EBERHARD KARLS UNIVERSITÄT TÜBINGEN

B470: Options and Futures

Some remarks about conic finance

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„Le présent serait plein de tous les avenirs,
si le passé n'y projetait déjà une histoire."

André Gide

Agenda

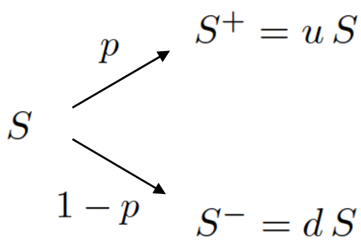
Introductory part:
The binomial model revisited
Two prices instead of one?!
Why „conic“ finance??

For the experts:
Conic Black-Scholes-Merton?
The Breeden-Litzenberger approach!

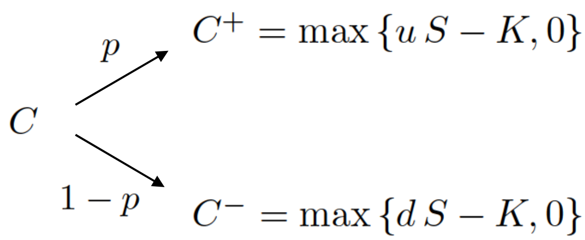
Creative part:
Update of insurance model (1984)
Update of corporate bond model (1998)
Update of stochastic volatility model (1999)
Update of fractional Brownian motion model (2006)
….
(„The future is wide open“)

Single-period binomial model

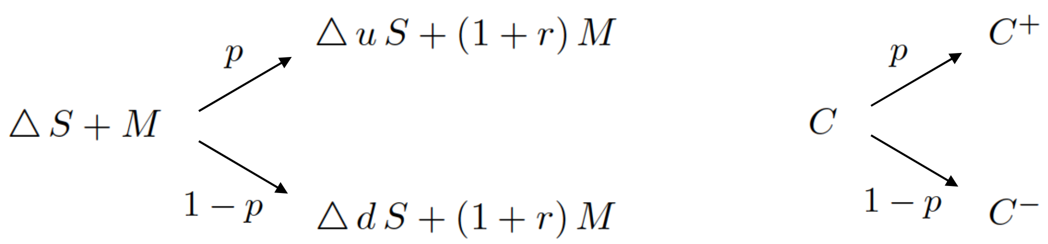
Recall the binomial model: The stock price moves up with probability *p* or down with *1-p* in a one-step tree.



The same does the option price. (We use the call option as our working horse.)



We construct a portfolio using Δ stocks *S* and the market account *M*, only.



We **compare the outcomes** with the option **and fix Δ and *M* appropriately**.

$$\begin{aligned} \triangle u S + (1+r) M &= C^+ \\ \triangle d S + (1+r) M &= C^- \end{aligned} \quad \longrightarrow \quad \boxed{C = \triangle S + M}$$

Because the outcomes are now identical, the portfolio is called the **duplicating portfolio**, which must have - in a perfect market - the same price as the option : **„law of one price“.**

Single-period binomial model

Solving the two equations for the unknowns Δ and *M,* we get the proportions of the duplicating portfolio.

$$\Delta = \frac{C^+ - C^-}{S\,(u-d)} \qquad M = \frac{1}{1+r}\left(\frac{C^+ d - C^- u}{u-d}\right)$$

$$\boxed{C = \triangle S + M}$$

Putting these into the pricing equation, we get…

$$\begin{aligned} C &= \left(\frac{C^+ - C^-}{S\,(u-d)}\right) S + \frac{1}{1+r}\left(\frac{C^+ d - C^- u}{u-d}\right) \\ &= \frac{1}{1+r}\left(\underbrace{\frac{(1+r)-d}{u-d}}_{p} C^+ + \underbrace{\frac{u-(1+r)}{u-d}}_{1-p} C^-\right) \end{aligned}$$

… the pricing equation for the **one-step binomial model**

$$C = \frac{1}{1+r}\left(p\,C^{+} + (1-p)\,C^{-}\right)$$

This is a **discounted expectation with probability**

$$p = \frac{(1+r)-d}{u-d}$$

Note that p is an artifical probability simply calculated from the other parameters of the model.

In a **complete market** model, like the binomial model here, **this probability is unique**, any other option price can be arbitraged away. The option price is completely independent of risk preferences. Therefore any choice of risk preferences leads to the same result, and the assumption of a risk-neutral world simplifies the calculation, which is still valid in general. That is the reason, why p is called – somewhat misleading - the **risk-neutral probability**.

However in an **incomplete market** things are different, as we will see later.

We multiply both sides by the compounding factor

$$(1+r)\,C = p\,C^{+} + (1-p)\,C^{-}$$

Now the **certainty equivalent** shows up on the left and the **expectation of the option's payoff** on the right side.

To prepare for the following we subtract the **certainty equivalent** from both sides.

$$0 = p\,C^{+} + (1-p)\,C^{-} - (1+r)\,C$$

This is called a **zero cost cash flow.** Indeed, if we use the **certainty equivalent** to adjust the cash flows accordingly, we get a **fairly priced futures contract**.

$$0 = p\,\left(C^{+} - (1+r)\,C\right) + (1-p)\,\left(C^{-} - (1+r)\,C\right)$$

Because its expectation is zero, this a **martingale** (fair game).

Numerical example

- Data:

$$
\begin{array}{lll}
S = 200 & r = (1.06)^{0.25} - 1 & u = 1.25 \\
K = 210 & = 0.01467385 & d = 0.80 \\
T = 0.25 & &
\end{array}
$$

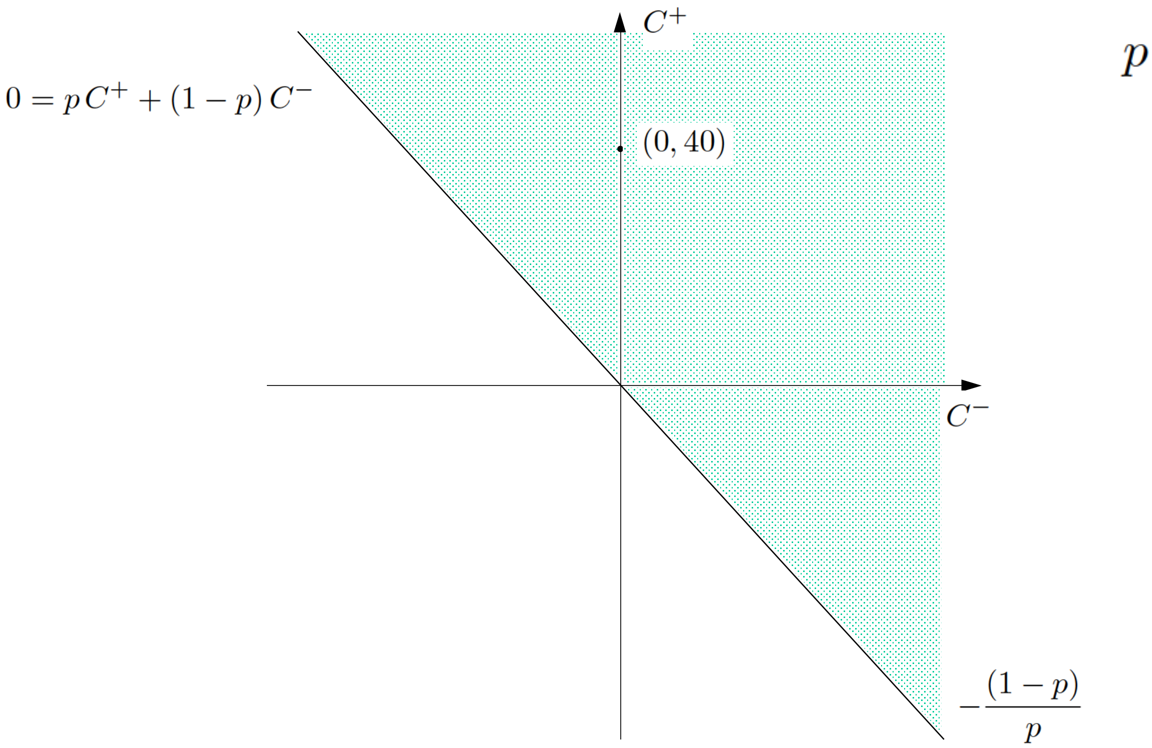
- Risk-adjusted probabilities, also often misleadingly called „risk-neutral“ probabilities

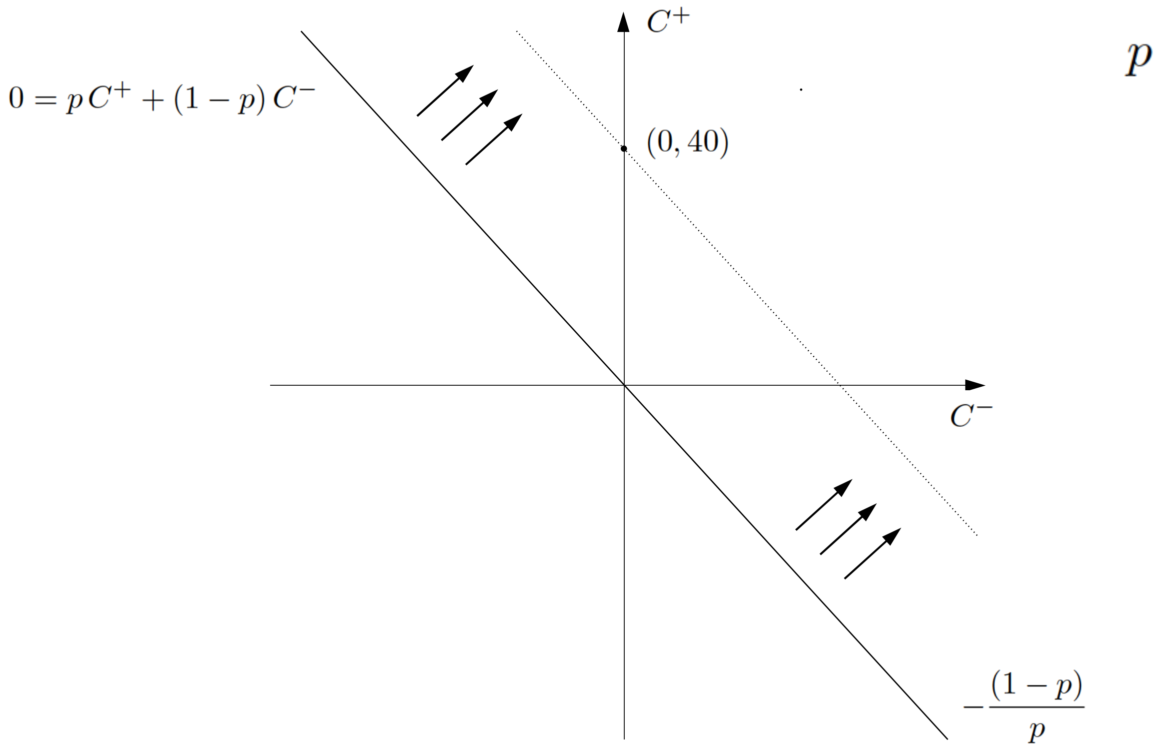
$$p = \left(\frac{1.0147 - 0.80}{1.25 - 0.80}\right) = 0.4771, \qquad 1 - p = 0.5229$$

