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Incompleteness of Intuitionistic Propositional Logic with Respect to Proof-Theoretic Semantics

Abstract. Prawitz proposed certain notions of proof-theoretic validity and conjectured that intuitionistic logic is complete for them [11,12]. Considering propositional logic, we present a general framework of five abstract conditions which any proof-theoretic semantics should obey. Then we formulate several more specific conditions under which the intuitionistic propositional calculus (IPC) turns out to be semantically incomplete. Here a crucial role is played by the generalized disjunction principle. Turning to concrete semantics, we show that prominent proposals, including Prawitz's, satisfy at least one of these conditions, thus rendering IPC semantically incomplete for them. Only for Goldfarb's [1] proof-theoretic semantics, which deviates from standard approaches, IPC turns out to be complete. Overall, these results show that basic ideas of proof-theoretic semantics for propositional logic are not captured by IPC.

Keywords: General proof theory, Proof-theoretic semantics, Intuitionistic logic, Prawitz's conjecture, Incompleteness, Logical constants, Kripke semantics.

In [5] it was shown that intuitionistic propositional logic is semantically incomplete for certain notions of proof-theoretic validity (see also [4]). This questioned a claim by Prawitz, who was the first to propose a proof-theoretic notion of validity, and claimed completeness for it [11,12].

In this paper we put these and related results into a more general context. We consider the calculus of intuitionistic propositional logic (IPC) and formulate, in Section 1, abstract semantic conditions for proof-theoretic validity which every proof-theoretic semantics is supposed to satisfy. They are so general that they cover most semantic approaches, even classical truth-theoretic semantics. In Section 2 we show that if in addition certain more special conditions are assumed, IPC fails to be complete. In Section 3 we study several concrete notions of proof-theoretic validity and investigate which of the conditions rendering IPC incomplete they meet. In Section 4 we consider Goldfarb's [1] semantic approach for which IPC is complete, but which is not a 'standard' notion of proof-theoretic validity compared to those proposed by Prawitz.

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1. Proof-Theoretic Validity in an Abstract Setting

We consider the intuitionistic propositional calculus (IPC) with the standard constants $\wedge$, $\vee$, $\rightarrow$ and $\neg$.

In validity-based proof-theoretic semantics, one normally considers the validity of atomic formulas to be determined by an atomic system S. This atomic system corresponds to what in truth-theoretic semantics is a structure $\mathfrak{A}$ (in propositional logic $\mathfrak{A}$ reduces to a truth-valuation of propositional variables). Via semantical clauses for the connectives, an atomic base then inductively determines the validity with respect to S, in short: S-validity $\vDash_S A$ of a formula A, as well as S-consequence $\Gamma \vDash_S A$ between a set of formulas Γ and a formula A. In our abstract setting we completely leave open the nature of S and just assume that a nonempty finite or infinite set $\mathscr{S}$ of entities called *bases* is given. We furthermore assume that for each base $S \in \mathscr{S}$ a consequence relation $\vDash_S$ is given, that is, a relation $\Gamma \vDash_S A$ between a set Γ of formulas and a formula A, such that the following conditions are satisfied:

(Reflexivity) $A \vDash_S A$.

(Monotonicity) If $\Gamma \vDash_S A$, then $\Gamma, B \vDash_S A$.

(Transitivity) If $\Gamma \vDash_S A$ and $\Gamma, A \vDash_S B$, then $\Gamma \vDash_S B$.

By $\Gamma \vDash_S \Delta$ we mean that for all $A \in \Delta$: $\Gamma \vDash_S A$.

The S-validity of A (i.e., $\vDash_S A$) is identified with the fact that A is an S-consequence of the empty set ($\emptyset \vDash_S A$). We expect that S-validity respects the intended meaning of the logical connectives, where we take only the positive connectives of conjunction, disjunction and implication into account (see, however, the remark on negation at the end of Section 2):

($\wedge$) $\vDash_S A \wedge B \iff \vDash_S A$ and $\vDash_S B$.

($\vee$) $\vDash_S A \vee B \iff \vDash_S A$ or $\vDash_S B$.

($\rightarrow$) $\vDash_S A \rightarrow B \iff A \vDash_S B$.

