The Geometry of Static Spacetimes in General Relativity

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Two Theories of Gravitation

Physics knows two major theories of gravitation:
the classical Newtonian one (NG, 17th century) and
Einstein's general relativity (GR, 20th century).

Both theories rely on PDEs connecting the matter content of a gravitating system to its gravitational variables:

Newtonian PDE

$$\triangle U = 4\pi G\rho$$

ρ: matter density
U: gravitational potential

Relativistic PDE (Einstein)

$$\mathrm{Ric} - \frac{1}{2}\,\mathrm{R}\,\mathrm{g} = \frac{8\pi G}{c^4} T$$

T: energy-momentum tensor
$\mathrm{g}, \mathrm{Ric}, \mathrm{R}$: geometric variables

G: Gravitational constant, c: speed of light

Two Theories of Gravitation ctd.

Both theories have their advantages and disadvantages:

Newtonian Theory

very accurate "in every day life"

INACCURATE for high speeds

comparably LITTLE effort for computations

centuries of experience/
fairly EASY to interpret/model

Relativistic Theory

very accurate in "every day life"

ACCURATE for high speeds

comparably LARGE effort for computations

relative lack of experience/
fairly HARD to interpret/model

Idea/Aim

Learn from our physical/mathematical knowledge of Newtonian gravity to gain

$\rightarrow$ a better understanding of relativistic gravitating systems

- their geodesics
- their asymptotic behavior
- equipotential surfaces
- local and total mass
- local and total center of mass
- the Newtonian limit

$\rightarrow$ a better intuition for GR

$\rightarrow$ new methods for proofs in GR

Setting

We specialize to the following (rather general) setting of

Physical Systems

- isolated
- static
- finite extension of source

Mathematical Models

- asymptotically flat
- timelike Killing vector, hypersurface-orthogonal
- matter tensor has spatially compact support

GR = *geometrodynamics* $\rightarrow$ static GR = *geometrostatics*

Contents

1 Geometrostatic Systems

2 Geometric and Physical Properties
- Equipotential Surfaces
- Uniqueness Properties

3 Mass and Center of Mass
- Pseudo-Newtonian Gravity
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Reminder: Formal Structure of GR

A relativistic system consists of

- a space-time 4-manifold M^4
- a symmetric matter tensor field T
- a Lorentzian metric ${}^4g$

satisfying the Einstein equations

$${}^4\mathrm{Ric} - \frac{1}{2}\,{}^4\mathrm{R}\,{}^4g = \frac{8\pi G}{c^4}\,T$$

with gravitational constant G, speed of light c.

3+1 Decomposition: Initial Value Problem Approach

Theorem (Choquet-Bruhat et al.)

The Einstein equations can be reformulated as a well-posed hyperbolic initial value problem (for suitable matter models).

Remarks:

- involves (non-canonical) 3+1 decomposition
- involves choice of coordinates (lapse and shift)
- gives rise to phenomena like gravitational waves
- IVP approach is used in numerical simulations

Static General Relativity

Definition

A relativistic system $(M^4, {}^4g_{\mu\nu}, T^{\mu\nu})$ is called static if it possesses a global timelike Killing vector field X that is hypersurface-orthogonal:

$$ {}^4\nabla_{(\alpha} X_{\beta)} = 0 \quad \text{and} \quad X_{[\alpha}\, {}^4\nabla_{\beta} X_{\gamma]} = 0 $$

Frobenius: hypersurface-orthogonality $\leftrightarrow$ integrability of distribution $X^{\perp}$

Lapse and 3-Metric

Proposition

Generic static spacetimes can be canonically decomposed into $M^4 = \mathbb{R} \times M^3$,

$$^4g = -N^2c^2dt^2 + {}^3g,$$

with induced Riemannian metric ^3g *and lapse function*

$$N := \sqrt{-{}^4g(X,X)} > 0;$$

X *is the hypersurface-orthogonal timelike Killing vector and* $dt := X^b$.

