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Proof-theoretic validity based on elimination rules[‡]

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Abstract

In the tradition of Dummett-Prawitz-style proof-theoretic semantics, which considers validity of derivations or proofs as one of its core notions, this paper sketches an approach to proof-theoretic validity based on elimination rules and assesses its merits and limitations. Some remarks are made on alternative approaches based on the idea of dualizing connectives and proofs, as well as on definitional reflection using elimination clauses.

1 Introduction

Following Gentzen's dictum that "the introductions represent so-to-speak the 'definitions' of the corresponding signs" (Gentzen, 1934/35, p. 189), many approaches to proof-theoretic semantics (see Schroeder-Heister, 2012b) consider introduction rules to be basic, meaning giving, or self-justifying, whereas the elimination inferences are justified as valid with respect to the given introduction rules. The roots of this conception are threefold: First there is a verificationist theory according to which assertibility conditions of a sentence constitute its meaning. This seems to underly not only to a large extent the semantic conceptions of Dummett and

[‡]I am grateful to Luiz Carlos Pereira for many fruitful discussions on issues of proof theory and proof-theoretic semantics over more than 35 years. As the systematics of natural deduction and the delineation of intuitionistic logic has been a main theme of Luiz Carlos's work, the topic of this paper seems to me appropriate for a volume in his honour. I am grateful to Thomas Piecha, Dag Prawitz and the editors for many helpful comments and suggestions. The research reported here has been supported by the French-German ANR-DFG projects "Hypothetical Reasoning" and "Beyond Logic" (DFG Schr 275/16-2 and Schr 275/17-1).

Prawitz, which are the most developed ones in this respect, but the whole movement of intuitionism. Even if it is not directly connected to the verificationism of the early Wittgenstein and the Vienna circle, there are strong reminiscences of their positions in verificationist proof-theoretic semantics. There is a justificationist and verificationist bias in certain branches of constructive semantics and philosophy of mathematics. The second point is the idea that we must distinguish between what constitutes the meaning and what are the consequences of this meaning, in order to cope with the 'paradox of inference' (Cohen & Nagel, 1934, Ch. 9, § 1); see the discussion in Dummett (1975). For an inference to be informative, not every inference can be definitional. The informative inferences are established by reflection on the meaning of the expressions involved, without being meaning-constituting themselves. Whereas introduction steps are meaning giving, the remaining valid inferences give novel insight beyond what is 'definitionally' already contained in the premisses. The third point, which is closely connected to the first, is the primacy of assertion over other speech acts such as assuming or denying, which is implicit in most approaches to proof-theoretic semantics. In Prawitz's definition of validity, and in intuitionistic semantics in general, assumptions are placeholders for proofs or constructions, and negation is reduced to implying absurdity. This yields a general bias towards positive forward reasoning, which is reflected in the primacy of forward-directed introductions (for a criticism of this approach see Schroeder-Heister, 2012*a*). To some extent this view is also implicit in the clause-based theory of definitional reflection (Hallnäs, 1991; Schroeder-Heister, 1993), as clauses are directed from bodies to heads, that is, from defining conditions towards defined atoms. The non-determinism in clauses, i.e., the fact that several clauses may define the same atom (which in logic we have, for example, with the introduction rules for disjunction) emphasizes this directedness. Whereas the use of a single clause is simply the application of a definition, definitional reflection extracts additional content from an expression with respect to the definition as a whole, which can be viewed as generating valid informative inferences.

The division between introductions and eliminations suggests to exchange their roles and thus to consider elimination rules rather than introduction rules as the basis of proof-theoretic semantics. Such an approach would be nearer to a falsificationist methodology in Popper's sense. The philosophical problems and shortcomings of verificationism, which cannot be discussed here, would be strong arguments in favour of this alternative. The second point mentioned in the previous paragraph is indifferent with respect to the primacy of introduction or elimination rules, as it only says that there must be *one part* of the rules which is meaning giving and *another one* which is informative, so one may as well choose the elimination rules as meaning giving. The third point, the primacy of assertion, would be replaced with the Popperian claim that conjectures and therefore assumptions are primary to assertions.

