Simplified Group Activity Selection[1]

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Abstract

Several real-world situations can be represented in terms of agents that have preferences over activities in which they may participate. Often, the agents can take part in at most one activity (for instance, since these take place simultaneously), and there are additional constraints on the number of agents that can participate in an activity. In such a setting we consider the task of assigning agents to activities in a reasonable way. We introduce the simplified group activity selection problem providing a general yet simple model for a broad variety of settings, and start investigating the case where upper and lower bounds of the groups have to be taken into account. We apply different solution concepts such as envy-freeness and core stability to our setting and provide a computational complexity study for the problem of finding such solutions.

1 Introduction

Several real-world situations can be represented in terms of agents that have preferences over activities in which they may participate, subject to some feasibility constraints on the way they are assigned to the different activities. In this respect, 'activity' should be taken in a wide sense; here are a few examples, each with its specificities which we will discuss further:

1. a group of co-workers may have to decide in which project to work, given that each project needs a fixed number of participants;
2. the participants to a big workshop, who are too numerous to fit all in a single restaurant, want to select a small number of restaurants (say, between two and four) out of a wider selection, with different capacities, and that serve different types of food, and to assign each participant to one of them;
3. a group of pensioners have to select two movies out of a wide selection, to be played simultaneously in two different rooms, and each of them will be able to see at most one of them;
4. a group of students have to choose one course each to follow out of a selection, given that each course opens only if it has a minimum number of registrants and has also an upper bound;
5. a set of voters want to select a committee of k representatives, given that each voter will be represented by one of the committee members.

While these examples seem to vary in several aspects, they share the same general structure: there is a set of *agents*, a set of available *activities*; each agent has preferences over the possible activities; there are constraints bearing on the selection of activities and the way agents are assigned to them; the goal is to assign each agent to one activity, respecting the constraints, and respecting as much as possible the agents' preferences.

Sometimes the set of selected activities is fixed (as Example 1), sometimes it will be determined by the agents' preferences. The nature of the constraints can vary: sometimes

[1] A previous version of this paper appeared in the proceedings of ADT'17 [8].

there are constraints that are *local* to each activity (typically, bounds on the number of participants, although we might imagine more complex constraints), as Examples 1, 2, 4, and also 3 if the rooms have a capacity smaller than the number of pensioners; sometimes there are *global* constraints, that bear on the whole assignment (typically, bounds on the number of activities that can be selected; once again, we may consider more complex constraints), as in Examples 2, 3, 5. Sometimes each agent *must* be assigned to an activity (as in Example 1), sometimes she has the option of not being assigned to any activity.

This class of problems can be seen as a simplified version of the *group activity selection problem* (GASP), which asks how to assign agents to activities in a "good" way. In the original form introduced by Darmann et al. [9], agents express their preferences both on the activities and on the number of participants for the latter; in general, these preferences are expressed by means of weak orders over pairs "(activity, group size)". Darmann [7] considers the variant of GASP in which the agents' preferences are strict orders over such pairs and analyzes the computational complexity of finding assignments that are stable or maximize the number of agents assigned to activities.

Our model considers a simplified version of the group activity selection problem, called s-GASP. Here, agents only express their preferences over the set of activities. However, the activities come with certain constraints, such as restrictions on the number of participants, concepts like balancedness, or more global restrictions. The goal is again to find a "good" assignment of agents to activities, respecting both the agents' preferences as well as the constraints.

But what is a good assignment? Clearly, this essentially depends on the application on hand, but there are several concepts in the social choice and game theory literature that propose for an evaluative solution. We consider two classes of criteria for assessing the quality of an assignment:

- *solution concepts* that mainly come from game theory and that aim at telling whether an assignment is stable enough (that is, immune to some types of deviations) to be implemented. First, *individual rationality* requires that each agent is assigned to an activity she likes better than not being assigned to any activity at all. Then, a solution concept considered both in hedonic games, where coalition building is studied, and in matching theory, is the notion of *stability*. It asks whether the assignment is stable in the sense that no agent would want to or be able to deviate from her coalition, her match, or in our case, her assigned activity. Besides considering different variants of *core stability*, it also makes sense in our setting to investigate variations of *virtual stability*, meaning that it is not possible that an agent deviates from her assigned activity due to the given constraints.