- Binomal option prices and certainty equivalents

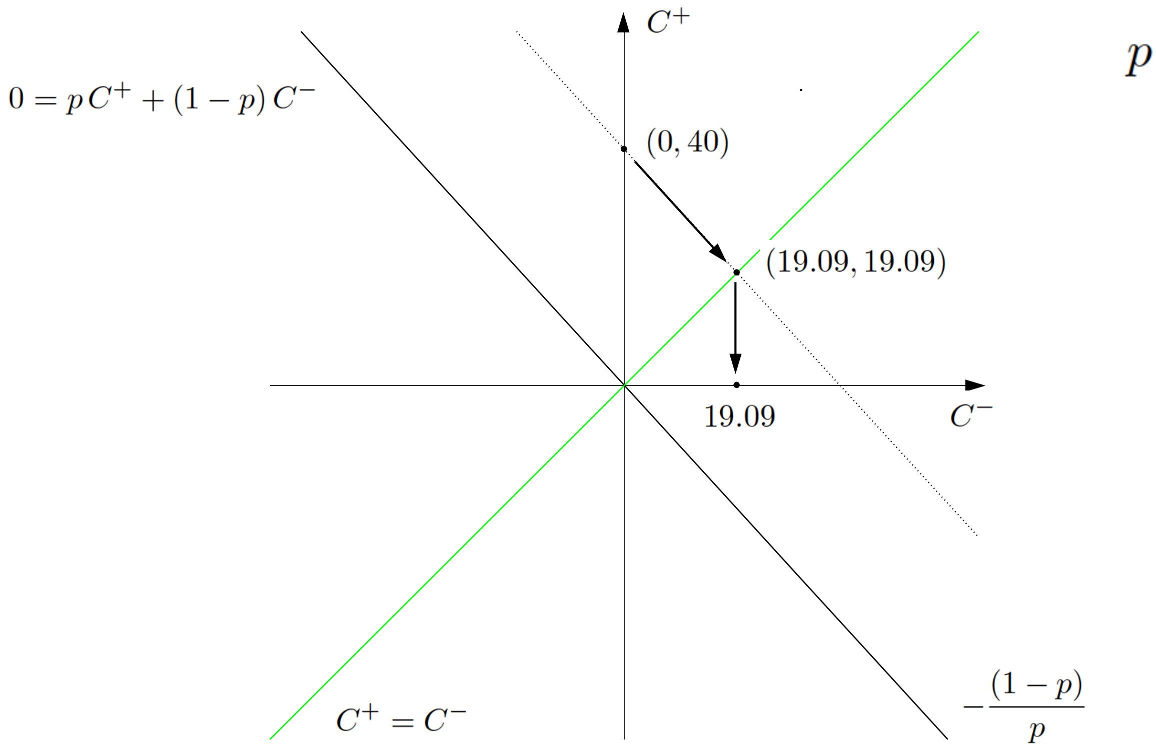
$$
\begin{aligned}
C &= \left(\frac{1}{1.0147}\right)(0.4771 \cdot 40 + 0.5229 \cdot 0) = 18.81 \\
P &= \left(\frac{1}{1.0147}\right)(0.4771 \cdot 0 + 0.5229 \cdot 50) = 25.77
\end{aligned}
\qquad
\begin{aligned}
(1+r)\,C &= 19.09 \\
(1+r)\,P &= 26.15
\end{aligned}
$$

Single-period binomial model





Single-period binomial model



Pricing by duplication …

…works well in (dynamically) **complete** markets.

The construction of a **riskless hedge** portfolio is essential for this method.

It is true for the

1) Binomial models (with traded risk factors)
2) Black-Scholes-Merton model
3) Diffusion models with no jumps, and traded risk factors.

These are **the only models** where pricing by duplication is (theoretically) feasible!

However, there is **an elephant in the room.**

Pricing by duplication …

…is **not feasible** in an **incomplete** market!

There is **no riskless hedge** portfolio in an incomplete market!

This is true for

1) Binomial and diffusion models with risk factors that are not traded.
2) Merton's jump-diffusion model
3) Stochastic volatility models
4) Madan's Variance Gamma model
5) Lévy jump-diffusion models with finite activity
6) Lévy jump-diffusion models with infinite activity
7) Time-changed Lévy models
8) …

Most advanced option pricing models nowadays are dealing with incomplete marktes. So **pricing by duplication seems to be somehow obsolete.**

??

The **elephant in the room** is called the „traditional" risk-neutral world, where the market prices of risk are considered to be simply zero.
(see Hull (2012), 8th ed., p. 630)

Conic pricing,…

… is a concept by **Cherny and Madan (2009, 2010)** which is based on a seminal paper by Artzner et.al. (1998) on coherent risk measures.

…finds the trade direction (buying or selling) to be important in real markets. Hence, there are always **two prices**: **bid and ask**.

Trading takes place against „the market“. However neither the ask nor the bid side of the market is willing to accept a **unique risk measure *p*.**

… starts from the fact, that risk cannot be eliminated completely. Therefore **acceptable risks** must be considered.

To fix an acceptable risk level the ask as well as the bid are **increasing subjectively the probability of a bad outcome.**

…is feasible for most models of uncertainty included in the class of (infinitely divisible) Lévy processes, as long as an expectation exists. This includes, of course, the Black-Scholes-Merton model.

Because bid and ask are on opposite sides of the market - „long“ and „short“ - , **they are concerned about opposite risky outcomes**.

This **risk averse behavior drives** a wedge between the bid and the ask price. For a call we get

$$C_{bid} \quad \leq \quad C \quad \leq \quad C_{ask}$$

The **ask gives away the call** against money and faces the risk, that the stock price goes up afterwards. He or she therefore distorts (increases**) the probability of an up move**.

$$C_{ask} \quad = \quad \frac{1}{1+r}\left(\Psi\left(p\right)C^{+} + \left(1-\Psi\left(p\right)\right)C^{-}\right)$$

The **bid offers to buy the call** for a price and faces the risk, that the stock price goes down afterwards. He or she therefore distorts **the probability of a down move**.

$$C_{bid} \quad = \quad \frac{1}{1+r}\left(\left(1-\Psi\left(1-p\right)\right)C^{+} + \Psi\left(1-p\right)C^{-}\right)$$

Conic binomial model

The **ask gives away the put** and faces the risk, that the stock price decreases afterwards. He or she therefore distorts **the probability of a down move**.

$$P_{ask} \quad = \quad \frac{1}{1+r}\left(\left(1-\Psi\left(1-p\right)\right)P^{+}+\Psi\left(1-p\right)P^{-}\right)$$

The bid offers to buy the put and faces the risk, that the stock price increases afterwards. He or she therefore distorts **the probability of an up move**.

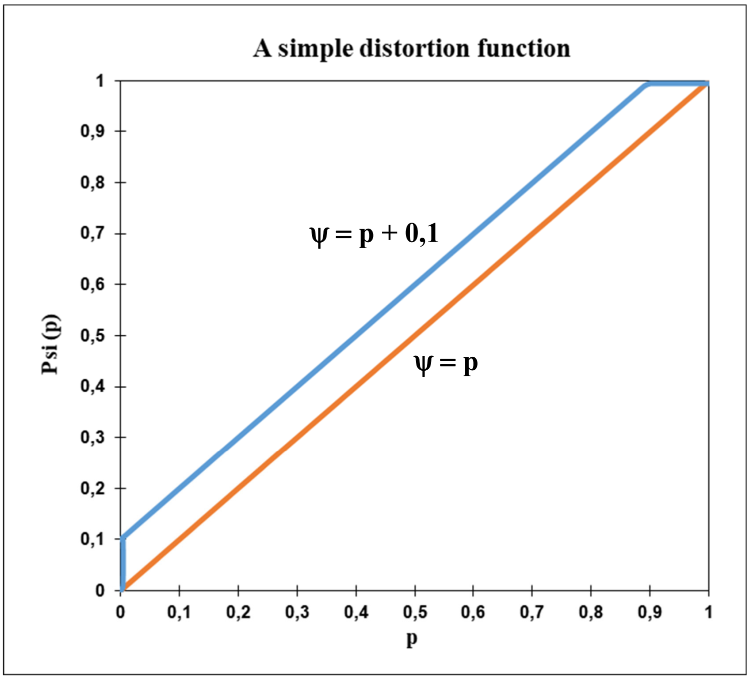
$$P_{bid} \quad = \quad \frac{1}{1+r}\left(\Psi\left(p\right)P^{+}+\left(1-\Psi\left(p\right)\right)P^{-}\right)$$

Again, this risk averse behavior drives a wedge between the bid and the ask price of the put.

$$P_{bid} \quad \leq \quad P \quad \leq \quad P_{ask}$$

A simple distortion function

$$\Psi\left(p\right) \quad = \quad min\left(p+0.1,\ 1\right), \qquad 0 \leq p \leq 1$$



Numerical example cont.