This possibility has been intensively discussed by the main advocates of verificationist semantics. In Dummett it often runs under the heading of a 'pragmatist' theory of meaning and has received considerable credit in some of his publications such as Dummett (1976)[1]. Some technical ideas towards a proof-theoretic semantics based on elimination rather than introduction rules have been sketched by Dummett (1991, Ch. 13). A precise definition of validity based on elimination inferences is due to Prawitz (1971, 2007). In slightly improved form, it will be presented in Section 3. As its background, we recall the validity definition based on introduction rules.

2 The introductions-based approach: Derivation structures, justifications and atomic systems

The definition of validity refers to a general notion of derivation structures and reductions that justify derivations, as well as to atomic systems. We refer to the version that is given in Schroeder-Heister (2006, 2012*b*), which is an interpretation of Prawitz's notion of proof-theoretic validity. We consider only the constants of positive propositional logic (conjunction, disjunction, implication). We assume that an atomic system S is given as determining the derivability of atomic formulas, which is the same as their validity. A formula over S is a formula built up by means of logical connectives starting with atoms from S. We want to define the validity of a derivation which proceeds from formulas over S as assumptions to a formula over S as conclusion. Such a derivation is not necessarily a derivation in a given formal system: We want to tell of an *arbitrary* derivation whether it is valid or not. We propose the term "derivation structure" for such an arbitrary derivation. (Prawitz uses various terminologies, such as "[argument or proof] schema" or "[argument or proof] skeleton".) Derivation structures are candidates for valid derivations. More precisely, a derivation structure is a formula tree which resembles a natural deduction tree with the difference that it is composed of arbitrary rules. Such rules can have arbitrary and arbitrarily many premisses, and each

[1]However, the main theses of this paper were withdrawn in the preface to Dummett (1993).

premiss may depend on assumptions which are discharged at this step. So the general form of an inference rule is the following, where the square brackets indicate assumptions which can be discharged at the application of the rule:

$$\frac{\begin{matrix}[C_{11},\ldots,C_{1m_1}] \\ A_1\end{matrix} \quad \ldots \quad \begin{matrix}[C_{n1},\ldots,C_{nm_n}] \\ A_n\end{matrix}}{B}\ , \qquad \text{in short: } \frac{\begin{matrix}[\Gamma_1] \\ A_1\end{matrix} \quad \ldots \quad \begin{matrix}[\Gamma_n] \\ A_n\end{matrix}}{B}\ .$$

Obviously, the standard introduction and elimination rules are particular cases of such rules. As a generalization of the standard reductions of maximal formulas it is supposed that certain reduction procedures are given. A reduction procedure transforms a given derivation structure into another one. A set of reduction procedures is called a *derivation reduction system* and denoted by $\mathcal{J}$. Reductions serve as justifying procedures for non-canonical steps. These are steps, which are not self-justifying, i.e., which are not introduction steps. Therefore a reduction system $\mathcal{J}$ is also called a *justification*. Reduction procedures must satisfy certain constraints such as closure under substitution. As the validity of a derivation not only depends on the atomic system S but also on the derivation reduction system used, we define the validity of a derivation structure with respect to the underlying atomic basis S and with respect to the justification $\mathcal{J}$. A *canonical* derivation structure is a derivation structure which uses an introduction rule in the last step. It is called *open*, if it depends on undischarged assumptions, otherwise it is called *closed*.

Definition: Validity based on introduction rules

(i) *Every closed derivation in S is S-valid with respect to $\mathcal{J}$ (for every $\mathcal{J}$).*

(ii) *A closed canonical derivation structure is S-valid with respect to $\mathcal{J}$, if its immediate substructure* $\begin{matrix} A \\ \mathcal{D} \\ B \end{matrix}$ *is S-valid with respect to $\mathcal{J}$.*

(iii) *A closed non-canonical derivation structure is S-valid with respect to $\mathcal{J}$, if it reduces, with respect to $\mathcal{J}$, to a canonical derivation structure, which is S-valid with respect to $\mathcal{J}$.*

(iv) *An open derivation structure* $\begin{matrix} A_1 \ldots A_n \\ \mathcal{D} \\ B \end{matrix}$*, where all open assumptions of $\mathcal{D}$ are among $A_1, \ldots, A_n$, is S-valid with respect to $\mathcal{J}$, if for* *every extension S' of S and every extension $\mathcal{J}'$ of $\mathcal{J}$, and for every list of closed derivation structures* $\begin{matrix}\mathcal{D}_i\\ A_i\end{matrix}$ $(1 \leq i \leq n)$*, which are S'-valid with respect to $\mathcal{J}'$,*

$$\begin{matrix}\mathcal{D}_1 & & \mathcal{D}_n\\ A_1 & \dots & A_n\\ & \mathcal{D} & \\ & B & \end{matrix}$$

is S'-valid with respect to $\mathcal{J}'$.

