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UNIQUENESS, DEFINABILITY AND INTERPOLATION

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This paper is meant to be a comment on Beth's definability theorem. In it we shall make the following points.

Implicit definability as mentioned in Beth's theorem for first-order logic is a special case of a more general notion of uniqueness. If α is a nonlogical constant, T_α a set of sentences, α^* an additional constant of the same syntactical category as α and T_{α^*} a copy of T_α with α^* instead of α, then for *implicit definability* of α in T_α one has, in the case of predicate constants, to derive $\alpha(x_1,\ldots,x_n) \leftrightarrow \alpha^*(x_1,\ldots,x_n)$ from $T_\alpha \cup T_{\alpha^*}$, and similarly for constants of other syntactical categories. For *uniqueness* one considers sets of *schemata* S_α and derivability from instances of $S_\alpha \cup S_{\alpha^*}$ in the language with both α and α^*, thus allowing mixing of α and α^* not only in logical axioms and rules, but also in nonlogical assumptions. In the first case, but not necessarily in the second one, explicit definability follows. It is crucial for Beth's theorem that mixing of α and α^* is allowed only inside logic, not outside. This topic will be treated in §1.

Let the structural part of logic be understood roughly in the sense of Gentzen-style proof theory, i.e. as comprising only those rules which do not specifically involve logical constants. If we restrict mixing of α and α^* to the structural part of logic which we shall specify precisely, we obtain a different notion of implicit definability for which we can demonstrate a general definability theorem, where α is not confined to the syntactical categories of nonlogical expressions of first-order logic. This definability theorem is a consequence of an equally general interpolation theorem. This topic will be treated in §§2, 3, and 4.

Finally, in §5 we shall show that under certain conditions, which in particular obtain in the case of implicit definability in the usual first-order case, the mixing of α and α^* in logic can be reduced to their mixing in the structural part of logic, which makes Beth's definability theorem a consequence of our general definability theorem.

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§1. Implicit definability and uniqueness. Let L_α be the set of formulae of a first-order language including a nonlogical constant α; let L_{α^*} differ from L_α by having instead of α the nonlogical constant α^* of the same syntactical category as α; and let $L_{\alpha\alpha^*}$ be the set of formulae of the first-order language with both α and α^*. By $\mathrm{PC}(L_\alpha)$ we shall denote the axioms and rules of the first-order predicate calculus in the language of L_α, and similarly for $\mathrm{PC}(L_{\alpha^*})$ and $\mathrm{PC}(L_{\alpha\alpha^*})$. Let T_α be a set of sentences of L_α, and let T_{α^*} be obtained from T_α by uniformly substituting α^* for α. Furthermore, let $A(\alpha)$ be a formula of L_α in which α may occur, and let $A(\alpha^*)$ be obtained from $A(\alpha)$ by uniformly substituting α^* for α. Then we introduce the following definitions:

DEFINITION 1. The constant α is *implicitly definable* in T_α iff for every $A(\alpha)$, from $\mathrm{PC}(L_{\alpha\alpha^*}) \cup T_\alpha \cup T_{\alpha^*}$ we can prove $A(\alpha) \leftrightarrow A(\alpha^*)$.

DEFINITION 2. The constant α is *explicitly definable* in T_α iff for every $A(\alpha)$ there is a formula B in L_α without α such that from $\mathrm{PC}(L_\alpha) \cup T_\alpha$ we can prove $A(\alpha) \leftrightarrow B$.

Beth's definability theorem states that these two notions of definability are equivalent, i.e., α is implicitly definable in T_α iff α is explicitly definable in T_α. That explicit definability implies implicit definability is rather trivial. The proof of the converse usually proceeds via *Craig's interpolation lemma*, which implies the following: If from $\mathrm{PC}(L_{\alpha\alpha^*})$ we can prove $A \to B^*$ for A in L_α and B^* in L_{α^*}, then there is a C in $L_\alpha \cap L_{\alpha^*}$ such that from $\mathrm{PC}(L_{\alpha\alpha^*})$ we can prove $A \to C$ and $C \to B^*$.

In Definition 1 we could have written only $A(\alpha) \to A(\alpha^*)$ instead of $A(\alpha) \leftrightarrow A(\alpha^*)$. It is clear that if we have $A(\alpha) \to A(\alpha^*)$ we have also its converse. Furthermore, since we have standard theorems for replacement of equivalents and identicals at our disposal, we might have restricted $A(\alpha)$ above to

$$
\begin{array}{ll}
x_1 = \alpha & \text{if } \alpha \text{ is an individual constant,}\\
x_k = \alpha(x_1, \ldots, x_{k-1}) & \text{if } \alpha \text{ is a functional constant,}\\
\alpha & \text{if } \alpha \text{ is a sentence constant,}\\
\alpha(x_1, \ldots, x_k) & \text{if } \alpha \text{ is a predicate constant,}
\end{array}
$$

where $x_1, \ldots, x_k$ are individual variables. Next, we might have assumed that B has no individual variables foreign to $A(\alpha)$. For suppose that the individual variables $y_1, \ldots, y_n$ foreign to $A(\alpha)$ occur in B. Then from $A(\alpha) \leftrightarrow B$ we easily obtain $A(\alpha) \leftrightarrow \exists y_1 \cdots \exists y_n B$.

What we have called *implicit definability* seems to correspond closely to *uniqueness*: Intuitively, a constant α is unique iff any constant α^* formally characterized as α is synonymous with α. We shall now give a more precise definition of uniqueness.

Let S_α be a set of axioms or axiom-schemata without free individual variables in L_α, and let S_{α^*} be obtained from S_α by uniformly substituting α^* for α. Then we have:

DEFINITION 3. The constant α is *unique* in S_α iff for every $A(\alpha)$, from $\mathrm{PC}(L_{\alpha\alpha^*}) \cup S_\alpha \cup S_{\alpha^*}$ we can prove $A(\alpha) \leftrightarrow A(\alpha^*)$.

This definition should be taken in the following sense: if in S_α there are axiom-schemata, then these axiom-schemata can be instantiated by formulae in $L_{\alpha\alpha^*}$. That is, an instance of an element of S_α or S_{α^*} may contain both α and α^*. Implicit definability is just a particular case of uniqueness when S_α is a set of sentences T_α.

If S_α is a set of axioms or axiom-schemata, let $T_\alpha(S_\alpha)$ be all the sentences in L_α which are instances of S_α. It is not difficult to show that if α is implicitly definable, i.e. unique, in $T_\alpha(S_\alpha)$, then it is unique in S_α. But not necessarily the other way round, as the following example shows. This example makes clear that in the proof of uniqueness we might need to instantiate the schemata of S_α in $L_{\alpha\alpha^*}$.

Let L_+ be the set of formulae of the language of first-order formal arithmetic with 0, s (successor) and $+$. Let S_+ consist of:

(P1) $\neg\exists x(sx = 0)$,

(P2) $\forall x\forall y(sx = sy \rightarrow x = y)$,

(I) $(B(0) \wedge \forall y(B(y) \rightarrow B(sy))) \rightarrow \forall x B(x)$,

$(+1)$ $\forall x(x + 0 = x)$,

$(+2)$ $\forall x\forall y(x + sy = s(x + y))$.

In S_+ we have four axioms ((P1), (P2), $(+1)$ and $(+2)$) which are sentences, and one axiom-schema (I). In S_{+^*} we shall have (P1), (P2), (I), $(+^*1)$ and $(+^*2)$, the last two axioms being obtained from $(+1)$ and $(+2)$ by substituting uniformly $+^*$ for $+$. In S_{++^*} we shall have (P1), (P2), (I), $(+1)$, $(+2)$, $(+^*1)$ and $(+^*2)$. Then we can show the following:

PROPOSITION 1. *The constant* $+$ *is unique in* S_+.

PROOF. Let $B(y)$ be $\forall x(x + y = x +^* y)$. Then using $(+1)$ and $(+^*1)$ it is easy to prove $B(0)$, and using $(+2)$ and $(+^*2)$ it is easy to prove $\forall y(B(y) \rightarrow B(sy))$. Hence, by (I) we obtain $\forall x\forall y(x + y = x +^* y)$, from which it follows that from $\mathrm{PC}(L_{++^*}) \cup S_+ \cup S_{+^*}$ for every $A(+)$ we can prove $A(+) \leftrightarrow A(+^*)$. □

PROPOSITION 2. *The constant* $+$ *is not unique in* $T_+(S_+)$.

PROOF. Let $M = \langle D, 0, s, + \rangle$ be a nonstandard model of arithmetic. The domain D of M has an initial segment H_0 isomorphic to the natural numbers, and after H_0 we have the blocks of nonstandard numbers: $\dots, H_i, \dots, H_j, \dots$, where $i, j, \dots$ are indices of these blocks. Let $M^* = \langle D, 0, s, +^* \rangle$, where $+^*$ is defined as follows:

if $a, b \in H_0$, then $a +^* b = a + b$;

if $a \in H_0$ and $b \in H_j$, then $a +^* b = a + b$ and $b +^* a = b + a$;

if $a \in H_i$ and $b \in H_j$, then $a +^* b = (a + b) - s0$.

Let $f\colon M \rightarrow M^*$ be defined as follows:

if $a \in H_0$, then $f(a) = a$;

if $a \in H_i$, then $f(a) = a + s0$.