The relation of *universal* or *logical consequence* is, as usual, understood as transmitting S-validity from the antecedents to the consequent. In our abstract setting, this is achieved by assuming that besides $\vDash_S$, there is a consequence relation $\vDash$ available, such that the following two conditions are satisfied:

($\vDash$) $\Gamma \vDash A \iff$ For all $S \in \mathscr{S}$: ($\vDash_S \Gamma \implies \vDash_S A$).

($\vDash'$) If $\Gamma \vDash A$, then $\Gamma \vDash_S A$ for any S.

Condition ($\models'$) expresses that $\models$ is a generalization of $\models_S$. It follows from condition ($\models$), if we assume that ($\models_S \Gamma \implies \models_S A$) implies $\Gamma \models_S A$, which we do not, however, want to presuppose as a necessary condition.

The five conditions $(\wedge), (\vee), (\to), (\models), (\models')$ constitute our abstract notion of a semantics. That is, if a non-empty set $\mathscr{S}$ of bases, and consequence relations $\models_S$ (for each $S \in \mathscr{S}$) and $\models$ are given such that these five conditions are met, we speak of a validity-based proof-theoretic semantics in the abstract sense, in short: a *semantics*.

Note that these conditions are also satisfied by classical truth-theoretic (or model-theoretic) semantics, if one defines $\Gamma \models_{\mathfrak{A}} A$ to mean: If $\mathfrak{A} \models \Gamma$, then $\mathfrak{A} \models A$.

Most concrete versions of proof-theoretic semantics, including those considered by Prawitz, are semantics in this abstract sense. Deviant proof-theoretic semantics we are aware of are only those, which challenge the fact that $\models_S$ or $\models$ are standard consequence relations, for example, by changing the principles of monotonicity or transitivity. However, even those semantics could be discussed in a modified framework of our kind (see [7,8]), but this is not our topic here.

The notions of valid rule and derivable rule will be used in what follows. It is $\vdash$ the derivability relation of IPC.

Definition 1.1. A rule

$$\frac{A_1 \qquad \ldots \qquad A_n}{B}$$

is called *valid* iff $A_1, \ldots, A_n \models B$. It is called *derivable* iff $A_1, \ldots, A_n \vdash B$.

The following standard results will play a prominent role.

Lemma 1.2. *1. Harrop's rule (see [2])*

$$\frac{\neg A \to (B_1 \vee B_2)}{(\neg A \to B_1) \vee (\neg A \to B_2)}$$

is not derivable in IPC (though it is admissible; cf. [3]).

2. For $\vdash$ the following generalized disjunction property holds:

GDP($\vdash$) *If $\Gamma \vdash A \vee B$, where $\vee$ does not occur in Γ, then $\Gamma \vdash A$ or $\Gamma \vdash B$.*

3. Disjunctions can always be removed from a negated formula, by the following principles:

$$(\vee\text{-}removal) \begin{cases} \neg(A \vee B) \dashv\vdash \neg A \wedge \neg B; \\ \neg(A \wedge B) \dashv\vdash \neg(\neg\neg A \wedge \neg\neg B); \\ \neg(A \to B) \dashv\vdash \neg\neg A \wedge \neg B. \end{cases}$$

Remark 1.3. A stronger version of GDP($\vdash$), in which it is only assumed that $\vee$ does not occur *positively* in Γ, was proven by Harrop [2] and, in a natural deduction setting, by Prawitz [9]. We here only need the stated weaker version GDP($\vdash$), where the stronger assumption is made that $\vee$ does not occur in Γ at all.

Soundness and completeness of IPC are understood in the usual way.

DEFINITION 1.4. *Soundness* of IPC means:

$$\text{For any } \Gamma \text{ and } A\text{: if } \Gamma \vdash A, \text{ then } \Gamma \vDash A;$$

and *completeness* of IPC means:

$$\text{For any } \Gamma \text{ and } A\text{: if } \Gamma \vDash A, \text{ then } \Gamma \vdash A.$$

LEMMA 1.5. *In view of* ($\vDash'$)*, soundness implies the following: For any* Γ *and* A*, if* $\Gamma \vdash A$*, then* $\Gamma \vDash_S A$ *for any* S*.*

2. Conditions for Incompleteness of IPC

We show that IPC turns out to be incomplete, if the semantics given satisfies certain special conditions.