(See Prawitz, 1973, p. 236; 1974.; p. 73; 2006).

In clause (iv), the reason for considering extensions $\mathcal{J}'$ of $\mathcal{J}$ and of extensions S' of S is a monotonicity constraint. Derivations should remain valid if one's knowledge incorporated in the atomic system and in the reduction procedures is extended. The consideration of such extensions, which can be found in Prawitz (1971), is a point of deviation of this exposition from later definitions given by Prawitz.

The S-validity of a generalized inference rule

$$\frac{\begin{matrix}[\Gamma_1] & & [\Gamma_n]\\ A_1 & \dots & A_n\end{matrix}}{B}$$

with respect to a justification $\mathcal{J}$ means that for all derivations $\begin{matrix}\Gamma_1\\ \mathcal{D}_1\\ A_1\end{matrix}, \dots, \begin{matrix}\Gamma_n\\ \mathcal{D}_n\\ A_n\end{matrix}$, which are S'-valid with respect to $\mathcal{J}'$ for extensions S' and $\mathcal{J}'$ of S and $\mathcal{J}$, respectively, the derivation

$$\frac{\begin{matrix}(1) & & (1)\\ [\Gamma_1] & & [\Gamma_n]\\ \mathcal{D}_1 & & \mathcal{D}_n\\ A_1 & \dots & A_n\end{matrix}}{B}\,(1)$$

is S'-valid with respect to $\mathcal{J}'$. For a simple inference rule

$$\frac{A_1 \ \dots \ A_n}{A}$$

this means that it is valid with respect to $\mathcal{J}$, if the one-step derivation structure of the same form is S-valid with respect to $\mathcal{J}$.

This gives rise to a corresponding notion of consequence (see also Prawitz, 1985). Instead of saying that the rule

$$\frac{A_1 \quad \dots \quad A_n}{A}$$

is S-valid with respect to $\mathcal{J}$, we may say that A is a *consequence* of $A_1, \ldots, A_n$ *with respect to* S *and* $\mathcal{J}$ ($A_1, \ldots, A_n \models_{S,\mathcal{J}} A$). If this holds for any S, we may speak of *universal consequence with respect to* $\mathcal{J}$ ($A_1, \ldots, A_n \models_{\mathcal{J}} A$); and finally, if there is some $\mathcal{J}$ such that we have universal consequence with respect to $\mathcal{J}$, then we may speak of *logical consequence* ($A_1, \ldots, A_n \models A$).

If for $\mathcal{J}$ we choose the standard reductions of intuitionistic logic, then all derivations in intuitionistic logic are valid with respect to $\mathcal{J}$, thus establishing the *soundness* of intuitionistic logic with respect to introductions-based proof-theoretic semantics. We may ask if the converse holds, namely whether, given that a derivation $\mathcal{D}$ is valid with respect to some $\mathcal{J}$, there is a derivation in intuitionistic logic with the same end-formula and without any open assumptions beyond those already open in $\mathcal{D}$. That intuitionistic logic is *complete* in this sense has been conjectured by Prawitz (see Prawitz, 1973, 2014). This conjecture is not without problems as results by Sandqvist (2009) and Piecha et al. (2014) indicate.

3 Validity based on elimination rules

In the approach based on elimination rules, the elimination inferences are considered 'self-justifying', and the introduction rules are justified with respect to them. The reductions need not to be changed for that purpose. The standard reductions for the logical constants can serve for the justification of the introductions from the eliminations as well. However, additional reductions must be considered which correspond to the permutative reductions in natural deduction. In this section, we speak of validity$_E$ as validity based on elimination rules in contradistinction to validity$_I$ as validity based on introduction rules.

