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WIRTSCHAFTS- UND  
SOZIALWISSENSCHAFTLICHE  
FAKULTÄT

Chair of Statistics, Econometrics and Empirical Economics

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**S414**  
**Advanced Mathematical Methods**  
Exercises

WS 2023/24

## INTEGRATION

## EXERCISE 1 Indefinite Integrals

Find the indefinite integrals. Assume that the respective domains are given appropriately.

$$\begin{array}{lll} \text{a)} \int x^{12} dx & \text{b)} \int x^{-4} dx & \text{c)} \int \frac{1}{x^3} dx \\ \text{d)} \int 5x^9 - 3x^7 dx & \text{e)} \int x \cdot \sqrt[3]{x} dx & \text{f)} \int 2e^{3x+1} - 1 dx \\ \text{g)} \int \frac{1}{1+x} dx & \text{h)} \int \frac{1}{1-x} dx & \text{i)} \int \frac{x+3}{x^2+x-6} dx \\ \text{j)} \int \frac{x-1}{\sqrt{x}-1} dx & \text{k)} \int \frac{(e^x)^2 - 1}{e^x + 1} dx & \end{array}$$

## EXERCISE 2 Definite Integrals

Compute the values of the definite integrals:

$$\text{a)} \int_0^2 2 - x dx \quad \text{b)} \int_{-2}^0 |x+1| dx$$

## EXERCISE 3 Integration by Parts

Use partial integration to solve the indefinite integrals.

$$\begin{array}{ll} \text{a)} \int x^2 \cdot \ln(x) dx & \text{b)} \int x \cdot \ln(x) dx \\ \text{c)} \int x^\rho \cdot \ln(x) dx \quad (\rho \neq -1) & \text{d)} \int \ln(x)^2 dx \\ \text{e)} \int x^2 \cdot e^x dx & \text{f)} \int \frac{x}{e^x} dx \end{array}$$

**EXERCISE 4 Integration by Parts**

Determine the values of the definite integrals.

$$\text{a) } \int_{-1}^1 x \ln(x+2) dx \quad \text{b) } \int_0^2 x 2^x dx$$

**EXERCISE 5 Integration by Substitution**

Solve the following indefinite integrals:

$$\begin{aligned} \text{a) } & \int \frac{x}{1+x^2} dx & \text{b) } & \int \frac{1}{x \cdot \ln(x)} dx \\ \text{c) } & \int \frac{\ln(x)^3}{x} dx & \text{d) } & \int \frac{x}{\sqrt{1+x}} dx \\ \text{e) } & \int x^5 \sqrt{4-x^3} dx \end{aligned}$$

**EXERCISE 6 Integration by Substitution**

Determine the values of the following definite integrals:

$$\text{a) } \int_1^{2,5} (2x-1)^6 dx \quad \text{b) } \int_{10}^{10.000} \frac{1}{x \cdot \ln(x)} dx \quad \text{c) } \int_a^b \frac{f'(x)}{f(x)} dx$$

**EXERCISE 7 Improper Integrals**

Determine the values of the following improper integrals:

$$\begin{array}{ll} \text{a) } \int_1^{\infty} \frac{1}{\sqrt[3]{x^2}} dx & \text{b) } \int_0^{\infty} x e^{-ax} dx \quad a > 0 \\ \text{c) } \int_{-\infty}^{-1} \frac{3}{x^3} dx & \text{d) } \int_0^1 \frac{1}{\sqrt{x}} dx \\ \text{e) } \int_0^1 \ln(x) dx & \end{array}$$

**EXERCISE 8 Multiple Integrals**

Solve the following multiple integrals:

$$\begin{array}{ll} \text{a) } \int_0^2 \int_0^1 (2x + 3y + 4) dx dy & \text{b) } \int_0^a \int_0^b (x - a)(x - b) dx dy \\ \text{c) } \int_1^3 \int_1^2 \frac{x - y}{x + y} dx dy & \text{d) } \int_0^{1/2} \int_0^{2\pi} y^3 \sin(xy^2) dx dy \end{array}$$

**EXERCISE 9 Multiple Integrals**

Find

$$I = \int_0^1 \int_0^1 \cdots \int_0^1 (x_1^2 + x_2^2 + \cdots + x_n^2) dx_1 dx_2 \cdots dx_n$$

**Solution Exercise 1:**

a)  $\frac{1}{13}x^{13} + C$

b)  $-\frac{1}{3}x^{-3} + C$

c)  $-\frac{1}{2x^2} + C$

d)  $\frac{1}{2}x^{10} - \frac{3}{8}x^8 + C$

e)  $\frac{3}{7}x^{\frac{7}{3}} + C$

f)  $\frac{2}{3}e^{3x+1} - x + C$

g)  $\ln(|1 + x|) + C$

h)  $-\ln(|1 - x|) + C$

i)  $\ln(|x - 2|) + C$

j)  $\frac{2}{3}x^{\frac{3}{2}} + x + C$

k)  $\int e^x - 1 dx = e^x - x + C$

**Solution Exercise 2:**

a) 2

b) 1

**Solution Exercise 3:**

a)  $\frac{1}{3}x^3 \ln(x) - \frac{1}{9}x^3 + C$

b)  $\frac{1}{2}x^2 \ln(x) - \frac{1}{4}x^2 + C$

c)  $\frac{1}{\rho+1}x^{\rho+1} \ln(x) - \frac{1}{(\rho+1)^2}x^{\rho+1} + C$

d)  $x(\ln(x)^2 - 2\ln(x) + 2) + C$

e)  $e^x(x^2 - 2x + 2) + C$

f)  $-e^{-x}(x + 1) + C$

**Solution Exercise 4:**

a)  $2 - \frac{3}{2} \ln 3$

b)  $\frac{8}{\ln 2} - \frac{3}{(\ln 2)^2}$

**Solution Exercise 5:**

- a)  $\frac{1}{2} \ln(|1 + x^2|) + C$
- b)  $\ln(|\ln(x)|) + C$
- c)  $\frac{1}{4} \ln(x)^4 + C$
- d)  $\frac{2}{3}(1+x)^{\frac{3}{2}} - 2\sqrt{1+x} + C$
- e)  $\frac{-8}{9}(4-x^3)^{\frac{3}{2}} + \frac{2}{15}(4-x^3)^{\frac{5}{2}} + C^*$

**Solution Exercise 6:**

- a)  $\frac{1}{14}(4^7 - 1^7) \approx 1,170.21$
- b)  $[\ln(\ln(x))]_{10}^{10.000} \approx 1.3863$
- c)  $\ln(|f(b)|) - \ln(|f(a)|)$

**Solution Exercise 7:**

- a) does not converge!
- b)  $\frac{1}{a^2}$
- c)  $-\frac{3}{2}$
- d) 2
- e) -1

**Solution Exercise 8:**

- a) 16
- b)  $\frac{1}{6}b^2a(3a - b)$
- c)  $8 \ln(4) - 5 \ln(5) - 3 \ln(3) = 16 \ln(2) - 5 \ln(5) - 3 \ln(3)$
- d)  $\frac{1}{8} - \frac{1}{4\pi}$

**Solution Exercise 9:**

- $I = \frac{n}{3}$