A crucial role is played by the *generalized disjunction property*, which was stated above for the derivability relation $\vdash$ of IPC. We are particularly interested in its semantical version. Therefore we formulate it for an arbitrary consequence relation $\Vdash$ in the language of IPC:

GDP($\Vdash$) If $\Gamma \Vdash A \vee B$, where $\vee$ does not occur in Γ, then $\Gamma \Vdash A$ or $\Gamma \Vdash B$.

We assume in the following that a semantics in the abstract sense of Section 1 is given, with respect to which IPC is sound.

LEMMA 2.1. *If* GDP($\vDash_S$) *for every* S*, then Harrop's rule*

$$\frac{\neg A \to (B_1 \vee B_2)}{(\neg A \to B_1) \vee (\neg A \to B_2)}$$

is valid.

PROOF.

$\vDash_S \neg A \to (B_1 \vee B_2) \implies \neg A \vDash_S B_1 \vee B_2$; by ($\to$)

$\implies A' \vDash_S B_1 \vee B_2$ for some $\vee$-free formula A' such that $A' \dashv\vdash \neg A$; by ($\vee$-*removal*), Lemma 1.5 and transitivity of $\vDash_S$

$\implies A' \vDash_S B_i$ for $i = 1$ or 2; by GDP($\vDash_S$)

$$\Longrightarrow \neg A \vDash_S B_i; \text{ by } (\vee\textit{-removal}), \text{ Lemma 1.5 and transitivity of } \vDash_S$$
$$\Longrightarrow \vDash_S \neg A \to B_i; \text{ by } (\to)$$
$$\Longrightarrow \vDash_S (\neg A \to B_1) \vee (\neg A \to B_2); \text{ by } (\vee).$$

As this holds for any S, condition ($\vDash$) gives us $\neg A \to (B_1 \vee B_2) \vDash (\neg A \to B_1) \vee (\neg A \to B_2)$. ■

This means that if we have GDP($\vDash_S$) for every S, then completeness fails, since Harrop's rule is not derivable in IPC.

Now consider the following property:

(Export) For every base S there is a set of $\vee$-free formulas S^* such that for all Γ and A: $\Gamma \vDash_S A \iff \Gamma, S^* \vDash A$.

This condition means that the base S of non-logical consequence $\vDash_S$ can be 'exported' as a set of assumptions (S^*) of logical consequence $\vDash$.

Lemma 2.2. *Assume completeness of IPC. Then Export implies* GDP($\vDash_S$) *for every S.*

Proof. Suppose completeness holds, and $\vee$ does not occur in Γ. Then we obtain GDP($\vDash_S$) as follows:

$$\begin{aligned}
\Gamma \vDash_S A_1 \vee A_2 &\Longrightarrow \Gamma, S^* \vDash A_1 \vee A_2; \text{ by Export} \\
&\Longrightarrow \Gamma, S^* \vdash A_1 \vee A_2; \text{ by completeness} \\
&\Longrightarrow \Gamma, S^* \vdash A_i \text{ for } i = 1 \text{ or } 2; \text{ by GDP}(\vdash) \\
&\Longrightarrow \Gamma, S^* \vDash A_i \text{ for } i = 1 \text{ or } 2; \text{ by soundness} \\
&\Longrightarrow \Gamma \vDash_S A_i \text{ for } i = 1 \text{ or } 2; \text{ by Export.}
\end{aligned}$$
■

This means that assuming completeness we obtain that Harrop's rule is valid. Again assuming completeness, this implies that Harrop's rule is derivable in IPC, which is not the case. Thus we have refuted completeness.

Note that we have not shown GDP($\vDash_S$) outright, but only under the assumption of completeness, which is, however, sufficient to refute completeness.

Now consider the condition

($\vDash_S$) $\quad \Gamma \vDash_S A \iff (\vDash_S \Gamma \implies \vDash_S A)$.

We obtain the following interesting result. (Note that the direction from left to right in ($\vDash_S$) follows already from the fact that $\vDash_S$ is a consequence relation.)