Static Geometry: the Facts

- All time-slices $(M^3, {}^3g)$ are isometric.
- They are embedded into $(M^4 = \mathbb{R} \times M^3, {}^4g)$ with vanishing second fundamental form.
- The lapse function $N : M^3 \to \mathbb{R}^+$ is independent of “time”.
- The matter tensor induces time-independent matter variables ρ (matter density) and S (stress tensor).

$\Rightarrow$ We think of a static spacetime as a tuple $(M^3, {}^3g, N, \rho, S)$.

What Are the Main Equations?

Static general relativity is governed by the

Static Metric Equations

$$
\begin{aligned}
{}^3\triangle N &= \frac{4\pi G}{c^2} N \left(\rho + \frac{{}^3\mathrm{tr} S}{c^2}\right) \\
N\ {}^3\mathrm{Ric} &= {}^3\nabla^2 N + \frac{4\pi G}{c^2} N \left(\rho\ {}^3g + \frac{2}{c^2}\left(S - \frac{{}^3\mathrm{tr} S}{2}{}^3g\right)\right)
\end{aligned}
$$

on M^3.

What Are the Main Equations?

Vacuum static general relativity is governed by the

Vacuum Static Metric Equations

$$\begin{aligned} {}^3\triangle N &= 0 \\ N\,{}^3\mathrm{Ric} &= {}^3\nabla^2 N \end{aligned}$$

on M^3.

Wave-Harmonic Coordinates and Regularity

The static metric equations form an elliptic system of PDEs in wave-harmonic coordinates:

Definition

Local coordinates x^i on M^3 are called wave-harmonic if they satisfy

$$^4\Box\, x^i = 0$$

with respect to $^4g = -N^2c^2dt^2 + {}^3g$.

Theorem (Müller zum Hagen)

Any solution of the static metric equations is real analytic with respect to local wave-harmonic coordinates (for suitable matter models).

Asymptotic Flatness

Theorem (Kennefick & O'Murchadha)

Asymptotically Euclidean solutions of the static metric equations with compactly supported matter and $N \to 1$ as $r \to \infty$ are automatically asymptotically Schwarzschildean:

$$N = 1 - \frac{mG}{rc^2} + \mathcal{O}(\frac{1}{r^2}) \quad \text{and} \quad {}^3g_{ij} = (1 + \frac{2mG}{rc^2})\,\delta_{ij} + \mathcal{O}(\frac{1}{r^2})$$

in asymptotically flat wave-harmonic coordinates.

Remark: m is the ADM-mass of ${}^3g$.

Geometrostatics

Definition (Geometrostatic Systems)

A geodesically complete asymptotically flat solution $(M^3, {}^3g, N, \rho, S)$ of the static metric equations with compactly supported ρ and S and $N \to 1$ as $r \to \infty$ is called a

geometrostatic system.

Remarks:

- asymptotic flatness is defined in weighted Sobolev spaces
- definition can be extended to include black hole solutions
- name stresses the geometric viewpoint taken

Formal Analogy to Newton

Newtonian

$$
\begin{aligned}
{}^{\delta}\triangle U &= 4\pi G \times \text{matter} \\
{}^{\delta}\mathrm{Ric} &= 0 \\
\\
U &= -mG/r + \mathcal{O}(r^{-2}) \\
\delta_{ij} &= \delta_{ij}
\end{aligned}
$$

Geometrostatics

$$
\begin{aligned}
{}^{3}\triangle N &= 4\pi G \times N/c^2 \times \text{matter} \\
{}^{3}\mathrm{Ric} &= {}^{3}\nabla^2 N/N + \text{matter} \\
\\
N &= 1 - mG/rc^2 + \mathcal{O}(r^{-2}) \\
{}^{3}g_{ij} &= \left(1 + 2mG/rc^2\right)\delta_{ij} \\
&\quad + \mathcal{O}(r^{-2})
\end{aligned}
$$

Question: How similar is U to N physically/geometrically?