It is not difficult to check that f is an isomorphism. Hence for every sentence $A(+)$ of L_+ we have $M \models A(+)$ iff $M^* \models A(+^*)$. Since M is a model of $T_+(S_+)$, we have that M^* is a model of $T_{+^*}(S_{+^*})$. So $\langle D, 0, s, +, +^* \rangle$ is a model of $T_+(S_+) \cup T_{+^*}(S_{+^*})$. But in this last model for $a \in H_i$ and $b \in H_j$ we have $a + b \neq a +^* b$. Hence $x + y = z \leftrightarrow x +^* y = z$ is not provable from $\mathrm{PC}(L_{++^*}) \cup T_+(S_+) \cup T_{+^*}(S_{+^*})$. □

In second-order logic the schema (I) can be replaced by the sentence

$$\forall P((P(0) \wedge \forall y(P(y) \rightarrow P(sy))) \rightarrow \forall x P(x)).$$

Let S_+^2 be the result of replacing (I) in S_+ by this sentence. Then, after extending Definitions 1–3 to second-order logic, we would obtain that the uniqueness of $+$ in S_+^2 amounts to implicit definability.

What we can conclude from our example with $+$ is that uniqueness in S_α and $T_\alpha(S_\alpha)$ need not coincide because in the first case, but not in the second, we can instantiate S_α in the mixed language $L_{\alpha\alpha^*}$. Roughly speaking, with implicit definability the "mixing" of α and α^* is confined to logic (where we may use $\mathrm{PC}(L_{\alpha\alpha^*})$, not only $\mathrm{PC}(L_\alpha) \cup \mathrm{PC}(L_{\alpha^*})$), whereas with uniqueness in general it might extend to the nonlogical assumptions (i.e., we may use $T_{\alpha\alpha^*}(S_\alpha \cup S_{\alpha^*})$, not only $T_\alpha(S_\alpha) \cup T_{\alpha^*}(S_{\alpha^*})$). Beth's definability theorem shows that uniqueness implies explicit definability if we confine this mixing to logic in the proof of uniqueness.

Our purpose in the sequel is to make this talk of mixing α and α^* in logic more articulate, and to show how Beth's theorem depends on mixing being confined to logic. More precisely, we shall consider restricting the mixing of α and α^* in the axioms and rules governing the logical constants, whereas mixing in the structural part of logic will always remain unrestricted. This will permit us to carry over the notions of definability and uniqueness to constants of other syntactical categories, in particular to logical constants.

Take, for example, Gentzen's sequent-calculus for intuitionistic logic, and add to this calculus rules for $\wedge^*$ and $\vee^*$, formally analogous to rules for conjunction ($\wedge$) and disjunction ($\vee$). Then we have the following derivations:

$$\dfrac{\dfrac{B \vdash B}{A \wedge B \vdash B} \qquad \dfrac{\dfrac{\dfrac{A \vdash A}{A \wedge B \vdash A} \qquad \dfrac{\dfrac{A \vdash A}{A, B \vdash A}\,\text{Thinning} \qquad \dfrac{B \vdash B}{A, B \vdash B}\,\text{Thinning}}{A, B \vdash A \wedge^* B}}{A \wedge B, B \vdash A \wedge^* B}\,\text{Cut}}{B, A \wedge B \vdash A \wedge^* B}\,\text{Permutation}}{\dfrac{A \wedge B, A \wedge B \vdash A \wedge^* B}{A \wedge B \vdash A \wedge^* B}\,\text{Contraction}}\,\text{Cut}$$

and

$$\dfrac{\dfrac{A \vdash A}{A \vdash A \vee^* B} \qquad \dfrac{B \vdash B}{B \vdash A \vee^* B}}{A \vee B \vdash A \vee^* B.}$$

So, both $\wedge$ and $\vee$ may be called unique in Gentzen's calculus. But note that we have derived $A \wedge B \vdash A \wedge^* B$ without mixing $\wedge$ and $\wedge^*$ in the rules governing $\wedge$ and $\wedge^*$: they were mixed only in applications of Cut, Permutation and Contraction. On the other hand, $\vee$ and $\vee^*$ are mixed in the application of the rule for introducing $\vee$ on the left, by which we have obtained $A \vee B \vdash A \vee^* B$, and this mixing seems to be unavoidable. So, we might conclude that $\wedge$ is implicitly definable in intuitionistic logic, whereas $\vee$ is not. This is reflected by the fact that $\wedge$ is explicitly definable, in the sense that $A \wedge B$ is equivalent to the sequence A, B on the left of the turnstile, whereas for $\vee$ no such sequence exists. This is in accordance with a general form of the interpolation and definability theorems which we are going to prove. The structural part of logic will be captured by a certain consequence relation, closely related to notions introduced by Tarski.

The idea that deductive systems specify consequence relations or consequence operations, and that one can study these systems abstractly by considering axiomatic theories for these relations or operations, stems from [2] (III–V). Our work here might be conceived as such an abstract study of definability and interpolation (§§ 3 and 4), and the application of the results obtained to first-order deductive systems (§5).

§2. Consequence relations. In Tarski's [2] one can find axiom systems for a consequence operation Cn. Our approach differs from Tarski's conception in at least two respects. First, we consider not only compact but also noncompact consequence relations. This gives our approach a greater generality which proves useful when applying our notions to certain logical constants (cf. the example of the universal quantifier in ω-arithmetic at the end of §3). Second, we deal with consequence relations between formulae and not only between sentences. This is essential if we want to interpret Craig's interpolation lemma, which is formulated by explicit reference to free variables, in a framework based on consequence relations.

Let L be a set of formulae of an arbitrary language, and let $X, Y, Z, U, V, W, \ldots, X', \ldots$ be subsets of L, and $A, B, C, D, \ldots, A', \ldots$ members of L. Then we introduce the following definitions:

DEFINITION 4. A *consequence relation* over L is any relation $\vdash$ between subsets of L on the left and members of L on the right (i.e., any subset of $2^L \times L$) which satisfies:

($\vdash$1) $\{A\} \vdash A$,

($\vdash$2) $X \vdash A \Rightarrow X \cup Y \vdash A$,

($\vdash$3) $((\forall A \in Y)X \vdash A \ \& \ Y \cup V \vdash B) \Rightarrow X \cup V \vdash B$.

DEFINITION 5. $X \Vdash Y \Leftrightarrow_{\mathrm{df}} (\forall A \in Y)X \vdash A$.

DEFINITION 6. $\mathrm{Cn}(X) =_{\mathrm{df}} \{A \mid X \vdash A\}$.

It is easy to show that $\Vdash$ satisfies:

($\Vdash$1) $X \Vdash X$,

($\Vdash$2) $X \Vdash Z \Rightarrow X \cup Y \Vdash Z$,

($\Vdash$3) $((\forall A \in Y)X \Vdash \{A\} \ \& \ Y \cup V \Vdash Z) \Rightarrow X \cup V \Vdash Z$,

and that Cn satisfies:

(Cn1) $X \subseteq \mathrm{Cn}(X)$,

(Cn2) $X \subseteq Y \Rightarrow \mathrm{Cn}(X) \subseteq \mathrm{Cn}(Y)$,

(Cn3) $\mathrm{Cn}(\mathrm{Cn}(X)) \subseteq \mathrm{Cn}(X)$.

If we start from $\Vdash$ as a primitive relation satisfying ($\Vdash$1)–($\Vdash$3) and define $\vdash$ and Cn as follows:

$$X \vdash A \Leftrightarrow_{\mathrm{df}} X \Vdash \{A\}, \qquad \mathrm{Cn}(X) =_{\mathrm{df}} \{A \mid X \Vdash \{A\}\},$$

we obtain ($\vdash$1)–($\vdash$3) and (Cn1)–(Cn3). (Note that ($\Vdash$3) can be replaced by

$$(X \Vdash Y \ \& \ Y \cup V \Vdash Z) \Rightarrow X \cup V \Vdash Z,$$
$$(\forall A \in Y)X \Vdash \{A\} \Rightarrow X \Vdash Y,$$

but not by the first principle alone.)

If we start from Cn as a primitive operation satisfying (Cn1)–(Cn3) and define $\vdash$ and $\Vdash$ as follows:

$$X \vdash A \Leftrightarrow_{\mathrm{df}} A \in \mathrm{Cn}(X), \qquad X \Vdash Y \Leftrightarrow_{\mathrm{df}} Y \subseteq \mathrm{Cn}(X),$$

we obtain ($\vdash$1)–($\vdash$3) and ($\Vdash$1)–($\Vdash$3).

For further applications it is useful to remember the following properties of $\Vdash$:

$$Y \subseteq X \Rightarrow X \Vdash Y,$$
$$X \Vdash Y \Leftrightarrow (\forall Z \subseteq Y) X \Vdash Z,$$
$$(\forall i) X \Vdash Y_i \Leftrightarrow X \Vdash \bigcup_i Y_i.$$

Let $X_{\text{fin}}, Y_{\text{fin}}, \ldots$ be finite subsets of L. Then we introduce the following definition:

DEFINITION 7. A consequence relation is *compact* iff it satisfies

($\vdash$4) $X \vdash A \Rightarrow (\exists X_{\text{fin}} \subseteq X) X_{\text{fin}} \vdash A$.