Lemma 2.3. *Suppose* $(\models_S)$. *Then, using classical logic in the metalanguage,* $\mathrm{GDP}(\models_S)$ *can be proved.*

Proof.

$$\begin{aligned}
&\Gamma \models_S A \vee B\\
&\qquad \Longrightarrow (\models_S \Gamma \Longrightarrow \models_S A \vee B);\ \text{by } (\models_S)\\
&\qquad \Longrightarrow (\models_S \Gamma \Longrightarrow (\models_S A \text{ or } \models_S B));\ \text{by } (\vee)\\
&\qquad \Longrightarrow (\models_S \Gamma \Longrightarrow \models_S A) \text{ or } (\models_S \Gamma \Longrightarrow \models_S B);\ \text{classical metalanguage}\\
&\qquad \Longrightarrow \Gamma \models_S A \text{ or } \Gamma \models_S B;\ \text{by } (\models_S).
\end{aligned}$$

(We do not need the supposition that $\vee$ does not occur in Γ.) ∎

However, to show $\mathrm{GDP}(\models_S)$ we do not have to rely on a classical metalanguage, if we can make use of the following principle:

(Import) For every S, every $\vee$-free Γ and every A there is a base $S+\Gamma$ such that: $\Gamma \models_S A \iff \models_{S+\Gamma} A$.

This condition means that any disjunction-free set of assumptions of logical consequence $\models$ can be 'imported' into a base S of non-logical consequence $\models_S$.

Lemma 2.4. *Import implies* $\mathrm{GDP}(\models_S)$.

Proof. Suppose $\vee$ does not occur in Γ.

$$\begin{aligned}
\Gamma \models_S A \vee B &\Longrightarrow \models_{S+\Gamma} A \vee B;\ \text{by Import}\\
&\Longrightarrow \models_{S+\Gamma} A \text{ or } \models_{S+\Gamma} B;\ \text{by } (\vee)\\
&\Longrightarrow \Gamma \models_S A \text{ or } \Gamma \models_S B;\ \text{by Import.}
\end{aligned}$$ ∎

Import is a condition that played a crucial role in [5], where we considered higher-level inference rules, but it is not needed in the general setting here.

Summary. For any semantics with respect to which IPC is sound, we have shown the following.

Theorem 2.5.

1. $\mathrm{GDP}(\models_S)$ *for all* $S \Longrightarrow$ *validity of Harrop's rule,*
   thus: $\mathrm{GDP}(\models_S)$ *for all* $S \Longrightarrow$ *incompleteness.*
2. *Export + completeness* $\Longrightarrow \mathrm{GDP}(\models_S)$ *for all* S,
   thus: Export $\Longrightarrow$ *incompleteness.*
3. *Condition* $(\models_S) \Longrightarrow \mathrm{GDP}(\models_S)$ *for all* S *(using classical metalanguage),*
   thus: Condition $(\models_S) \Longrightarrow$ *incompleteness.*

4. Import $\Longrightarrow$ GDP($\vDash_S$) *for all S,*

thus: Import $\Longrightarrow$ *incompleteness.*

Remark 2.6. Theorem 2.5(1) and (2) continue to hold if for GDP($\vDash_S$) and GDP($\vdash$) it is only assumed that $\vee$ does not occur *positively* in Γ.

Therefore, in order to establish the incompleteness of IPC for a semantics, for which IPC is sound, we only need to establish one of the four conditions stated in the clauses of Theorem 2.5.

Remark on Negation. The counterexample to completeness, which is based on Harrop's rule, relies heavily on negation being available, in addition to implication and disjunction. The reason why there is no principle explicitly required of negation in our abstract semantics in Section 1 nor in the conditions for incompleteness in the current section is due to the fact that for our incompleteness results we throughout assume soundness, which means that the principles governing negation with respect to $\vDash$ and $\vDash_S$ are inherited from those derivable in IPC. If we wanted to establish incompleteness results for semantics for which IPC is not even required to be sound, we would have to formulate explicit semantic principles for negation or absurdity.