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Physical Interpretation

From definition:

- $(M^3, {}^3g)$ is a time-slice, i.e. the status of a static system at any point of time in the eyes of the chosen "observer" X
- N describes how to measure time in order to "see" staticity.

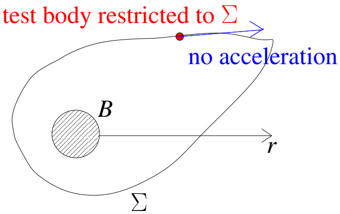
New focus/interpretation:

- The level sets of N relate to the dynamics of test bodies!

Equipotential Surfaces in Newtonian Gravity

A surface $\Sigma \subset \mathbb{R}^3$ is an equipotential surface in NG if it is a level set of the Newtonian potential U.

- Fact 1: Test bodies constrained to equipotential surfaces are not accelerated.
- Fact 2: Level sets of U are the only surfaces.



Equipotential Surfaces in Geometrostatics

Definition (Mimicking Newtonian Gravity, C.)

A timelike curve in $(\mathbb{R} \times M^3, {}^4g = -N^2c^2dt^2 + {}^3g)$ is called a constrained test body if it is a critical point of the time functional

$$\mathcal{T}(\mu) := \int_a^b |\dot{\mu}(\tau)| \, d\tau$$

among all timelike curves of the form $\mu(\tau) = (t(\tau), x(\tau))$ with $x(\tau) \in \Sigma$.

A surface $\Sigma \subset M^3$ is called an equipotential surface in $(M^3, {}^3g, N)$ if the spatial component $x(\tau)$ of every constrained test body is a geodesic in Σ with respect to the induced 2-metric.

The Role of the Level Sets of N

Theorem (C.)

A hypersurface $\Sigma \subset M^3$ is an equipotential surface in $({}^3g, N)$ if and only if $N \equiv$ const on Σ, i.e. iff Σ is a level set of N.

Proof: Calculus of variations with Lagrange multipliers.

Newtonian Gravity Versus Geometrostatics

Newtonian

levels of U are the only equipotential surfaces

Geometrostatic

levels of N are the only equipotential surfaces

So: the level sets of U, N have the same physical interpretation!
$\rightarrow$ definition of force $\vec{F}$ (on test bodies); $\vec{F} = m\vec{a}$

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Uniqueness in Newtonian Gravity

Newton's Equation

$$\begin{aligned} \triangle U &= 4\pi G\rho && \text{in } \mathbb{R}^3 \\ U &\to 0 && \text{as } r \to \infty \end{aligned}$$

is

- an elliptic PDE (Poisson equation)
- with "Dirichlet boundary conditions" at ∞
  $\to$ formally modelled by weighted Sobolev spaces
- uniquely solvable in suitably chosen function spaces

Uniqueness of N in Geometrostatics

Y. Choquet-Bruhat's famous ▸ theorem on the Cauchy problem implies

Corollary (C.)

The equipotential surfaces (and the Lorentzian metric) of a static gravitational system are in fact independent of the lapse function N.

In other words, if $(\mathbb{R} \times M^3, {}^4g)$ is a static as. flat solution of Einstein's equation, then it is uniquely characterized by its induced 3-metric ${}^3g$.

Proof: The constraint equations reduce to ${}^3\mathrm{R} = 0$ in our case ($K \equiv 0$). Equipotential surfaces only depend on ${}^4g$. □

Uniqueness of N ctd.

Theorem (C.)

Let $({}^3g, N)$ and $({}^3g, \widetilde{N})$ solve the static vacuum Einstein equations with $N, \widetilde{N} \to 1$ as $r \to \infty$ and suppose that ${}^3g$ is non-flat. Then $N \equiv \widetilde{N}$.