- risk-adjusted probabilities and distortions

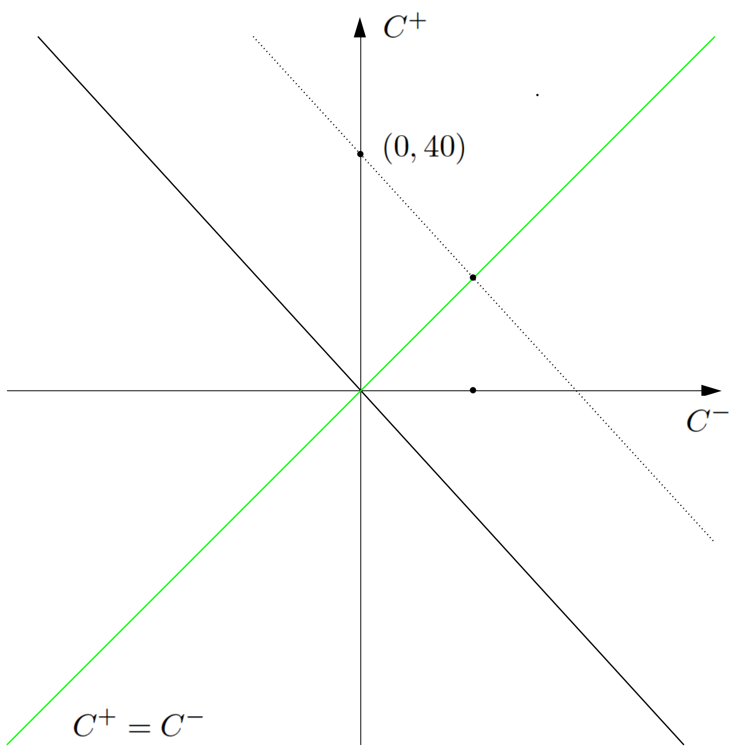
$$
\begin{array}{rclrcl}
p & = & 0.4771 & 1-p & = & 0.5229 \\
\Psi(p) & = & 0.5771 & \Psi(1-p) & = & 0.6229 \\
1-\Psi(p) & = & 0.4229 & 1-\Psi(1-p) & = & 0.3771
\end{array}
$$

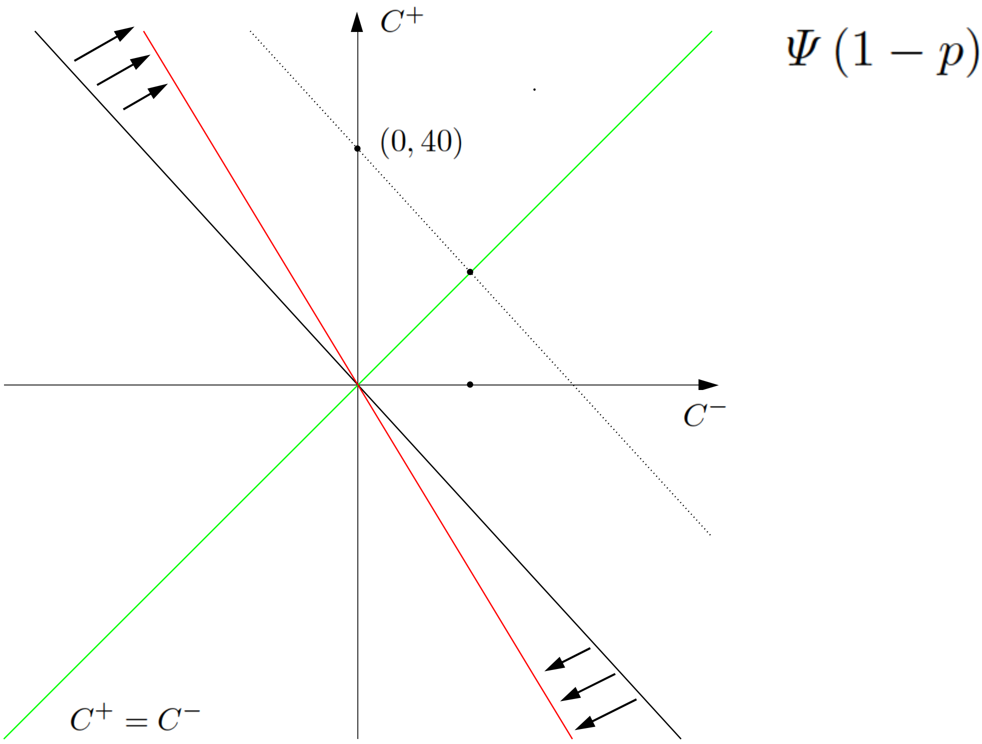
- Conic binomial bid and ask option prices and certainty equivalents

$$
\begin{array}{rclcl}
C_{ask} & = & \left(\frac{1}{1.0147}\right)(0.5771 \cdot 40 + 0.4229 \cdot 0) & = & 22.75 \\
C_{bid} & = & \left(\frac{1}{1.0147}\right)(0.3771 \cdot 40 + 0.6229 \cdot 0) & = & 14.86 \\
P_{ask} & = & \left(\frac{1}{1.0147}\right)(0.3771 \cdot 0 + 0.6229 \cdot 50) & = & 30.69 \\
P_{bid} & = & \left(\frac{1}{1.0147}\right)(0.5771 \cdot 0 + 0.4229 \cdot 50) & = & 20.84
\end{array}
$$

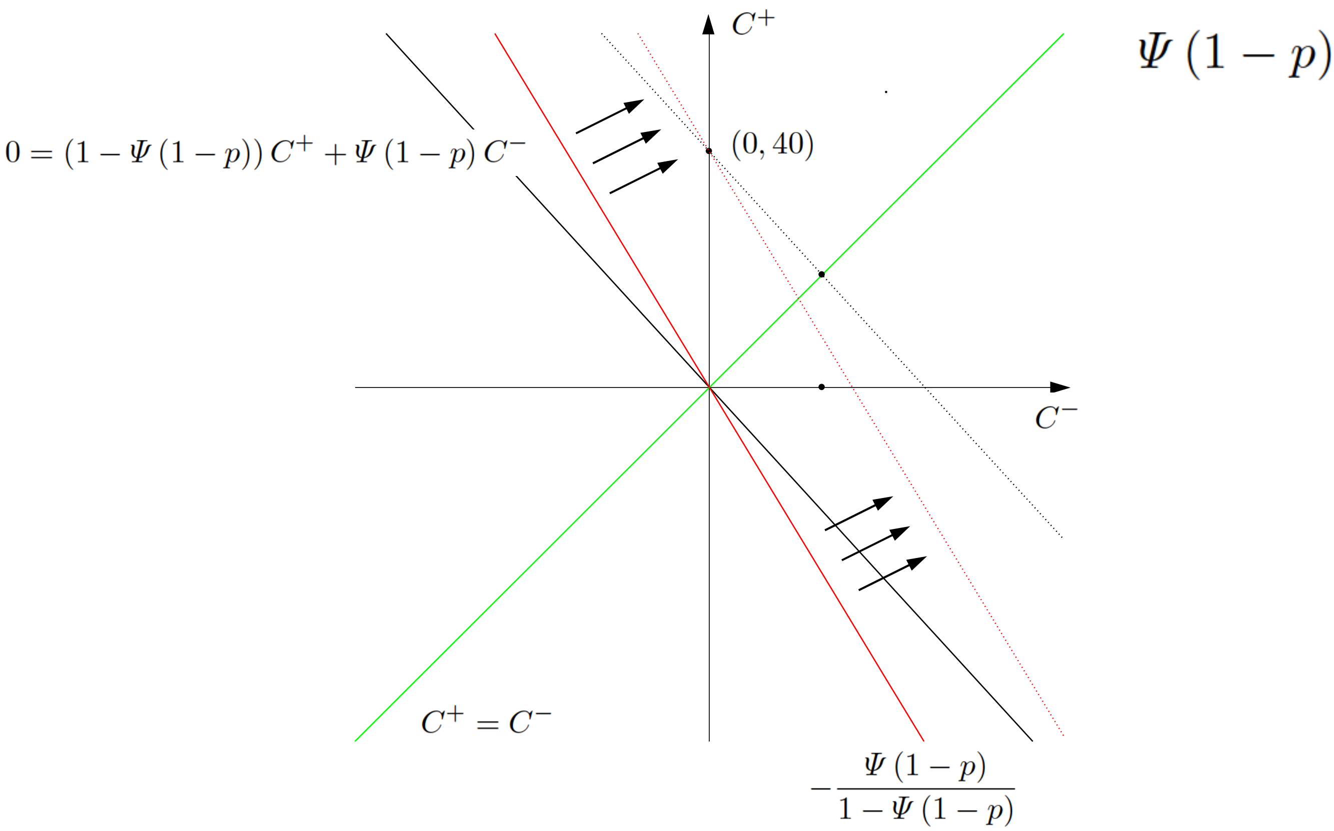
$$
\begin{array}{rcl}
(1+r)\,C_{ask} & = & 23.08 \\
(1+r)\,C_{bid} & = & 15.08 \\
(1+r)\,P_{ask} & = & 31.15 \\
(1+r)\,P_{bid} & = & 21.15
\end{array}
$$