3. Incompleteness Results for Concrete Proof-Theoretic Semantics

By a concrete semantics we understand a semantical approach in which bases S are explicitly specified, and in which consequence relations $\vDash$ and $\vDash_S$ are defined in such a way that the result is a semantics in the abstract sense of Section 1. The specification of $\vDash$ and $\vDash_S$ can proceed by explicit or inductive definition. Another possibility would be to start with a different fundamental concept in terms of which $\vDash_S$ is then defined. The latter is the case in Prawitz's definition of the validity of a derivation or derivation structure, on which the definition of valid consequence is based.

We consider certain types of concrete semantics. All of them are proof-theoretic semantics in the sense that bases are understood as atomic systems generating valid atomic formulas by means of inference rules.

Definition 3.1. An *atomic system S* is a deductive system with rules of the form

$$\frac{a_1 \quad \ldots \quad a_n}{b}$$

where $a_1, \ldots, a_n, b$ are *atoms*. As a limiting case, n can be 0, in which case we have a rule without premisses, that is, an *axiom*. The set of rules may be empty, in which case S takes the form $\emptyset$.

The atoms $a_1, \ldots, a_n, b$ can be of a specific form different from the atomic formulas (= propositional variables) of IPC. In that case, in order to interpret IPC semantically, one would have to consider valuations which interpret propositional variables by such atoms. Alternatively, one might consider the atoms of an atomic system S to be just the propositional variables. By giving rules for propositional variables and a notion of derivability $\vdash_S p$ of propositional variables p in S one obtains a way of interpreting propositional variables in an atomic system S in analogy to truth valuations in classical logic. This is how we proceed in the following. That is, the atomic formulas derived by an atomic system are propositional variables.

The S-validity of an atomic formula a is defined as the derivability of a in S:

(At) $\quad \vDash_S a \;:\Longleftrightarrow\; \vdash_S a.$

The set $\mathscr{S}$ of bases considered is the set of all atomic systems, where atomic systems are identified with the sets of their rules. The systems within $\mathscr{S}$ are ordered in the usual way by set inclusion $\subseteq$.

Remark 3.2. Different kinds of concrete semantics are obtained from different kinds of atomic systems. Besides the kind of atomic systems considered here, one may, for example, consider systems of higher-level rules (see [5]). Furthermore, for a given kind of atomic systems different kinds of derivability relations $\vdash_S$ can be examined. For example, an interpretation of atomic systems as definitions justifies additional principles for deductions in atomic systems, which yields a derivability relation which is no longer monotone with respect to extensions of atomic systems (see [7,8]). We will not treat such variants here.

Concrete semantics based on (any kind of) atomic systems can be classified into so-called extension semantics and non-extension semantics, depending on how they interpret implication.

In *extension semantics*, S-consequence is defined using extensions $S' \supseteq S$, for atomic systems S and S':

($\vDash_S^{\text{ext}}$) $\;\Gamma \vDash_S A \;:\Longleftrightarrow\;$ For all $S' \supseteq S$: $(\vDash_{S'} \Gamma \implies \vDash_{S'} A)$.

In *non-extension semantics*, S-consequence is just defined by

($\vDash_S$) $\quad \Gamma \vDash_S A \;:\Longleftrightarrow\; (\vDash_S \Gamma \implies \vDash_S A)$.

Remark 3.3. Concerning ($\rightarrow$) this means that with ($\vDash_S^{\text{ext}}$) we have

$$\vDash_S A \rightarrow B \iff \text{For all } S' \supseteq S\text{: } (\vDash_{S'} A \implies \vDash_{S'} B),$$

whereas with ($\vDash_S$) we have $\vDash_S A \rightarrow B \iff (\vDash_S A \implies \vDash_S B)$.

Whether one should prefer one or the other kind of concrete semantics depends on how atomic systems are to be interpreted. Note that for non-extension semantics S-consequence fails to be monotone with respect to atomic systems, whereas extension semantics guarantee monotonicity (for details see [7,8]).

Remark 3.4. Obviously, in extension semantics $\vDash_S$ is monotone with respect to atomic systems, that is, $\Gamma \vDash_S A \implies \Gamma \vDash_{S \cup S'} A$ for any S and S'.