If we define $\Vdash$ and Cn in terms of a compact consequence relation $\vdash$, then we obtain:

($\Vdash$4) $X \Vdash Y_{\text{fin}} \Rightarrow (\exists X_{\text{fin}} \subseteq X) X_{\text{fin}} \Vdash Y_{\text{fin}}$,

(Cn4) $\mathrm{Cn}(X) \subseteq \bigcup_{X_{\text{fin}} \subseteq X} \mathrm{Cn}(X_{\text{fin}})$,

and as before we could start with a primitive $\Vdash$ or Cn satisfying ($\Vdash$4) or (Cn4) and obtain ($\vdash$4). Note that in the presence of ($\vdash$4), we can replace ($\vdash$3) by

($\vdash$3.1) $(X \vdash A \;\&\; \{A\} \cup V \vdash B) \Rightarrow X \cup V \vdash B$

or by

($\vdash$3.2) $(X \vdash A \;\&\; \{A\} \cup X \vdash B) \Rightarrow X \vdash B)$.

If instead of (Cn4) we assume $\mathrm{Cn}(X) = \bigcup_{X_{\text{fin}} \subseteq X} \mathrm{Cn}(X_{\text{fin}})$, as in [2, pp. 31, 64], then (Cn2) becomes derivable.

Let L_1 and L_2 be two sets of formulae which are subsets of L, and let $L_1 \cap L_2 = L_0$. We shall use $X_1, Y_1, Z_1, U_1, V_1, W_1, \ldots, X'_1, \ldots$ for subsets of L_1, and $A_1, B_1, C_1, D_1, \ldots, A'_1, \ldots$ for elements of L_1; and we use analogous notation with 2 and 0 instead of 1. If $(X)_i =_{\text{df}} X \cap L_i$, it is easy to check that $(X_1)_1 = X_1$, $(X_2)_2 = X_2$, $(X_1)_2 = (X_1)_0$ and $(X_2)_1 = (X_2)_0$.

Let now $\vdash_1$ be a consequence relation over L_1, and $\vdash_2$ a consequence relation over L_2. We define $\Vdash_1$, $\Vdash_2$, Cn_1, and Cn_2 as before. An *extension* of $\vdash_1$ over L is any relation $R \subseteq 2^L \times L$ such that $X_1 \vdash_1 A_1 \Rightarrow X_1 R A_1$. We proceed analogously to define extensions of $\vdash_2$. There are always consequence relations which extend both $\vdash_1$ and $\vdash_2$, since the trivial relation which holds between any X and any A is a consequence relation, which is evidently compact. Let $\vdash^i$ be a consequence relation over L which is an extension of both $\vdash_1$ and $\vdash_2$, and let $X \vdash_c A \Leftrightarrow_{\text{df}} (\forall i) X \vdash^i A$. Then we can easily check the following lemma:

LEMMA 1.1. *The relation $\vdash_c$ is the minimal consequence relation over L which is an extension of the consequence relations $\vdash_1$ and $\vdash_2$.* □

If $\vdash_1$ and $\vdash_2$ are compact, and $\vdash^j$ is a compact consequence relation over L which is an extension of both $\vdash_1$ and $\vdash_2$, then

$$X \vdash_{cc} A \Leftrightarrow_{\text{df}} (\exists X_{\text{fin}} \subseteq X)(\forall j) X_{\text{fin}} \vdash^j A.$$

It is again easy to check the following lemma:

LEMMA 1.2. *The relation $\vdash_{cc}$ is the minimal compact consequence relation over L which is an extension of the compact consequence relations $\vdash_1$ and $\vdash_2$.* □

The relationship between the axioms for compact consequence relations and the structural rules in Gentzen-style proof theory is obvious. If one takes the left-hand side of a sequent to be a set rather than a sequence of formulae, the structural rules in the standard sequent calculi are just the axioms $A \vdash A$, Thinning and Cut, which obviously correspond to ($\vdash$1), ($\vdash$2) and ($\vdash$3.1) if X, Y and V are finite. Thus compact consequent relations are a slight generalization of this idea, allowing for infinite sets on the left of the turnstile, as in the standard semantic first-order consequence relation (which is compact). Arbitrary consequence relations (which are not necessarily compact) are a further generalization which may be useful, e.g., in the consideration of infinitary systems.

What at the end of §1 was loosely called "the structural part of logic", we shall from now on identify with $\vdash_c$ and $\vdash_{cc}$. As a matter of fact, $\vdash_c$ and $\vdash_{cc}$ could more appropriately be called "the minimal (compact) structural logic extending two given consequence relations"; so, when we say "the structural part of logic", this should be understood in a specific way.

§3. Interpolation and definability. First we introduce the following definitions:

DEFINITION 8. A relation $R \subseteq 2^L \times L$ satisfies the *set-interpolation property* with respect to $\vdash_1$ and $\vdash_2$ iff

$$X_1 R A_2 \Rightarrow \exists Y_0 (X_1 \Vdash_1 Y_0 \;\&\; Y_0 \vdash_2 A_2),$$
$$X_2 R A_1 \Rightarrow \exists Y_0 (X_2 \Vdash_2 Y_0 \;\&\; Y_0 \vdash_1 A_1).$$

If Y_0 is finite, we have the *finite set-interpolation property*, and if Y_0 is a singleton, we have the *formula-interpolation property* with respect to $\vdash_1$ and $\vdash_2$.

DEFINITION 9. A relation $R \subseteq 2^L \times L$ is *conservative* with respect to $\vdash_1$ (or $\vdash_2$) iff $X_1 R A_1 \Rightarrow X_1 \vdash_1 A_1$ (or $X_2 R A_2 \Rightarrow X_2 \vdash_2 A_2$).

Our aim in this section is to show that if we assume for $\vdash_1$ and $\vdash_2$ that

$$(0) \qquad X_0 \vdash_1 A_0 \Leftrightarrow X_0 \vdash_2 A_0,$$

i.e., that $\vdash_1$ and $\vdash_2$ agree on L_0, then $\vdash_c$ satisfies the set-interpolation property and is conservative with respect to $\vdash_1$ and $\vdash_2$. For compact consequence relations $\vdash_1$ and $\vdash_2$ we shall show that $\vdash_{cc}$ satisfies the finite set-interpolation property and that it is conservative with respect to $\vdash_1$ and $\vdash_2$. With some additional assumptions we shall show that $\vdash_{cc}$ also satisfies the formula-interpolation property.

Our proof proceeds as follows. First we introduce two auxiliary relations $\vdash'$, $\vdash'' \subseteq 2^L \times L$ by use of certain set operations P_1 and P_2. The relation $\vdash'$ is defined in such a way that it is easy to show that $\vdash'$ satisfies the set-interpolation property and is conservative with respect to $\vdash_1$ and $\vdash_2$ (Lemma 3.1), and furthermore that $\vdash'$ is a consequence relation over L (Lemma 4.1). It can then be proved that $\vdash'$ is an extension of $\vdash_1$ and $\vdash_2$, provided (0) holds (Lemma 5.1). This immediately yields the desired result for $\vdash_c$ (Theorem 1.1), since $\vdash_c$, as the *minimal* consequence relation over L which extends $\vdash_1$ and $\vdash_2$ (cf. Lemma 1.1), a fortiori satisfies the set-interpolation and conservativeness properties obtained for $\vdash'$. The proof for $\vdash_{cc}$ starting from $\vdash''$ proceeds parallel to the one for $\vdash_c$.

The auxiliary notions P_1, P_2, $\vdash'$ and $\vdash''$ mentioned above are defined as follows:

$$P_1(X) =_{\mathrm{df}} \{Y_1 \mid Y_1 \Vdash_1 (X)_1 \;\&\; (\mathrm{Cn}_1(Y_1))_0 \Vdash_2 (X)_2\},$$
$$P_2(X) =_{\mathrm{df}} \{Y_2 \mid Y_2 \Vdash_2 (X)_2 \;\&\; (\mathrm{Cn}_2(Y_2))_0 \Vdash_1 (X)_1\},$$
$$X \vdash' A \Leftrightarrow_{\mathrm{df}} (P_1(X) \subseteq P_1(\{A\}) \;\&\; P_2(X) \subseteq P_2(\{A\})),$$
$$X \vdash'' A \Leftrightarrow_{\mathrm{df}} (\exists X_{\mathrm{fin}} \subseteq X) X_{\mathrm{fin}} \vdash' A.$$

Intuitively, the set operation P_1 may be viewed as defining a relation between sets Y_1

and X such that some kind of conservativeness with respect to $\Vdash_1$ holds when X is restricted to L_1 and some kind of set-interpolation property holds when X is restricted to L_2, and analogously for P_2 with the indices "1" and "2" interchanged. The definition of $\vdash'$ is then chosen in a way that makes both of these relations hereditary in some sense with respect to $\vdash'$. This kind of heredity is used in particular when we prove transitivity, i.e., property ($\vdash$3), for $\vdash'$ (see Lemma 4.1 below).