The idea behind validity$_E$ is that, if all applications of elimination rules to the complex end-formula A of a derivation structure $\mathcal{D}$ yield S-valid$_E$ derivation structures or reduce to such (with respect to a justification $\mathcal{J}$), then $\mathcal{D}$ is itself S-valid$_E$ (with respect to $\mathcal{J}$). This suggests the following definition for positive propositional logic:

Definition: Validity based on elimination rules

(i) *Every closed derivation in S is S-valid$_E$ with respect to $\mathcal{J}$ (for every $\mathcal{J}$).*

(ii$\wedge$) *A closed derivation structure* $\begin{matrix}\mathcal{D}\\ A\wedge B\end{matrix}$ *is S-valid$_E$ with respect to $\mathcal{J}$,* *if the closed derivation structures* $\dfrac{\overset{\mathcal{D}}{A \wedge B}}{A}$ *and* $\dfrac{\overset{\mathcal{D}}{A \wedge B}}{B}$ *are* S*-valid*$_E$ *with respect to* $\mathcal{J}$*, or reduce to derivation structures, which are* S*-valid*$_E$ *with respect to* $\mathcal{J}$*.*

(ii$\rightarrow$) *A closed derivation structure* $\overset{\mathcal{D}}{A \rightarrow B}$ *is* S*-valid*$_E$ *with respect to* $\mathcal{J}$*, if for every extension* S' *of* S *and for every extension* $\mathcal{J}'$ *of* $\mathcal{J}$*, and for every closed derivation structure* $\overset{\mathcal{D}'}{A}$ *, which is* S'*-valid*$_E$ *with respect to* $\mathcal{J}'$*, the (closed) derivation structure* $\dfrac{\overset{\mathcal{D}}{A \rightarrow B} \qquad \overset{\mathcal{D}'}{A}}{B}$ *is* S'*-valid*$_E$ *with respect to* $\mathcal{J}'$*, or reduces to a derivation structure, which is* S'*-valid*$_E$ *with respect to* $\mathcal{J}'$*.*

(ii$\vee$) *A closed derivation structure* $\overset{\mathcal{D}}{A \vee B}$ *is* S*-valid*$_E$ *with respect to* $\mathcal{J}$*, if for every extension* S' *of* S *and every extension* $\mathcal{J}'$ *of* $\mathcal{J}$*, and for all derivation structures* $\begin{matrix} A \\ \mathcal{D}_1 \\ C \end{matrix}$ *and* $\begin{matrix} B \\ \mathcal{D}_2 \\ C \end{matrix}$ *with atomic* C*, which are* S'*-valid*$_E$ *with respect to* $\mathcal{J}'$ *and which depend on no assumptions beyond* A *and* B*, respectively, the (closed) derivation structure*

$$\dfrac{\overset{\mathcal{D}}{A \vee B} \qquad \begin{matrix} (1) \\ [A] \\ \mathcal{D}_1 \\ C \end{matrix} \qquad \begin{matrix} (1) \\ [B] \\ \mathcal{D}_2 \\ C \end{matrix}}{C}\,(1)$$

is S'*-valid*$_E$ *with respect to* $\mathcal{J}'$*, or reduces to a derivation structure, which is* S'*-valid*$_E$ *with respect to* $\mathcal{J}'$*.*

(iii) *A closed derivation structure* $\overset{\mathcal{D}}{A}$ *of an atomic formula* A*, which is not a derivation in* S*, is* S*-valid*$_E$ *with respect to* $\mathcal{J}$*, if it reduces with respect to* $\mathcal{J}$ *to a derivation in* S*.*

(iv) *An open derivation structure* $\begin{matrix} A_1 \ldots A_n \\ \mathcal{D} \\ B \end{matrix}$ *, where all open assumptions of* $\mathcal{D}$ *are among* $A_1, \ldots, A_n$*, is* S*-valid*$_E$ *with respect to* $\mathcal{J}$*, if for every extension* S' *of* S *and every extension* $\mathcal{J}'$ *of* $\mathcal{J}$*, and for every list of closed derivation structures* $\overset{\mathcal{D}_i}{A_i}$ $(1 \leq i \leq n)$*, which are*

S'-valid$_E$ with respect to $\mathcal{J}'$, $\begin{array}{ccc} \mathcal{D}_1 & & \mathcal{D}_n \\ A_1 & \ldots & A_n \\ & \mathcal{D} & \\ & B & \end{array}$ *is S'-valid$_E$ with respect to $\mathcal{J}'$.*

