The Power Mean Laplacian for Multilayer Graph Clustering

Pedro Mercado[1] **Antoine Gautier**[1] **Francesco Tudisco**[2] **Matthias Hein**[1]
[1]Department of Mathematics and Computer Science, Saarland University, Germany
[2]Department of Mathematics and Statistics, University of Strathclyde, G11XH Glasgow, UK

1 Proofs for the Stochastic Block Model analysis

This section has two parts corresponding to the Case 1 and 2 of the stochastic block model analysis. At the beginning of each of these sections, we first state what will be proved and discuss further refinements implied by the results presented here. For convenience we recall the notation where needed.

The correspondence between the results of the main paper and those proved here is as follows: In Section 1.1 we discuss and prove Lemma 1, 2, Theorem 1 and Corollary 1 of the main paper. These results are directly implied by Lemma 1, Theorem 1 and Corollaries 1, 2, respectively, of the present manuscript. Then, in Section 1.2, we prove Theorem 2 and Theorem 3 of the main paper which are respectively equivalent to Theorems 2 and 3 below.

Before proceeding to the proofs, let us recall the setting. Let $V = \{v_1, \dots, v_n\}$ be a set of nodes and let T be the number of layers, represented by the adjacency matrices $\mathbb{W} = \{W^{(1)}, \dots, W^{(T)}\}$. For each matrix $W^{(t)}$ we have a graph $G^{(t)} = (V, W^{(t)})$ and, overall, a multilayer graph $\mathbb{G} = (G^{(1)}, \dots, G^{(T)})$. We denote the ground truth clusters by $\mathcal{C}_1, \dots, \mathcal{C}_k$ and assume that they all have the same size, i.e. $|\mathcal{C}_i| = |\mathcal{C}|$ for $i = 1, \dots, k$.

In the following, we denote the identity matrix in $\mathbb{R}^m$ by I_m. Furthermore, for a matrix $X \in \mathbb{R}^{m\times m}$, we denote its eigenvalues by $\lambda_1(X), \dots, \lambda_m(X)$.

1.1 All layers have the same clustering structure

For $t = 1, \dots, T$, let $p_{\text{in}}^{(t)}$ (resp. $p_{\text{out}}^{(t)}$) denote the probability that there exists an edge in layer $G^{(t)}$ between nodes that belong to the same (resp. different) clusters. Suppose that for $t = 1, \dots, T$, the expected adjacency matrix $\mathcal{W}^{(t)} \in \mathbb{R}^{n\times n}$ of $G^{(t)}$ is given for $i, j = 1, \dots, n$ as

$$\mathcal{W}_{ij}^{(t)} = \begin{cases} p_{\text{in}}^{(t)} & \text{if } v_i, v_j \text{ belong to the same cluster} \\ p_{\text{out}}^{(t)} & \text{otherwise.} \end{cases}$$

Furthermore, for every $t = 1, \dots, T$, and $\epsilon \geq 0$, let

$$\mathcal{D}^{(t)} = \operatorname{diag}(\mathcal{W}^{(t)}\mathbf{1}), \quad \rho_t = \frac{p_{\text{in}}^{(t)} - p_{\text{out}}^{(t)}}{p_{\text{in}}^{(t)} + (k-1)p_{\text{out}}^{(t)}},$$

$$\mathcal{L}_{\text{sym}}^{(t)} = I_n - (\mathcal{D}^{(t)})^{-1/2}\mathcal{W}^{(t)}(\mathcal{D}^{(t)})^{-1/2} + \epsilon I_n$$

Observe that $\mathcal{L}_{\text{sym}}^{(t)}$ is the normalized Laplacian of the expected graph plus a diagonal shift. The diagonal shift is necessary to enforce this matrix to be positive definite for the cases $p \leq 0$, as stated in [1].

We consider the vectors $\boldsymbol{\chi}_1, \dots, \boldsymbol{\chi}_k \in \mathbb{R}^n$ defined as

$$\boldsymbol{\chi}_1 = \mathbf{1}, \quad \boldsymbol{\chi}_i = (k-1)\mathbf{1}_{\mathcal{C}_i} - \mathbf{1}_{\overline{\mathcal{C}_i}}, \quad i = 2, \dots, k.$$

By construction, $\boldsymbol{\chi}_1, \dots, \boldsymbol{\chi}_k$ are all eigenvectors of $\mathcal{W}^{(t)}$ for every $t = 1, \dots, T$. These eigenvectors are precisely the vectors allowing to recover the ground truth clusters. Let

$$\mathcal{L}_p = M_p\big(\mathcal{L}_{\text{sym}}^{(1)}, \dots, \mathcal{L}_{\text{sym}}^{(T)}\big)$$

where we assume that $\epsilon > 0$ if $p \leq 0$. We prove the following:

Theorem 1. *Let $p \in [-\infty, \infty]$, and assume that $\epsilon > 0$ if $p \leq 0$. Then, there exists λ_i such that $\mathcal{L}_p\boldsymbol{\chi}_i = \lambda_i\boldsymbol{\chi}_i$ for all $i = 1, \dots, k$. Furthermore, $\lambda_1, \dots, \lambda_k$ are the k-smallest eigenvalues of $\mathcal{L}_p$ if and only if $m_p(\boldsymbol{\mu} + \epsilon\mathbf{1}) < 1 + \epsilon$, where $\boldsymbol{\mu} = (1-\rho_1, \dots, 1-\rho_T)$.*

Before giving a proof of Theorem 1 we discuss some of its implications in order to motivate the result. First, we note that it implies that if $\boldsymbol{\chi}_1, \dots, \boldsymbol{\chi}_k$ are among the smallest eigenvectors of $\mathcal{L}_p$ then they are among the smallest eigenvectors of $\mathcal{L}_q$ for any $q \leq p$.

Corollary 1. *Let $q \leq p$ and assume that $\epsilon > 0$ if $\min\{p, q\} \leq 0$. If $\boldsymbol{\chi}_1, \dots, \boldsymbol{\chi}_k$ correspond to the k-smallest eigenvalues of $\mathcal{L}_p$, then $\boldsymbol{\chi}_1, \dots, \boldsymbol{\chi}_k$ correspond to the k-smallest eigenvalues of $\mathcal{L}_q$.*

Proof. If $\lambda_1, \dots, \lambda_k$ are among the k-smallest eigenvalues of $\mathcal{L}_p$, then by Theorem 1, we have $m_p(\boldsymbol{\mu}+\epsilon\mathbf{1}) < 1+\epsilon$. As $m_q(\boldsymbol{\mu}+\epsilon\mathbf{1}) \leq m_p(\boldsymbol{\mu}+\epsilon\mathbf{1})$, Theorem 1 concludes the proof. $\square$

The next corollary deals with the extreme cases where $p = \pm\infty$. In particular, it implies that whenever at least one layer $G^{(t)}$ is informative then the eigenvectors of $\mathcal{L}_{-\infty}$ allow to recover the clusters. This contrasts with $p = \infty$ where the clusters can be recovered from the eigenvectors of $\mathcal{L}_\infty$ if and only if all layers are informative.

Corollary 2. *Let $p \in [-\infty, \infty]$ and $\epsilon > 0$ if $p \leq 0$.*

1. *If $p = \infty$, then $\boldsymbol{\chi}_1, \dots, \boldsymbol{\chi}_k$ correspond to the k-smallest eigenvalues of $\mathcal{L}_\infty$ if and only if all layers are informative, i.e. $p_{\text{in}}^{(t)} > p_{\text{out}}^{(t)}$ holds for all $t \in \{1, \dots, T\}$.*
2. *If $p = -\infty$, then $\boldsymbol{\chi}_1, \dots, \boldsymbol{\chi}_k$ correspond to the k-smallest eigenvalues of $\mathcal{L}_{-\infty}$ if and only if there is at least one informative layer, i.e. there exists a $t \in \{1, \dots, T\}$ such that $p_{\text{in}}^{(t)} > p_{\text{out}}^{(t)}$.*

Proof. Recall that $\lim_{p\to\infty} m_p(\mathbf{v}) = \max_{i=1,\dots,m} v_i$ and $\lim_{p\to-\infty} m_p(\mathbf{v}) = \min_{i=1,\dots,m} v_i$ for any $\mathbf{v} \in \mathbb{R}^m$ with nonnegative entries. Hence, we have $m_{\pm\infty}(\boldsymbol{\mu}+\epsilon\mathbf{1}) = m_{\pm\infty}(\boldsymbol{\mu})+\epsilon$ and thus the condition $m_p(\boldsymbol{\mu}+\epsilon\mathbf{1}) < 1+\epsilon$ of Theorem 1 reduces to $m_{\pm\infty}(\boldsymbol{\mu}) < 1$ for $p = \pm\infty$. Furthermore, note that we have $\mu_t = 1-\rho_t < 1$ if and only if $p_{\text{in}}^{(t)} > p_{\text{out}}^{(t)}$. To conclude, note that $m_\infty(\boldsymbol{\mu}) = \max_{t=1,\dots,T} \mu_t < 1$ if and only if $\mu_t < 1$ for all $t = 1, \dots, T$ and $m_{-\infty}(\boldsymbol{\mu}) = \min_{t=1,\dots,T} \mu_t < 1$ if and only if there exists $t \in \{1, \dots, T\}$ such that $\mu_t < 1$. $\square$

For the proof of Theorem 1, we give an explicit formula for eigenvalues of $\mathcal{L}_p$ in terms of the eigenvalues of $\mathcal{L}_{\text{sym}}^{(1)}, \dots, \mathcal{L}_{\text{sym}}^{(T)}$. Then, we discuss the ordering of these eigenvalues. Furthermore, we show that $\boldsymbol{\chi}_i$ are all eigenvectors of $\mathcal{L}_p$ and compute their corresponding eigenvalues.