Next we prove the following easy lemma:

LEMMA 2. *For any X_1 and X_2 we have $X_1 \in P_1(X_1)$ and $X_2 \in P_2(X_2)$.*

PROOF. Since $(X_1)_1 = X_1$, we have $X_1 \Vdash_1 (X_1)_1$. And since $X_1 \subseteq \mathrm{Cn}_1(X_1)$, we have $(X_1)_0 \subseteq (\mathrm{Cn}_1(X_1))_0$. Using $(X_1)_0 = (X_1)_2$, and properties of $\Vdash$, we obtain $(\mathrm{Cn}_1(X_1))_0 \Vdash_2 (X_1)_2$. We proceed similarly with 2. □

Then we can show the following:

LEMMA 3.1. *The relation $\vdash'$ satisfies the set-interpolation property and is conservative with respect to the consequence relations $\vdash_1$ and $\vdash_2$.*

PROOF. Suppose $X_1 \vdash' A_2$. This means that $P_1(X_1) \subseteq P_1(\{A_2\})$. Using Lemma 2 we obtain $X_1 \in P_1(\{A_2\})$, which implies $(\mathrm{Cn}_1(X_1))_0 \Vdash_2 \{A_2\}$. It follows from the properties of $\Vdash$ that $X_1 \Vdash_1 (\mathrm{Cn}_1(X_1))_0$. Proceeding analogously with $X_2 \vdash' A_1$ we obtain set-interpolation. Now suppose $X_1 \vdash' A_1$. Again, by Lemma 2, $X_1 \in P_1(\{A_1\})$, which yields $X_1 \Vdash_1 \{A_1\}$. Proceeding analogously with $X_2 \vdash' A_2$ we obtain conservativeness. □

Quite analogously we can prove the following lemma:

LEMMA 3.2. *The relation $\vdash''$ satisfies the finite set-interpolation property and is conservative with respect to the compact consequence relations $\vdash_1$ and $\vdash_2$.*

PROOF. Suppose $X_1 \vdash'' A_2$. Then $X_{\mathrm{fin}} \vdash' A_2$ for $X_{\mathrm{fin}} \subseteq X_1$. Thus by Lemma 3.1 there is a Y_0 such that $X_{\mathrm{fin}} \Vdash_1 Y_0$ and $Y_0 \vdash_2 A_2$. By ($\Vdash$2), X_{fin} can be replaced by X_1, and since $\vdash_2$ is compact, Y_0 can be chosen finite. Arguing similarly for $X_2 \vdash'' A_1$, we obtain finite set-interpolation. Suppose $X_1 \vdash'' A_1$. Then $X_{\mathrm{fin}} \vdash' A_1$ for $X_{\mathrm{fin}} \subseteq X_1$; thus, by Lemma 3.1, $X_{\mathrm{fin}} \vdash_1 A_1$. Therefore, by ($\Vdash$2), $X_1 \vdash_1 A_1$. Arguing similarly for $X_2 \vdash'' A_2$, we obtain conservativeness. □

Next we prove the following lemmata:

LEMMA 4.1. *The relation $\vdash'$ is a consequence relation over L.*

PROOF. For ($\vdash$1) we have $P_1(\{A\}) \subseteq P_1(\{A\})$ & $P_2(\{A\}) \subseteq P_2(\{A\})$. For ($\vdash$2), suppose $X \vdash' A$, i.e., $P_1(X) \subseteq P_1(\{A\})$ and $P_2(X) \subseteq P_2(\{A\})$. Then it is easy to check that $P_1(X \cup Y) \subseteq P_1(X)$ and $P_2(X \cup Y) \subseteq P_2(X)$, which immediately yields $X \cup Y \vdash' A$.

For ($\vdash$3), suppose $(\forall A \in Y) X \vdash' A$ and $Y \cup V \vdash' B$, i.e.,

$$(\forall A \in Y)(P_1(X) \subseteq P_1(\{A\}) \;\&\; P_2(X) \subseteq P_2(\{A\})), \tag{1}$$

$$P_1(Y \cup V) \subseteq P_1(\{B\}) \;\&\; P_2(Y \cup V) \subseteq P_2(\{B\}). \tag{2}$$

Then we can easily check the following properties of P_1:

$$\bigcap_{A \in Y} P_1(\{A\}) = P_1(Y), \tag{i}$$

$$P_1(X) \subseteq P_1(Y) \Rightarrow P_1(X \cup V) \subseteq P_1(Y \cup V). \tag{ii}$$

Then, using (i), from (1) we obtain $P_1(X) \subseteq P_1(Y)$, which by (ii) yields $P_1(X \cup V) \subseteq P_1(Y \cup V)$. Proceeding similarly for P_2, and using (2), we obtain $X \cup V \vdash' B$. □

LEMMA 4.2. *The relation $\vdash''$ is a compact consequence relation over L.*

PROOF. To show that $\vdash''$ satisfies ($\vdash$1), ($\vdash$2) and ($\vdash$4) is quite easy. We shall only demonstrate that it satisfies also ($\vdash$3.1), which as we have said above can replace ($\vdash$3) in the presence of ($\vdash$4). So suppose $X \vdash'' A$ and $\{A\} \cup V \vdash'' B$, i.e.,

$$(3) \qquad (\exists X_{\mathrm{fin}} \subseteq X)(P_1(X_{\mathrm{fin}}) \subseteq P_1(\{A\})\ \&\ P_2(X_{\mathrm{fin}}) \subseteq P_2(\{A\})),$$

$$(4) \qquad (\exists Z_{\mathrm{fin}} \subseteq \{A\} \cup V)(P_1(Z_{\mathrm{fin}}) \subseteq P_1(\{B\})\ \&\ P_2(Z_{\mathrm{fin}}) \subseteq P_2(\{B\})).$$

Let $V_{\mathrm{fin}} =_{\mathrm{df}} Z_{\mathrm{fin}} \cap V$. Since $Z_{\mathrm{fin}} \subseteq \{A\} \cup V$, we easily obtain that $Z_{\mathrm{fin}} \subseteq \{A\} \cup V_{\mathrm{fin}}$. Using the fact that for any Y and W we have $P_1(Y \cup W) \subseteq P_1(Y)$, we obtain

$$(5) \qquad P_1(\{A\} \cup V_{\mathrm{fin}}) \subseteq P_1(Z_{\mathrm{fin}}).$$

Next, from (3) we have $P_1(X_{\mathrm{fin}}) \subseteq P_1(\{A\})$, which together with (ii) of the previous proof yields

$$(6) \qquad P_1(X_{\mathrm{fin}} \cup V_{\mathrm{fin}}) \subseteq P_1(\{A\} \cup V_{\mathrm{fin}}).$$

Now, (4), (5) and (6) give $P_1(X_{\mathrm{fin}} \cup V_{\mathrm{fin}}) \subseteq P_1(\{B\})$. Proceeding similarly with P_2, and using the fact that $X_{\mathrm{fin}} \cup V_{\mathrm{fin}} \subseteq X \cup V$, we obtain $X \cup V \vdash'' B$. □

LEMMA 5.1. *The relation $\vdash'$ is an extension of the consequence relations $\vdash_1$ and $\vdash_2$, provided $\vdash_1$ and $\vdash_2$ satisfy* (0).

PROOF. Suppose $X_1 \vdash_1 A_1$. Next suppose $Z_1 \in P_1(X_1)$. It follows immediately that $Z_1 \Vdash_1 X_1$, which together with $X_1 \vdash_1 A_1$ using ($\vdash$3) gives $Z_1 \vdash_1 A_1$, i.e.

$$(7) \qquad Z_1 \Vdash_1 (\{A_1\})_1.$$

This means that $A_1 \in \mathrm{Cn}_1(Z_1)$, i.e. $\{A_1\} \subseteq \mathrm{Cn}_1(Z_1)$, which yields that $(\{A_1\})_0 \subseteq (\mathrm{Cn}_1(Z_1))_0$. Then $(\mathrm{Cn}_1(Z_1))_0 \Vdash_2 (\{A_1\})_2$, which together with (7) implies $Z_1 \in P_1(\{A_1\})$.

Now suppose $Z_2 \in P_2(X_1)$. It follows immediately that $(\mathrm{Cn}_2(Z_2))_0 \Vdash_1 X_1$, which together with $X_1 \vdash_1 A_1$ using ($\vdash$3) gives $(\mathrm{Cn}_2(Z_2))_0 \vdash_1 A_1$, i.e.

$$(8) \qquad (\mathrm{Cn}_2(Z_2))_0 \Vdash_1 (\{A_1\})_1.$$

If $A_1 \in L_0$, using the assumption (0) $X_0 \vdash_1 A_0 \Leftrightarrow X_0 \vdash_2 A_0$, we obtain

$$(9) \qquad (\mathrm{Cn}_2(Z_2))_0 \Vdash_2 (\{A_1\})_2.$$

If $A_1 \notin L_0$, then $(\{A_1\})_2$ is empty and (9) is trivially satisfied. Since $Z_2 \Vdash_2 (\mathrm{Cn}_2(Z_2))_0$, with (9) using ($\Vdash$3) we obtain $Z_2 \Vdash_2 (\{A_1\})_2$. This together with (8) implies that $Z_2 \in P_2(\{A_1\})$.