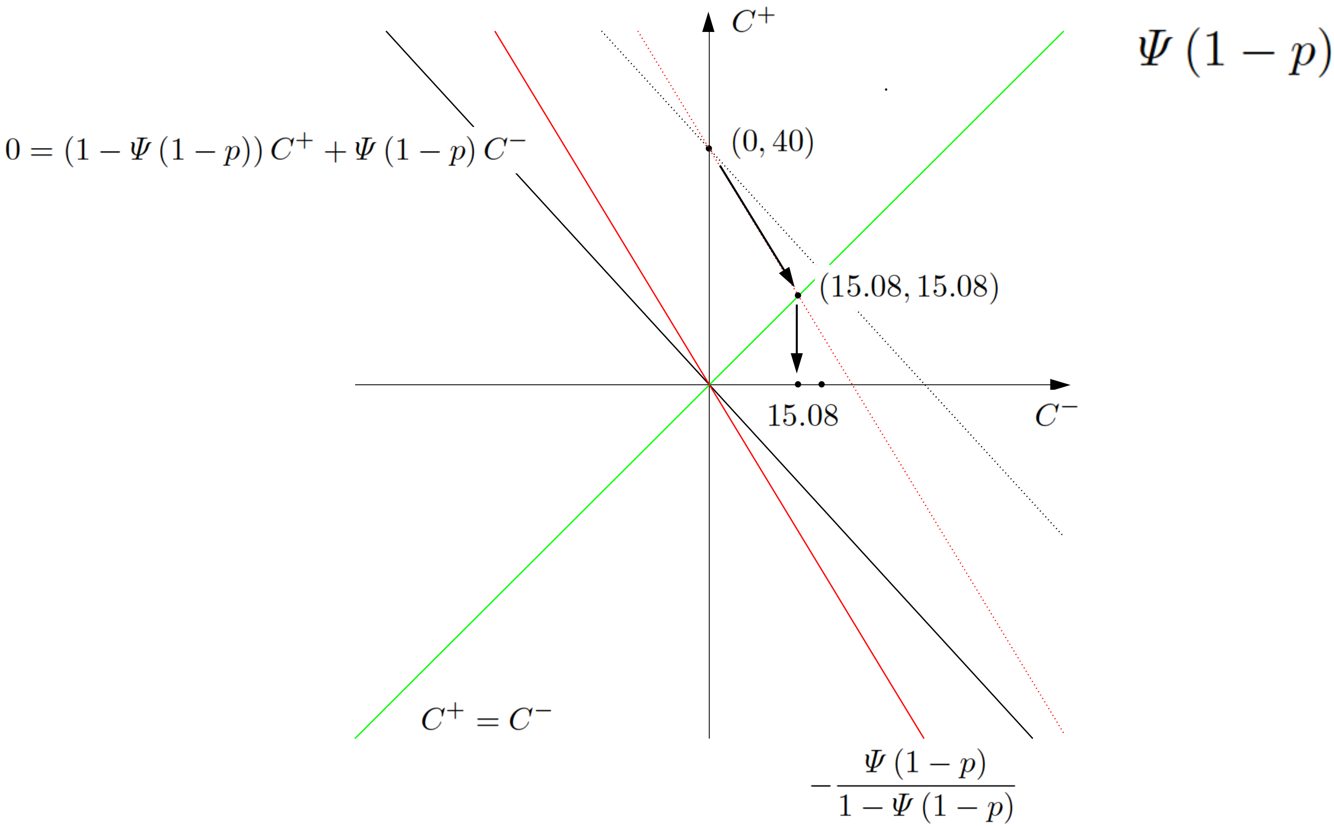
Single-period binomial model



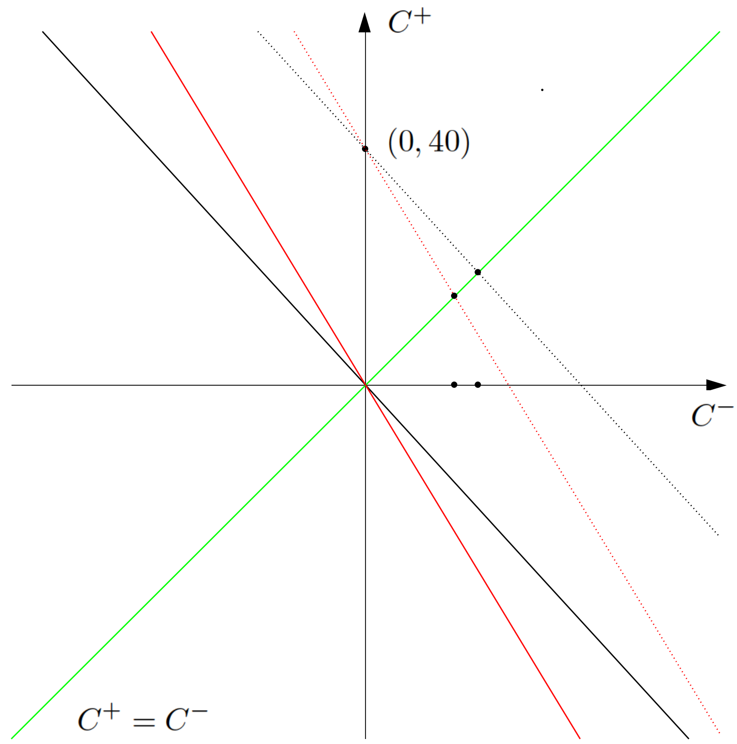


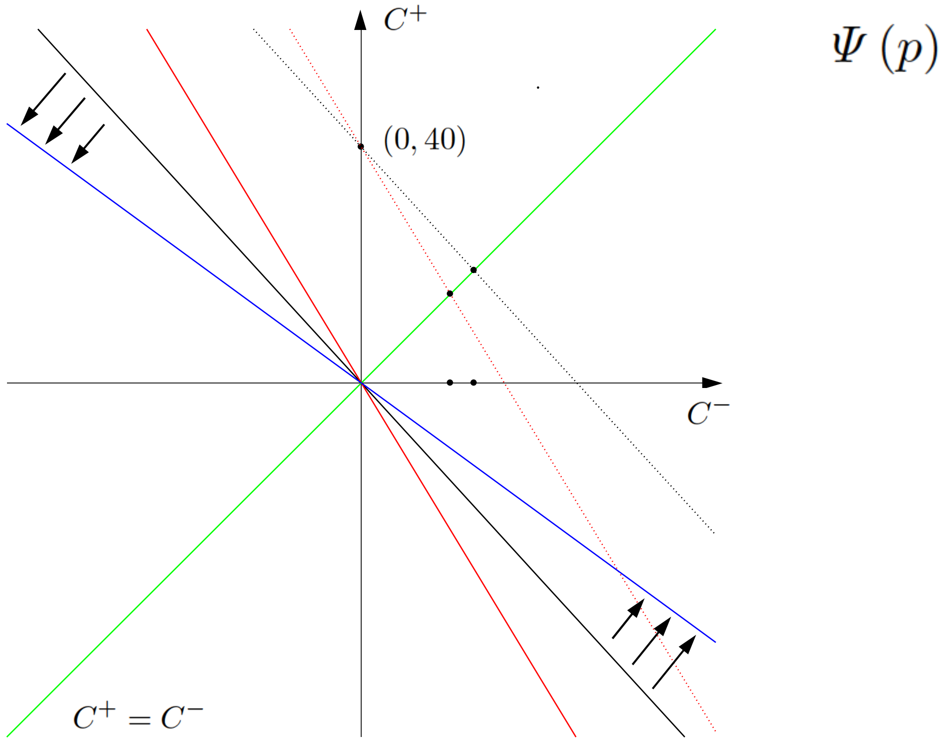
Single-period binomial model



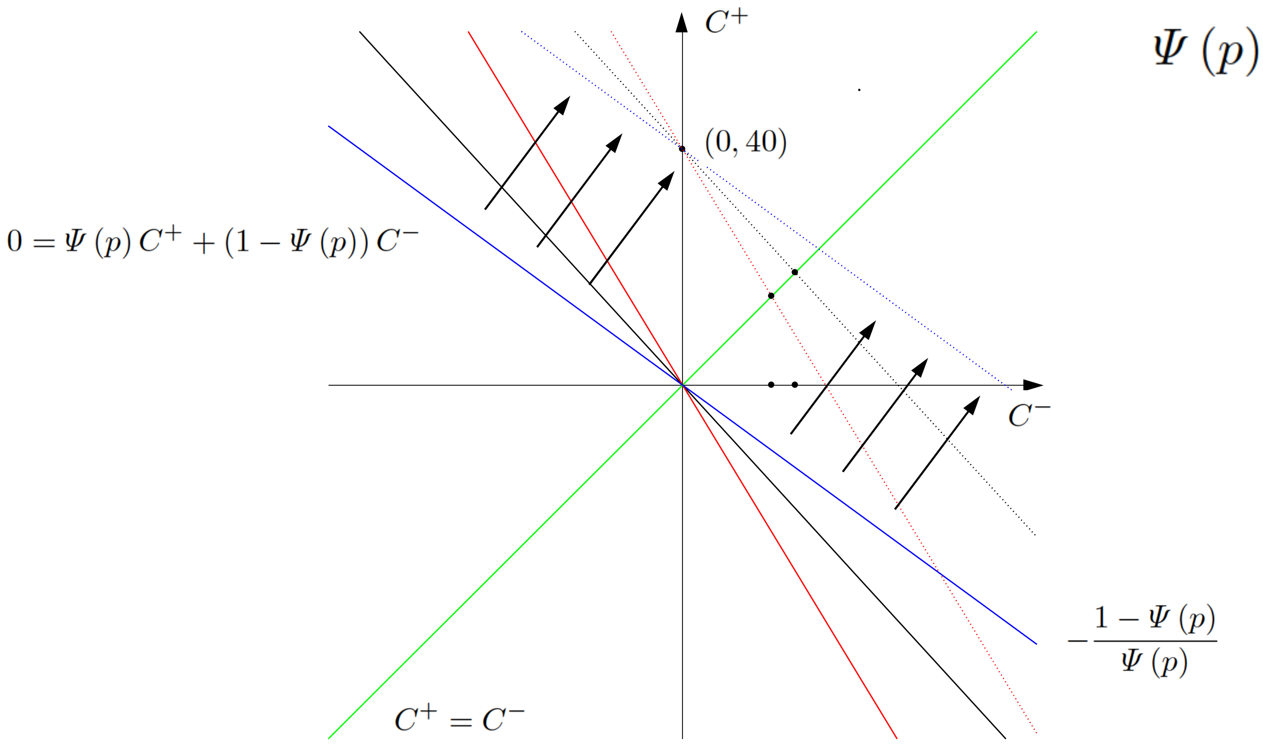


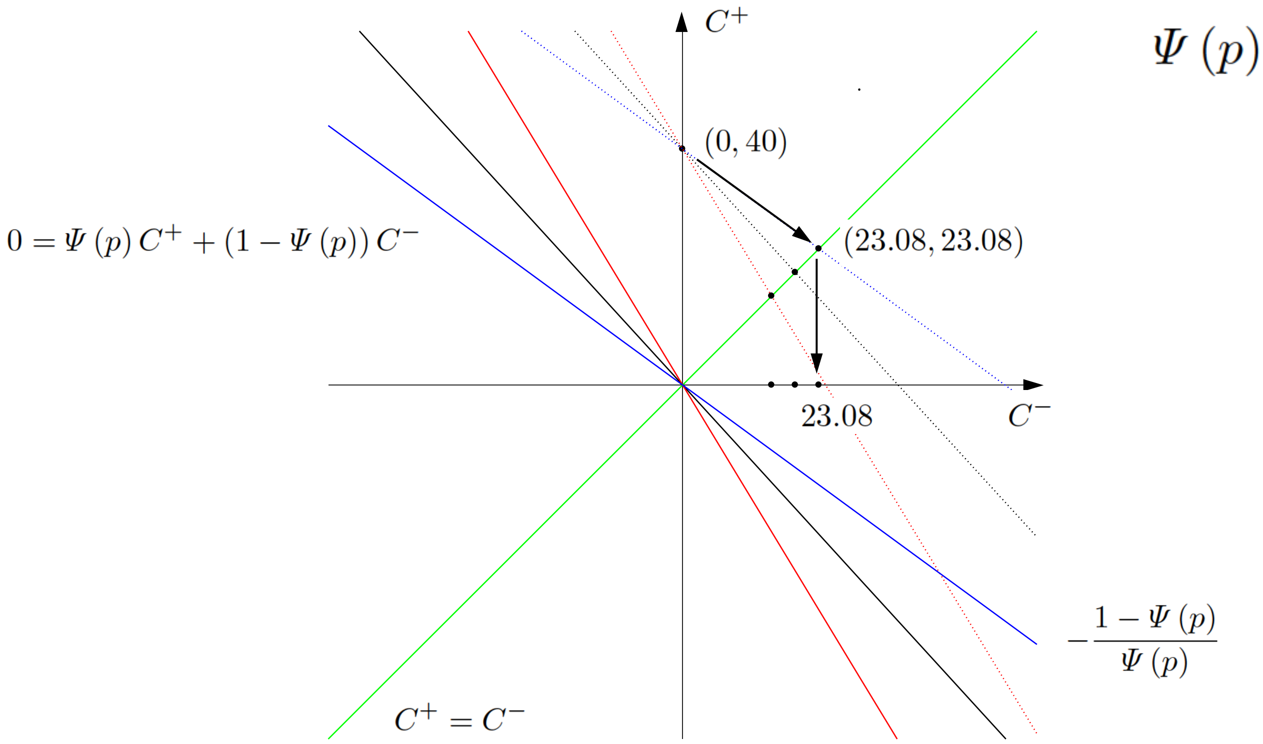
Single-period binomial model

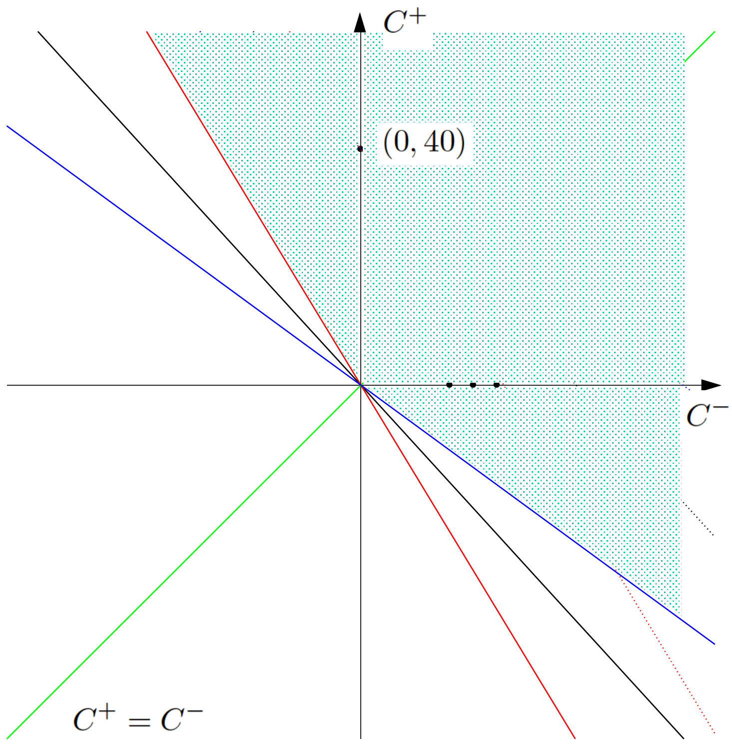




Single-period binomial model




C^+
$\Psi(p)$
$(0, 40)$
$0 = \Psi(p)\, C^+ + (1 - \Psi(p))\, C^-$
$(23.08, 23.08)$
23.08
C^-
$-\frac{1-\Psi(p)}{\Psi(p)}$
$C^+ = C^-$


C^+
$(0, 40)$
C^-
$C^+ = C^-$

„Economists are aware that reality is always more complicated; but they are also aware that to come up with a mathematical model, one always has **to make the world into a bit of a cartoon**.“

David Graeber, *Debt: the first 5,000 years*

The (extended) Black-Scholes-Merton formulas for $t \leq T$:

$$c(S,t) = Se^{(h-r)(T-t)}N(d_1) - Ke^{-r(T-t)}N(d_2)$$

$$p(S,t) = Ke^{-r(T-t)}N(-d_2) - Se^{(h-r)(T-t)}N(-d_1)$$

$$d_1 = \frac{ln(S/K) + \left(h + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}$$

$$d_2 = d_1 - \sigma\sqrt{T-t}$$

Special cases**:**

$h = r$: Plain vanilla stock option without dividends (Black/Scholes (1973))
$h = r - q$: Stock option, with a constant dividend rate (Merton (1973))
$h = r - r_f$: Currency option (Garman/Kohlhagen (1983))

An example:

- Data:

$$\begin{aligned} S &= 200 & r &= ln\,(1.06) \\ K &= 210 & h &= r \\ T &= 0.75 & \sigma^2 &= 0.2 \\ t &= 0 \end{aligned}$$

- The arguments of $N\,(d)$

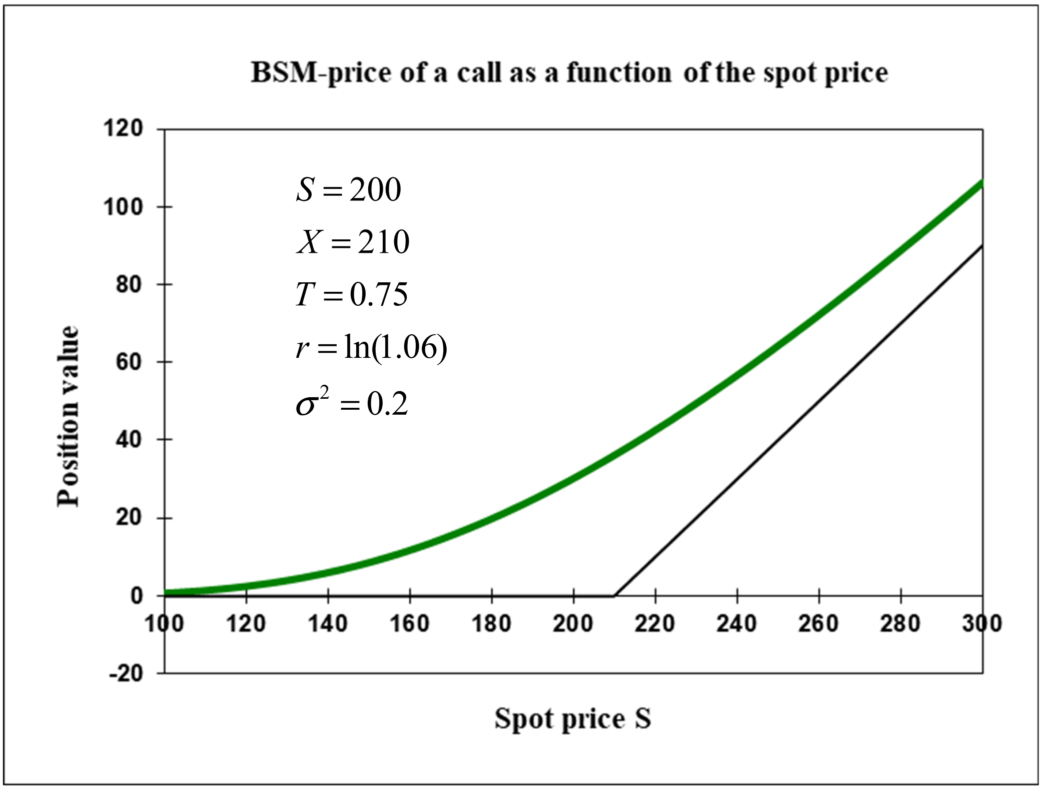
$$\begin{aligned} d_1 &= \frac{ln\,(200/210) + \left(ln\,(1.06) + \frac{1}{2}0.2\right)0.75}{\sqrt{0.2 * 0.75}} \\ &= \frac{-0.04879 + (0.05827 + 0.1)\,0.75}{0.3873} && = 0.1805 \\ d_2 &= d_1 - \sqrt{0.2 * 0.75} && = -0.2068 \end{aligned}$$