By construction, there are k eigenvectors $\boldsymbol{\chi}_i$ of $\mathcal{W}^{(t)}$ corresponding to a possibly nonzero eigenvalue $\lambda_i^{(t)}$. These are given by

$$\boldsymbol{\chi}_1 = \mathbf{1}, \qquad \lambda_1^{(t)} = |\mathcal{C}|\,(p_{\text{in}}^{(t)} + (k-1)p_{\text{out}}^{(t)}),$$
$$\boldsymbol{\chi}_i = (k-1)\mathbf{1}_{\mathcal{C}_i} - \mathbf{1}_{\overline{\mathcal{C}_i}}, \quad \lambda_i^{(t)} = |\mathcal{C}|\,(p_{\text{in}}^{(t)} - p_{\text{out}}^{(t)})$$

for $i = 2, \dots, k$. It follows that $\boldsymbol{\chi}_1, \dots, \boldsymbol{\chi}_k$ are eigenvectors of $\mathcal{L}_{\text{sym}}^{(t)}$ with eigenvalues $\lambda_i(\mathcal{L}_{\text{sym}}^{(t)})$. Furthermore, we have

$$\lambda_1(\mathcal{L}_{\text{sym}}^{(t)}) = \epsilon, \quad \lambda_i(\mathcal{L}_{\text{sym}}^{(t)}) = 1-\rho_t+\epsilon, \quad i = 2, \dots, k,$$
$$\lambda_j(\mathcal{L}_{\text{sym}}^{(t)}) = 1+\epsilon, \quad j = k+1, \dots, n \tag{1}$$

Let
$$\mathcal{L}_p = M_p\big(\mathcal{L}_{\text{sym}}^{(1)}, \dots, \mathcal{L}_{\text{sym}}^{(T)}\big).$$

The following lemma will be helpful to show that $\boldsymbol{\chi}_1, \dots, \boldsymbol{\chi}_k$ are all eigenvectors $\mathcal{L}_p$ and gives a formula for their corresponding eigenvalue.

Lemma 1. *Let $A_1, \dots, A_T \in \mathbb{R}^{n\times n}$ be symmetric positive semi-definite matrices and let $p \in \mathbb{R}$. Suppose that $A_1, \dots, A_T$ are positive definite if $p \leq 0$. If $\mathbf{u}$ is an eigenvector of A_i with corresponding eigenvalue λ_i for all $i = 1, \dots, T$, then $\mathbf{u}$ is an eigenvector of $M_p(A_1, \dots, A_T)$ with eigenvalue $m_p(\lambda_1, \dots, \lambda_T)$.*

Proof. First, note that $M = M_p(A_1, \dots, A_T)$ is symmetric positive (semi-)definite as it is a positive sum of such matrices. In particular, M is diagonalizable and thus the eigenvectors of M and M^p are the same for every p. Now, as $A_i\mathbf{u} = \lambda_i\mathbf{u}$ for $i = 1, \dots, T$, we have $A_i^p\mathbf{u} = \lambda_i^p\mathbf{u}$ for all i and thus

$$M_p^p(A_1, \dots, A_T)\mathbf{u} = \frac{1}{T}\sum_{i=1}^T A_i^p\mathbf{u} = \frac{1}{T}\sum_{i=1}^T \lambda_i^p\mathbf{u}$$
$$= m_p^p(\lambda_1, \dots, \lambda_T)\mathbf{u}.$$

Thus, $\mathbf{u}$ is an eigenvector of $M_p(A_1, \dots, A_T)$ with eigenvalue $m_p(\lambda_1, \dots, \lambda_T)$. $\square$

The above lemma, allows to obtain an explicit formula for $\mathcal{L}_p$ which fully describes its spectrum. Indeed, we have the following

Corollary 3. *Let X and $\Lambda^{(1)}$ be matrices such that $\mathcal{L}_{\text{sym}}^{(1)} = X\Lambda^{(1)}X^T$, X is orthogonal and $\Lambda^{(1)} = \operatorname{diag}(\lambda_1(\mathcal{L}_{\text{sym}}^{(1)}), \dots, \lambda_n(\mathcal{L}_{\text{sym}}^{(1)}))$. Then, we have $\mathcal{L}_p = X\Lambda X$ where Λ is the diagonal matrix $\Lambda = \operatorname{diag}(\lambda_1(\mathcal{L}_p), \dots, \lambda_n(\mathcal{L}_p))$ with $\lambda_i(\mathcal{L}_p) = m_p(\lambda_i(\mathcal{L}_{\text{sym}}^{(1)}), \dots, \lambda_i(\mathcal{L}_{\text{sym}}^{(T)}))$, for all $i = 1, \dots, n$.*

Proof. As $\mathcal{L}_{\text{sym}}^{(t)}$ have the same eigenvectors for every $t = 1, \dots, T$, it follows by Lemma 1 that $\mathcal{L}_pX = X\Lambda$ and thus $\mathcal{L}_p = X\Lambda X^\top$. $\square$

We note that on top of providing information on the spectral properties of $\mathcal{L}_p$, Corollary 3 ensures the existence of $\mathcal{L}_{\pm\infty} \in \mathbb{R}^{n\times n}$ such that $\lim_{p\to\pm\infty}\mathcal{L}_p = \mathcal{L}_{\pm\infty}$.

Combining Lemma 1 with equation (1) we obtain the following

Lemma 2. *The limits $\lim_{p\to\pm\infty}\mathcal{L}_p = \mathcal{L}_{\pm\infty}$ exist. Furthermore, for $p \in [-\infty, \infty]$, we have $\mathcal{L}_p\boldsymbol{\chi}_i = \lambda_i\boldsymbol{\chi}_i$ with*

$$\lambda_1 = \epsilon, \quad \lambda_i = m_p(\boldsymbol{\mu}+\epsilon\mathbf{1}), \quad i = 2, \dots, k$$

where $\boldsymbol{\mu} = (1-\rho_1, \dots, 1-\rho_T)$. Furthermore, the remaining eigenvalues satisfy $\lambda_{k+1} = \dots = \lambda_n = 1+\epsilon$.

Proof. With Corollary 3 and Equation (1) we directly obtain for $i = 2, \dots, k$ and $j = k+1, \dots, n$,

$$\lambda_1 = \Big(\frac{1}{T}\sum_{t=1}^{T}\epsilon^p\Big)^{1/p} = m_p(\epsilon\mathbf{1}) = \epsilon$$

$$\lambda_i = \Big(\frac{1}{T}\sum_{t=1}^{T}(1-\rho_t+\epsilon)^p\Big)^{1/p} = m_p(\boldsymbol{\mu}+\epsilon\mathbf{1})$$

$$\lambda_j = \Big(\frac{1}{T}\sum_{t=1}^{T}(1+\epsilon)^p\Big)^{1/p} = m_p((1+\epsilon)\mathbf{1}) = 1+\epsilon$$

□

We are now ready to prove Theorem 1.

Proof of Theorem 1. Clearly, $\lambda_1, \dots, \lambda_k$ are among the k-smallest eigenvalues of $\mathcal{L}_p$ if and only if $\lambda_1 < \lambda_{k+1} \leq \dots \leq \lambda_n$ and $\lambda_2 \leq \dots \leq \lambda_k < \lambda_{k+1} \leq \dots \lambda_n$ where $\lambda_1, \dots, \lambda_n$ are all eigenvalues of $\mathcal{L}_p$. By Lemma 2, we have $\lambda_1 = \epsilon$, $\lambda_2 = \dots = \lambda_k = m_p(\boldsymbol{\mu}+\epsilon\mathbf{1})$ and $\lambda_{k+1} = \dots = \lambda_n = 1+\epsilon$. Clearly $\lambda_1 = \epsilon < 1+\epsilon = \lambda_k \leq \dots \leq \lambda_n$, thus, the first condition holds. Hence, $\lambda_1, \dots, \lambda_k$ correspond to the k-smallest eigenvalues of $\mathcal{L}_p$ if and only if $m_p(\boldsymbol{\mu}+\epsilon\mathbf{1}) < 1+\epsilon$ which concludes the proof. □

1.2 No layer contains full information of the Graph

In this setting, we fix the number k of cluster to $k = 3$.

For convenience, we slightly overload the notation for the remaining of this section: we denote by n the size of each cluster $\mathcal{C}_1, \dots, \mathcal{C}_k$, i.e. $|\mathcal{C}_i| = |\mathcal{C}| = n$ for $i = 1, \dots, k$. Thus, the size of the graph is expressed in terms of the number and size of clusters, i.e. $|V| = nk$.