Clause (iv) is identical with clause (iv) in the definition of introductions-based validity in Section 2, i.e., open assumptions in derivations are interpreted in the same way as before, namely as placeholders for closed valid derivations. Note that clause (iii) is needed, as we do not have here the notion of a canonical derivation. In the definition of validity based on introduction rules, the case considered in clause (iii) was a special case of non-canonical derivations. Clauses (i) and (iii) can be conjoined to form the single clause

(i/iii) *A closed derivation structure* $\begin{array}{c} \mathcal{D} \\ A \end{array}$ *of an atomic formula A is S-valid$_E$ with respect to $\mathcal{J}$, if it reduces with respect to $\mathcal{J}$ to a derivation in S.*

The validity$_E$ of an inference rule as well as the notions of consequence and logical consequence are defined exactly as in the introductions-based approach of Section 2.

It is crucial that the minor premisses C in the application of $\vee$-elimination (and similarly for $\exists$-elimination, if we deal with quantifiers) are atomic, otherwise the induction over the end-formulas of derivations, on which this definition is based, would break down. Prawitz (1971, Appendix A.2), eliminations-based definition of validity was without clauses for disjunction (and existential quantification), as he had not been aware at the time that for the purpose of defining validity the restriction to atomic C is sufficient (repeated in Schroeder-Heister, 2006). The revised proposal with atomic C was published in Prawitz (2007). There he refers to the fact that also Dummett (1991, Ch. 13), in his remarks on a "pragmatist" theory of meaning with an inverse justification based on elimination rules uses atomic C. The fact that one can do without complex C is closely related to the fact that the definability of first-order logical constants in second-order propositional $\forall\rightarrow$-logic, which was first observed by Prawitz (1965, Ch. 5), can already be obtained in *predicative* second-order $\forall\rightarrow$-logic in the sense that the latter proves the introduction and elimination rules for the defined connectives as shown by Ferreira (2006, see also Ferreira & Ferreira 2013)[2].

The condition "or reduces to a derivation structure, which is S'-valid$_E$ with respect to $\mathcal{J}'$" at the end of clauses (ii$\rightarrow$), (ii$\wedge$), (ii$\vee$) is called the 'reduction condition'. It corresponds to the basic intuition of proof-theoretic validity semantics that a derivation is valid, if it is of a certain form *or reduces to such a form*. It simplifies certain proofs such as that of the validity$_E$ of the introduction inferences. However, it can be omitted without loss of definitional power, since both with and without the reduction condition we can show that a derivation structure is valid$_E$ if and only if it reduces to a valid$_E$ derivation structure. In fact, in the original notion of validity$_E$ envisaged by Dummett and defined by Prawitz (and also in corresponding notions of computability) the notion of reduction does not come in until the atomic stage is reached. In any case a reduction condition must be contained in clause (iii) which governs derivations of atomic sentences.

The standard reductions, which remove maximum formulas, are not sufficient to show that all introduction and elimination rules are valid. Due to the restriction that C must be atomic, we now have to justify in particular the $\vee$-elimination rule for *nonatomic C*. For that we need reductions, which correspond to permutative reductions in natural deduction. For example, in order to show that

$$\frac{\begin{matrix}\mathcal{D}\\ A\vee B\end{matrix}\qquad \begin{matrix}\overset{(1)}{[A]}\\ \mathcal{D}_1\\ C_1\wedge C_2\end{matrix}\qquad \begin{matrix}\overset{(1)}{[B]}\\ \mathcal{D}_2\\ C_1\wedge C_2\end{matrix}}{C_1\wedge C_2}\,(1)$$

is valid, given that $\mathcal{D}$, $\mathcal{D}_1$ and $\mathcal{D}_2$ are valid, we need to use reductions according to which

$$\cfrac{\cfrac{\begin{matrix}\mathcal{D}\\ A\vee B\end{matrix}\qquad \begin{matrix}\overset{(1)}{[A]}\\ \mathcal{D}_1\\ C_1\wedge C_2\end{matrix}\qquad \begin{matrix}\overset{(1)}{[B]}\\ \mathcal{D}_2\\ C_1\wedge C_2\end{matrix}}{C_1\wedge C_2}\,(1)}{C_i} \quad\text{reduces to}\quad \frac{\begin{matrix}\mathcal{D}\\ A\vee B\end{matrix}\qquad \cfrac{\begin{matrix}\overset{(1)}{[A]}\\ \mathcal{D}_1\\ C_1\wedge C_2\end{matrix}}{C_i}\qquad \cfrac{\begin{matrix}\overset{(1)}{[B]}\\ \mathcal{D}_2\\ C_1\wedge C_2\end{matrix}}{C_i}}{C_i}\,(1)\ .$$

[2]This fact was independently discovered by Sandqvist.