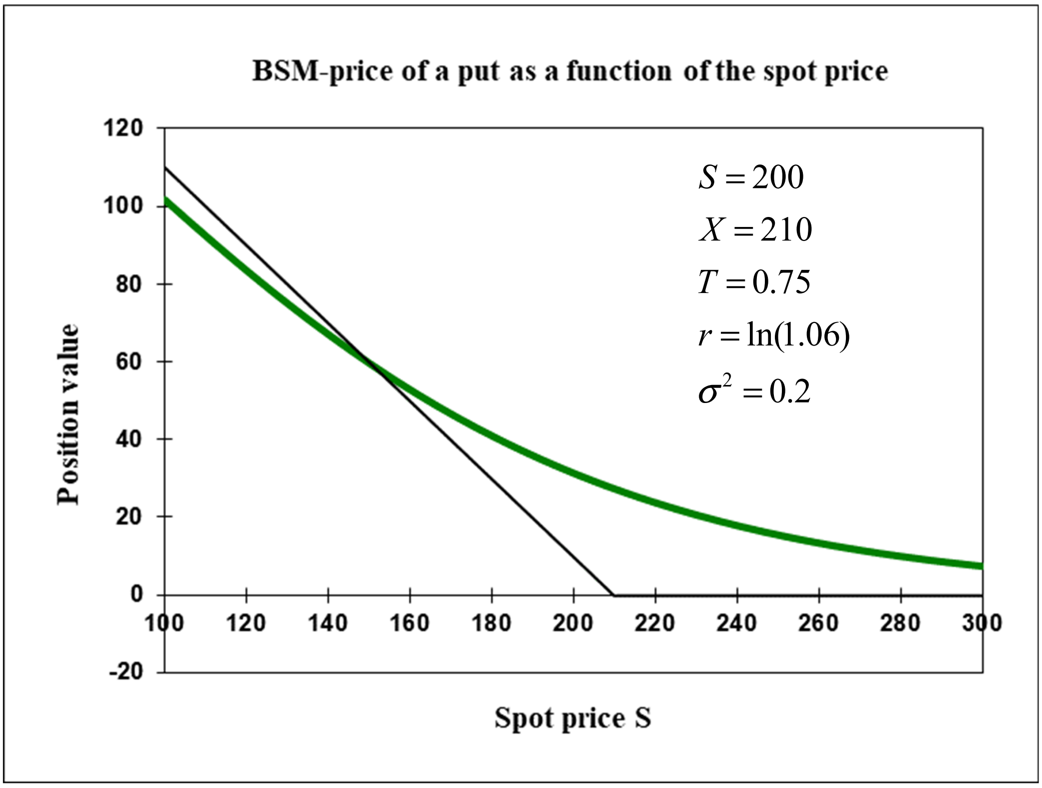
- Determination of the risk-adjusted probabilities $N\,(d)$

$$\begin{aligned} N\,(d_1) &= N\,(0.1805) &&= 0.5716 && N\,(-d_1) &= 0.4284 \\ N(d_2) &= N\,(-0.2068) \\ &= 1 - N\,(0.2086) &&= 0.41805 && N\,(-d_2) &= 0.58195 \end{aligned}$$

- European call and put values: $c\,(S,t)$ and $p\,(S,t)$

$$\begin{aligned} c\,(S,t) &= 200 * 0.5716 - 210 * \left((1.06)^{-0.75}\right) * 0.41805 \\ &= 114.32 - 84.04 && = 30.28 \\ p\,(S,t) &= 210 * \left((1.06)^{-0.75}\right) * 0.58195 - 200 * 0.4284 && = 31.30 \end{aligned}$$


BSM-price of a call as a function of the spot price
$S = 200$
$X = 210$
$T = 0.75$
$r = \ln(1.06)$
$\sigma^2 = 0.2$
Position value
Spot price S
120
100
80
60
40
20
0
-20
100
120
140
160
180
200
220
240
260
280
300


BSM-price of a put as a function of the spot price
$S = 200$
$X = 210$
$T = 0.75$
$r = \ln(1.06)$
$\sigma^2 = 0.2$
Position value
Spot price S
120
100
80
60
40
20
0
-20
100
120
140
160
180
200
220
240
260
280
300

Unfortunately the BSM-formula seems to be not suitable for distortion. It is not clear, how to proceed with the partial means $N(d_1)$ and $N(-d_1)$, and the probabilities $N(d_2)$ and $N(-d_2)$, respectively.

$$
\begin{aligned}
c(S,t) &= Se^{(h-r)(T-t)}N(d_1) - Ke^{-r(T-t)}N(d_2) \\
p(S,t) &= Ke^{-r(T-t)}N(-d_2) - Se^{(h-r)(T-t)}N(-d_1) \\
d_1 &= \frac{ln(S/K) + \left(h + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}} \\
d_2 &= d_1 - \sigma\sqrt{T-t}
\end{aligned}
$$

At least we know, that for the call $N(d_2)$ is the **probability to end up in the money** under the martingale measure. The same applies for the put and $N(-d_2)$.