Furthermore, we suppose that for $t = 1, 2, 3$, the expected adjacency matrix $\mathcal{W}^{(t)} \in \mathbb{R}^{3n\times 3n}$ of $G^{(t)}$, are given, for all $i, j = 1, \dots, 3n$, as

$$\mathcal{W}^{(t)}_{ij} = \begin{cases} p_{in} & \text{if } v_i, v_j \in \mathcal{C}_t \text{ or } v_i, v_j \in \overline{\mathcal{C}_t} \\ p_{out} & \text{otherwise,} \end{cases}$$

where $0 < p_{out} \leq p_{in} \leq 1$. For $t = 1, 2, 3$ and $\epsilon \geq 0$, let $\mathcal{D}^{(t)} = \operatorname{diag}(\mathcal{W}^{(t)}\mathbf{1})$,

$$\mathcal{L}^{(t)}_{\text{sym}} = I - (\mathcal{D}^{(t)})^{-1/2}\mathcal{W}^{(t)}(\mathcal{D}^{(t)})^{-1/2} + \epsilon I,$$

and for a nonzero integer p let

$$\mathcal{L}_p = M_p(\mathcal{L}^{(1)}_{\text{sym}}, \mathcal{L}^{(2)}_{\text{sym}}, \mathcal{L}^{(3)}_{\text{sym}}),$$

where we assume that $\epsilon > 0$ if $p < 0$. Consider further $\boldsymbol{\chi}_1, \boldsymbol{\chi}_2, \boldsymbol{\chi}_3 \in \mathbb{R}^{3n}$ the vectors defined as

$$\boldsymbol{\chi}_1 = \mathbf{1}, \quad \boldsymbol{\chi}_2 = \mathbf{1}_{\mathcal{C}_1} - \mathbf{1}_{\mathcal{C}_2}, \quad \boldsymbol{\chi}_3 = \mathbf{1}_{\mathcal{C}_1} - \mathbf{1}_{\mathcal{C}_3}.$$

In opposition to the previous model, it turns out that $\mathcal{L}^{(1)}_{\text{sym}}, \mathcal{L}^{(2)}_{\text{sym}}$ and $\mathcal{L}^{(3)}_{\text{sym}}$ do not commute and thus do not share the same eigenvectors. Hence, we can not derive an explicit expression for $\mathcal{L}_p$ as in Corollary 3. In particular this implies that we need to use different mathematical tools in order to study the eigenpairs of $\mathcal{L}_p$.

The first main result of this section, presented in Theorem 2, shows that, in general, the ground truth clusters can not be reconstructed from the 3 smallest eigenvectors of $\mathcal{L}^{(t)}_{\text{sym}}$ for any $t = 1, 2, 3$.

Theorem 2. *If $1 \geq p_{\text{in}} > p_{\text{out}} > 0$, then for any $t = 1, 2, 3$, there exist scalars $\alpha > 0$ and $\beta > 0$ such that the eigenvectors of $\mathcal{L}^{(t)}_{\text{sym}}$ corresponding to the two smallest eigenvalues are*

$$\boldsymbol{\varkappa}_1 = \alpha\mathbf{1}_{\mathcal{C}_t} + \mathbf{1}_{\overline{\mathcal{C}_t}} \quad \text{and} \quad \boldsymbol{\varkappa}_2 = -\beta\mathbf{1}_{\mathcal{C}_t} + \mathbf{1}_{\overline{\mathcal{C}_t}}$$

whereas any vector orthogonal to both $\boldsymbol{\varkappa}_1$ and $\boldsymbol{\varkappa}_2$ is an eigenvector for the third smallest eigenvalue.

In fact, we prove even more by giving a full description of the eigenvectors of $\mathcal{L}^{(t)}_{\text{sym}}$ as well as the ordering of their corresponding eigenvalues. These results can be found in Lemma 12 below.

Our second main result is the following Theorem 3. It shows that the ground truth clusters can always be recovered from the three smallest eigenvectors of $\mathcal{L}_p$.

Theorem 3. *Let p be any nonzero integer and assume that $\epsilon > 0$ if $p < 0$. Furthermore, suppose that $0 < p_{out} < p_{in} \leq 1$. Then, there exists λ_i such that $\mathcal{L}_p\boldsymbol{\chi}_i = \lambda_i\boldsymbol{\chi}_i$ for $i = 1, 2, 3$ and $\lambda_1, \lambda_2, \lambda_3$ are the three smallest eigenvalues of $\mathcal{L}_p$.*

Again, we actually prove more than just Theorem 3. In fact, a full description of the eigenvectors of $\mathcal{L}_p$ and of the ordering of their corresponding eigenvalues is given in Lemma 18 below.

For the proof of Theorems 2 and 3, and the corresponding additional results, we proceed as follows. First we assume that $n = |\mathcal{C}_i| = 1$ and prove our claims. Then, we generalize these results to the case $n > 1$. For the sake of clarity, as we will need to refer to the case $n = 1$ for the proofs of the case $n > 1$, we put a tilde on the matrices in $\mathbb{R}^{3\times 3}$.

<u>*The case $n = 1$:*</u>

Suppose that $n = 1$, then $\tilde{\mathcal{L}}_{\text{sym}} = \mathcal{L}^{(1)}_{\text{sym}}$ is given by

$$\tilde{\mathcal{L}}_{\text{sym}} = \tau I_3 - \tilde{\mathcal{D}}^{-1/2}\tilde{\mathcal{W}}\tilde{\mathcal{D}}^{-1/2} = \tau I_3 - \tilde{\mathcal{M}},$$

where $\tau = 1+\epsilon$, $\tilde{\mathcal{W}} = \mathcal{W}^{(1)}$, $\tilde{\mathcal{D}} = \operatorname{diag}(\tilde{\mathcal{W}}\mathbf{1})$,

$$\tilde{\mathcal{D}} = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \beta \end{pmatrix}, \qquad \tilde{\mathcal{M}} = \begin{pmatrix} a & b & b \\ b & c & c \\ b & c & c \end{pmatrix}, \qquad (2)$$

and $\alpha, \beta, a, b, c > 0$ are given by

$$\alpha = p_{in} + 2p_{out}, \qquad \beta = 2p_{in} + p_{out},$$
$$a = \frac{p_{in}}{\alpha}, \qquad b = \frac{p_{out}}{\sqrt{\alpha\beta}}, \qquad c = \frac{p_{in}}{\beta}.$$

Moreover, note that for any $(\lambda, \mathbf{v}) \in \mathbb{R} \times \mathbb{R}^3$ we have

$$\tilde{\mathcal{M}}\mathbf{v} = \lambda\mathbf{v} \qquad \iff \qquad \tilde{\mathcal{L}}_{\mathrm{sym}}\mathbf{v} = (\tau - \lambda)\mathbf{v}. \quad (3)$$

This implies that we can study the spectrum of $\tilde{\mathcal{M}}$ in order to obtain the spectrum of $\tilde{\mathcal{L}}_{\mathrm{sym}}$. We have the following lemma:

Lemma 3. *Suppose that $p_{out} > 0$ and let $\Delta > 0$ be defined as $\Delta = \sqrt{(a-2c)^2 + 8b^2}$. Then the eigenvalues of $\tilde{\mathcal{M}}$ are*

$$\tilde{\lambda}_1 = 0, \qquad \tilde{\lambda}_2 = \frac{a + 2c - \Delta}{2}, \qquad \tilde{\lambda}_3 = 1,$$

and it holds $\tilde{\lambda}_1 < \tilde{\lambda}_2 < \tilde{\lambda}_3$. Furthermore, the corresponding eigenvectors are given by

$$\mathbf{u}_1 = (0, -1, 1)^\top, \qquad \mathbf{u}_2 = \Big(\frac{a - 2c - \Delta}{2b}, 1, 1\Big)^\top,$$
$$\mathbf{u}_3 = \big(\sqrt{\alpha}, \sqrt{\beta}, \sqrt{\beta}\big)^\top,$$

and it holds $\frac{a-2c-\Delta}{2b} < 0$.

Proof. The equality $\tilde{\mathcal{M}}\mathbf{u}_1 = 0$ follows from a direct computation. Furthermore, note that $\mathbf{u}_3 = \tilde{\mathcal{D}}^{1/2}\mathbf{1}$ and so

$$\tilde{\mathcal{M}}\mathbf{u}_3 = \tilde{\mathcal{D}}^{-1/2}\tilde{\mathcal{W}}\tilde{\mathcal{D}}^{-1/2}\mathcal{D}^{1/2}\mathbf{1} = \tilde{\mathcal{D}}^{-1/2}\tilde{\mathcal{W}}\mathbf{1} = \mathbf{u}_3$$

implying $\tilde{\mathcal{M}}\mathbf{u}_3 = \mathbf{u}_3$. Now, let $s_\pm = \frac{a-2c\pm\Delta}{2b}$. Then s_+ and s_- are the solutions of the quadratic equation $bs^2 + (2c - a)s - 2b = 0$ which can be rearranged as $as + 2b = (bs + 2c)s$. The latter equation is equivalent to

$$\begin{cases} as + 2b = \lambda s \\ bs + 2c = \lambda \end{cases} \qquad \iff \qquad \tilde{\mathcal{M}}\begin{pmatrix} s \\ 1 \\ 1 \end{pmatrix} = \lambda \begin{pmatrix} s \\ 1 \\ 1 \end{pmatrix}.$$

Hence, $\mathbf{u}_\pm = (s_\pm, 1, 1)$ are both eigenvectors of $\tilde{\mathcal{M}}$ corresponding to the eigenvalues

$$\lambda_\pm = b\, s_\pm + 2c = \frac{a - 2c \pm \Delta}{2} + 2c = \frac{a + 2c \pm \Delta}{2}.$$

Note in particular that we have $\mathbf{u}_2 = \mathbf{u}_-$ and $\tilde{\lambda}_2 = \lambda_-$. This concludes the proof that $(\lambda_i, \mathbf{u}_i)$ are eigenpairs of $\tilde{\mathcal{M}}$ for $i = 1, 2, 3$. We now show that $\tilde{\lambda}_1 < \tilde{\lambda}_2 < \tilde{\lambda}_3$ and $(a - 2c - \Delta)/2b < 0$.