Hence, from $X_1 \vdash_1 A_1$ it follows that $P_1(X_1) \subseteq P_1(\{A_1\})$ and $P_2(X_1) \subseteq P_2(\{A_1\})$, i.e. $X_1 \vdash' A_1$. We proceed analogously with $X_2 \vdash_2 A_2$. □

LEMMA 5.2. *The relation $\vdash''$ is an extension of the compact consequence relations $\vdash_1$ and $\vdash_2$, provided $\vdash_1$ and $\vdash_2$ satisfy* (0).

PROOF. If $X_1 \vdash_1 A_1$, then $X_{\mathrm{fin}} \vdash_1 A_1$ for some $X_{\mathrm{fin}} \subseteq X_1$ since $\vdash_1$ is compact. Then $X_{\mathrm{fin}} \vdash' A_1$ by Lemma 5.1, which means that $X_1 \vdash'' A_1$. We proceed analogously with $X_2 \vdash_2 A_2$. □

Now we can prove the main results of this section.

THEOREM 1.1. *The consequence relation* $\vdash_c$ *satisfies the set-interpolation property and is conservative with respect to the consequence relations* $\vdash_1$ *and* $\vdash_2$, *provided* $\vdash_1$ *and* $\vdash_2$ *satisfy* (0).

PROOF. By Lemmata 1.1, 4.1 and 5.1, we have that $X \vdash_c A \Rightarrow X \vdash' A$, and then it suffices to apply Lemma 3.1. □

THEOREM 1.2. *The compact consequence relation* $\vdash_{cc}$ *satisfies the finite set-interpolation property and is conservative with respect to the compact consequence relations* $\vdash_1$ *and* $\vdash_2$, *provided* $\vdash_1$ *and* $\vdash_2$ *satisfy* (0).

PROOF. By Lemmata 1.2, 4.2. and 5.2, we have that $X \vdash_{cc} A \Rightarrow X \vdash'' A$, and then it suffices to apply Lemma 3.2. □

The consequence relations $\vdash'$ and $\vdash''$ are not identical with $\vdash_c$ and $\vdash_{cc}$, respectively, as one might perhaps suppose. For example, $X \vdash' A$ and $X \vdash'' A$ hold for arbitrary X if $L_1 \cup L_2 \neq L$ and $A \notin L_1 \cup L_2$, whereas $\varnothing \vdash_c A$ and $\varnothing \vdash_{cc} A$ do not necessarily hold. But even if $A \in L_1 \cup L_2$, the following counterexample can be constructed: Assume $L = \{p, q, r\}$, $L_1 = \{p, q)$ and $L_2 = \{r\}$; thus $L_0 = \varnothing$. Let $\vdash_1$ and $\vdash_2$ be given by $\{p\} \vdash_1 p$, $\{q\} \vdash_1 q$, $\{p,q\} \vdash_1 p$, $\{p,q\} \vdash_1 q$ and $\{r\} \vdash_2 r$. Then $(\mathrm{Cn}_1(\{p,q\}))_0 = (\mathrm{Cn}_2(\{r\}))_0 = \varnothing$; thus $P_1(\{p,r\}) = P_2(\{p,r\}) = \varnothing$, and therefore $\{p,r\} \vdash' q$ and $\{p,r\} \vdash'' q$. However, we do not have $\{p,r\} \vdash_c q$ and $\{p,r\} \vdash_{cc} q$, since $\vdash_c$ (which equals $\vdash_{cc}$) can be given by $X_1 \vdash_1 A_1 \Rightarrow X_1 \cup X_2 \vdash_c A_1$ and $X_2 \vdash_2 A_2 \Rightarrow X_1 \cup X_2 \vdash_c A_2$, and neither $\{p\} \vdash_i q$ nor $\{r\} \vdash_i q$ holds for $i = 1$ or 2.

Now suppose that in L_0 we have a conjunction connective $\wedge$ and a sentence constant $\top$ for which it holds that

$$\{A_0, B_0\} \vdash_1 A_0 \wedge B_0, \qquad \{A_0 \wedge B_0\} \vdash_1 A_0, \qquad \{A_0 \wedge B_0\} \vdash_1 B_0, \qquad \varnothing \vdash_1 \top,$$

and analogously with $\vdash_2$. Then as a corollary of Theorem 1.2 we have the following:

THEOREM 1.3. *If in* L_0 *we have* $\wedge$ *and* $\top$, *then* $\vdash_{cc}$ *satisfies the formula-interpolation property and is conservative with respect to the compact consequence relations* $\vdash_1$ *and* $\vdash_2$, *provided* $\vdash_1$ *and* $\vdash_2$ *satisfy* (0). □

Now let L_1 differ from L_2 only in having a constant α, of an arbitrary syntactical category, where L_2 has a constant α^* of the same syntactical category as α, and vice versa. Let $\vdash_1$ and $\vdash_2$ be consequence relations over L_1 and L_2, respectively, which upon uniform substitution of α for α^*, and vice versa, become identical. Then we can prove the following theorem:

THEOREM 2 (DEFINABILITY THEOREM). *If* $A(\alpha) \in L_1$, *then*

$$\{A(\alpha)\} \vdash_c A(\alpha^*) \quad \textit{iff} \quad \exists Y_0(\{A(\alpha)\} \Vdash_1 Y_0 \ \& \ Y_0 \vdash_1 A(\alpha)).$$

If $\vdash_1$ *and* $\vdash_2$ *are compact, then the same holds for* $\vdash_{cc}$ *and finite* Y_0. *If moreover* L_0 *contains* $\wedge$ *and* $\top$ *as above,* Y_0 *can be chosen as a singleton.*

PROOF. Assume $\{A(\alpha)\} \vdash_c A(\alpha^*)$. It is easy to check that $\vdash_1$ and $\vdash_2$ satisfy the assumption (0). So we can use Theorem 1.1 to obtain

$$\exists Y_0(\{A(\alpha)\} \Vdash_1 Y_0 \ \& \ Y_0 \vdash_2 A(\alpha^*)),$$

from which the right-hand side follows. If now we assume the right-hand side, we obtain that there is a Y_0 such that $\{A(\alpha)\} \Vdash_1 Y_0$ and $Y_0 \vdash_2 A(\alpha^*)$. Then using Lemma 1.1 we obtain $\{A(\alpha)\} \Vdash_c Y_0$ and $Y_0 \vdash_c A(\alpha^*)$, and apply ($\vdash$3). The results for compact $\vdash_1$ and $\vdash_2$ follow analogously by use of Theorem 1.2 and Lemma 1.2, and for a singleton Y_0 from Theorem 1.3. $\square$

It is clear that under the assumptions of this theorem, we could have written on the left-hand side $\{A(\alpha)\} \vdash_c A(\alpha^*)$ & $\{A(\alpha^*)\} \vdash_c A(\alpha)$: if we have one of these conjuncts we also have the other, because α and α^* are exactly parallel.

Our definability theorem is an analogue of Beth's definability theorem which makes fewer assumptions about logic, and leaves the syntactical category of α quite undetermined. In particular, it captures the case of logical constants which are now considered explicitly definable by *sets* of formulae, provided they are implicitly definable. For example, the universal quantifier $\bigwedge$ in arithmetic with the ω-rule is explicitly definable by

$$\left\{\bigwedge_x B(x)\right\} \Vdash_1 \{B(1), B(2), B(3), \ldots\} \;\&\; \{B(1), B(2), B(3), \ldots\} \vdash_1 \bigwedge_x B(x),$$

where $\vdash_1$ denotes derivability in this theory.

Logic represented by $\vdash_c$ or $\vdash_{cc}$ in our definability theorem is reduced to its structural part. So the notion of implicit definability of α in $\vdash_1$ as $\{A(\alpha)\} \vdash_c A(\alpha^*)$ or $\{A(\alpha)\} \vdash_{cc} A(\alpha^*)$ for every $A(\alpha)$ is stronger than the one used in Beth's Theorem for first-order logic, where "mixing" of α and α^* in proofs of implicit definability is allowed in any rule of logic. Therefore, our definability theorem is not simply a generalization of Beth's theorem in the sense that the latter is a trivial corollary of the former. It can however be shown that under certain conditions Beth's theorem follows from our definability theorem, since the mixing of α and α^* in the "operational" part of logic involving the logical constants can in many cases be dispensed with, as we shall try to show in the last section.

First, however, we show that much more information about $\vdash_{cc}$ is available if the language under consideration contains at least an implication connective obeying certain standard principles.

§4. Explicit characterization of $\vdash_{cc}$ in the presence of implication. In §2 we characterized $\vdash_c$ [$\vdash_{cc}$] as the minimal [compact] consequence relation over L which is an extension of the [compact] consequence relations $\vdash_1$ and $\vdash_2$ (cf. Lemmata 1.1 and 1.2). In the case of $\vdash_{cc}$, an explicit characterization can be given, provided a connective of implication is at one's disposal. It can be motivated as follows. By Theorem 1.2 we know already that if $X_1 \vdash_{cc} A_2$, then $(\mathrm{Cn}_1(X_1))_0 \vdash_2 A_2$, i.e., what is essentially needed of X_1 to obtain A_2 as a $\vdash_{cc}$-consequence of X_1 are the $\vdash_1$-consequences of X_1 in L_0 (and then it suffices to continue with $\vdash_2$ alone). Lemma 6 below then says that this also holds if additional assumptions from L_2 are present, i.e., if $X_2 \cup X_1 \vdash_{cc} A_2$ then $X_2 \cup (\mathrm{Cn}_1(X_1))_0 \vdash_2 A_2$, and similarly for A_1. Since by Lemma 6 also the converse holds, one obtains a characterization of $\vdash_{cc}$.