Using the Breeden-Litzenberger approach

European call value, given **any arbitrary density** for the underlying stock:

$$c(K) = e^{-r(T-t)} \int_K^{\infty} (\xi - K) f(\xi)\, d\xi.$$

1st derivative of the call with respect to *K:*

$$c'(K) = -e^{-r(T-t)} \int_K^{\infty} f(\xi)\, d\xi.$$

2nd derivative of the call with respect to *K:*

$$c''(K) = e^{-r(T-t)} f(K).$$

The 1st derivative is of special interest here

$$c'(K) = -e^{-r(T-t)} \int_K^\infty f(\xi)\, d\xi .$$

For **lognormally distributed** stock prices we get

$$c'(K) = -e^{-r(T-t)} \left(N\left(d_2(\infty)\right) - N\left(d_2(K)\right) \right).$$

Note, that

$$d_2(x) = \frac{lnS - lnx + \left(h - \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}} \quad \text{and} \quad N\left(d_2(\infty)\right) = N(-\infty) = 0 .$$

Hence the 1st derivative is the **discounted probability** of the call **to end up in the money**:

$$c'(K) = e^{-r(T-t)} N\left(d_2(K)\right).$$

Integrating back, we find

$$c(K) = e^{-r(T-t)} \int_K^\infty N(d_2(x))\, dx$$

or

$$c(K) = e^{-r(T-t)} \int_0^\infty N(d_2(K+x))\, dx ,$$

In any case, it is **a discounted integral along the probability to end up in the money!**

Trivially, we get the same result by integrating over

$$c(K) = e^{-r(T-t)} \int_0^\infty \left(1 - \left(1 - N(d_2(K+x))\right)\right) dx .$$

However, in the conic two-price world this is not the same any more.

The **ask goes short** and faces the risk, that **the call ends up in the money** and will be exercised against him. He or she will therefore distort (increase) the probability

$$N(d_2) \Rightarrow \Psi_\lambda(N(d_2)).$$

$$c_{ask}(K) = e^{-r(T-t)} \int_0^\infty \Psi_\lambda(N(d_2(K+x)))\, dx.$$

The **bid goes long** and fears the risk, that **the call will end up out of the money**. He or she distorts therefore the corresponding probability

$$1 - N(d_2) \Rightarrow \Psi_\lambda(1 - N(d_2)).$$

$$c_{bid}(K) = e^{-r(T-t)} \int_0^\infty (1 - \Psi_\lambda(1 - N(d_2(K+x))))\, dx.$$

The **ask goes short in the put** and faces the risk, that it **ends up in the money** and will be exercised against him. He or she will therefore distort the probability

$$N(-d_2) \Rightarrow \Psi_\lambda(N(-d_2)).$$

$$p_{ask}(K) = e^{-r(T-t)} \int_0^K \Psi_\lambda(N(-d_2(K-x)))\, dx.$$

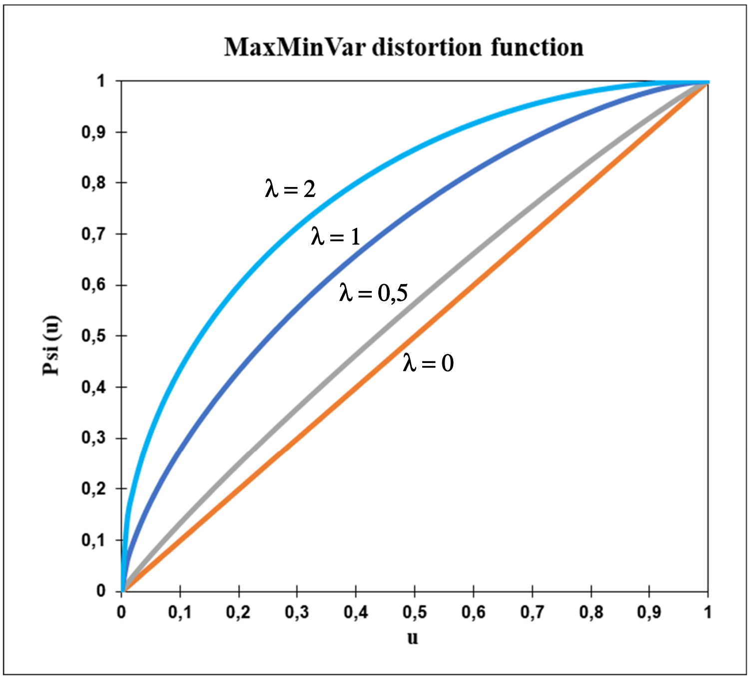
The **bid goes long in the put** an fears the risk, that it will **end up out of the money**. He or she distorts therefore the corresponding probability

$$1 - N(-d_2) \Rightarrow \Psi_\lambda(1 - N(-d_2)).$$

$$p_{bid}(K) = e^{-r(T-t)} \int_0^K (1 - \Psi_\lambda(1 - N(-d_2(K-x))))\, dx.$$

Distortion functions

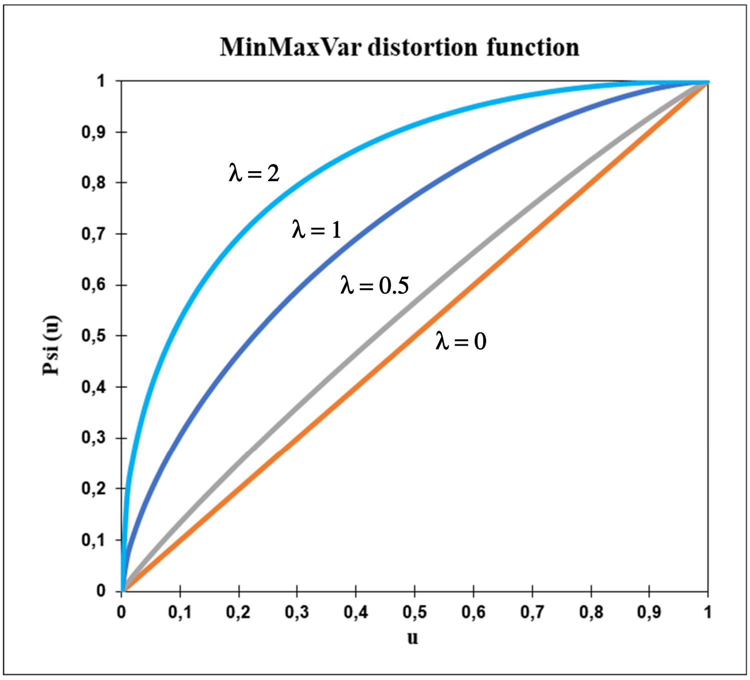
$$\Psi_\lambda^{MaxMinVar}(u) = \left(1-(1-u)^{1+\lambda}\right)^{\frac{1}{1+\lambda}}, \qquad \lambda \geq 0$$



Cerny and Madan (2009)

Distortion functions

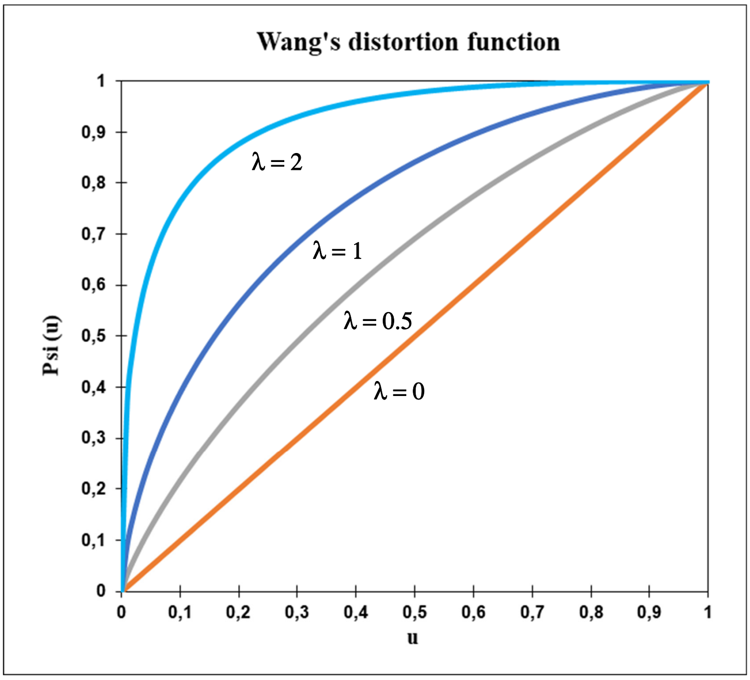
$$\Psi_\lambda^{MinMaxVar}(u) = 1-\left(1-u^{\frac{1}{1+\lambda}}\right)^{1+\lambda}, \qquad \lambda \geq 0$$



Cerny and Madan (2009)