Remark 3.5. In extension semantics we can strengthen (At) to

(At′) $a_1, \dots, a_n \vDash_S a \iff a_1, \dots, a_n \vdash_S a$.

The direction from right to left is trivial. For the direction from left to right we consider any extension S' of S, which has $a_1, \dots, a_n$ as additional axioms. Then $a_1, \dots, a_n \vDash_S a$ implies $\vDash_{S'} a$, and thus $\vdash_{S'} a$ by (At). Now $\vdash_{S'} a$ means the same as $a_1, \dots, a_n \vdash_S a$.

LEMMA 3.6. *For extension semantics we can establish*

(Export) For every S there is a set of $\vee$-free formulas S^* such that for all Γ and A: $\Gamma \vDash_S A \iff \Gamma, S^* \vDash A$.

PROOF. First note that every atomic system S can be represented by a set of $\vee$-free formulas S^*:

1. Axioms $\overline{\ a\ }$ are represented by the atom a.
2. Rules $\dfrac{a_1 \quad \dots \quad a_n}{b}$ are represented by formulas $a_1 \wedge \dots \wedge a_n \to b$.

Obviously $\vDash_S S^*$ holds true.

To show $\Gamma, S^* \vDash A \implies \Gamma \vDash_S A$ for sets S and S^* as described, assume $\Gamma, S^* \vDash A$. By monotonicity with respect to atomic systems (Remark 3.4), this implies $\Gamma, S^* \vDash_S A$ for any S. Since $\vDash_S S^*$, we get $\Gamma \vDash_S A$ by transitivity of $\vDash_S$.

To show $\Gamma \vDash_S A \implies \Gamma, S^* \vDash A$, assume $\Gamma \vDash_S A$. By ($\vDash_S^{\text{ext}}$) it holds that

$$\Gamma \vDash_S A \iff \text{For all } S_1\text{: } (\vDash_{S \cup S_1} \Gamma \implies \vDash_{S \cup S_1} A).$$

Now assume $\vDash_{S_2} \Gamma$, for an arbitrary S_2. Thus $\vDash_{S \cup S_2} \Gamma$ by monotonicity with respect to atomic systems (Remark 3.4). Hence $\vDash_{S \cup S_2} A$, and $\Gamma \vDash_{S \cup S_2} A$ by ($\vDash_S^{\text{ext}}$).

Assuming $\vDash_{S_2} S^*$, we can infer $\Gamma \vDash_{S_2} A$ by using

$$(\vDash_{S_2} S^* \text{ and } \Gamma \vDash_{S \cup S_2} A) \implies \Gamma \vDash_{S_2} A.$$

We prove this implication by induction on the joint complexity of Γ and A.

For atomic formulas we have $\vDash_{S \cup S_2} a \iff \vdash_{S \cup S_2} a$ by (At), and $a_1, \ldots, a_n \vDash_{S_2} b \iff a_1, \ldots, a_n \vdash_{S_2} b$ by (At$'$). The latter implies for $\vDash_{S_2} S^*$ that all rules of S are derivable in S_2. Hence $\vdash_{S_2} a$, and thus $\vDash_{S_2} a$ by (At).

For non-atomic formulas A we consider the case where A is an implication $B \to C$ (the cases where A has the form $B \wedge C$ or $B \vee C$ are similar):

$$\begin{aligned} \vDash_{S \cup S_2} B \to C &\iff B \vDash_{S \cup S_2} C;\ \text{by } (\to) \\ &\implies B \vDash_{S_2} C;\ \text{by } \vDash_{S_2} S^* \text{ and induction hypothesis} \\ &\implies \vDash_{S_2} B \to C;\ \text{by } (\to). \end{aligned}$$

For S-consequence we have the following:

$$\begin{aligned} \Gamma \vDash_{S \cup S_2} A &\iff \text{For all } S_3 \supseteq (S \cup S_2)\text{: } (\vDash_{S_3} \Gamma \implies \vDash_{S_3} A) \\ &\iff \text{For all } S_4\text{: } (\vDash_{S_4 \cup S \cup S_2} \Gamma \implies \vDash_{S_4 \cup S \cup S_2} A) \\ &\iff \text{For all } S_4\text{: } (\vDash_{S_4 \cup S_2} \Gamma \implies \vDash_{S_4 \cup S_2} A);\ \text{by } \vDash_{S_4 \cup S_2} S^*, \text{ i.h.} \\ &\iff \Gamma \vDash_{S_2} A;\ \text{by } (\vDash). \end{aligned}$$

