \quad \mathbf{B} = \begin{pmatrix} 3 & -5 \\ -1 & 2 \end{pmatrix}$$

$$\text{f) } \mathbf{A} = \begin{pmatrix} 1 & 5 & 0 \\ 2 & 6 & 6 \\ 3 & 7 & 4 \\ 0 & 8 & 2 \end{pmatrix} \quad \mathbf{B} = \mathbf{A}'$$

EXERCISE 5 Matrix Algebra

Apply the calculation rules for matrix algebra in order to simplify the following expressions:

$$\text{a) } \mathbf{A}(\mathbf{BA})^{-1}\mathbf{B} \quad \text{b) } (\mathbf{AB}')'(\mathbf{BA}')^{-1}\mathbf{C} \quad \text{c) } \mathbf{AB}'(\mathbf{B}^{-1})'\mathbf{A}^{-1}$$

EXERCISE 6 Matrix Algebra

Solve each of the following matrix equations for \mathbf{X} applying the calculation rules for matrices:

$$\begin{aligned} \text{a) } & \mathbf{A}'\mathbf{I} + \mathbf{X}' = [\mathbf{A}(\mathbf{I} + \mathbf{B})]' \\ \text{b) } & (\mathbf{XA} + \mathbf{IX})' = \mathbf{A}' + \mathbf{I} \\ \text{c) } & \mathbf{X}(\mathbf{A} + \mathbf{I}) = \mathbf{I} + \mathbf{A}^{-1} \end{aligned}$$

\mathbf{I} : appropriate identity matrix

EXERCISE 7 **Determinant**

Calculate the determinant for the following matrices:

$$\begin{array}{ll} \text{a) } \mathbf{A} = \begin{pmatrix} 4 & 1 \\ -4 & 2 \end{pmatrix} & \text{b) } \mathbf{B} = \begin{pmatrix} 1 & -4 & -10 \\ 10 & -8 & 2 \\ 0 & -1 & 6 \end{pmatrix} \\ \\ \text{c) } \mathbf{C} = \begin{pmatrix} -7 & 1 & -10 \\ 1 & 10 & 2 \\ 1 & 0 & 6 \end{pmatrix} & \text{d) } \mathbf{D} = \begin{pmatrix} -3 & 0 & -8 & 7 \\ -7 & 1 & -4 & -10 \\ 1 & 10 & -8 & 2 \\ 1 & 0 & -1 & 6 \end{pmatrix} \\ \\ \text{e) } \mathbf{E} = \mathbf{D}^{-1} & \text{f) } \mathbf{F} = \begin{pmatrix} 3 & 0 & 8 & -7 \\ 7 & -1 & 4 & 10 \\ -1 & -10 & 8 & -2 \\ -1 & 0 & 1 & -6 \end{pmatrix} \end{array}$$

EXERCISE 8 **Determinant**

The matrices

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 3 \\ 2 & 5 & 7 \\ 3 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \mathbf{B} = \begin{pmatrix} 2 & 3 & 1 \\ 0 & 2 & 5 \\ 1 & 0 & 1 \end{pmatrix}$$

are given.

- Compute the determinant of \mathbf{A} .
- Compute the determinant of a matrix that we receive by interchanging the first and the third column of \mathbf{A} (second column \mathbf{A} unchanged). Compare your result to a).
- Compute the determinant of \mathbf{A}' . Compare your result to a).
- Compute the determinant of $2 \cdot \mathbf{A}$. How can you compute this determinant more quickly?
- Compute the determinant of \mathbf{B} .
- Compute the determinant of $\mathbf{A}\mathbf{B}$. Compare your result to a) and e).
- Compute the determinant of $\mathbf{A} + \mathbf{B}$. Compare your result to $\det(\mathbf{A}) + \det(\mathbf{B})$.

EXERCISE 9 Calculation of the Inverse

Calculate the inverse if possible:

$$\begin{array}{ll} \text{a) } \mathbf{A} = \begin{pmatrix} 4 & 1 \\ -4 & 2 \end{pmatrix} & \text{b) } \mathbf{B} = \begin{pmatrix} 4 & 1 \\ -2 & -0,5 \end{pmatrix} \\ \text{c) } \mathbf{C} = \begin{pmatrix} 1 & 2 \\ 3 & -4 \end{pmatrix} & \text{d) } \mathbf{D} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & -3 \\ 0 & -1 & 3 \end{pmatrix} \end{array}$$

EXERCISE 10 Rank, Regularity and Inverse

The following matrix is given:

$$\mathbf{A} = \begin{pmatrix} 1 & -1 & 2 \\ 1 & 0 & 3 \\ 4 & 0 & 2 \end{pmatrix}$$

- a) Determine the rank of the matrix.
- b) Is the matrix \mathbf{A} regular or singular?
- c) Calculate the inverse of \mathbf{A} , if possible.