Let $\vdash_1$ and $\vdash_2$ be compact consequence relations which fulfill condition (0). Let them contain an implication connective $\rightarrow$ in the sense that for $i = 1, 2$, $A_i \rightarrow B_i$ is in L_i for every A_i and B_i, and the following principles hold:

(MP) $\{A_i, A_i \rightarrow B_i\} \vdash_i B_i$,

(DT) $X_i \cup \{A_i\} \vdash_i B_i \Rightarrow X_i \vdash_i A_i \rightarrow B_i$.

For $\vdash_{cc}$ we can now give the following characterization:

LEMMA 6. *If* $A \in L_1 \cup L_2$, *then* $X \vdash_{cc} A$ *iff*

$$(*) \qquad (X)_1 \cup (\mathrm{Cn}_2((X)_2))_0 \Vdash_1 (\{A\})_1 \;\&\; (X)_2 \cup (\mathrm{Cn}_1((X)_1))_0 \Vdash_2 (\{A\})_2.$$

We shall prove this lemma in steps. First we demonstrate the following auxiliary lemma:

LEMMA 6.1. *The following statements are equivalent*:
(1) $X_1 \cup W_0 \cup (\mathrm{Cn}_2(Y_2 \cup W_0))_0 \vdash_1 B_1$,
(2) $X_1 \cup (\mathrm{Cn}_2(Y_2 \cup W_0))_0 \vdash_1 B_1$,
(3) $X_1 \cup W_0 \cup (\mathrm{Cn}_2(Y_2))_0 \vdash_1 B_1$.

PROOF. The equivalence of (1) and (2) follows from the fact that $W_0 \subseteq (\mathrm{Cn}_2(Y_2 \cup W_0))_0$. Now suppose (1). If $W_0 = \varnothing$, we immediately have (3). If $W_0 \neq \varnothing$, then using ($\vdash$4) there is a nonempty $W_{\mathrm{fin}} \subseteq W_0$ and a nonempty $Z_{\mathrm{fin}} \subseteq (\mathrm{Cn}_2(Y_2 \cup W_0))_0$ such that $X_1 \cup W_{\mathrm{fin}} \cup Z_{\mathrm{fin}} \vdash_1 B_1$. If $Z_{\mathrm{fin}} = \{D_0^1, D_0^2, \ldots, D_0^m\}$, then for all i, $1 \leq i \leq m$, $Y_2 \cup W_0 \vdash_2 D_0^i$; hence, by ($\vdash$4) and ($\vdash$2), $Y_2 \cup W_{\mathrm{fin}}^i \vdash_2 D_0^i$ for some $W_{\mathrm{fin}}^i \subseteq W_0$. Let $W_{\mathrm{fin}}^* = W_{\mathrm{fin}} \cup W_{\mathrm{fin}}^1 \cup \cdots \cup W_{\mathrm{fin}}^m$. Then, by ($\vdash$2), $X_1 \cup W_{\mathrm{fin}}^* \cup Z_{\mathrm{fin}} \vdash_1 B_1$ and $Y_2 \cup W_{\mathrm{fin}}^* \vdash_2 D_0^i$ for all i, $1 \leq i \leq m$. If $W_{\mathrm{fin}}^* = \{A_0^1, A_0^2, \ldots, A_0^k\}$, let $V_0 =_{\mathrm{df}} \{A_0^1 \to (A_0^2 \to \cdots \to (A_0^k \to D_0^i)\cdots) \mid 1 \leq i \leq m\}$. It is easy to show using (DT) that $V_0 \subseteq (\mathrm{Cn}_2(Y_2))_0$. It is also easy to show using (MP) that $W_{\mathrm{fin}}^* \cup V_0 \Vdash_1 Z_{\mathrm{fin}}$, which together with $X_1 \cup W_{\mathrm{fin}}^* \cup Z_{\mathrm{fin}} \vdash_1 B_1$ by ($\vdash$3) gives $X_1 \cup W_{\mathrm{fin}}^* \cup V_0 \vdash_1 B_1$. From this last statement we obtain (3) by ($\vdash$2), since $W_{\mathrm{fin}}^* \subseteq W_0$ and $V_0 \subseteq (\mathrm{Cn}_2(Y_2))_0$. Now suppose (3). Since $(\mathrm{Cn}_2(Y_2))_0 \subseteq (\mathrm{Cn}_2(Y_2 \cup W_0))_0$, we obtain (1) by using ($\vdash$2). □

Quite analogously to Lemma 6.1 we can prove the following lemma:

LEMMA 6.2. *The following statements are equivalent*:
(1) $X_2 \cup W_0 \cup (\mathrm{Cn}_1(Y_1 \cup W_0))_0 \vdash_2 B_2$,
(2) $X_2 \cup (\mathrm{Cn}_1(Y_1 \cup W_0))_0 \vdash_2 B_2$,
(3) $X_2 \cup W_0 \cup (\mathrm{Cn}_1(Y_1))_0 \vdash_2 B_2$. □

Let $X\mathbf{R}A$ be an abbreviation for $(*)$. Then we prove the following lemmata:

LEMMA 6.3. *The relation* $\mathbf{R}$ *is an extension over L of the compact consequence relations* $\vdash_1$ *and* $\vdash_2$.

PROOF. Suppose $X_1 \vdash_1 A_1$. Then the first conjunct of $X_1\mathbf{R}A_1$ follows by using ($\vdash$2). The second conjunct either is trivial if $A_1 \notin L_2$, or $A_1 \in (\mathrm{Cn}_1((X_1)_1))_0$, and we obtain this conjunct again by using ($\vdash$2). We proceed analogously with $X_2 \vdash_2 A_2$. □

LEMMA 6.4. *The relation* $\mathbf{R}$ *is a compact consequence relation over L.*

PROOF. It is quite easy to show that $\mathbf{R}$ satisfies ($\vdash$1) and ($\vdash$2). For ($\vdash$4) assume that $X\mathbf{R}A$. Using ($\vdash$4) for $\vdash_1$, we obtain that for some $Y'_{\mathrm{fin}} \subseteq (X)_1$ and $Z_{\mathrm{fin}} \subseteq (\mathrm{Cn}_2((X)_2))_0$ the following holds: $Y'_{\mathrm{fin}} \cup Z_{\mathrm{fin}} \Vdash_1 (\{A\})_1$. By ($\vdash$4) for $\vdash_2$, for some $V'_{\mathrm{fin}} \subseteq (X)_2$ we have that $Z_{\mathrm{fin}} \subseteq (\mathrm{Cn}_2((V'_{\mathrm{fin}})_2))_0$. Therefore

$$Y'_{\mathrm{fin}} \cup (\mathrm{Cn}_2((V'_{\mathrm{fin}})_2))_0 \Vdash_1 (\{A\})_1.$$

Proceeding analogously with $\vdash_2$, we obtain, for some $Y''_{\mathrm{fin}} \subseteq (X)_2$ and $V''_{\mathrm{fin}} \subseteq (X)_1$,

$$Y''_{\mathrm{fin}} \cup (\mathrm{Cn}_1((V''_{\mathrm{fin}})_1))_0 \Vdash_2 (\{A\})_2.$$

Taking X_{fin} to be $Y'_{\mathrm{fin}} \cup V'_{\mathrm{fin}} \cup Y''_{\mathrm{fin}} \cup V''_{\mathrm{fin}}$, which is a subset of X, we obtain $X_{\mathrm{fin}}\mathbf{R}A$.

It remains only to demonstrate that $\mathbf{R}$ satisfies ($\vdash$3.2), i.e., $(X\mathbf{R}A \;\&\; X \cup \{A\}\mathbf{R}B) \Rightarrow X\mathbf{R}B$, which as we have remarked can replace ($\vdash$3) in the presence of ($\vdash$2) and ($\vdash$4). So suppose $X\mathbf{R}A$, i.e.

(i) $$(X)_1 \cup (\mathrm{Cn}_2((X)_2))_0 \Vdash_1 (\{A\})_1,$$

(ii) $$(X)_2 \cup (\mathrm{Cn}_1((X)_1))_0 \Vdash_2 (\{A\})_2,$$

and suppose $\{A\} \cup X\mathbf{R}B$, i.e.