Having assumed $\vDash_{S_2} \Gamma$ and $\vDash_{S_2} S^*$, we can conclude $\vDash_{S_2} A$, and thus $\Gamma, S^* \vDash A$ by ($\vDash_S^{\text{ext}}$). ■

Remark 3.7. For extension semantics one can also establish Import. However, this presupposes atomic systems of higher-level rules, that is, rules which allow at least for the discharge of atomic assumptions (cf. [7,8]). Whereas in Export we proceed, in the first place, from rules to implicational formulas, we proceed in Import from implicational formulas to rules. To establish Import we thus have to be able to translate left-iterated implications into rules, which in general can only be done by using higher-level rules.

Remark 3.8. Suppose we formulate completeness in a stronger way as

$$\Gamma \vDash_S A \iff \Gamma, S^* \vdash A$$

where S^* is chosen as in Export, that is, as a set of $\vee$-free formulas. Then this strong completeness (which implies completeness in the sense of Definition 1.4) is refuted outright, since it gives us GDP($\vDash_S$), which can be directly inferred from GDP($\vdash$).

Corollary 3.9. *In view of Theorem 2.5 we have the following results:*

1. *By Lemma 2.3 we obtain by classical reasoning that IPC is incomplete with respect to non-extension semantics.*
2. *By Lemma 3.6 we obtain that IPC is incomplete with respect to extension semantics.*

3. By Remark 3.7 IPC is incomplete with respect to extension semantics based on higher-level atomic systems.

These results pertain to proof-theoretic semantics that do not directly specify $\vDash_S$ and $\vDash$, but define some other basic concept which then leads to relations $\vDash_S$ and $\vDash$ in the sense of non-extension or extension semantics. A prominent example is Prawitz's definition of validity for derivations or derivation structures (which was adopted to some extent by Dummett). Here one defines the S-validity of a derivation structure, that is, a tree structure of formulas which results from the application of rules in natural deduction style. These rules can be any rules, possibly discharging free assumptions, and do not necessarily have to be the introduction and elimination rules used in standard natural deduction.

A definition of S-validity along these lines can be given as follows:

1. Every closed derivation in S is S-valid.
2. A closed canonical derivation structure is S-valid, if all its immediate substructures are S-valid. ("Canonical" means ending with an introduction rule.)
3. A closed non-canonical derivation structure is S-valid, if it reduces to an S-valid canonical derivation structure.
4. An open derivation structure

$$\begin{array}{c} A_1 \dots A_n \\ \mathcal{D} \\ B \end{array}$$

where all open assumptions of $\mathcal{D}$ are among $A_1, \dots, A_n$, is S-valid, if for every extension $S' \supseteq S$ and for every list of closed S'-valid derivation structures

$$\begin{array}{c} \mathcal{D}_i \\ A_i \end{array}$$

the closed derivation structure

$$\begin{array}{c} \mathcal{D}_1 \qquad \mathcal{D}_n \\ A_1 \dots A_n \\ \mathcal{D} \\ B \end{array}$$

is S'-valid.

Once the notion of an S-valid derivation structure has been defined, the consequence relation $\Gamma \vDash_S A$ can be defined by requiring that there be an S-valid derivation structure from Γ to A. Then $\Gamma \vDash A$ means that $\Gamma \vDash_S A$

holds for every S. Depending on whether in the definition of an S-valid derivation structure we define the validity of an open derivation structure by reference to extensions $S' \supseteq S$ of atomic systems S or not, we obtain an extension semantics or a non-extension semantics. If we define the validity of an open derivation by using extensions, which is certainly what Prawitz intended in his first proposal [10] of proof-theoretic validity, then by Corollary 3.9(2) we obtain the incompleteness of IPC for this semantics, thus refuting Prawitz's conjecture that IPC is complete for this semantics. If we understand Prawitz's semantics as a non-extension semantics (which is not without problems, see [8]), then by Corollary 3.9(1) we again obtain incompleteness of IPC, albeit by means of classical reasoning in the meta-language, which, as a negative result, is as devastating for the completeness conjecture as a constructive proof.