EXERCISE 11 Linear Equation Systems

Determine for the following linear equation system the solution vector using

- a) Gaussian Elimination
- b) Matrix Inversion
- c) Cramer's Rule

$$\begin{array}{rcl} & x_2 & + & x_3 & = & -1 \\ 3x_1 & + & 4x_2 & + & 5x_3 & = & -2 \\ 4x_1 & + & 6x_2 & + & 8x_3 & = & -4 \end{array}$$

EXERCISE 12 Rank of a matrix

Determine the rank of the following matrices using Gaussian Elimination:

$$\text{a) } \mathbf{A} = \begin{pmatrix} 2 & 1 & 0 & 1 & 2 \\ -1 & 2 & 1 & 0 & -1 \\ 0 & 4 & 1 & 2 & 1 \end{pmatrix} \quad \text{b) } \mathbf{B} = \begin{pmatrix} 4 & 2 & 1 \\ -2 & 1 & 2 \\ 1 & 0 & 3 \\ -1 & -1 & -1 \end{pmatrix}$$

EXERCISE 13 Inverse of a matrix

Given the following matrix

$$\mathbf{A} = \begin{pmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{pmatrix}$$

Calculate the inverse A^{-1} using Gaussian Elimination.

Solution Exercise 1:

a) 1

b) 14

c)
$$\begin{pmatrix} -23 \\ 0 \\ 69 \\ -46 \end{pmatrix}$$

d)
$$\begin{pmatrix} 2 \\ 3 \\ -1 \\ -3 \end{pmatrix}$$

Solution Exercise 2:

$$x=8, z=2, y=-3$$

Solution Exercise 3:

$$-6 \cdot \mathbf{v}_1 + 3 \cdot \mathbf{v}_2 + 2 \cdot \mathbf{v}_3 = \mathbf{w}$$

Solution Exercise 4:

a) Possible.
$$\mathbf{C} = \begin{pmatrix} -1 & 1 & 4 & 3 \\ 4 & 8 & -2 & 12 \end{pmatrix}$$

b) Not possible.
$$\underset{(2 \times 2)}{\mathbf{B}} \cdot \underset{(2 \times 3)}{\mathbf{A}} = \begin{pmatrix} -4 & 7 & 0 \\ 0 & 5 & -4 \end{pmatrix}$$

c) Not possible.
$$\underset{(2 \times 3)}{\mathbf{B}} \cdot \underset{(3 \times 1)}{\mathbf{A}} = \begin{pmatrix} 0 \\ 6 \end{pmatrix}$$

d) Possible, both sides. $7 \cdot \mathbf{I}$ e) Possible, both sides. \mathbf{I}

f) Possible, both sides.

$$\begin{pmatrix} 26 & 32 & 38 & 40 \\ 32 & 76 & 72 & 60 \\ 38 & 72 & 74 & 64 \\ 40 & 60 & 64 & 68 \end{pmatrix}$$
 and
$$\begin{pmatrix} 14 & 38 & 24 \\ 38 & 174 & 80 \\ 24 & 80 & 56 \end{pmatrix}$$

Solution Exercise 5:a) \mathbf{I} b) \mathbf{C} c) \mathbf{I}

Solution Exercise 6:

a) $\mathbf{X} = \mathbf{AB}$

b) $\mathbf{X} = \mathbf{I}$

c) $\mathbf{X} = \mathbf{A}^{-1}$

Solution Exercise 7:

a) $|\mathbf{A}| = 12$

b) $|\mathbf{B}| = 294$

c) $|\mathbf{C}| = -324$

d) $|\mathbf{D}| = 989$

e) $|\mathbf{E}| = \frac{1}{989}$

f) $|\mathbf{F}| = 989$

Solution Exercise 8:

a) $|\mathbf{A}| = -26$

b) $\begin{vmatrix} 3 & 1 & 0 \\ 7 & 5 & 2 \\ 1 & 0 & 3 \end{vmatrix} = 26$

c) $|\mathbf{A}'| = -26$

d) $|2\mathbf{A}| = -208$

e) $|\mathbf{B}| = 17$

f) $|\mathbf{AB}| = -442$

g) $|\mathbf{A} + \mathbf{B}| = 92$

Solution Exercise 9:

a) $\mathbf{A}^{-1} = \frac{1}{12} \cdot \begin{pmatrix} 2 & -1 \\ 4 & 4 \end{pmatrix}$

b) \mathbf{B} is singular $\Leftrightarrow \mathbf{B}^{-1}$ doesn't exist!

c) $\mathbf{C}^{-1} = -\frac{1}{10} \begin{pmatrix} -4 & -2 \\ -3 & 1 \end{pmatrix}$

d) $\mathbf{D}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & \frac{1}{3} & \frac{2}{3} \end{pmatrix}$

Solution Exercise 10:a) $Rg(\mathbf{A}) = 3$.b) \mathbf{A} is regular.

c) $\mathbf{A}^{-1} = \begin{pmatrix} 0 & -0,2 & 0,3 \\ -1 & 0,6 & 0,1 \\ 0 & 0,4 & -0,1 \end{pmatrix}$.

Solution Exercise 11:

Solution vector: $x = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$

Solution Exercise 12:

a) The rank of the matrix is 3.

b) The rank of the matrix is 3.

Solution Exercise 13:

$$\mathbf{A}^{-1} = \begin{pmatrix} 6,75 & -2,75 & 0,75 \\ -2,75 & 1,25 & -0,25 \\ 0,75 & -0,25 & 0,25 \end{pmatrix}$$