- *criteria* that mainly come from social choice theory and that measure, qualitatively or quantitatively, the welfare of agents. A common quality measure in terms of efficiency of an assignment is the notion of *Pareto optimality*: there should be no feasible assignment in which there is an agent that is strictly better off, while the remaining agents do not change for the worse. More generally, one may wish to *optimize social welfare*, for some notion of utility derived from the agents' preferences: for instance, one may simply be willing to maximize the number of agents assigned to an activity. If fairness is important in the design, the notion of *envy-freeness* makes sense: an assignment respecting the constraints is envy-free if no agent strictly prefers the activity another agent is assigned to.

Related Work.

Apart from GASP, our model is related to various streams of work:

Course allocation, e.g. [5, 11, 17, 22]. Students bear preferences over courses they would like to be enrolled in (these preferences are typically strict orders), and there are usually constraints given on the size of the courses. Courses will only be offered if a minimum number of participants is found, and there are upper bounds due to space or capacity limitations. In particular, Cechlárová and Fleiner [5] consider a course-allocation framework, so for them it makes sense that one agent can be matched to more than one activity (course), while Kamiyama [17] and Monte and Tumennnasan [22] consider the case in which an agent can be assigned to at most one activity (project). The latter works are very close to our setting with constraints over group sizes. In contrast to the above works however, our setting contains a dedicated outside option (the *void activity*), and agents' preferences are represented by weak orders over activities instead of strict rankings. In addition, our setting has a certain vicinity to the assignment problems considered by Arulselvan et al. [2] and Garg et al. [10].

Hedonic games (see the recent survey by Aziz and Savani [3]) are coalition formation games where each agent has preferences over coalitions containing her. The stability notions we will focus on are derived from those for hedonic games. However, in our model, agents do not care about who else is assigned to the same activity as them, but only on the activity to which they are assigned to.[2]

In *multiwinner elections*, there is a set of candidates, voters have preferences over single candidates, and a subset of k candidates has to be elected. In some approaches to multi-winner elections, each voter is assigned to one of the members of the elected committee, who is supposed to represent her. Sometimes there are no constraints on the number of voters assigned to a given committee member (as is the case for the *Chamberlin-Courant* rule [6]), in which case each voter is assigned to her most preferred committee member; on the other hand, for the *Monroe* rule [21], the assignment has to be balanced. A more general setting, with more general constraints, has been defined by Skowron et al. [24]. Note that multiwinner elections can also be interpreted as *resource allocation* with items that come in several units ([24]) and as *group recommendation* (Lu and Boutilier [20]). While assignment-based multiwinner elections problems are similar to simplified group activity selection, an important difference is that for the former, stability notions play no role, as the voters are not assumed to be able to deviate from their assigned representatives.

Contents and Outline.

In this work, we will take into account various solution concepts and ask two questions: First, do "good" assignments exist? Can we decide this efficiently? And if they exist, can we find them efficiently? Our second concern is optimization: we are looking for desirable assignments that maximize the number of agents which can be assigned to an activity. Again, we may ask whether an assignment that is optimal in this sense exists, and we can try to find it.

We will focus on one family of constraints concerning the size of the groups—we assume that each activity comes with a lower and an upper bound on the number of participants—and give a detailed analysis of the described problems for this class.

Our results for this class are twofold. First, we show that it is often possible to find assignments with desirable properties in an efficient way: we propose several polynomial time algorithms to find good assignments or to optimize them. We complement these findings with NP- and coNP-completeness results for certain solution concepts. Whenever

[2] Still, it is possible to express simplified group activity selection within the setting of hedonic games, by adding special agents corresponding to activities, who are indifferent between all locally feasible coalitions. See the work by Darmann et al. [9] for such a translation for the more general group activity selection problem. But it is a rather artificial, and overly complex, representation of our model, which moreover does not help characterizing and computing solution concepts.

we encounter computational hardness, we identify tractable special cases: we will see that basically all our problems can be solved in polynomial time if there is no restriction on the minimum number of participants for the activities to take place. An overview of our computational complexity results is given in Table 1 in Section 3; due to space constraints, we do not elaborate all proofs. Second, we show that also in this class of problems considered, there is a certain tension between the concepts of envy-freeness and Pareto optimality, even for small instances.

The remainder of this work is organized as follows. In Section 2, we formally introduce the simplified model as well as possible constraints and several solution concepts. Section 3 is the main part of the paper and provides an analysis of the computational complexity of the questions described above. Section 4 deals with the tension between envy-freeness and Pareto optimality. In Section 5, we conclude and discuss future directions of research.