As $\Delta > 0$, we have $\lambda_- < \lambda_+$. We prove $\lambda_- > 0$. As $p_{in} > p_{out}$ by assumption, the definition of $a, b, c > 0$ implies that

$$\begin{aligned} b^2 &= \frac{p_{out}^2}{(2p_{in} + p_{out})(p_{in} + 2p_{out})} \\ &< \frac{p_{in}^2}{(2p_{in} + p_{out})(p_{in} + 2p_{out})} = ac. \end{aligned}$$

And from $ac > b^2$ it follows that $a^2 + 4ac + 4c^2 > a^2 - 4ac + 4c^2 + 8b^2$ which implies that $(a+2c)^2 > (a-2c)^2 + 8b^2 = \Delta^2$. Hence, $a + 2c - \Delta > 0$ and thus $\lambda_- > 0$. Thus we have $0 < \lambda_- < \lambda_+$. Now, as $\tilde{\mathcal{M}}$ has strictly positive entries, the Perron-Frobenius theorem (see for instance Theorem 1.1 in [3]) implies that $\tilde{\mathcal{M}}$ has a unique nonnegative eigenvector $\mathbf{u}$. Furthermore, $\mathbf{u}$ has positive entries and its corresponding eigenvalue is the spectral radius of $\tilde{\mathcal{M}}$. As $\mathbf{u}_3 = \tilde{\mathcal{D}}^{1/2}\mathbf{1}$ has positive entries and is an eigenvector of $\tilde{\mathcal{M}}$, we have $\mathbf{u} = \mathbf{u}_3$. It follows that $\rho(\tilde{\mathcal{M}}) = \lambda_+ = \tilde{\lambda}_3$. Furthermore, $\mathbf{u}_2$ must have a strictly negative entry and thus it holds $s_- < 0$. □

Combining the results of Lemma 3 and Equation (3) we directly obtain the following corollary which fully describes the eigenvectors of $\tilde{\mathcal{L}}_{\mathrm{sym}}$ as well as the ordering of the corresponding eigenvalues:

Corollary 4. *There exists $\tilde{\lambda} \in (0, 1)$ and $s_- < 0 < s_+ < 1$ such that $\big(\tau - 1, (s_+, 1, 1)^\top\big), \big(\tau - \tilde{\lambda}, (s_-, 1, 1)^\top\big), \big(\tau, (0, -1, 1)^\top\big)$ are the eigenpairs of $\tilde{\mathcal{L}}_{\mathrm{sym}}$.*

Proof. The only thing which is not directly implied by Lemma 3 and Equation (3) is that $s_+ < 1$. But this follows again from Lemma 3. Indeed, as $(s_+, 1, 1)$ and $(\sqrt{\alpha}, \sqrt{\beta}, \sqrt{\beta})$ must span the same line, we have

$$s_+ = \sqrt{\frac{\alpha}{\beta}} = \sqrt{\frac{p_{in} + 2p_{out}}{2p_{in} + p_{out}}}.$$

As $p_{out} < p_{in}$, we get $0 < s_+ < 1$. □

Now, we study the spectral properties of $\tilde{\mathcal{L}}_p = \mathcal{L}_p \in \mathbb{R}^{3\times3}$. To this end, for $t = 1, 2, 3$ let $\tilde{\mathcal{W}}^{(t)} = \mathcal{W}^{(t)}, \tilde{\mathcal{L}}_{\mathrm{sym}}^{(t)} = \mathcal{L}_{\mathrm{sym}}^{(t)} \in \mathbb{R}^{3\times3}$. Furthermore, consider the permutation matrices $\tilde{P}_1, \tilde{P}_2, \tilde{P}_3 \in \mathbb{R}^{3\times3}$ defined as

$$\tilde{P}_1 = I_3, \quad \tilde{P}_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \tilde{P}_3 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then, we have $\tilde{\mathcal{W}}^{(t)} = \tilde{P}_t\tilde{\mathcal{W}}\tilde{P}_t$ for $t = 1, 2, 3$. The following lemma relates $\tilde{\mathcal{L}}_{\mathrm{sym}}^{(t)}$ and $\tilde{\mathcal{L}}_{\mathrm{sym}}$.

Lemma 4. *For $t = 1, 2, 3$, we have $\tilde{P}_t = \tilde{P}_t^{-1} = \tilde{P}_t^\top$ and $\tilde{\mathcal{L}}_{\mathrm{sym}}^{(t)} = \tilde{P}_t\tilde{\mathcal{L}}_{\mathrm{sym}}\tilde{P}_t$.*

Proof. The identity $\tilde{P}_t = \tilde{P}_t^{-1} = \tilde{P}_t^\top$ follows by a direct computation. Now, as $\tilde{P}_t\mathbf{1} = \mathbf{1}$, we have $\tilde{P}_t\tilde{\mathcal{W}}\tilde{P}_t\mathbf{1} = \tilde{P}_t\tilde{\mathcal{W}}\mathbf{1}$. Assuming the exponents on the vector in the following expressions are taken component wise, we have $\operatorname{diag}(\tilde{\mathcal{W}}\mathbf{1})^{-1/2} = \operatorname{diag}\big((\tilde{\mathcal{W}}\mathbf{1})^{-1/2}\big)$ and thus

$$\begin{aligned}\operatorname{diag}(\tilde{P}_t\tilde{\mathcal{W}}\tilde{P}_t\mathbf{1})^{-1/2} &= \operatorname{diag}\big((\tilde{P}_t\tilde{\mathcal{W}}\tilde{P}_t\mathbf{1})^{-1/2}\big)\\ &= \operatorname{diag}\big(\tilde{P}_t(\tilde{\mathcal{W}}\mathbf{1})^{-1/2}\big) = \tilde{P}_t\operatorname{diag}\big((\tilde{\mathcal{W}}\mathbf{1})^{-1/2}\big)\tilde{P}_t\\ &= \tilde{P}_t\operatorname{diag}(\tilde{\mathcal{W}}\mathbf{1})^{-1/2}\tilde{P}_t = \tilde{P}_t\tilde{\mathcal{D}}^{-1/2}\tilde{P}_t.\end{aligned}$$

It follows that

$$\begin{aligned}\tilde{\mathcal{L}}_{\text{sym}}^{(t)} &= \tau\tilde{P}_t\tilde{P}_t - \tilde{P}_t\tilde{\mathcal{D}}^{-1/2}\tilde{P}_t\tilde{P}_t\tilde{\mathcal{W}}\tilde{P}_t\tilde{P}_t\tilde{\mathcal{D}}^{-1/2}\tilde{P}_t\\ &= \tilde{P}_t\tilde{\mathcal{L}}_{\text{sym}}\tilde{P}_t,\end{aligned}$$

which concludes our proof. □

Combining Corollary 4 with Lemma 4, we directly obtain the following

Corollary 5. *There exists $\tilde{\lambda} \in (0,1)$ and $s_- < 0 < s_+$ such that $\big(\tau - 1, \tilde{P}_t(s_+,1,1)^\top\big), \big(\tau - \lambda, \tilde{P}_t(s_-,1,1)^\top\big), \big(\tau, \tilde{P}_t(0,-1,1)^\top\big)$ are the eigenpairs of $\tilde{\mathcal{L}}_{\text{sym}}^{(t)}$ for $t = 1,2,3$.*

A similar argument as in the proof of Lemma 1 implies that the eigenvectors of $\tilde{\mathcal{L}}_p$ coincide with those of the matrix $\tilde{\mathsf{L}}_p \in \mathbb{R}^{3\times 3}$ defined as

$$\tilde{\mathsf{L}}_p = (\tilde{\mathcal{L}}_{\text{sym}}^{(1)})^p + (\tilde{\mathcal{L}}_{\text{sym}}^{(2)})^p + (\tilde{\mathcal{L}}_{\text{sym}}^{(3)})^p = 3\tilde{\mathcal{L}}_p^p.$$

We study the spectral properties of $\tilde{\mathsf{L}}_p$. To this end, we consider the following subspaces of matrices:

$$\mathcal{U}_3 = \Big\{ \begin{pmatrix} s_1 & s_2 & s_2\\ s_3 & s_5 & s_4\\ s_3 & s_4 & s_5\end{pmatrix} \;\Big|\; s_1,\dots,s_5 \in \mathbb{R}\Big\},$$
$$\mathcal{Z}_3 = \Big\{ \begin{pmatrix} t_1 & t_2 & t_2\\ t_2 & t_1 & t_2\\ t_2 & t_2 & t_1\end{pmatrix} \;\Big|\; t_1,t_2 \in \mathbb{R}\Big\}.$$