Proof: (vacuum part here, remainder follows from elliptic theory)

- Levels of $N, \widetilde{N}$ are each the only equipotential surfaces $\Rightarrow \widetilde{N} = f \circ N$ for some function $f : \mathbb{R} \to \mathbb{R}$.
- $0 = {}^3\triangle \widetilde{N} = {}^3\triangle (f \circ N) = f'' \circ N \, \|{}^3\mathrm{grad} N\|^2_{{}^3g} + f' \circ N \underbrace{{}^3\triangle N}_{=0}$ so that $f'' = 0$ and thus $\widetilde{N} = \alpha N + \beta$ with $\alpha, \beta \in \mathbb{R}$.
- ${}^3\nabla^2 \widetilde{N} = \widetilde{N}\, {}^3\mathrm{Ric} = (\alpha N + \beta)\, {}^3\mathrm{Ric} = \alpha\, {}^3\nabla^2 N + \beta\, {}^3\mathrm{Ric} = {}^3\nabla^2 \widetilde{N} + \beta\, {}^3\mathrm{Ric}$ and ${}^3\mathrm{Ric} \neq 0$ gives $\beta = 0$.
- Finally $N, \widetilde{N} \to 1$ as $r \to \infty \Rightarrow \alpha = 1 \Rightarrow N \equiv \widetilde{N}$. $\square$

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- Finally $N, \widetilde{N} \to 1$ as $r \to \infty \Rightarrow \alpha = 1 \Rightarrow N \equiv \widetilde{N}$. $\square$

Newtonian Gravity Versus Geometrostatics

Newtonian

For given matter and Galilei coordinates (i.e. flat metric), U is unique.

Geometrostatic

For given matter and metric, N is unique.

- U, N have very similar uniqueness properties
- justifies name "static potential" used in the literature
- can also be proved with purely analytic methods
- and with a 3-geodesic method combined with an open-closed argument

Even More: Uniqueness of ${}^3g$

Theorem (C.)

${}^3g$ is unique for given N and prescribed wave-harmonic asymptotically flat coordinates.

Proof: Asymptotic analysis in weighted Sobolev spaces using the static metric equations. Properties of homogeneous harmonic polynomials.

- For given coordinates, N and ${}^3g$ are "dual".
- Geometrostatic systems have 4 degrees of freedom
  $\rightarrow$ plausibility check for Bartnik's conjecture on mass

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Pseudo-Newtonian Gravity

Definition (Pseudo-Newtonian System)

Let $(M^3, {}^3g, N, \rho, S)$ be a geometrostatic system. Let

$$\begin{aligned} U &:= c^2 \log N \\ \gamma &:= N^2\, {}^3g = e^{2U/c^2}\, {}^3g \end{aligned}$$

and call U the associated pseudo-Newtonian potential and $(M^3, \gamma, U, \rho, S)$ the associated pseudo-Newtonian system.

Pseudo-Newtonian Equations

Proposition

$(M^3, {}^3g, N, \rho, S)$ satisfies the static metric equations iff $(M^3, \gamma, U, \rho, S)$ satisfies the pseudo-Newtonian equations

$$\begin{aligned} {}^{\gamma}\triangle U &= 4\pi G \left(\frac{\rho}{e^{2c^{-2}U}} + \frac{{}^{\gamma}\mathrm{tr}S}{c^2} \right) \\ {}^{\gamma}\mathrm{Ric} &= \frac{2}{c^4}\, dU \otimes dU + \frac{8\pi G}{c^4} \left(S - {}^{\gamma}\mathrm{tr}S\, \gamma \right). \end{aligned}$$

In vacuum, these read

$$\begin{aligned} {}^{\gamma}\triangle U &= 0 \\ {}^{\gamma}\mathrm{Ric} &= \frac{2}{c^4}\, dU \otimes dU. \end{aligned}$$

Pseudo-Newtonian Fall-Off

Proposition

U and γ inherit the fall-off

$$U = -\frac{mG}{r} + \mathcal{O}(\frac{1}{r^2}) \quad \text{and} \quad \gamma_{ij} = \delta_{ij} + \mathcal{O}(\frac{1}{r^2})$$

in asymptotically flat γ-harmonic coordinates.