2 Model, Constraints, and Solution Concepts

We start with defining our model and with introducing the solution concepts we want to consider.

2.0.1 Simplified Group Activity Selection, Constraints.

An instance (N, A, P, R) of the *simplified group activity selection problem* (s-GASP) is given as follows. The set $N = \{1, \dots, n\}$ denotes a set of agents and $A = A^* \cup \{a_\emptyset\}$ a set of activities with $A^* = \{a_1, \dots, a_m\}$, where $a_\emptyset$ stands for the *void activity.* An agent who is assigned to $a_\emptyset$ can be thought of as not participating in any activity. The preference profile $P = \langle \succsim_1, \dots, \succsim_n \rangle$ consists of n votes (one for each agent), where $\succsim_i$ is a weak order over A (with strict part $\succ_i$ and indifference part $\sim_i$) for each $i \in N$. The set R is a set of side constraints that restricts the set of assignments.

A mapping $\pi : N \to A$ is called an *assignment.* Given assignment π, $\#(\pi) = |\{i \in N : \pi(i) \neq a_\emptyset\}|$ denotes the number of agents π assigns to a non-void activity; for activity $a \in A$, $\pi^a := \{i \in N : \pi(i) = a\}$ is the set of agents π assigns to a.

The goal will be to find "good" assignments that satisfy the constraints in R. The structure of the set R depends on the application. Some typical kinds of constraints are (combinations of) the following cases:

1. each activity comes with a lower and/or upper bound on the number of participants;
2. no more than k activities can have some agent assigned to them;
3. the number of voters per activity should be balanced in some way.

Intuitively, if there are no constraints or the constraints are flexible enough, then agents go where they want and the problem becomes trivial. If the constraints are tight enough (e.g., perfect balancedness, provided $|A|$ and $|V|$ allow for it), then some agents are generally not happy, but they are unable to deviate because most deviations violate the constraints. The interesting cases can therefore be in between these two extreme cases.

In this work, we will start investigations for s-GASP for the first class of constraints: We assume that each activity $a \in A^*$ comes with a lower bound $\ell(a)$ and an upper bound $u(a)$, and all constraints in R are of the following type: for each $a \in A^*$, $|\pi^a| \in \{0\} \cup [\ell(a), u(a)]$. We lay particular focus on the special cases of $\ell(a) = 1$ and $u(a) = n$ respectively.

2.0.2 Feasible Assignments, Solution Concepts.

Let an instance (N, A, P, R) of s-GASP be given. A *feasible assignment* is an assignment meeting the constraints in R. We will consider the following properties. A feasible assignment π is

- *envy-free* if there is no pair of agents $(i, j) \in N \times N$ with $\pi(j) \in A^*$ such that $\pi(j) \succ_i \pi(i)$ holds;
- *individually rational* if for each $i \in N$ we have $\pi(i) \succsim a_\emptyset$;
- *individually stable* if there is no agent i and no activity $a \in A$ such that (i) $a \succ_i \pi(i)$ and (ii) the mapping π' defined by $\pi'(i) = a$ and $\pi'(k) = \pi(k)$ for $k \in N \setminus \{i\}$ is a feasible assignment;
- *core stable* if there is no set $E \subseteq N$ and no activity $a \in A$ such that (i) $a \succ_i \pi(i)$ for all $i \in E$, (ii) $\pi^a \subset E$ holds if $a \in A^*$, and (iii) the mapping π' defined by $\pi'(i) = a$ for $i \in E$ and $\pi'(k) = \pi(k)$ for $k \in N \setminus E$ is a feasible assignment; (Note that the respective activity a to which the set E of agents wishes to deviate must be either $a_\emptyset$ or currently unused.)
- *strictly core stable* if there is no set $E \subseteq N$ and no activity $a \in A$ such that (i) $a \succsim_i \pi(i)$ for all $i \in E$ where $a \succ_i \pi(i)$ for at least one $i \in E$, (ii) $\pi^a \subset E$ holds if $a \in A^*$, and (iii) the mapping π' defined by $\pi'(i) = a$ for all $i \in E$ and $\pi'(k) = \pi(k)$ for $k \in N \setminus E$ is a feasible assignment;
- *Pareto optimal* if there is no feasible assignment $\pi' \neq \pi$ such that $\pi'(i) \succsim_i \pi(i)$ for all $i \in N$ and $\pi'(i) \succ_i \pi(i)$ for at least one $i \in N$;

For the class of constraints we consider, the notion of *virtual stability* is interesting. It requires that any deviation from the assigned towards a more preferred activity a violates the capacity constraints of a. Formally, we define the following stability concepts (for the sake of conciseness we set $\ell(a_\emptyset) = 1$ and $u(a_\emptyset) = n$).