The rule of importation

$$(R_{imp})\quad \frac{A\rightarrow(B\rightarrow C)}{A\wedge B\rightarrow C}$$

is an instructive example to compare the justifications based on validity$_I$ vs. validity$_E$. For validity$_E$, we would now need the following reduction as a justification:

$$\dfrac{\dfrac{\begin{matrix}\mathcal{D}\\ A\rightarrow(B\rightarrow C)\end{matrix}}{A\wedge B\rightarrow C}\qquad \begin{matrix}\mathcal{D}'\\ A\wedge B\end{matrix}}{\text{C}} \quad\text{reduces to}\quad \dfrac{\dfrac{\begin{matrix}\mathcal{D}\\ A\rightarrow(B\rightarrow C)\end{matrix}\qquad \dfrac{\begin{matrix}\mathcal{D}'\\ A\wedge B\end{matrix}}{A}}{B\rightarrow C}\qquad \dfrac{\begin{matrix}\mathcal{D}'\\ A\wedge B\end{matrix}}{B}}{C}\ .$$

We do not need any of the standard reductions, which means that importation is valid with respect to the justification consisting of this reduction alone.

In order to justify (R_{imp}) with respect to validity$_I$, we would rely on a similar reduction: $\dfrac{\begin{matrix}\mathcal{D}\\ A\rightarrow(B\rightarrow C)\end{matrix}}{A\wedge B\rightarrow C}$

$$\text{reduces to}\quad \dfrac{\dfrac{\dfrac{\begin{matrix}\mathcal{D}\\ A\rightarrow(B\rightarrow C)\end{matrix}\qquad \dfrac{\overset{(1)}{[A\wedge B]}}{A}}{B\rightarrow C}\qquad \dfrac{\overset{(1)}{[A\wedge B]}}{B}}{C}}{A\wedge B\rightarrow C}\,(1)\ .$$

However, we would need to use in addition the standard reductions of conjunction and implication in order to justify the ($\wedge$E) and ($\rightarrow$E) steps involved (see the supplement to Schroeder-Heister, 2012*b*). In both cases we must use a reduction that unfolds a single step into a succession of more elementary steps.

It is not entirely clear which logic we obtain by the eliminations-based approach. From the remarks above it is clear that we can justify the rules of intuitionistic logic, which means that intuitionistic logic is sound with respect to this semantics. In view of our definitional clause for disjunction, it is natural to consider atomic second-order propositional logic $\mathbf{F_{at}}$ as the formal system corresponding to eliminations-based semantics. This system, in which $A\vee B$ is interpreted as $\forall X((A\rightarrow X)\rightarrow((B\rightarrow X)\rightarrow X))$, where the universal quantifier runs over atomic propositions only, has been studied by Ferreira (2006). Though it does not contain disjunction as a primitive sign, it satisfies the disjunction property for the second-order interpretation of disjunction. In fact, Ferreira & Ferreira (2014) could show that it is equivalent to intuitionistic logic. However, whether $\mathbf{F_{at}}$ and thus intuitionistic logic is complete with respect to eliminations-based semantics is not obvious and can be questioned. As mentioned before, there are arguments that problematize Prawitz's completeness conjecture for intuitionistic logic with respect to introductions-based proof-theoretic semantics. Depending on certain assumptions about the form of atomic systems and on the way of dealing with hypothetical proofs[3], there are actually counterexamples to completeness (see Piecha et al., 2014; Piecha, 2015). These counterexamples can be adapted to the eliminations-based approach, as the handling of atomic systems and hypothetical proofs does not differ between the two approaches.

It should be remarked that there is a notion of computability based on elimination rules used in proofs of (strong) normalization which corresponds to the eliminations-based notion of validity. Actually, this notion is more common in presentations of this topic than computability based on introductions-based(see, for example, Troelstra & Schwichtenberg, 2000, Ch. 6.6).