- Coordinates are harmonic w.r.t. γ iff they are wave-harmonic.
- $m =$ ADM-mass of ${}^3g$

Formal Analogy to Newton

Newtonian

$$\begin{aligned} {}^{\delta}\triangle U &= 4\pi G \times \rho \\ {}^{\delta}\mathrm{Ric} &= 0 \\ U &= -mG/r + \mathcal{O}(r^{-2}) \\ \delta_{ij} &= \delta_{ij} \end{aligned}$$

Pseudo-Newtonian

$$\begin{aligned} {}^{\gamma}\triangle U &= 4\pi G \times \text{matter} \\ {}^{\gamma}\mathrm{Ric} &= 2c^{-4}\, dU \otimes dU \\ &\quad + c^{-4}\,\text{matter} \\ U &= -mG/r + \mathcal{O}(r^{-2}) \\ \gamma_{ij} &= \delta_{ij} + \mathcal{O}(r^{-2}) \end{aligned}$$

Formally: $c \to \infty \Rightarrow$ equations converge (if variables converge)

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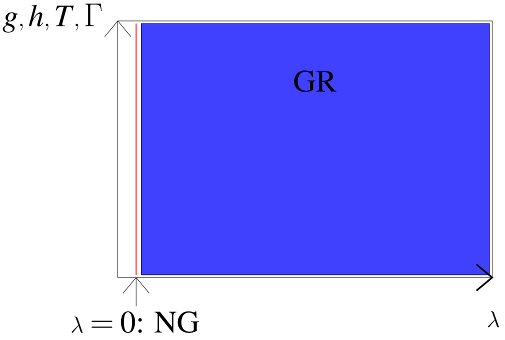
Newtonian Limit

Ehlers constructed Frame Theory encompassing GR and NG

→ Frame Theory allows rigorous definition of Newtonian Limit

→ Rendall, Olyinyk, etc. showed existence of converging families



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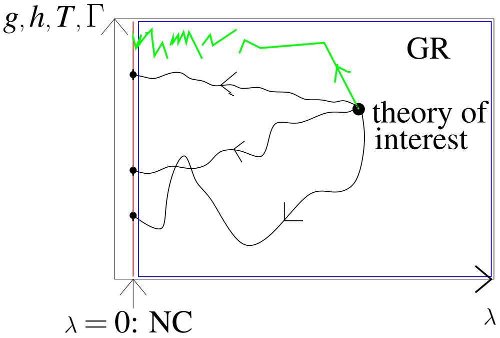

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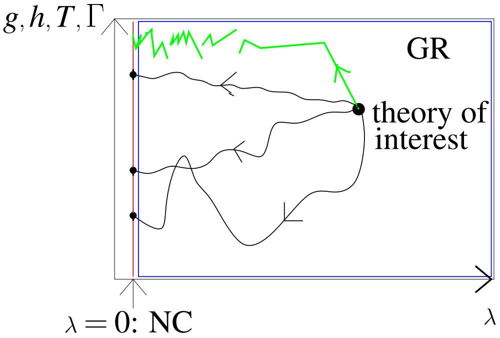

Newtonian Limit ctd.

Theorem (C.)

Using Ehlers' Frame Theory, one finds that in the Newtonian limit

- *the pseudo-Newtonian potential $U(c)$ converges to the Newtonian potential U and*
- *the metric $\gamma(c)$ converges to the flat metric δ.*

Remarks:

- in suitable asymptotically flat coordinates
- in suitably chosen weighted Sobolev spaces
- for families with a Newtonian limit

Difficulties

- Definition of staticity in Frame Theory
  (Killing vector fields, hypersurface-orthogonality)
- Definitions of U, γ in Frame Theory
- Definition of a (uniform, C^1) Newtonian Limit
- Handling coordinate dependence

Physical Properties and the Newtonian Limit

- How do physical properties behave under the Newtonian limit along a family of pseudo-N. systems that has a Newtonian limit?
- Can we stretch the analogy between Newtonian and pseudo-Newtonian theories further?