A feasible assignment π is

- *virtually individually stable* if there is no agent i and no activity $a \in A$ with $\ell(a) \leq |\pi^a| + 1 \leq u(a)$ such that $a \succ_i \pi(i)$ holds;
- *virtually core stable* if there is no set $E \subseteq N$ and no activity $a \in A$ with $\ell(a) \leq |E| \leq u(a)$ such that (i) $a \succ_i \pi(i)$ for all $i \in E$, and (ii) $\pi^a \subset E$ holds if $a \in A^*$;
- *virtually strictly core stable* if there is no set $E \subseteq N$ and no activity $a \in A$ with $\ell(a) \leq |E| \leq u(a)$ such that (i) $a \succsim_i \pi(i)$ for all $i \in E$ where $a \succ_i \pi(i)$ for at least one $i \in E$, and (ii) $\pi^a \subset E$ holds if $a \in A^*$.

Note that as in the definition of core stability, also in virtual core stability the respective activity a to which the set E of agents wishes to deviate must be either the void activity $a_\emptyset$ or currently unused.

Finally, an individually rational assignment π is *maximum individually rational* if for all individually rational assignments π' we have $\#(\pi) \geq \#(\pi')$. Analogously, maximum feasible/envy-free/.../virtually strictly core stable assignments are defined.

The relationships between the solution concepts are shown in Figure 1. Notably for none of the relations the converse holds as well. As the following theorem shows, the technical subtleties of the solution concepts lead to some surprising results. This also fixes an error in the corresponding figure in our previous work [6] where we wrongly claimed that (virtual) core stability implies (virtual) individual stability.

Theorem 1 *There are s-GASP-instances with assignments which are*

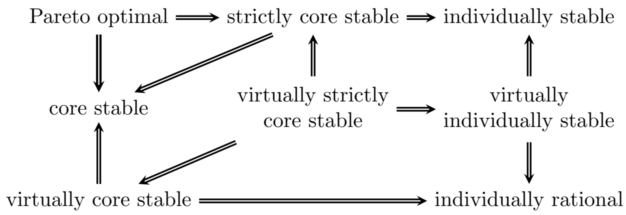

Figure 1: Relations between the solution concepts we consider, where an arrow directed from A to B means A implies B; e.g., a Pareto optimal assignment is also core stable.

1. *(virtually) core stable but neither (virtually) individually stable nor (virtually) strictly core stable,*
2. *virtually strictly core stable but not Pareto optimal,*
3. *Pareto optimal but not virtually strictly core stable,*
4. *individually stable but not individually rational,*
5. *individually stable and individually rational but not virtually individually stable.*

Proof: We give a proof for the first statement and refer to the Appendix for the remaining ones. Let $A^* = \{a, b\}$, $N = \{1, 2, 3, 4\}$, $a \succ_i b \succ_i a_\emptyset$ for $i \in \{1, 2\}$ and $b \succ_j a \succ_j a_\emptyset$ for $j \in \{3, 4\}$. Furthermore $\ell(a) = 1$ and $\ell(b) = 2$. The assignment π with $\pi^a = \{1\}$ and $\pi^b = \{2, 3, 4\}$ is not individually stable: the assignment π_* with $\pi_*^a = \{1, 2\}$ and $\pi_*^b = \{3, 4\}$ is an improvement for agent 2. The assignment π_* also contradicts strict core stability of π, as activity a and the coalition $E = \{1, 2\}$ meet the conditions. The assignment π is core stable: the only possible coalition would be $E = \{2\}$, but the transition to activity a is not permitted as a is neither the void activity nor unused under π. Analogously, it follows that π is also virtually core stable but neither virtually individually stable nor virtually strictly core stable. ∎

3 Computational Complexity for s-GASP with Group Size Constraints

We will now consider the computational complexity of s-GASP for various solution concepts. An overview of our results is given in Table 1.