4 Co-implication and other alternatives

The intuition behind the approach based on elimination rules is that a derivation is valid, if the result of the application of each possible elimination rule to its end-formula is valid. This means that even a closed derivation is not valid due to its actual form or to the form to which it can be reduced (as in the introductions-based approach), but due to appending further inference steps to it. Its validity depends on that of the immediate consequences we can reach starting with this derivation. So one might call it a consequentialist view of validity. This is an original approach, which brings a fresh idea into proof-theoretic validity. It must be noted, however, that basic tenets of introductions-based validity concepts are kept. Among those is the primacy of closed derivatons and the interpretation of open derivations. In both validity conceptions the definition of validity starts with closed derivations. And in both conceptions the validity of an open derivation is defined via the substitution of closed derivations for the open assumptions in open derivations, as expressed by the fact that clause (iv) of the definition of validity, which deals with open derivations, is identical

[3]All counterexamples assume that we consider arbitrary *extensions* of atomic systems when interpreting hypothetical proofs — an assumption now longer made by Prawitz.

in both of them. This means that both are biased towards assertions (by means of closed derivations), whereas assumptions are just placeholders for what can be asserted by means of closed derivations. It is assertions which, in the eliminations-based approach, are justified by their consequences. It is definitely not the case that assumptions receive a stronger stance in this sort of theory.

Therefore the approach sketched here is not the only possible and perhaps not even the most genuine way of putting elimination rules first. An eliminations-based approach which reverses the conceptual priority between assertions and assumptions would be one which considers derivations from assumptions to be primary. Such an approach can be obtained by dualizing the introductions-based approach by putting "deriving from" rather than "deriving of" in front. One would then develop ideas such as the following: A *closed derivation from* A should be a derivation of absurdity from A (corresponding to the fact that a closed derivation in the standard conception can be viewed as a derivation from truth), and a derivation $\begin{matrix} A \\ \mathcal{D} \\ B \end{matrix}$ should be justified, if, for every closed valid derivation $\begin{matrix} B \\ \mathcal{D}' \end{matrix}$ from B, $\begin{matrix} A \\ \mathcal{D} \\ B \\ \mathcal{D}' \end{matrix}$ is a closed valid derivation from A. A full dualization would even lead to some variant of a single-assumption/multiple-conclusion logic, whose derivations are branching downwards rather than upwards. A closed derivation from A, in which all downward branches end with absurdity, might be called a closed refutation of A. If one of these branches ends with a formula B different from absurdity, it is an open refutation of A in the sense that replacing B with a closed refutation of B yields a closed refutation of A. Such approaches would lead to rules for logical constants which are dual to the standard ones. Conjunction (as the dual of disjunction) would be the constant that is canonically refuted by a refutation of A as well as by one of B, disjunction (as the dual of conjunction) would be the constant that is canonically refuted by a refutation of both A and B etc. Co-implication would come in as the dual of implication, which is canonically refuted by an open refutation of B to A, that is, of B given a refutation of A, and so on. This leads essentially to an approach in which usual derivation trees are written upside down, the concept of derivation is interchanged with that of refutation, etc. It corresponds to a system of dual-intuitionistic logic, in which connectives are replaced with their duals, and in particular implication by co-implication. However, structurally, the standard approach and its dual are the same — writing derivations upside down is not really an essential change. So if we want to obtain any conceptual gain from the consideration of dual concepts, we should be able to develop a joint system for both notions. A genuine elimination-rule approach might be desirable if one wanted to logically elaborate ideas like Popper's falsificationism by establishing refutation as the basis of reasoning. However, it is still not entirely clear what such an approach, which was already discussed by Popper (1948) and whose proof theory was initiated by Prawitz (1965, Appendix B.2), should look like formally. It would be a justification of what is now called 'bi-intuitionistic logic', which incorporates both implication and co-implication (see Wansing, 2008, 2013; Tranchini, 2012; Kapsner, 2014, and the references therein).[4]

Under normal circumstances, multiple-conclusion proof systems go beyond intuitionistic logic. However, by means of certain restrictions concerning the dependencies between formulas, constructivity in the intuitionistic sense can be enforced. Such systems have been studied by de Paiva & Pereira (1995, 2005). It would be a worthwhile task to turn this idea into some form of proof-theoretic semantics, which keeps track of such dependencies in the form of semantic conditions.