4. Observations on Semantics for Which IPC is Complete

Our focus has been on incompleteness. We established semantical conditions under which IPC is incomplete and which are satisfied by basic notions of proof-theoretic validity. Applying these results to semantics for which IPC is not incomplete but complete, gives us nonetheless substantial insight, in particular for the condition of Export. Theorem 2.5 tells us that if we have a semantics for which IPC is sound and complete, then Export cannot hold. In other words, if IPC is sound and complete, the bases S of the semantics, which constitute S-consequence $\ldots \vDash_S \ldots$, cannot be represented by means of sets of $\vee$-free formulas Γ functioning as assumptions of logical consequence $\ldots, \Gamma \vDash \ldots$. In particular it is not necessarily true that the S-validity of a formula A (i.e., $\vDash_S A$) can be expressed as the universal validity of A with respect to some set of $\vee$-free assumptions Γ (i.e., $\Gamma \vDash A$). This is a significant result, for example for Kripke semantics of IPC.

It can easily be seen that Kripke semantics for IPC is a semantics in the abstract sense of Section 1. A base is an entity $\langle \mathfrak{W}, \geq, v, w \rangle$, where $\langle \mathfrak{W}, \geq \rangle$ is a Kripke frame for intuitionistic propositional logic, v is a valuation assigning a truth value to every propositional variable in any reference point, and w is a reference point. All five conditions for a semantics are satisfied if we define logical consequence as follows:

$$\Gamma \vDash_{\langle \mathfrak{W}, \geq, v, w \rangle} A \;:\Longleftrightarrow\; \text{For all } w' \geq w\text{: } (\vDash_{\langle \mathfrak{W}, \geq, v, w' \rangle} \Gamma \implies \vDash_{\langle \mathfrak{W}, \geq, v, w' \rangle} A).$$

The non-validity of Export for Kripke semantics means in particular that we cannot internalize, that is, code the validity of A in a reference point w

(i.e., $\vDash_{\langle \mathfrak{W}, \geq, v, w \rangle} A$) as the derivability from a suitably chosen set of $\vee$-free assumptions Γ.

Within the realm of proof-theoretic semantics, Goldfarb [1] has given a semantics for which IPC is complete. It can be reconstructed in our framework as follows. Take a base S to be a pair $\langle \mathcal{R}, \alpha \rangle$, where α is a set of propositional variables, and $\mathcal{R}$ is an atomic system (in the sense of Definition 3.1), such that α is closed under the rules of $\mathcal{R}$. Then we obtain a semantics in the sense of Section 1, if we define

$$\vDash_{\langle \mathcal{R}, \alpha \rangle} p \;:\Longleftrightarrow\; p \in \alpha, \text{ for propositional variables } p$$

and

$$\Gamma \vDash_{\langle \mathcal{R}, \alpha \rangle} A \;:\Longleftrightarrow\; \text{For all } \beta \supseteq \alpha\text{: } (\vDash_{\langle \mathcal{R}, \beta \rangle} \Gamma \implies \vDash_{\langle \mathcal{R}, \beta \rangle} A).$$

Goldfarb [1] was able to show that IPC is complete for this semantics, by interpreting standard Kripke semantics in it. Our results then show that Export cannot hold for this semantics, that is, we cannot code $\langle \mathcal{R}, \alpha \rangle$-validity as universal validity with respect to a set of $\vee$-free assumptions.

Final Remark. From the point of view of proof-theoretic semantics, intuitionistic logic has always been considered the main alternative to classical logic. However, in view of the results discussed here, intuitionistic logic does not capture basic ideas of proof-theoretic semantics. Given the fact that a semantics should be primary over a syntactic specification of a logic, we observe that intuitionistic logic falls short of what is valid according to proof-theoretic semantics. The incompleteness of intuitionistic logic with respect to such a semantics therefore raises the question of whether there is an intermediate logic between intuitionistic and classical logic which is complete with respect to it.

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