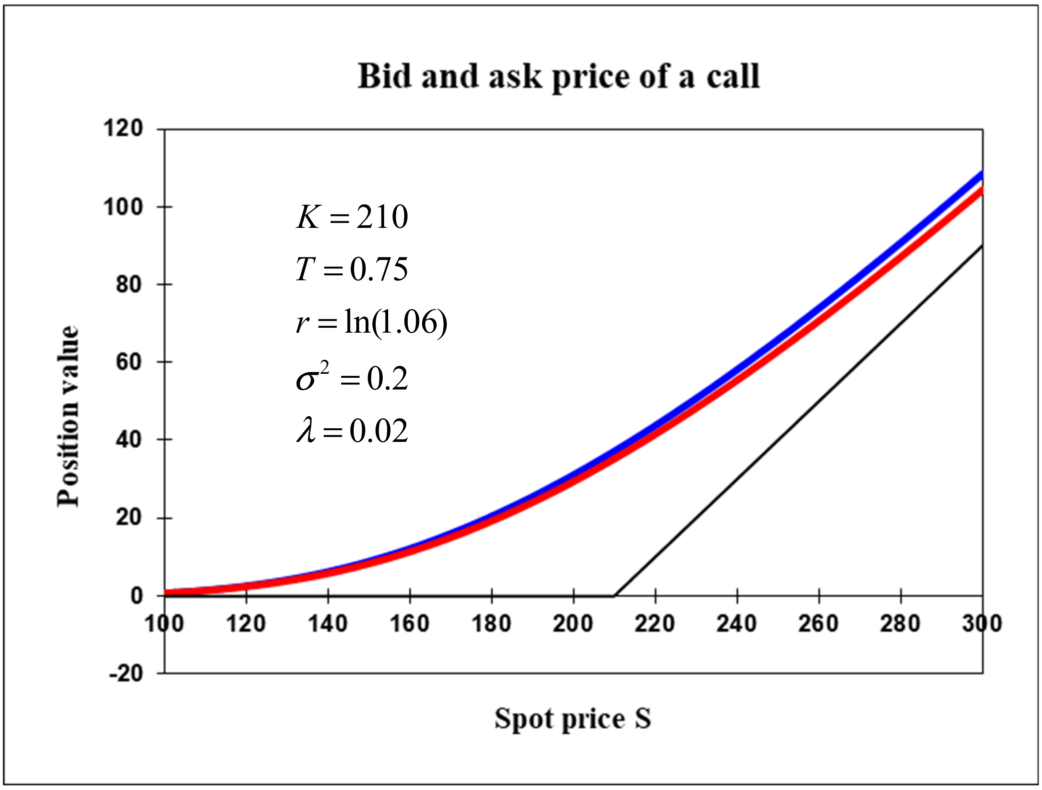
Distortion functions

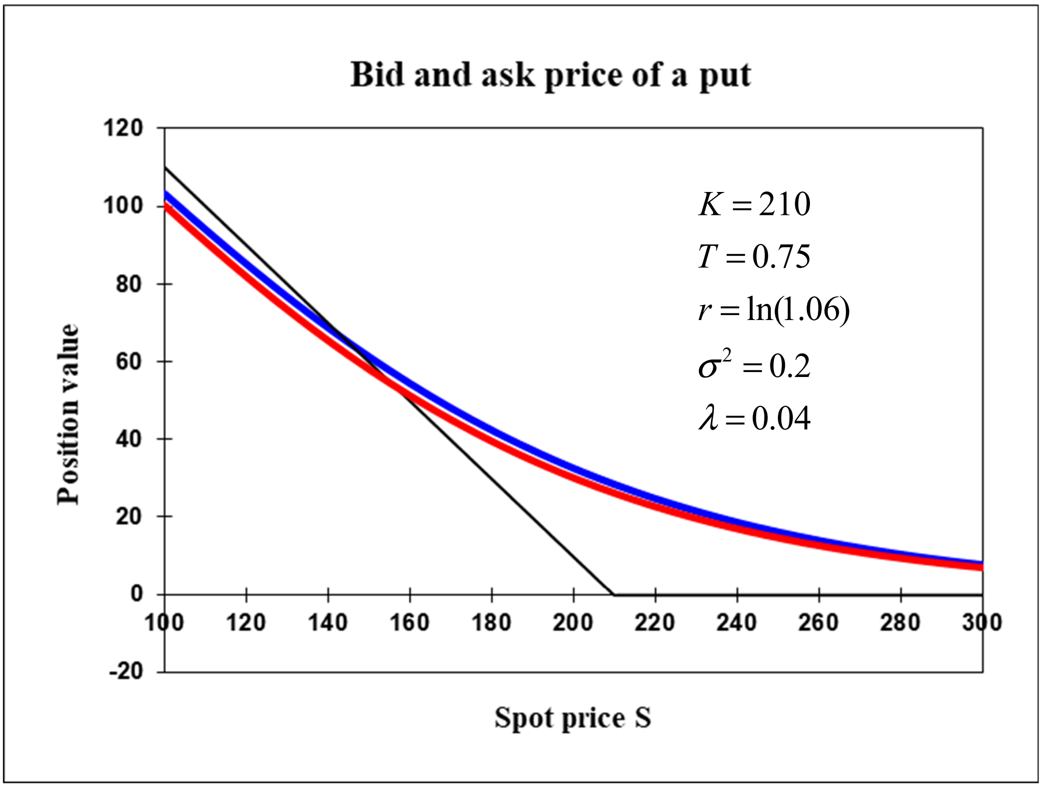
$$\Psi_\lambda^{Wang}(u) = \mathcal{N}\left(\mathcal{N}^{-1}(u) + \lambda\right), \qquad \lambda \geq 0$$

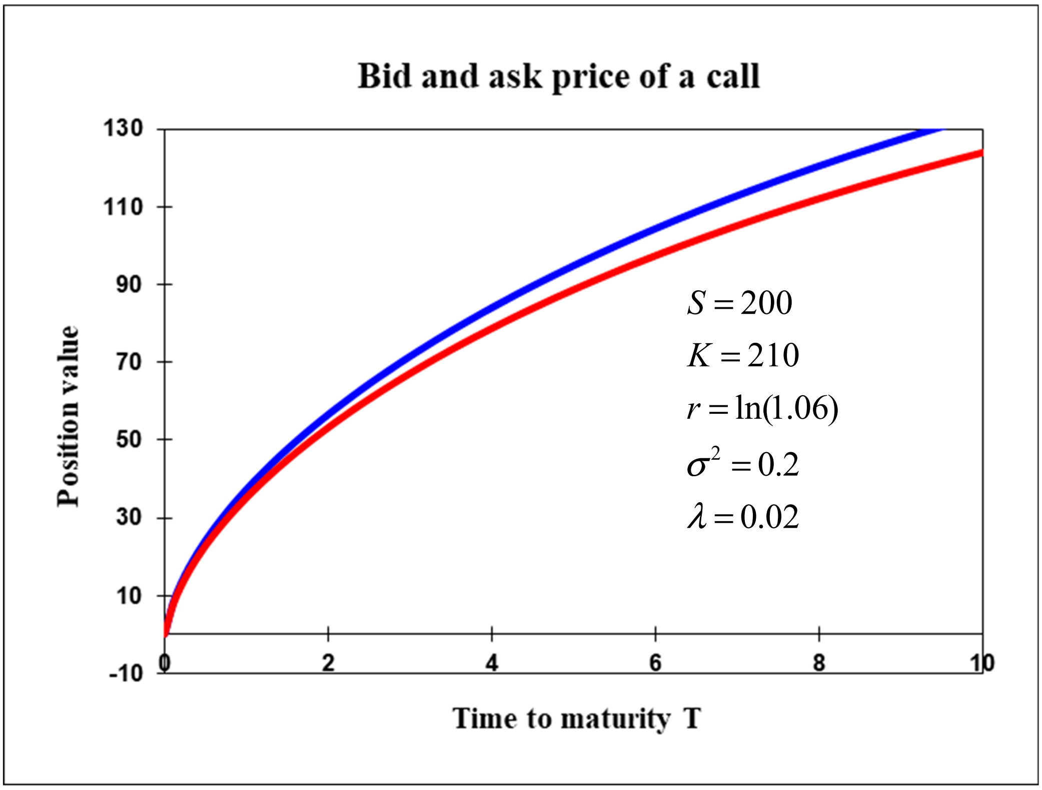


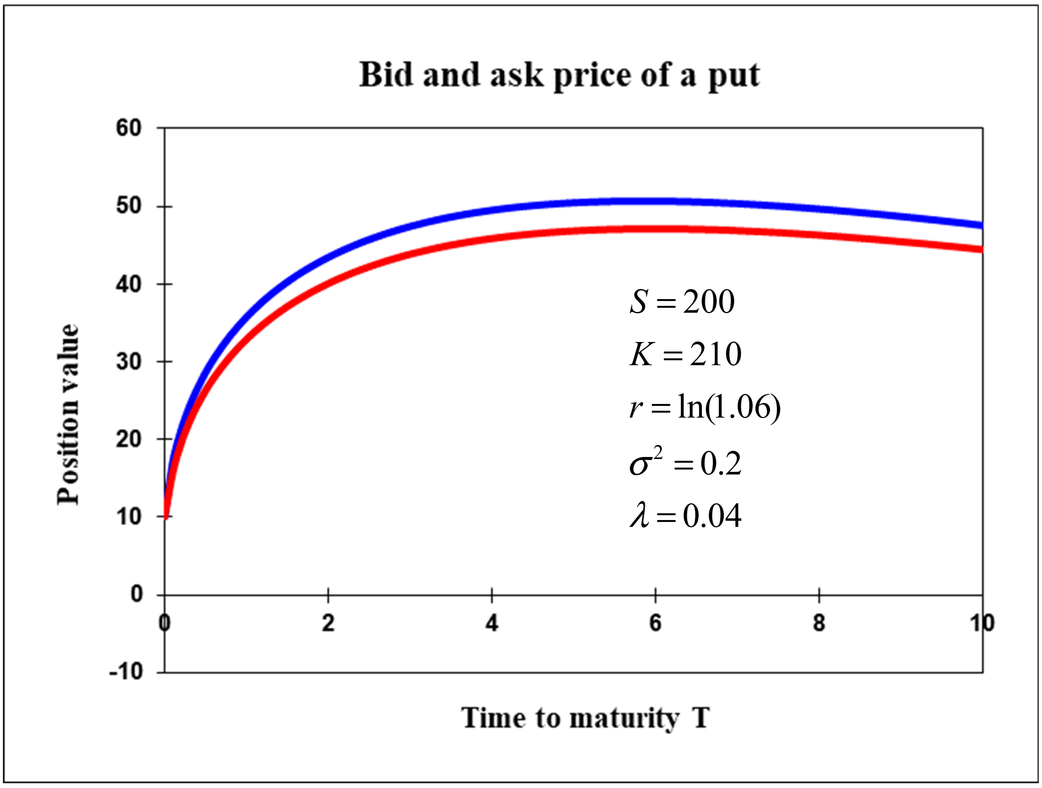
Wang (2000)

Conic Black-Scholes-Merton model




Bid and ask price of a put
120
100
80
60
40
20
0
-20
100
120
140
160
180
200
220
240
260
280
300
Position value
Spot price S
$K = 210$
$T = 0.75$
$r = \ln(1.06)$
$\sigma^2 = 0.2$
$\lambda = 0.04$