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The Mass of a Newtonian System

- The *mass* of a system with density ρ is defined as

$$m_N := \int_{\mathbb{R}^3} \rho \, dV.$$

- By $\triangle U = 4\pi G\rho$ and the divergence theorem rewrite

$$m_N = \int_{\mathbb{R}^3} \rho \, dV = \int_C \rho \, dV = \frac{1}{4\pi G} \int_C \triangle U \, dV = \frac{1}{4\pi G} \int_{\partial C} \frac{\partial U}{\partial \nu} \, d\sigma$$

where C is any compact domain with ∂C smooth and $\operatorname{supp} \rho \subset C$.

- Here: dV volume measure, $d\sigma$ surface measure of ∂C, ν outer unit normal to ∂C

The Mass of a Newtonian System Revisited

Quasilocal Newtonian Mass

For a Newtonian system with potential U and a smooth surface $\Sigma \subset \mathbb{R}^3$ with outer unit normal ν define

$$m_N(\Sigma) := \frac{1}{4\pi G} \int_\Sigma \frac{\partial U}{\partial \nu}\, d\sigma.$$

This proves the classical

Total Mass Theorem

If Σ encloses $\operatorname{supp} \rho$, then $m_N(\Sigma) = m_N$.

The Pseudo-Newtonian Mass

In this spirit, we define

Definition (Quasilocal Pseudo-Newtonian Mass)

Let (γ, U) be as before. Let all geometric notions refer to γ. For any smooth surface $\Sigma \subset M^3$ define

$$m_{PN}(\Sigma) := \frac{1}{4\pi G} \int_{\Sigma} \frac{\partial U}{\partial \nu}\, d\sigma.$$

Here, ν is the γ-outer unit normal to and $d\sigma$ is the γ-surface measure on Σ.

The integral $\frac{1}{4\pi G} \int_{S_\infty} \frac{\partial U}{\partial \nu}\, d\sigma$ is well-known as the Komar-mass.

The Newtonial Limit of Mass

Just as in the Newtonian setting, we have

Theorem (Pseudo-Newtonian Total Mass Theorem, C.)

If Σ encloses the support of the matter, then $m_{PN}(\Sigma) = m_{ADM}({}^3g)$. In particular, m_{PN} is independent of Σ and can be calculated "locally".

Proof: Recall $U = -mG/r + \mathcal{O}(r^{-2})$ with $m = m_{ADM}({}^3g)$ and use suitable weighted Sobolev spaces. □

Theorem (Newtonian Limit of Mass Theorem, C.)

For any sequence of space-times with a Newtonian limit,

$$m_{ADM}(c) = m_{PN}(c) \to m_N \text{ as } c \to \infty.$$

Proof: $U(c) \to U$ and $\gamma(c) \to \delta$ as $c \to \infty$, see above. □

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The CoM of a Newtonian System

- The *center of mass* (CoM) of a system with density ρ is defined as

$$\vec{z}_N := \frac{1}{m_N} \int\limits_{\mathbb{R}^3} \rho\, \vec{x}\, dV$$

  w.r.t. Galilei coordinates (x^i).

- For Σ enclosing $\operatorname{supp}\rho$, we can rewrite

$$4\pi G\, m_N \vec{z}_N = \int\limits_{\Sigma} \left(\frac{\partial U}{\partial \nu}\, \vec{x} - U\, \frac{\partial \vec{x}}{\partial \nu} \right) d\sigma$$

  using Green's formula.

The CoM of a Newtonian System Revisited

Definition (Quasi-local Newtonian CoM)

For a Newtonian system with potential U define

$$\vec{z}_N(\Sigma) := \frac{1}{4\pi G m_N} \int\limits_{\Sigma} \left(\frac{\partial U}{\partial \nu}\,\vec{x} - U\,\frac{\partial \vec{x}}{\partial \nu} \right) d\sigma,$$

where ν is the outer unit normal to Σ.