We prove that for every p, it holds $(\tilde{\mathcal{L}}_{\text{sym}}^{(1)})^p \in \mathcal{U}_3$ and $\tilde{\mathsf{L}}_p \in \mathcal{Z}_3$. We need the following lemma:

Lemma 5. *The following holds:*

1. *For all $\tilde{A}, \tilde{B} \in \mathcal{U}_3$ we have $\tilde{A}\tilde{B} \in \mathcal{U}_3$.*
2. *If $\tilde{A} \in \mathcal{U}_3$ and $\det(\tilde{A}) \neq 0$, then $\tilde{A}^{-1} \in \mathcal{U}_3$.*
3. *$\mathcal{Z}_3 = \tilde{P}_1\mathcal{U}_3\tilde{P}_1 + \tilde{P}_2\mathcal{U}_3\tilde{P}_2 + \tilde{P}_3\mathcal{U}_3\tilde{P}_3$.*

Proof. Let $\tilde{A} \in \mathcal{U}_3, \tilde{C} \in \mathcal{Z}_3$ be respectively defined as

$$\tilde{A} = \begin{pmatrix} s_1 & s_2 & s_2\\ s_3 & s_5 & s_4\\ s_3 & s_4 & s_5\end{pmatrix}, \quad \tilde{C} = \begin{pmatrix} t_1 & t_2 & t_2\\ t_2 & t_1 & t_2\\ t_2 & t_2 & t_1\end{pmatrix}$$

1. Follows from a direct computation.
2. If $\det(\tilde{A}) \neq 0$, then $\tilde{A}$ is invertible and

$$\det(\tilde{A})\tilde{A}^{-1} = \begin{pmatrix} s_5^2 - s_4^2 & s_2(s_4 - s_5) & s_2(s_4 - s_5)\\ s_3(s_4 - s_5) & s_1s_5 - s_2s_3 & s_2s_3 - s_1s_4\\ s_3(s_4 - s_5) & s_2s_3 - s_1s_4 & s_1s_5 - s_2s_3\end{pmatrix}.$$

It follows that $\tilde{A}^{-1} \in \mathcal{U}_3$.

3. We have

$$\sum_{i=1}^{3}\tilde{P}_i\tilde{A}\tilde{P}_i = \begin{pmatrix} s_1 + 2s_5 & s_2+s_3+s_4 & s_2+s_3+s_4\\ s_2+s_3+s_4 & s_1+2s_5 & s_2+s_3+s_4\\ s_2+s_3+s_4 & s_2+s_3+s_4 & s_1+2s_5\end{pmatrix} \tag{4}$$

and conversely, there clearly exists $s_1,\dots,s_4$ such that $s_1 + 2s_5 = t_1$ and $s_2 + s_3 + s_4 = t_2$, so we have $\sum_{i=1}^3 \tilde{P}_i\tilde{A}\tilde{P}_i = \tilde{C}$ implying the reverse inclusion. □

Now, we show that $\tilde{\mathsf{L}}_p \in \mathcal{Z}_3$ for all nonzero integer p.

Lemma 6. *For every integer $p \neq 0$ we have $\tilde{\mathsf{L}}_p \in \mathcal{Z}_3$.*

Proof. From (2), we know that $\tilde{\mathcal{L}}_{\text{sym}}^{(1)} \in \mathcal{U}_3$. By point 2 in Lemma 5, this implies that $(\tilde{\mathcal{L}}_{\text{sym}}^{(1)})^{\operatorname{sign}(p)} \in \mathcal{U}_3$. Now point 1 of Lemma 5 implies that $(\tilde{\mathcal{L}}_{\text{sym}}^{(1)})^p = \big((\tilde{\mathcal{L}}_{\text{sym}}^{(1)})^{\operatorname{sign}(p)}\big)^{|p|} \in \mathcal{U}_3$. Finally, by Lemma 4 and point 3 in Lemma 5, we have

$$\tilde{\mathsf{L}}_p = \sum_{t=1}^{3}(\tilde{\mathcal{L}}_{\text{sym}}^{(t)})^p = \sum_{t=1}^{3}\tilde{P}_t(\tilde{\mathcal{L}}_{\text{sym}}^{(1)})^p\tilde{P}_t \in \mathcal{Z}_3,$$

which concludes the proof. □

Matrices in $\mathcal{Z}_3$ have the interesting property that they have a simple spectrum and they all share the same eigenvectors. Indeed we have the following:

Lemma 7. *Let $\tilde{C} \in \mathcal{Z}_3$ and t_1, t_2 be such that $\tilde{C} = (t_1 - t_2)I_3 + t_2\tilde{E}$ where $\tilde{E} \in \mathbb{R}^{3\times 3}$ is the matrix of all ones. Then the eigenpairs of $\tilde{C}$ are given by:*

$$\big(t_1 - t_2, (-1,0,1)^\top\big), \quad \big(t_1 - t_2, (-1,1,0)^\top\big),$$
$$\big(t_1 + 2t_2, (1,1,1)^\top\big).$$

Proof. Follows from a direct computation. □

So, the last thing we need to discuss is the order of the eigenvalues of $\tilde{\mathsf{L}}_p$. To this end, we study the sign pattern of the powers of this matrix.

Lemma 8. *For every positive integer $p > 0$ we have $(\tilde{\mathcal{L}}^p_{\mathrm{sym}})_{i,j} < 0 < (\tilde{\mathcal{L}}^p_{\mathrm{sym}})_{i,i} < \tau^p$ for all $i, j = 1, 2, 3$ with $i \neq j$. For every negative integer $p < 0$ we have $(\tilde{\mathcal{L}}^p_{\mathrm{sym}})_{i,j} > 0$ for all $i, j = 1, 2, 3$.*

Proof. First, assume that $p > 0$ and let $\tilde{S} = \tilde{\mathcal{D}}^{-1}\tilde{\mathcal{W}}$. We have

$$\begin{aligned}\tilde{\mathcal{L}}^p_{\mathrm{sym}} &= (\tau I_3 - \tilde{\mathcal{D}}^{-1/2}\tilde{\mathcal{W}}\tilde{\mathcal{D}}^{-1/2})^p \\ &= \sum_{r=0}^{p}\binom{p}{r}\tau^{p-r}(-1)^r(\tilde{\mathcal{D}}^{-1/2}\tilde{\mathcal{W}}\tilde{\mathcal{D}}^{-1/2})^r \\ &= \tilde{\mathcal{D}}^{1/2}\Big(\sum_{r=0}^{p}\binom{p}{r}\tau^{p-r}(-1)^r(\tilde{\mathcal{D}}^{-1}\tilde{\mathcal{W}})^r\Big)\tilde{\mathcal{D}}^{-1/2} \\ &= \tilde{\mathcal{D}}^{1/2}(\tau I_3 - \tilde{S})^p\tilde{\mathcal{D}}^{-1/2}.\end{aligned}$$

As $\tilde{\mathcal{D}}^{1/2}$ and $\tilde{\mathcal{D}}^{-1/2}$ are diagonal with positive diagonal entries, the sign of the entries of $\tilde{\mathcal{L}}^p_{\mathrm{sym}}$ coincide with those of $(\tau I_3 - \tilde{S})^p$. Furthermore, we have $(\tilde{\mathcal{L}}^p_{\mathrm{sym}})_{i,i} = ((\tau I_3 - \tilde{S})^p)_{i,i}$ for all i. Now the matrix $\tilde{S}$ is row stochastic, that is $\tilde{S}\mathbf{1} = \mathbf{1}$ and has the following form

$$\tilde{S} = \begin{pmatrix}1-2\hat{a} & \hat{a} & \hat{a}\\ 1-2\hat{b} & \hat{b} & \hat{b}\\ 1-2\hat{b} & \hat{b} & \hat{b}\end{pmatrix} \qquad \hat{a} = \frac{\hat{\alpha}}{1+2\hat{\alpha}}, \quad \hat{b} = \frac{1}{2+\hat{\alpha}}$$

where $\hat{\alpha} = p_{out}/p_{in} \in (0,1)$. Let

$$\gamma = (\hat{a} - \hat{b}) = \frac{p_{out}^2 - p_{in}^2}{(2p_{in}+p_{out})(2p_{out}+p_{in})} < 0,$$

$$\mu = (1-2\hat{b}) = \frac{p_{in}}{2p_{out}+p_{in}} > 0$$

For all positive integer p, we have

$$(\tau I_3 - \tilde{S})^p = \frac{1}{2\gamma+1}\begin{pmatrix}q_p & r_p & r_p\\ s_p & t_p & u_p\\ s_p & u_p & t_p\end{pmatrix}$$

where q_p, r_p, s_p, t_p, u_p are given by

$$\begin{aligned}q_p &= \mu(\tau-1)^p + 2\hat{a}(2\gamma+\tau)^p,\\ r_p &= \hat{a}\big[(\tau-1)^p - (2\gamma+\tau)^p\big],\\ s_p &= \frac{\mu}{\hat{a}}r_p,\\ t_p &= \hat{a}\big[(\tau-1)^p + \tau^p\big] + \frac{\mu}{2}\big[\tau^p + (2\gamma+\tau)^p\big],\\ u_p &= \hat{a}\big[(\tau-1)^p - \tau^p\big] - \frac{\mu}{2}\big[\tau^p - (2\gamma+\tau)^p\big].\end{aligned} \tag{5}$$