Bid and ask price of a call
130
110
90
70
50
30
10
-10
0
2
4
6
8
10
Position value
Time to maturity T
$S = 200$
$K = 210$
$r = \ln(1.06)$
$\sigma^2 = 0.2$
$\lambda = 0.02$



Our stochastic volatility model (Schöbel and Zhu (1999)) …

$$dS = rS\,dt + vS\,dw_1$$

$$dv = \kappa\,(\theta - v)\;dt + \sigma\,dw_2$$

$$dw_1 dw_2 = \rho\,dt$$

… has a quite general solution

$$c\,(K) = SF_1\,(K) - Ke^{-r(T-t)}F_2\,(K)$$

with a quite general **probability to end up in the money**

$$F_2\,(K) = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty Re\left(f_2\,(\phi)\,\frac{exp\,(-i\phi\,ln\,K)}{i\phi}\right)\,d\phi$$

But the special part is the **characteristic function!**

Our stochastic volatility model has the characteristic function

$$f_2(\phi) = \mathbb{E}^Q\left[\exp\{i\phi x(T)\}\right]$$

$$= \exp\left\{i\phi\left(r\left(T-t\right)+x(t)\right)-\tfrac{1}{2}i\phi\rho\left[\sigma^{-1}v^2(t)+\sigma\left(T-t\right)\right]\right\}\times$$

$$\times\exp\left\{\tfrac{1}{2}D(t,T;\hat{s}_1,\hat{s}_3)v^2(t)+B\left(t,T;\hat{s}_1,\hat{s}_2,\hat{s}_3\right)v(t)+C(t,T;\hat{s}_1,\hat{s}_2,\hat{s}_3)\right\}$$

with constants …

$$\hat{s}_1 = \tfrac{1}{2}\phi^2\left(1-\rho^2\right)+\tfrac{1}{2}i\phi\left(1-2\kappa\rho\sigma^{-1}\right),$$

$$\hat{s}_2 = i\phi\kappa\theta\rho\sigma^{-1},$$

$$\hat{s}_3 = \tfrac{1}{2}i\phi\rho\sigma^{-1}.$$

… with auxiliary functions

$$D(t,T) = \frac{1}{\sigma^2}\left(\kappa-\gamma_1\frac{\sinh\{\gamma_1(T-t)\}+\gamma_2\cosh\{\gamma_1(T-t)\}}{\cosh\{\gamma_1(T-t)\}+\gamma_2\sinh\{\gamma_1(T-t)\}}\right)$$

$$B(t,T) = \frac{1}{\sigma^2\gamma_1}\left(\frac{(\kappa\theta\gamma_1-\gamma_2\gamma_3)+\gamma_3\left(\sinh\{\gamma_1(T-t)\}+\gamma_2\cosh\{\gamma_1(T-t)\}\right)}{\cosh\{\gamma_1(T-t)\}+\gamma_2\sinh\{\gamma_1(T-t)\}}-\kappa\theta\gamma_1\right)$$

$$C(t,T) = -\tfrac{1}{2}\ln\left(\cosh\{\gamma_1(T-t)\}+\gamma_2\sinh\{\gamma_1(T-t)\}\right)+\tfrac{1}{2}\kappa(T-t)+$$

$$+\frac{(\kappa^2\theta^2\gamma_1^2-\gamma_3^2)}{2\sigma^2\gamma_1^3}\left(\frac{\sinh\{\gamma_1(T-t)\}}{\cosh\{\gamma_1(T-t)\}+\gamma_2\sinh\{\gamma_1(T-t)\}}-\gamma_1(T-t)\right)+$$

$$+\frac{(\kappa\theta\gamma_1-\gamma_2\gamma_3)\,\gamma_3}{\sigma^2\gamma_1^3}\left(\frac{\cosh\{\gamma_1(T-t)\}-1}{\cosh\{\gamma_1(T-t)\}+\gamma_2\sinh\{\gamma_1(T-t)\}}\right)$$

and more constants

$$\gamma_1=\sqrt{2\sigma^2 s_1+\kappa^2},\qquad \gamma_2=\frac{1}{\gamma_1}\left(\kappa-2\sigma^2 s_3\right),\qquad \gamma_3=\kappa^2\theta-s_2\sigma^2.$$

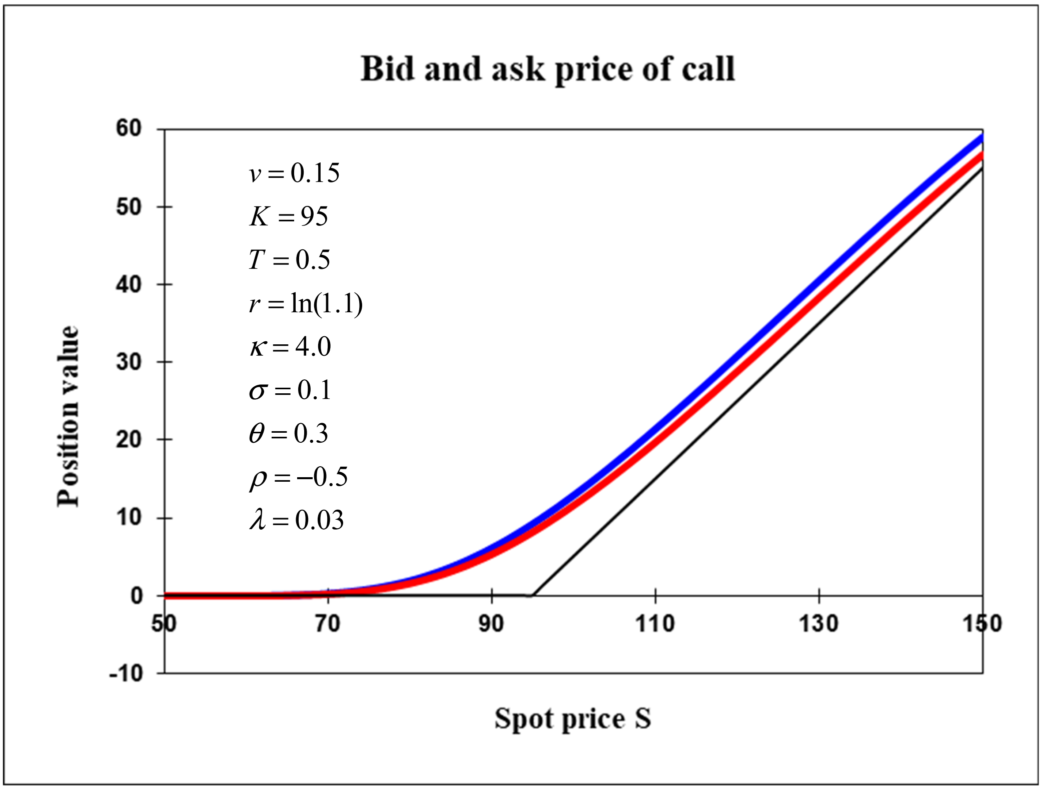
For the distorted option prices we may now easily apply our results from above. („cum grano salis")

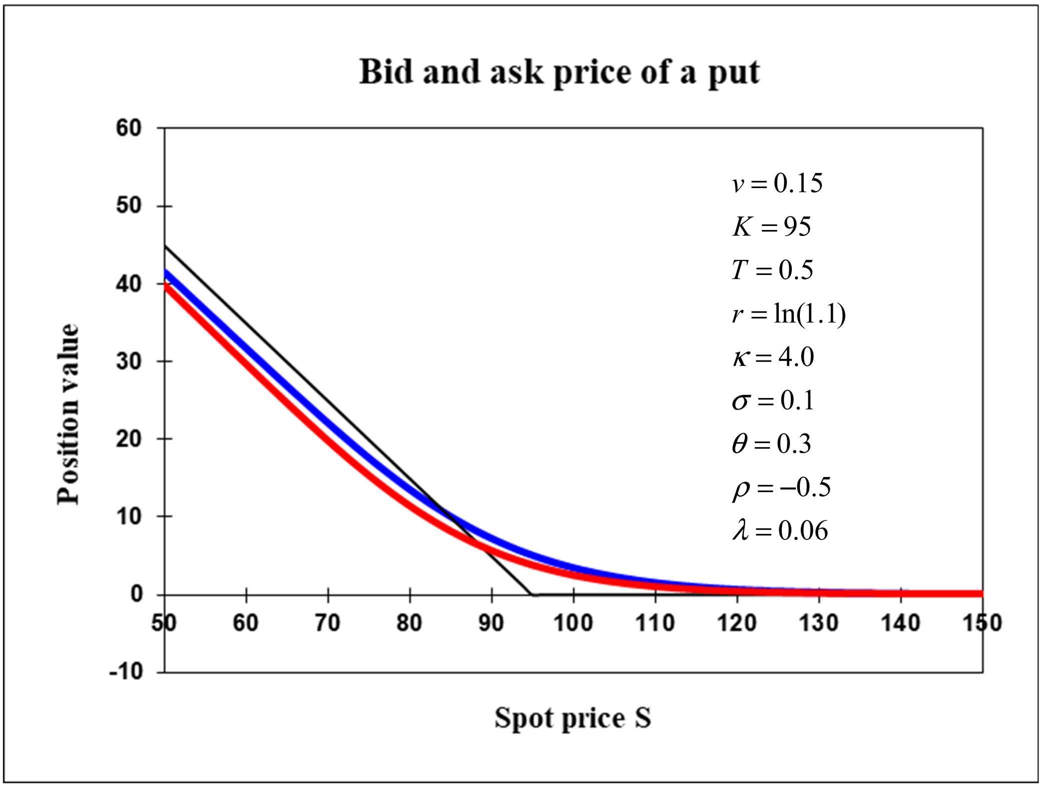
$$c_{ask}(K) = e^{-r(T-t)} \int_0^\infty \Psi_\lambda \left(F_2 (K+x)\right) dx$$

$$c_{bid}(K) = e^{-r(T-t)} \int_0^\infty \left(1 - \Psi_\lambda \left(1 - F_2 (K+x)\right)\right) dx$$

$$p_{ask}(K) = e^{-r(T-t)} \int_0^K \Psi_\lambda \left(1 - F_2 (K-x)\right) dx$$

$$p_{bid}(K) = e^{-r(T-t)} \int_0^K \left(1 - \Psi_\lambda \left(F_2 (K-x)\right)\right) dx$$





My conclusion

The era of pricing by duplication in (dynamically) complete markets, using a **riskless hedge** portfolio to justify a preference free pricing relationship **will be over soon**.

Instead of hiding the fact, that a riskless hedge arises only in very special and unrealistic model situations, a **robust theory of derivative pricing** should be based on the fact that there is **always some risk remaining**.

Conic finance uses the **concept of acceptable risks** to overcome the shortcomings of the classical approach. **This could be the dawn of a new and more consistent foundation of financial engineering.**