Newtonian CoM Theorem

If Σ encloses $\operatorname{supp}\rho$, then $\vec{z}_N(\Sigma) = \vec{z}_N$. If Σ is a level set of U, then

$$\vec{z}_N = \frac{1}{4\pi G m_N} \int\limits_{\Sigma_U} \frac{\partial U}{\partial \nu}\,\vec{x}\,d\sigma.$$

The Pseudo-Newtonian CoM

Definition (Quasilocal Pseudo-Newtonian CoM)

Let (γ, U) be as before. Take all geometric notions w.r.t. γ and use γ-harmonic coordinates: ${}^{\gamma}\triangle x^i = 0$. Then we define

$$\vec{z}_{PN}(\Sigma) := \frac{1}{4\pi G m_{PN}} \int_{\Sigma} \left(\frac{\partial U}{\partial \nu}\, \vec{x} - U\, \frac{\partial \vec{x}}{\partial \nu} \right) d\sigma.$$

Here, ν and σ are the outer unit normal to and surface measure on Σ w.r.t. γ, respectively.

Remark: γ-harmonic coordinates are also wave-harmonic.

Facts on the Pseudo-Newtonian CoM

Again, we can prove theorems similar to the Newtonian one:

1st Pseudo-Newtonian CoM Theorem, C.

$$\vec{z}_{PN} := \vec{z}_{PN}(\Sigma)$$

is independent of the specific surface Σ enclosing the support of the matter and can be calculated "locally". If Σ is a level set of U, then

$$\vec{z}_{PN} = \frac{1}{4\pi G m_{PN}} \int_{\Sigma_U} \frac{\partial U}{\partial \nu}\, \vec{x}\, d\sigma.$$

Facts on the Pseudo-Newtonian CoM ctd.

2nd Pseudo-Newtonian CoM Theorem, C.

The (pseudo-Newtonian) CoM $\vec{z}$ can be read off the asymptotics of U:

$$U = -\frac{mG}{r} - \frac{mG\,\vec{z}\cdot\vec{x}}{r^3} + \mathcal{O}(\frac{1}{r^3}).$$

This expression transforms adequately under change of asymptotically flat γ-harmonic coordinates.

Proof: theory of weighted Sobolev spaces, regularity arguments.

Faster Fall-Off

Faster Fall-Off Theorem

Let $k \in \mathbb{N}_0$, $1 < p < \infty$, $\delta < 0$ with $\delta \notin \mathbb{Z}$, and $f \in C^\infty(\mathbb{R}^3)$. Assume

$$f \in W^{k+2,p}_{\delta+1} \text{ and } {}^{\delta}\triangle f \in W^{k,p}_{\delta-2}.$$

Then there exists a harmonic rescaled polynomial p of degree $d \leq \lceil \delta \rceil$ with $f - p \in V^{k+2,p}_{\delta}$.

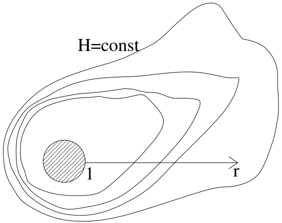
- In $\mathcal{O}$-notation: $f = \mathcal{O}(r^{-l+1}), \triangle f = \mathcal{O}(r^{-l-2}) \Rightarrow f - p = \mathcal{O}(r^{-l})$
- Example: $U - {}^{S}U = \mathcal{O}(r^{-2}), {}^{\delta}\triangle(U - {}^{S}U) = \mathcal{O}(r^{-5})$ gives $U - {}^{S}U = \vec{z} \cdot \vec{x}/r^3 + \mathcal{O}(r^{-3})$.
- This is also used to show uniqueness of ${}^{3}g$ given N and for a different proof of the uniqueness of N given ${}^{3}g$.

Facts on the Pseudo-Newtonian CoM ctd.

3rd Pseudo-N. CoM Theorem, C.