Note that as $p_{in} > p_{out} > 0$, we have

$$\begin{aligned}\delta &= 2\gamma + 1 = 2(\hat{a}-\hat{b}) + 1 = 2\hat{a} + \mu \\ &= \frac{5p_{in}p_{out} + 4p_{out}^2}{2p_{in}^2 + 5p_{in}p_{out} + 2p_{out}^2} \in (0,1),\end{aligned}$$

Furthermore, as $\tau \geq 1$ and $\gamma < 0$, we have $\delta \leq (2\gamma + \tau) < \tau$. It follows that

$$\begin{aligned}&0 < \mu(\tau-1)^p + 2\hat{a}\delta^p \leq q_p < \mu(\tau-1)^p + 2\hat{a}\tau^p \\ &\quad \leq \delta\tau^p < \tau^p \\ &0 < \hat{a}\big[(\tau-1)^p + \tau^p\big] + \frac{\mu}{2}(\tau^p + \delta^p) \leq t_p \\ &\quad < 2\hat{a}\tau^p + \mu\tau^p = \delta\tau^p < \tau^p.\end{aligned}$$

Finally, we have

$$r_p = \hat{a}\big[(\tau-1)^p - \big(\delta + (\tau-1)\big)^p\big] < 0, \quad s_p = \frac{\mu}{\hat{a}}r_p < 0.$$

Now, suppose that $p < 0$, then we have $\tau > 1$ and

$$(\tau I_3 - \tilde{\mathcal{D}}^{-1/2}\tilde{\mathcal{W}}\tilde{\mathcal{D}}^{1/2})^{-1} = \sum_{k=0}^{\infty}(\tilde{\mathcal{D}}^{-1/2}\tilde{\mathcal{W}}\tilde{\mathcal{D}}^{1/2})^k.$$

As $\tilde{\mathcal{M}} = \tilde{\mathcal{D}}^{-1/2}\tilde{\mathcal{W}}\tilde{\mathcal{D}}^{1/2}$ is a matrix with strictly positive entries, this implies that $\tilde{\mathcal{L}}_{\mathrm{sym}}$ has positive entries as well. Furthermore, it also implies that $\tilde{\mathcal{L}}^p_{\mathrm{sym}} = (\tilde{\mathcal{L}}^{-1}_{\mathrm{sym}})^{|p|}$ is positive for every $p < 0$. □

Observation 1. *Numerical evidences strongly suggest that the formulas in (5) for the coefficients of $(\tau I_3 - \tilde{S})^p$ hold for any real $p \in \mathbb{R} \setminus \{0\}$.*

We can now use the above lemma to determine the ordering of the eigenvalues of $\tilde{\mathsf{L}}_p$.

Lemma 9. *Let $t_1, t_2 \in \mathbb{R}$ be such that it holds $\tilde{\mathsf{L}}_p = (t_1 - t_2)I_3 + t_2\tilde{E}$. Furthermore, for any nonzero integer p, it holds $0 < t_1 - t_2 < t_1 + 2t_2$ if $p < 0$ and $t_1 - t_2 > t_1 + 2t_2$ otherwise.*

Proof. If $p < 0$, then we must have $\tau > 1$ for $\tilde{\mathsf{L}}_p$ to be well defined. By Lemma 8, $(\tilde{\mathcal{L}}^{(1)}_{\mathrm{sym}})^p$ has strictly positive entries. Hence, $\tilde{\mathsf{L}}_p = \sum_{t=1}^{3}\tilde{P}_t(\tilde{\mathcal{L}}^{(1)}_{\mathrm{sym}})^p\tilde{P}_t$ is also a matrix with positive entries. It follows that $t_1 - t_2 > 0$ and $t_2 > 0$ so that $0 < t_1 - t_2 < t_1 + 2t_2$. Now assume that $p > 0$, Lemma 8 implies that $(\tilde{\mathcal{L}}^{(1)}_{\mathrm{sym}})^p$ with positive diagonal elements and negative off-diagonal. It follows from (4) that $\tilde{\mathsf{L}}_p$ also has positive diagonal elements and negative off-diagonal. Hence, we have $t_2 < 0 < t_1$ and thus $t_1 - t_2 > t_1 + 2t_2$ which concludes the proof. □

We have the following corollary on the spectral properties of the Laplacian p-mean.

Corollary 6. *Let p be a nonzero integer and let $\epsilon \geq 0$ if $p > 0$ and $\epsilon > 0$ if $p < 0$. Define*

$$\tilde{\mathcal{L}}_p = \Big(\frac{(\tilde{\mathcal{L}}^{(1)}_{\mathrm{sym}})^p + (\tilde{\mathcal{L}}^{(2)}_{\mathrm{sym}})^p + (\tilde{\mathcal{L}}^{(3)}_{\mathrm{sym}})^p}{3}\Big)^{1/p},$$

then there exists $0 \leq \tilde{\lambda}_1 < \tilde{\lambda}_2$ *such that the eigenpairs of* $\tilde{\mathcal{L}}_p$ *are given by*

$$\big(\tilde{\lambda}_1, (-1,0,1)^\top\big), \qquad \big(\tilde{\lambda}_1, (-1,1,0)^\top\big),$$
$$\big(\tilde{\lambda}_2, (1,1,1)^\top\big).$$

Proof. First, note that $\tilde{\mathcal{L}}_p = \big(\frac{1}{3}\tilde{\mathsf{L}}_p\big)^{1/p}$ hence as they are positive semi-definite matrices, $\tilde{\mathcal{L}}_p$ and $\tilde{\mathsf{L}}_p$ share the same eigenvectors. Precisely, we have $\tilde{\mathsf{L}}_p \mathbf{v} = \lambda \mathbf{v}$ if and only if $\tilde{\mathcal{L}}_p \mathbf{v} = f(\lambda)\mathbf{v}$ where $f(t) = (t/3)^{1/p}$. Now, by Lemmas 6 and 7 we know all eigenvectors of $\tilde{\mathsf{L}}_p$ and the corresponding eigenvalues are $\theta_1 = t_1 - t_2$ and $\theta_2 = t_1 + 2t_2$. Finally, using Lemma 9 and the fact that f is increasing if $p > 0$ and decreasing if $p < 0$ we deduce the ordering of $\tilde{\lambda}_i = f(\theta_i)$. $\square$

The case $n > 1$:

We now generalize the previous results to the case $n > 1$. To this end, we use mainly the properties of the Kronecker product $\otimes$ which we recall is defined for matrices $A \in \mathbb{R}^{m_1 \times m_2}, B \in \mathbb{R}^{m_3 \times m_4}$ as the block matrix $A \otimes B \in \mathbb{R}^{m_1 m_3 \times m_2 m_4}$ with $m_1 m_2$ blocks of the form $A_{i,j} B \in \mathbb{R}^{m_3 \times m_4}$ for all i, j. In particular, for $n > 1$, if E denotes the matrix of all ones in $\mathbb{R}^{n \times n}$, we have then $\mathcal{W}^{(t)} = \tilde{\mathcal{W}}^{(t)} \otimes E$ for every $t = 1,2,3$. Furthermore, let us define $\mathcal{W} = \tilde{\mathcal{W}} \otimes E$ and $P_t = \tilde{P}_t \otimes I_n$ for $t = 1,2,3$ so that $\mathcal{W}^{(t)} = P_t \mathcal{W} P_t$ for $t = 1,2,3$. Finally, let $\mathcal{L}_{\mathrm{sym}} = \tau I_{3n} - \mathcal{D}^{-1/2}\mathcal{W}\mathcal{D}^{-1/2}$ where we recall that $\tau = 1 + \epsilon$ and $\mathcal{D} = \mathrm{diag}(\mathcal{W}\mathbf{1})$. The normalized Laplacians of $\mathcal{W}$ and $\tilde{\mathcal{W}}$ are related in the following lemma:

Lemma 10. *It holds*

$$\mathcal{L}_{\mathrm{sym}} = \tau I_{3n} - \big[\tfrac{1}{n}\tilde{\mathcal{D}}^{-1/2}\tilde{\mathcal{W}}\tilde{\mathcal{D}}^{-1/2}\big] \otimes E.$$

Proof. First, note that $\mathcal{D} = n\tilde{\mathcal{D}} \otimes I_n$, as $(A_1 \otimes B_1)(A_2 \otimes B_2) = (A_1 A_2 \otimes B_1 B_2)$ for any compatible matrices A_1, A_2, B_1, B_2. We have

$$\begin{aligned}\mathcal{D}^{-1/2}\mathcal{W}\mathcal{D}^{-1/2} &= \frac{(\tilde{\mathcal{D}}^{-1/2} \otimes I_n)(\tilde{\mathcal{W}} \otimes E)(\tilde{\mathcal{D}}^{-1/2} \otimes I_n)}{n} \\ &= \tfrac{1}{n}\tilde{\mathcal{D}}^{-1/2}\tilde{\mathcal{W}}\tilde{\mathcal{D}}^{-1/2} \otimes E,\end{aligned}$$

which concludes the proof. $\square$

In order to study the eigenpairs of $\mathcal{L}_{\mathrm{sym}}$, we combine Lemma 4 with the following theorem from [2] which implies that eigenpairs of Kronecker products are Kronecker products of the eigenpairs:

Theorem 4 (Theorem 4.2.12, [2])**.** *Let* $A \in \mathbb{R}^{m \times m}$ *and* $B \in \mathbb{R}^{n \times n}$*. Let* (λ, x) *and* (μ, y) *be eigenpairs of* A *and* B *respectively. Then* $(\lambda\mu, x \otimes y)$ *is an eigenpair of* $A \otimes B$*.*

Indeed, the above theorem implies that the eigenpairs of $\mathcal{D}^{-1/2}\mathcal{W}\mathcal{D}^{-1/2}$ are Kronecker products of the eigenpairs of $\tilde{\mathcal{D}}^{-1/2}\tilde{\mathcal{W}}\tilde{\mathcal{D}}^{-1/2}$ and E. As we already know those of $\tilde{\mathcal{D}}^{-1/2}\tilde{\mathcal{W}}\tilde{\mathcal{D}}^{-1/2}$, we briefly describe those of E:

Lemma 11. *Let* $E \in \mathbb{R}^{n \times n}$*,* $n \geq 2$ *be the matrix of all ones, then the eigenpairs of* E *are given by* $(n, \mathbf{1})$ *and* $(0, \mathbf{v}_1), \ldots, (0, \mathbf{v}_{n-1})$ *where* $\mathbf{v}_k \in \mathbb{R}^n$ *is given as*

$$(\mathbf{v}_k)_j = \begin{cases} 1 & \text{if } j \leq k, \\ -k & \text{if } j = k+1, \\ 0 & \text{otherwise.} \end{cases} \tag{6}$$

Proof. As $E = \mathbf{1}\mathbf{1}^\top$, it is clear that $(n, \mathbf{1})$ is an eigenpair of E. Now, for every i we have $E\mathbf{v}_i = (\mathbf{1}^\top v_i)\mathbf{1}$ and $\mathbf{1}^\top \mathbf{v}_i = i - i = 0$. $\square$

We can now describe the spectral properties of $\mathcal{L}_{\mathrm{sym}}^{(t)}$ for $t = 1,2,3$.

Lemma 12. *There exists* $\lambda \in (0,1)$ *and* $s_- < 0 < s_+ < 1$ *such that, for* $t = 1,2,3$*, the eigenpairs of* $\tilde{\mathcal{L}}_{\mathrm{sym}}^{(t)}$ *are given by*

$$\begin{gathered}\big(\tau - 1, P_t(s_+,1,1)^\top \otimes \mathbf{1}\big),\ \big(\tau - \lambda, P_t(s_-,1,1)^\top \otimes \mathbf{1}\big),\\ \big(\tau, P_t(0,-1,1)^\top \otimes \mathbf{1}\big), \qquad \big(\tau, P_t(s_+,1,1)^\top \otimes \mathbf{v}_k\big),\\ \big(\tau, P_t(s_-,1,1)^\top \otimes \mathbf{v}_k\big), \qquad \big(\tau, P_t(0,-1,1)^\top \otimes \mathbf{v}_k\big),\end{gathered}$$

for $k = 1, \ldots, n-1$*, where* $\mathbf{v}_k$ *is defined as in* (6)*.*

Proof. Follows from Lemmas 3, 11 and Theorem 4. $\square$

Similarly to the case $n = 1$, let us consider $\mathsf{L}_p \in \mathbb{R}^{3n \times 3n}$ defined as

$$\mathsf{L}_p = (\mathcal{L}_{\mathrm{sym}}^{(1)})^p + (\mathcal{L}_{\mathrm{sym}}^{(2)})^p + (\mathcal{L}_{\mathrm{sym}}^{(3)})^p = 3\mathcal{L}_p^p.$$

Again, we note that the eigenvectors of L_p and $3\mathcal{L}_p^p$ are the same. Now, let us consider the sets $\mathcal{U}_{3n} \subset \mathbb{R}^{3n \times 3n}$ and $\mathcal{Z}_{3n} \subset \mathbb{R}^{3n \times 3n}$ defined as

$$\begin{aligned}\mathcal{U}_{3n} &= \{s_0 I_{3n} - \tilde{A} \otimes E \mid \tilde{A} \in \mathcal{U}_3, s_0 \in \mathbb{R}\},\\ \mathcal{Z}_{3n} &= \{t_0 I_{3n} - \tilde{C} \otimes E \mid \tilde{C} \in \mathcal{Z}_3, s_0 \in \mathbb{R}\}.\end{aligned}$$

Note that, as $s_0 I_3 + \mathcal{U}_3 = \mathcal{U}_3$ and $s_0 I_3 + \mathcal{Z}_3 = \mathcal{Z}_3$ for all $s_0 \in \mathbb{R}$, the definitions of $\mathcal{U}_{3n}$ and $\mathcal{Z}_{3n}$ reduce to that of $\mathcal{U}_3$ and $\mathcal{Z}_3$ when $n = 1$. We prove that $\mathsf{L}_p \in \mathcal{Z}_{3n}$ for all nonzero integer p. To this end, we first prove the following lemma which generalizes Lemma 5.

Lemma 13. *The following holds:*

1. $\mathcal{U}_{3n}$ *is closed under multiplication, i.e. for all* $A, B \in \mathcal{U}_{3n}$ *we have* $AB \in \mathcal{U}_{3n}$*.*
2. *If* $A \in \mathcal{U}_{3n}$ *satisfies* $\det(A) \neq 0$*, then* $A^{-1} \in \mathcal{U}_{3n}$*.*
3. $\mathcal{Z}_3 = P_1\mathcal{U}_{3n}P_1 + P_2\mathcal{U}_{3n}P_2 + P_3\mathcal{U}_{3n}P_3$*.*

Proof. Let $A, B \in \mathcal{U}_{3n}, C \in \mathcal{Z}_{3n}$ and $s_0, r_0, t_0 \in \mathbb{R}$, $\tilde{A}, \tilde{B} \in \mathcal{U}_3, \tilde{C} \in \mathcal{Z}_3$ such that $A = s_0 I_{3n} - \tilde{A} \otimes E$, $B = r_0 I_{3n} - \tilde{B} \otimes E$ and $C = t_0 I_{3n} - \tilde{C} \otimes E$.

1. We have

$$A\,B = s_0 r_0 I_{3n} + (n\tilde{A}\tilde{B} - s_0\tilde{B} - r_0\tilde{A}) \otimes E$$

As $\tilde{A}\tilde{B} \in \mathcal{U}_3$ by Lemma 5, (1), we have $(n\tilde{A}\tilde{B} - s_0\tilde{B} - r_0\tilde{A}) \in \mathcal{U}_3$ and so $AB \in \mathcal{U}_{3n}$.

2. First note that as A is invertible, it holds $s_0 \neq 0$. Furthermore, using von Neumann series, we have

$$\begin{aligned}(s_0 I_{3n} - \tilde{A} \otimes E)^{-1} &= \sum_{k=0}^{\infty} s_0^{k-1} (\tilde{A} \otimes E)^k \\ &= \sum_{k=0}^{\infty} s_0^{k-1} n^k (\tilde{A}^k \otimes E).\end{aligned}$$

As $\tilde{A}^k \in \mathcal{U}_3$ for all k by Lemma 5, (1) we have that $S_\nu = \sum_{k=0}^{\nu} s_0^{k-1} nk(\tilde{A}^k \otimes E) \in \mathcal{U}_{3n}$ for all $\nu = 0, 1, \ldots$ As $\lim_{\nu\to\infty} S_\nu = A^{-1}$ and $\mathcal{U}_{3n}$ is closed, it follows that $A^{-1} \in \mathcal{U}_{3n}$.

3. Note that for $i = 1, 2, 3$ it holds

$$P_i\,A\,P_i = s_0 I_{3n} - (\tilde{P}_i \tilde{A} \tilde{P}_i \otimes E).$$

Hence, we have

$$\sum_{i=1}^{3} P_i\,A\,P_i = 3s_0 I_{3n} - \Big(\sum_{i=1}^{3} \tilde{P}_i \tilde{A} \tilde{P}_i\Big) \otimes E.$$

We know from Lemma 5, (3) that $\sum_{i=1}^{3} \tilde{P}_i \tilde{A} \tilde{P}_i \in \mathcal{U}_3$ and thus $\sum_{i=1}^{3} \tilde{P}_i\, \tilde{A}\, \tilde{P}_i \in \mathcal{Z}_3$. Finally, note that by choosing the coefficients in $\tilde{A}$ in the same way as in the proof of Lemma 5, (3), we have $A = C$ with $s_0 = t_0$. This concludes the proof.

□

We can now prove that $\mathsf{L}_p \in \mathcal{Z}_{3n}$.