5 Consequentialism and definitional reflection

Definitional reflection adapts basic ideas concerning harmony and inversion from the logical realm to the realm of clausal definitions of atoms, inspired by a proof-theoretic interpretation of logic programming (Hallnäs, 1991; Schroeder-Heister, 1993, 2012*b*). In the simplest case, a definition is a finite list of clauses of the form

$$b_1, \ldots, b_n \Rightarrow a$$

where $b_1, \ldots, b_n, a$ are atoms. A finite set of clauses with the same head a

$$\left\{ \begin{array}{r} b_{11}, \ldots, b_{1n_1} \Rightarrow a \\ \vdots \\ b_{k1}, \ldots, b_{kn_k} \Rightarrow a \end{array} \right.$$

is called a *definition* of a. Then the rules of *definitional closure* says that we may pass along any definitional clause from its body to its head,

[4] A proof-theoretic approach, which does not consider co-implication but mixes standard implication with conjunction and disjunction (which are dual to one another) is hinted at in Dummett (1991, Ch. 13), and has been worked out (and improved) in detail by Litland (2012).

yielding rules

$$\frac{b_{i1} \ \dots \ b_{in_1}}{a} \qquad \dots \qquad \frac{b_{k1} \ \dots \ b_{kn_k}}{a}$$

which correspond to introduction rules in logic. The powerful rule of *definitional reflection* says that anything that can be obtained from each defining condition of a can be obtained from a itself, which corresponds to the idea of elimination inferences:

$$\frac{a \qquad \begin{matrix}[b_{11}, \dots, b_{1n_1}] \\ C\end{matrix} \qquad \dots \qquad \begin{matrix}[b_{k1}, \dots, b_{kn_k}] \\ C\end{matrix}}{C} \quad .$$

Even though the rules of definitional closure and reflection come as a pair, without any of them primary over the other, there is some implicit bias towards introductions since clauses are directed. Definitional closure is interpreted as expressing the direction from definiens to definiendum, and definitional reflection as expressing the opposite direction. Changing this bias and inverting it, would have to be a radical reform of what a definition looks like. We would then have to consider 'consequential' clauses which determine the consequences of a given atom, such as

$$\begin{cases} a \Rightarrow b_1 \\ \quad \vdots \\ a \Rightarrow b_m \end{cases} .$$

Definitional closure would then express reasoning along these consequential clauses

$$\frac{a}{b_1} \qquad \dots \qquad \frac{a}{b_m}$$

which correspond to elimination inferences, and definitional reflection would be an introduction rule telling that a can be introduced from all possible definitional consequences taken together

$$\frac{b_1 \ \dots \ b_m}{a} \quad .$$

To make this approach reasonably expressive, we would have to consider also complex conclusions of consequential clauses rather than just atoms b_i. A multiple-conclusion clause

$$a \ \Rightarrow c_1, \dots, c_n$$

would then be interpreted by a reflection rule like

$$\frac{a \quad \begin{array}{c}[c_1]\\ C\end{array} \quad \dots \quad \begin{array}{c}[c_n]\\ C\end{array}}{C}\ .$$

If we have more than one multiple-conclusion clause, we would have to consider an appropriate list of reflection (elimination) rules. Alternatively, we could just consider single clauses, but with structural implications (rules) as conclusion, be means of which we can code in principle what can be expressed, e.g., by disjunction. Such a clause would, e.g. be

$$a \Rightarrow ((c_1 \Rightarrow p), \dots, (c_n \Rightarrow p) \Rightarrow p)$$

for a fresh variable p. This leads to an eliminations-based approach for atomic rules corresponding to the one discussed in Section 3 for the case of logical constants. It dualizes the notion of a definition, but not the concept of a derivation, which is, as before, a concept of derivation of single formulas from (possibly) multiple assumptions. A complete dualization using single-assumption / multiple-conclusion derivations would trivialize the whole notion by just exchanging the right and left sides of definitions and the assumptions and conclusions of derivations, as does the dual-intuitionistic approach mentioned in Section 4. Using multiple-assumptions / multiple-conclusion derivations based on both standard definitional clauses and consequential ones should lead to an atomic framework corresponding to bi-intuitionistic logic. Such a theory still needs to be worked out. Particular attention should be paid to the question, which connections to logic programming and inductive definitions remain, and, therefore, how far definitional reasoning keeps the computational content which is inherent in forward-directed clauses.[5]

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[5] Some remarks on dual frameworks can be found in Schroeder-Heister (2011).

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