3.1 Finding "Good" Assignments

The first interesting question is whether "good" assignments exist and how to find them. Obviously, assigning the void activity to every agent results in a feasible, individually rational and envy-free assignment. However, this is not a satisfying solution in terms of stability because agents will want to deviate. The good news is that for several stability concepts, a corresponding assignment always exists and can efficiently be found.

Theorem 2 *A strictly core stable assignment always exists and can be found in polynomial time.*

Recall that a strictly core stable assignment is also core stable and individually stable. Hence, as a consequence of the above theorem, also a core stable and an individually stable assignment always exist and can efficiently be found. As it turns out, an analogous result holds for virtually individually stable assignments.

find assignment that is	general	$u(a) = n$	$\ell(a) = 1$
feasible	in P (Prop 6)	in P (Prop 6)	in P (Prop 6)
individually rational	in P (Thm 3)	in P (Thm 3)	in P (Cor 19)
envy-free	in P (trivial)	in P (trivial)	in P (trivial)
individually stable	in P (Thm 2)	in P (Thm 2)	in P (Cor 19)
core stable	in P (Thm 2)	in P (Thm 2)	in P (Cor 19)
strictly core stable	in P (Thm 2)	in P (Thm 2)	in P (Cor 19)
virtually individually stable	in P (Thm 3)	in P (Thm 3)	in P (Cor 19)
virtually core stable	NP-c (Cor 5)	NP-c (Cor 5)	in P (Cor 19)
virtually strictly core stable	NP-c (Thm 4)	NP-c (Thm 4)	in P (Cor 19)
Pareto optimal	NP-h (Thm 17)	NP-h (Thm 17)	in P (Thm 18)
is there an assignment π with $\#(\pi) \geq k$ ($k \in \mathbb{N}$) that is	**general**	**$u(a) = n$**	**$\ell(a) = 1$**
feasible	in P (Prop 6)	in P (Prop 6)	in P (Prop 6)
individually rational	NP-c (Thm 7; Thm 10 of [5])	NP-c (Thm 7)	in P (Thm 8)
envy-free	NP-c (Thm 16)	in P (trivial)	?
individually stable	?	?	in P (Cor 19)
core stable	?	?	in P (Cor 19)
strictly core stable	NP-c (Thm 15)	NP-c (Thm 15)	in P (Cor 19)
virtually individually stable	NP-c (Thm 14)	NP-c (Thm 14)	in P (Cor 19)
virtually core stable	NP-c (Cor 5)	NP-c (Cor 5)	in P (Cor 19)
virtually strictly core stable	NP-c (Thm 4)	NP-c (Thm 4)	in P (Cor 19)
Pareto optimal	NP-h (Thm 20)	?	in P (Thm 18)

Table 1: Overview of results for constraints $|\pi^a| \in \{0\} \cup [\ell(a), u(a)]$, $a \in A^*$.

Theorem 3 *A virtually individually stable assignment always exists and can be found in polynomial time.*

Proof: In an instance (N, A, P, R) of s-GASP, we initially assign each agent to $a_\emptyset$, i.e., set $\pi(i) := a_\emptyset$ for $i \in N$. For $a \in A^*$ with $\ell(a) \geq 2$, if no agent is assigned to such a, then $\ell(a) \leq |\pi^a| + 1$ cannot hold. Hence, in what follows, we only consider activities $a \in A^*$ with $\ell(a) = 1$. For $1 \leq i \leq n$, assign agent i to the best ranked such activity $a \succ_i a_\emptyset$ with $|\pi^a| < u(a)$ and update π (i.e., set $\pi(i) := a$ while $\pi(j)$ remains unchanged for $j \in N \setminus \{i\}$). It is easy to see that the resulting assignment π is virtually individually stable. ■

In contrast, a virtually core stable (and thus a virtually strictly core stable) assignment does not always exist as Example 26 (see appendix) shows; in particular, the problem to decide whether or not a virtually strictly core stable assignment exists turns out to be computationally difficult.

Theorem 4 *It is* NP*-complete to decide if there is a virtually strictly core stable assignment, even when for each activity $a \in A^*$ we have $u(a) = n$.*

In the instance considered in the proof of the above theorem (see Appendix), an assignment is virtually strictly core stable if and only if it is virtually core stable. As a consequence, we get the following corollary.