Lemma 14. *For every nonzero integer p, we have $\mathsf{L}_p \in \mathcal{Z}_{3n}$.*

Proof. As $\mathcal{L}_{\text{sym}} = \mathcal{L}_{\text{sym}}^{(1)} \in \mathcal{U}_{3n}$, we have $\mathcal{L}_{\text{sym}}^p \in \mathcal{U}_{3n}$ by Lemma 13, (1) and (2). We prove that $\mathsf{L}_p = \sum_{t=1}^{3} P_t \mathcal{L}_{\text{sym}}^p P_t$. To this end, note that, with the convention that powers on vectors are considered component wise, for $t = 1, 2, 3$, we have

$$\begin{aligned}\operatorname{diag}(P_t \mathcal{W} P_t \mathbf{1})^{1/2} &= \operatorname{diag}\big(P_t (\mathcal{W}\mathbf{1})^{-1/2}\big) \\ &= P_t \operatorname{diag}\big((\mathcal{W}\mathbf{1})^{-1/2}\big) P_t = P_t \mathcal{D} P_t.\end{aligned}$$

Furthermore,

$$\begin{aligned}&\operatorname{diag}(P_t \mathcal{W} P_t \mathbf{1})^{-1/2} P_t \mathcal{W} P_t \operatorname{diag}(P_t \mathcal{W} P_t \mathbf{1})^{-1/2} \\ &\quad= P_t \mathcal{D}^{-1/2} P_t^2 \mathcal{W} P_t^2 \mathcal{D}^{-1/2} P_t = P_t \mathcal{D}^{-1/2} \mathcal{W} \mathcal{D}^{-1/2} P_t.\end{aligned}$$

This implies that $\mathcal{L}_{\text{sym}}^{(t)} = P_t \mathcal{L}_{\text{sym}} P_t$ for $t = 1, 2, 3$ and thus we obtain the desired expression for L_p. Lemma 13, (3) finally imply that $\mathsf{L}_p \in \mathcal{Z}_{3n}$. □

We combine Theorem 4 and Lemmas 7, 11 to obtain the following:

Lemma 15. *Let $C \in \mathcal{Z}_{3n}$ and t_0, t_1, t_2 such that $C = t_0 I_{3n} - ((t_1 - t_2) I_3 + t_2 E) \otimes E$. Then, the eigenpairs of C are given by*

$$\begin{aligned}&\big(t_0 - n(t_1 - t_2), (-1, 0, 1)^\top \otimes \mathbf{1}\big), \\ &\big(t_0 - n(t_1 - t_2), (-1, 1, 0)^\top \otimes \mathbf{1}\big), \\ &\big(t_0 - n(t_1 + 2t_2), (1, 1, 1)^\top \otimes \mathbf{1}\big).\end{aligned}$$

and, with $\mathbf{v}_i$ defined as in (6),

$$\begin{aligned}&\big(t_0, (-1, 0, 1)^\top \otimes \mathbf{v}_i\big), \qquad \big(t_0, (-1, 1, 0)^\top \otimes \mathbf{v}_i\big), \\ &\big(t_0, (1, 1, 1)^\top \otimes \mathbf{v}_i\big), \qquad i = 1, \ldots, n-1.\end{aligned}$$

Similar to Lemma 9, we have following lemma for deciding the order of the eigenvectors of L_p.

Lemma 16. *For every positive $p > 0$ we have $(\mathcal{L}_{\text{sym}}^p)_{i.j} < 0 < (\mathcal{L}_{\text{sym}}^p)_{i.i} < \tau^p$ for all $i, j = 1, \ldots, 3n$ with $i \neq j$. For every negative $p < 0$ we have $(\mathcal{L}_{\text{sym}}^p)_{i,j} > 0$ for all $i, j = 1, \ldots, 3n$.*

Proof. Let $\mathcal{M} = \mathcal{D}^{-1/2} \mathcal{W} \mathcal{D}^{-1/2}$, then by Lemma 10, we have $\mathcal{M} = \frac{1}{n}(\tilde{\mathcal{M}} \otimes E)$. Now, for $p > 0$, it holds:

$$\begin{aligned}\mathcal{L}_{\text{sym}}^p &= \tau^p I_{3n} + \sum_{k=1}^{p} \binom{p}{k} \tau^{p-k} (-1)^k n^{-k} (\tilde{\mathcal{M}}^k \otimes E^k) \\ &= \tau^p I_{3n} + \Big(\sum_{k=1}^{p} \binom{p}{k} \tau^{p-k} (-1)^k \tilde{\mathcal{M}}^k\Big) \otimes E \\ &= \tau^p I_{3n} + \big(\tilde{\mathcal{L}}_{\text{sym}}^p - \tau^p I_3\big) \otimes E. \qquad (7)\end{aligned}$$

By Lemma 8, we know that $(\tilde{\mathcal{L}}_{\text{sym}}^p)_{i,j} < 0$ if $i \neq j$ and $(\tilde{\mathcal{L}}_{\text{sym}}^p)_{i,i} - \tau^p < 0$ for all i. Hence, the matrix $\tilde{Q} = \tilde{\mathcal{L}}_{\text{sym}}^p - \tau^p I_3$ has strictly negative entries. Thus, all the off-diagonal elements of $\mathcal{L}_{\text{sym}}^p$ are strictly negative. Finally, note that

$$(\mathcal{L}_{\text{sym}}^p)_{i,i} = \tau^p + (\tilde{\mathcal{L}}_{\text{sym}}^p \otimes E)_{i,i} - \tau^p = (\tilde{\mathcal{L}}_{\text{sym}}^p \otimes E)_{i,i} > 0.$$

This concludes the proof for the case $p > 0$. The case $p < 0$ can be proved in the same way as for the case $n = 1$ (see Lemma 8). □

Observation 2. *We note that Equation* (7) *implies the following relation between* L_p *and* $\tilde{\mathsf{L}}_p$*:*

$$\mathsf{L}_p = 3\tau^p I_{3n} + \big(\tilde{\mathsf{L}}_p - \tau^p I_3\big) \otimes E. \tag{8}$$

Lemma 17. *Let* $t_0, t_1, t_2 \in \mathbb{R}$ *be such that* $\mathsf{L}_p = t_0 I_{3n} - ((t_1 - t_2) I_3 + t_2 E_3) \otimes E_n$*. Furthermore, for any integer* $p \neq 0$*, it holds* $t_0 < t_0 - n(t_1 - t_2) < t_0 - n(t_1 + 2t_2)$ *if* $p < 0$ *and* $t_0 > t_0 - n(t_1 - t_2) > t_0 - n(t_1 + 2t_2)$ *otherwise.*

Proof. The proof is essentially the same as that of Lemma 9. Indeed, if $p < 0$, then L_p is strictly positive and thus $t_2 < 0$ as $(\mathsf{L}_p)_{1,3n} > 0$, $t_1 - t_2 < 0$ as $(\mathsf{L}_p)_{1,n} > 0$ and $t_0 - nt_1 > 0$ as $(\mathsf{L}_p)_{1,1} > 0$. This means that $t_1 - t_2 > t_1 + 2t_2$ and so $t_0 - n(t_1 - t_2) < t_0 - n(t_1 + 2t_2)$. Furthermore, this shows that $t_0 - n(t_1 - t_2) > t_0$. Now, if $p > 0$, by Lemma 16 we have $t_2 > 0$ as $(\mathsf{L}_p)_{1,3n} < 0$, $t_1 - t_2 > 0$ as $(\mathsf{L}_p)_{1,n} < 0$ and $t_0 - nt_1 > 0$ as $(\mathsf{L}_p)_{1,1} > 0$. It follows that $t_1 - t_2 < t_1 + 2t_2$ and thus $t_0 - n(t_1 - t_2) > t_0 - n(t_1 + 2t_2)$. Finally, as $t_1 - t_2 > 0$, we have $t_0 > t_0 - n(t_1 - t_2)$ which concludes the proof. $\square$

We conclude by giving a description of the spectral properties of $\mathcal{L}_p$.

Lemma 18. *Let* p *be any nonzero integer and assume that* $\epsilon > 0$ *if* $p < 0$*. Define*

$$\mathcal{L}_p = \Big(\frac{(\mathcal{L}^{(1)}_{\mathrm{sym}})^p + (\mathcal{L}^{(2)}_{\mathrm{sym}})^p + (\mathcal{L}^{(3)}_{\mathrm{sym}})^p}{3}\Big)^{1/p},$$

then there exists $0 \leq \lambda_1, \lambda_2 < \lambda_3$ *such that the eigenpairs of* $\mathcal{L}_p$ *are given by*

$$\begin{array}{ll} \big(\lambda_1, (-1,0,1)^\top \otimes \mathbf{1}\big), & \big(\lambda_3, (-1,0,1)^\top \otimes \mathbf{v}_i\big) \\ \big(\lambda_1, (-1,1,0)^\top \otimes \mathbf{1}\big), & \big(\lambda_3, (-1,1,0)^\top \otimes \mathbf{v}_i\big) \\ \big(\lambda_2, (1,1,1)^\top \otimes \mathbf{1}\big), & \big(\lambda_3, (1,1,1)^\top \otimes \mathbf{v}_i\big), \end{array}$$

and $i = 1, \ldots, n-1$*, where* $\mathbf{v}_i$ *is defined in* (6)*.*

Proof. The proof is the same as that of Corollary 6 where one uses Lemmas 14, 15, 17 instead of Lemmas 6, 7, 9. $\square$

References

[1] K. V. Bhagwat and R. Subramanian. Inequalities between means of positive operators. *Mathematical Proceedings of the Cambridge Philosophical Society*, 83(3):393401, 1978.

[2] R. Horn and C. Johnson. *Topics in Matrix Analysis.* Cambridge University Press, 1991.

[3] F. Tudisco, V. Cardinali, and C. Fiore. On complex power nonnegative matrices. *Linear Algebra Appl.*, 471:449–468, 2015.