(iii) $$(\{A\})_1 \cup (X)_1 \cup (\mathrm{Cn}_2((X)_2 \cup (\{A\})_2))_0 \Vdash_1 (\{B\})_1,$$

(iv) $$(\{A\})_2 \cup (X)_2 \cup (\mathrm{Cn}_1((X)_1 \cup (\{A\})_1))_0 \Vdash_2 (\{B\})_2.$$

If $A \in L_0$, from (iii) and (iv) by applying Lemmata 6.1 and 6.2 we obtain

$$(\{A\})_1 \cup (X)_1 \cup (\mathrm{Cn}_2((X)_2))_0 \Vdash_1 (\{B\})_1,$$
$$(\{A\})_2 \cup (X)_2 \cup (\mathrm{Cn}_1((X)_1))_0 \Vdash_2 (\{B\})_2,$$

which together with (i) and (ii) by two applications of ($\vdash$3) yields $X\mathbf{R}B$. If $A \notin L_0$ and $A \in L_1$, then

(v) $$(X)_1 \cup (\mathrm{Cn}_2((X)_2))_0 \Vdash_1 (\{B\})_1$$

follows from (i) and (iii) by ($\vdash$3) (note that $(\{A\})_2 = \varnothing$). On the other hand, from (i) we have

$$(X)_1 \cup (\{A\})_1 \subseteq \mathrm{Cn}_1((X)_1 \cup (\mathrm{Cn}_2((X)_2))_0),$$
$$\mathrm{Cn}_1((X)_1 \cup (\{A\})_1) \subseteq \mathrm{Cn}_1(\mathrm{Cn}_1((X)_1 \cup (\mathrm{Cn}_2((X)_2))_0)),$$
$$\mathrm{Cn}_1((X)_1 \cup (\{A\})_1) \subseteq \mathrm{Cn}_1((X)_1 \cup (\mathrm{Cn}_2((X)_2))_0),$$
$$(\mathrm{Cn}_1((X)_1 \cup (\{A\})_1))_0 \subseteq (\mathrm{Cn}_1((X)_1 \cup (\mathrm{Cn}_2((X)_2))_0))_0,$$

which together with (iv) and ($\vdash$2), since $(\{A\})_2 = \varnothing$, gives

$$(X)_2 \cup (\mathrm{Cn}_1((X)_1 \cup (\mathrm{Cn}_2((X)_2))_0))_0 \Vdash_2 (\{B\})_2.$$

From this last statement by Lemma 6.2 we obtain

$$(X)_2 \cup (\mathrm{Cn}_2((X)_2))_0 \cup (\mathrm{Cn}_1((X)_1))_0 \Vdash_2 (\{B\})_2.$$

Since $(X)_2 \Vdash_2 (\mathrm{Cn}_2((X)_2))_0$, by ($\vdash$3) we obtain $(X)_2 \cup (\mathrm{Cn}_1((X)_1))_0 \Vdash_2 (\{B\})_2$, which together with (v) gives $X\mathbf{R}B$.

We proceed analogously in the case $A \notin L_0$ and $A \in L_2$. □

From Lemmata 1.2, 6.3 and 6.4 it follows immediately that $X \vdash_{cc} A \Rightarrow X\mathbf{R}A$. To show the converse, suppose $X\mathbf{R}A$. Since $\vdash_{cc}$ is an extension of $\vdash_1$ and $\vdash_2$, and since $(X)_1 \Vdash_{cc} (\mathrm{Cn}_1((X)_1))_0$ and $(X)_2 \Vdash_{cc} (\mathrm{Cn}_2((X)_2))_0$, we easily obtain $(X)_1 \cup (X)_2 \Vdash_{cc} (\{A\})_1$ and $(X)_1 \cup (X)_2 \Vdash_{cc} (\{A\})_2$, from which $(X)_1 \cup (X)_2 \vdash_{cc} A$ follows because of $A \in L_1 \cup L_2$. $X \vdash_{cc} A$ is obtained by use of ($\vdash$2), if necessary. Hence, $X\mathbf{R}A \Rightarrow X \vdash_{cc} A$, which proves Lemma 6. □

From Lemma 6 we obtain the following characterization of $X \vdash_{cc} A_0$:

LEMMA 7. $X \vdash_{cc} A_0 \Leftrightarrow (\mathrm{Cn}_1((X)_1))_0 \cup (\mathrm{Cn}_2((X)_2))_0 \vdash_1 A_0$.

PROOF. Suppose $X \vdash_{cc} A_0$. By Lemma 6, we have that $(X)_1 \cup (\mathrm{Cn}_2((X)_2))_0 \vdash_1 A_0$, which implies $(\mathrm{Cn}_1((X)_1 \cup (\mathrm{Cn}_2((X)_2))_0))_0 \vdash_1 A_0$. Then using (0) and Lemma 6.2 we obtain the right-hand side. For the converse we use (0), $(X)_1 \Vdash_1 (\mathrm{Cn}_1((X)_1))_0$, $(X)_2 \Vdash_2 (\mathrm{Cn}_2((X)_2))_0$, and Lemma 6. □

We know from §3 that Theorem 1.2 holds for $\vdash_{cc}$. But with the help of the characterization of Lemma 6 this can be proved independently and rather more directly. For interpolation assume $X_1 \vdash_{cc} A_2$. Hence, $(X_1)_2 \cup (\mathrm{Cn}_1(X_1))_0 \vdash_2 A_2$, from which we easily obtain $(\mathrm{Cn}_1(X_1))_0 \vdash_2 A_2$. For conservativeness assume $X_1 \vdash_{cc} A_1$. Hence,

$$(X_1)_1 \cup (\mathrm{Cn}_2((X_1)_2))_0 \vdash_1 A_1.$$

With $(X_1)_2 = (X_1)_0$ and (0), i.e. $(\mathrm{Cn}_2(Y_0))_0 = (\mathrm{Cn}_1(Y_0))_0$, it easily follows that $X_1 \vdash_1 A_1$.

What we have proved in this section depends on the assumption that an implication connective $\rightarrow$ is obtainable for $\vdash_1$ and $\vdash_2$ such that (MP) and (DT) hold. That this requirement cannot be dispensed with is shown by the following example: Let $\vdash_1$ be derivability in propositional logic with $\supset$ as the only connective and *modus ponens* for $\supset$ as the only inference rule, $\vdash_2$ the same with $\supset^*$ instead of $\supset$, L_0 therefore having no connective. Then for $A, B, C \in L_0$, from $\{A, A \supset B\} \vdash_1 B$ and $\{B, B \supset^* C\} \vdash_2 C$ it follows by ($\vdash$3) that $\{A, A \supset B, B \supset^* C\} \vdash_{cc} C$, but not necessarily that $\{A, A \supset B\} \cup (\mathrm{Cn}_2(\{A, B \supset^* C\}))_0 \vdash_1 C$ as required by Lemma 6. (The other conjunct required by Lemma 6, viz. $\{A, B \supset^* C\} \cup (\mathrm{Cn}_1(\{A, A \supset B\}))_0 \vdash_2 C$, does hold, since $B \in (\mathrm{Cn}_1(\{A, A \supset B\}))_0$.)

It is clear that for $A, B \in L_1$ or $A, B \in L_2$, the relation $\vdash_{cc}$ satisfies (MP). We shall also show that with these assumptions it satisfies (DT):

LEMMA 8. *If $A, B \in L_1$ or $A, B \in L_2$, then $X \cup \{A\} \vdash_{cc} B \Rightarrow X \vdash_{cc} A \rightarrow B$.*

PROOF. Suppose $X \cup \{A_1\} \vdash_{cc} B_1$. Then, by Lemma 6, we have

(i) $$(X)_1 \cup (\{A_1\})_1 \cup (\mathrm{Cn}_2((X)_2 \cup (\{A_1\})_2))_0 \Vdash_1 (\{B_1\})_1,$$

(ii) $$(X)_2 \cup (\{A_1\})_2 \cup (\mathrm{Cn}_1((X)_1 \cup (\{A_1\})_1))_0 \Vdash_2 (\{B_1\})_2.$$

If $A_1 \in L_0$, then we obtain by Lemmata 6.1 and 6.2

$$(X)_1 \cup \{A_1\} \cup (\mathrm{Cn}_2((X)_2))_0 \Vdash_1 (\{B_1\})_1,$$
$$(X)_2 \cup \{A_1\} \cup (\mathrm{Cn}_1((X)_1))_0 \Vdash_2 (\{B_1\})_2,$$

which together with (DT) for $\vdash_1$ and $\vdash_2$ yields $X \vdash_{cc} A_1 \rightarrow B_1$ (if $B_1 \notin L_2$, then $(X)_2 \cup (\mathrm{Cn}_1((X)_1))_0 \Vdash_2 (\{A_1 \rightarrow B_1\})_2$ is trivially satisfied because $(\{A_1 \rightarrow B_1\})_2 = \varnothing$). If $A_1 \notin L_0$, then from (i) with the help of (DT) for $\vdash_1$ we obtain

$$(X)_1 \cup (\mathrm{Cn}_2((X)_2))_0 \Vdash_1 (\{A_1 \rightarrow B_1\})_1.$$

Since $(\{A_1 \rightarrow B_1\})_2 = \varnothing$, we have that $(X)_2 \cup (\mathrm{Cn}_2((X)_2))_0 \Vdash_2 (\{A_1 \rightarrow B_1\})_2$ is trivially satisfied. Hence, $X \vdash_{cc} A_1 \rightarrow B_1$. We proceed analogously with $X \cup \{A_2\} \vdash_{cc} B_2$. □

§5. Interpolation and definability for first-order logic. Let L now be the set of formulae of a first-order language which has besides the usual logical constants arbitrarily many nonlogical constants. The subsets L_1 and L_2 of L are obtained by specifying which nonlogical constants may occur in them. Note that $L_1 \cup L_2$ is in general not closed under binary logical functors, and is hence a proper subset of L.