This CoM coincides with the CoM given by a foliation by spheres of constant mean curvature (Huisken & Yau, Metzger) and with the ADM center of mass:

$$\vec{z}_{PN} = \vec{z}_{CMC} = \vec{z}_{ADM}.$$



Proof: Uses Huang's result $\vec{z}_{CMC} = \vec{z}_{ADM}$ and asymptotics proved above.

The Newtonian Limit of the Pseudo-Newtonian CoM

Newtonian Limit of Pseudo-Newtonian CoM Theorem

The Newtonian limit of the pseudo-Newtonian CoM and therefore also of the CMC center of mass is the CoM of the Newtonian limit along any sequence of space-times with a Newtonian limit:

$$\vec{z}_{CMC}(c) = \vec{z}_{ADM}(c) = \vec{z}_{PN}(c) \;\to\; \vec{z}_N \text{ as } c \to \infty.$$

Generalizations: other spacelike slices in static GR

- graphs over $\{t = 0\}$-slice
  - with Christopher Nerz:
  - asymptotic decay becomes more delicate
  - global notion of CoM can break down even in Schwarzschild slices
- general asymptotically flat slices (e.g. boosted ones)
- general asymptotically hyperbolic slices

Possible Applications of Geometrostatics

- Static n-Body Problem?
- Bartnik's Static Metric Extension Conjecture?

Summary

$$\triangle U = 4\pi G \rho$$

- static Newtonian Gravity with potential U
- Geometrostatics $({}^3g, N)$ or equivalently pseudo-Newtonian Gravity (γ, U)
- equipotential surfaces
- mass as surface integral
- expressions for CoM
- Newtonian limit.

Summary

$$
\begin{aligned}
{}^{3}\triangle N &= \frac{4\pi G}{c^2} N \times \text{matter} \\
N\,{}^{3}\mathrm{Ric} &= {}^{3}\nabla^2 N + N \times \text{matter}
\end{aligned}
$$

$$
\begin{aligned}
{}^{\gamma}\triangle U &= 4\pi G \times \text{matter} \\
{}^{\gamma}\mathrm{Ric} &= c^{-4}(2dU \otimes dU + \text{matter})
\end{aligned}
$$

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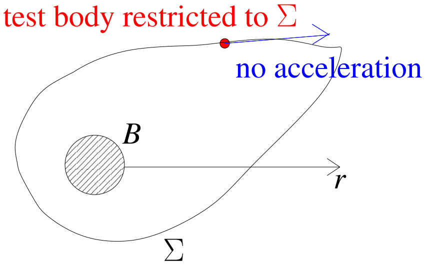
Summary



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Summary

$$m_{PN} := \frac{1}{4\pi G} \int\limits_{\Sigma} \frac{\partial U}{\partial \nu}\, d\sigma$$

Theorem

$m_{PN} = m_{ADM}$

- static Newtonian Gravity with potential U
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Summary

$$\vec{z}_{PN} := \frac{1}{4\pi G m_{PN}} \times \int_{\Sigma} \left(\frac{\partial U}{\partial \nu}\,\vec{x} - U \frac{\partial \vec{x}}{\partial \nu} \right) d\sigma$$

Theorem

$U = -\frac{mG}{r} - \frac{mG\,\vec{z}\cdot\vec{x}}{r^3} + \mathcal{O}(\frac{1}{r^3})$

Theorem

$\vec{z}_{PN} = \vec{z}_{CMC} = \vec{z}_{ADM}$

- static Newtonian Gravity with potential U
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Summary

Theorem

$$\begin{aligned} U(c) &\rightarrow U \\ \gamma(c) &\rightarrow \delta \\ m_{PN}(c) &\rightarrow m_N \\ \vec{z}_{PN}(c) &\rightarrow \vec{z}_N \end{aligned}$$

- static Newtonian Gravity with potential U
- Geometrostatics $({}^3g, N)$ or equivalently pseudo-Newtonian Gravity (γ, U)
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Thank you for your attention!