Let $\vdash$ be the usual relation of deducibility from hypotheses in either classical or intuitionistic logic. The relation $\vdash$ is a compact consequence relation. We assume that the deduction theorem holds in the following form:

$$X \cup \{A\} \vdash B \Rightarrow X \vdash A \to B.$$

This can be achieved by restricting the use of rules like universal generalization in proofs from hypotheses.

Let V_1 and V_2 be chosen in such a way that for every X_0 and A_0 we have $X_0 \cup V_1 \vdash A_0 \Leftrightarrow X_0 \cup V_2 \vdash A_0$. Then we define:

$$\begin{aligned} X_1 \vdash_1 A_1 &\Leftrightarrow_{\mathrm{df}} X_1 \cup V_1 \vdash A_1, \\ X_2 \vdash_2 A_2 &\Leftrightarrow_{\mathrm{df}} X_2 \cup V_2 \vdash A_2, \\ X \vdash^* A &\Leftrightarrow_{\mathrm{df}} X \cup V_1 \cup V_2 \vdash A. \end{aligned}$$

V_1 and V_2 need not be sets of sentences. Thus, unlike $\vdash$, the relations $\vdash_1$ and $\vdash_2$ are not necessarily closed under replacement of individual variables by terms; this substitution is not necessary in our framework.

It is easy to check that $\vdash_1$ and $\vdash_2$ are compact consequence relations over L_1 and L_2, respectively, for which (MP) and (DT) hold. Moreover the assumption (0) is satisfied for $\vdash_1$ and $\vdash_2$, and since in L_0 we have $\wedge$ and $\top$ (if $\top$ is not primitive it can be defined as $\forall x(x = x)$), both Theorem 1.2 and Theorem 1.3 hold for $\vdash_{\mathrm{cc}}$. Furthermore, we have the explicit characterization of $\vdash_{\mathrm{cc}}$ given in the previous section.

In general $X \vdash_{\mathrm{cc}} A$ is not equivalent to $X \vdash^* A$. For example, if $V_1 = V_2 = \varnothing$, then $\{A_1 \wedge A_2\} \vdash A_1$ holds, but not necessarily $\{A_1 \wedge A_2\} \vdash_{\mathrm{cc}} A_1$. Consider however the following restricted equivalence of $X \vdash_{\mathrm{cc}} A$ and $X \vdash^* A$:

(Equ) $X \cup \{A\} \subseteq L_1 \cup L_2 \Rightarrow (X \vdash^* A \Leftrightarrow X \vdash_{\mathrm{cc}} A)$.

We shall show that (Equ) is equivalent to the following Interpolation Property:

(Interp) $X_1 \cup X_2 \vdash^* A_2 \Rightarrow \exists B_0(X_1 \vdash_1 B_0 \;\&\; X_2 \cup \{B_0\} \vdash_2 A_2)$ and
$X_1 \cup X_2 \vdash^* A_1 \Rightarrow \exists B_0(X_2 \vdash_2 B_0 \;\&\; X_1 \cup \{B_0\} \vdash_1 A_1)$.

THEOREM 3. (Equ) $\Leftrightarrow$ (Interp).

PROOF. ($\Rightarrow$) Assume (Equ) and $X_1 \cup X_2 \vdash^* A_2$. With ($\vdash$4) and (DT) we can obtain a B_2 such that $X_1 \vdash^* B_2$, and then by (Equ) we get $X_1 \vdash_{\mathrm{cc}} B_2$. Then by Theorem 1.3 there is a B_0 such that $X_1 \vdash_1 B_0$ and $\{B_0\} \vdash_2 B_2$. Using (MP) and properties of $\vdash_2$, it follows easily from the second conjunct that $X_2 \cup \{B_0\} \vdash_2 A_2$. We proceed analogously with $X_1 \cup X_2 \vdash^* A_1$.

($\Leftarrow$) Assume (Interp). That $X \vdash_{\mathrm{cc}} A$ implies $X \vdash^* A$ is shown as follows. The relation $\vdash^*$ is a compact consequence relation extending $\vdash_1$ and $\vdash_2$, and we apply Lemma 1.2. Now assume $X \cup \{A\} \subseteq L_1 \cup L_2$ and $X \vdash^* A$. From the second conjunct we have $(X)_1 \cup (X)_2 \vdash^* A$. Let $A \in L_2$. Then by (Interp) there is a B_0 such that $(X)_1 \vdash_1 B_0$ and $(X)_2 \cup \{B_0\} \vdash_2 A$. Hence, by Lemma 1.2, $(X)_1 \vdash_{\mathrm{cc}} B_0$ and $(X)_2 \cup \{B_0\} \vdash_{\mathrm{cc}} A$, which by ($\vdash$3) gives $X \vdash_{\mathrm{cc}} A$. We proceed analogously for $A \in L_1$. $\square$

It is not difficult to show that (Interp) is equivalent to Craig's interpolation lemma. As a direct consequence of the first implication of (Interp) we have, for $X_1 = \{A_1\}$

and $X_2 = V_1 = V_2 = \varnothing$,

$$\{A_1\} \vdash A_2 \Rightarrow \exists B_0(\{A_1\} \vdash B_0 \;\&\; \{B_0\} \vdash A_2),$$

whereas from Craig's interpolation lemma using (DT) and (MP) for $\vdash$ we easily obtain (Interp).

By proving (Equ) for $V_1 = V_2 = \varnothing$ independently of (Interp) we would obtain an independent proof of Craig's lemma. Such a proof of (Equ) might proceed as follows. Using the characterization of Lemma 6, it is possible to prove that $\vdash_{cc}$ satisfies the rules of a sequent or natural deduction calculus formalizing first-order logic, if these rules are restricted to formulae of $L_1 \cup L_2$, i.e. if no mixing of L_1 and L_2 occurs within formulae. In Lemma 8 this is shown for the rule of $\rightarrow$-introduction, and along the lines of its proof it can be carried out for the other rules as well. If for this calculus we could demonstrate that

(Norm)

if $X \cup \{A\} \subseteq L_1 \cup L_2$ and $X \vdash A$ is provable, then there is a proof of $X \vdash A$ in which only members of $L_1 \cup L_2$ occur,

then we would have that $X \cup \{A\} \subseteq L_1 \cup L_2$ and $X \vdash A$ imply $X \vdash_{cc} A$ (that $X \vdash_{cc} A$ implies $X \vdash A$ follows from Lemma 1.2).

Since a proof of (Norm) would involve a cut-elimination or normalization procedure, which yields Craig's lemma more directly and more informatively, we shall not try to produce such a proof here. The point of this section was not to find an independent proof of Craig's lemma, but to demonstrate that with (Equ) Beth's definability theorem in a consequence of our definability theorem.

Let L_1 now be L_α of §1, and let L_2 be L_{α^*}. Suppose that α is implicitly definable in T_α, i.e. for every $A(\alpha)$ we have $T_\alpha \cup T_{\alpha^*} \cup \{A(\alpha)\} \vdash A(\alpha^*)$. Then if $V_1 = T_\alpha$ and $V_2 = T_{\alpha^*}$, by (Equ) we have $\{A(\alpha)\} \vdash_{cc} A(\alpha^*)$. It is not difficult to check that the conditions of our definability theorem are fulfilled, and hence there is a B_0 such that $\{A(\alpha)\} \vdash_1 B_0$ and $\{B_0\} \vdash_1 A(\alpha)$, i.e. $T_\alpha \cup A(\alpha) \vdash B_0$ and $T_\alpha \cup \{B_0\} \vdash A(\alpha)$. From this we conclude that α is explicitly definable.

For our definability theorem we did not make any assumption about the syntactical category of α, whereas in this section, as in Beth's theorem, α is restricted to the syntactical categories of nonlogical expressions of a first-order language. This restriction on α comes in via (Equ), which is equivalent to Craig's lemma. To prove Craig's lemma, we determine L_1 and L_2 by their nonlogical vocabulary.

The sets V_1 and V_2 used for defining $\vdash_1$ and $\vdash_2$ were not necessarily sets of sentences, as T_α and T_{α^*} are in Beth's theorem. We could take this freedom with V_1 and V_2, because we introduce implicit and explicit definability with the clause "for every $A(\alpha)$", and hence need not prove in Beth's theorem the universal closures of formulae like $\alpha(x_1,\ldots,x_n) \leftrightarrow \alpha^*(x_1,\ldots,x_n)$ and $\alpha(x_1,\ldots,x_n) \leftrightarrow B(x_1,\ldots,x_n)$. Usually, implicit and explicit definability are presented with universal closures of such formulae.

If from the hypotheses $T_\alpha \cup T_{\alpha^*} \cup \{A(\alpha)\}$ we have deduced $A(\alpha^*)$ with the help of first-order logic in $L = L_{\alpha\alpha^*}$, we might have applied logical laws involving logical constants to formulae with both α and α^*. Then (Equ), which guarantees that we also have $\{A(\alpha)\} \vdash_{cc} A(\alpha^*)$, shows that this mixing of α and α^* is unnecessary. With (Equ) we can reduce the mixing of α and α^* in logic, involving possible mixing on the "operational" level with logical constants, to mixing on the "structural" level only. This reduction enables us to view Beth's definability theorem as a consequence of our definability theorem.

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