Corollary 5 *It is* NP*-complete to decide if there is a virtually core stable assignment, even if for each activity $a \in A^*$ we have $u(a) = n$.*

However, for the case of $\ell(a) = 1$ for each $a \in A^*$, we get a positive complexity result (see Section 3.2). In particular, we can show that in this case a virtually strictly core stable assignment that maximizes the number of agents assigned to a non-void activity can be found in polynomial time.

Turning to Pareto optimality, in the special case of $\ell(a) = 1$ for each $a \in A^*$, there is a simple algorithm to compute a Pareto optimal assignment. In that case, it is easy to see that a Pareto optimal assignment is always individually rational. Thus, neglecting activities ranked below $a_\emptyset$, we start with the assignment $\pi(i) = a_\emptyset$ for each $i \in N$ and iteratively assign an agent to the best-ranked among the activities a with $|\pi^a| < u(a)$. However, in the case of $\ell(a) = 1$ for each $a \in A^*$ we can even find a Pareto optimal assignment that maximizes the number of agents assigned to a non-void activity in polynomial time (see Section 3.2).

3.2 Maximizing the Number of Agents Assigned to a Non-Void Activity

We now turn to an optimization problem: Among all feasible assignments that feature a certain property, one is usually interested in finding one that maximizes the number of agents that are assigned to a non-void activity, thus keeping the number of agents who cannot be enrolled in any activity low.

3.2.1 Feasible and Individually Rational Assignments

On the positive side, if we are only interested in a feasible assignment maximizing the number of agents assigned to a non-void activity, we can find such an assignment in polynomial time.

Proposition 6 *In polynomial time we can find a feasible assignment that maximizes the number of agents assigned to a non-void activity.*

But already for individual rational assignments it is hard to decide whether all agents can be assigned to a non-void activity.

Theorem 7 *It is* NP*-complete to decide if there is an individually rational assignment that assigns each agent to some $a \in A^*$, even if for each activity $a \in A^*$ we have $u(a) = n$.*

However, if we assume that each activity admits a group size of 1, then we can find an optimal individually rational assignment efficiently.

Theorem 8 *If for each activity $a \in A^*$ we have $\ell(a) = 1$, then in polynomial time we can find a maximum individually rational assignment.*

Proof: Reduction to max integer flow with upper bounds which is solvable in polynomial time (see, e.g., Ahuja et al. [1]) . Given an instance $\mathcal{I} = (N, A, P, R)$ of s-GASP with $\ell(a) = 1$ for all $a \in A^*$, we construct an instance $\mathcal{M}$ of max integer flow with directed graph $G = (V, E)$. Set $V := \{s, t\} \cup N \cup A^*$, and let the edges and their capacities be given as follows: for each $i \in N$, introduce edge (s, i) with capacity 1; for each $a \in A^*$ and $i \in N$ introduce an edge (i, a) of capacity 1 if $a \succsim_i a_\emptyset$ holds; for each $a \in A^*$, introduce edge (a, t) of capacity $u(a)$. It is easy to see that a max integer flow from s to t induces a maximum individually rational assignment in $\mathcal{I}$ and vice versa. ■

In the remainder of this subsection, we consider a special kind of constraints. We assume that for each activity, the upper bound equals the lower bound, i.e., for all activities $a \in A^*$ we have $\ell(a) = u(a) = q$ for some $q \in \mathbb{N}$—an activity can only take place if exactly q agents sign up for it. In this case, maximizing the number of agents assigned to an activity is the same as maximizing the number of activities that can take place. These constraints can also be thought of as asking for some kind of balancedness (hence can also be seen as belonging to type 3 of the constraints presented in Section 2.0.1) and seem natural in many applications: For example, for $q = 1$, the time slots available for taking an oral exam with a teacher can be seen as activities on which students express their preferences. Each student can only be assigned to one of these dates. For $q = 2$, the activities could be squash courts or climbing routes and can only be used by pairs of agents. In both settings, one is interested in satisfying as many students/players as possible (as well as in charging the maximum number of time slots/courts or routes).

Finding maximum individually rational assignments for such settings turns out to be solvable in polynomial time for $q = 1$. The same holds for two-player activities ($q = 2$). In contrast, for $q \geq 3$, the problem becomes NP-complete.

We will also make use of the following notation: Let $A' \subseteq A^*$ be a subset of non-void activities. The notation $A' \succsim_i a_\emptyset$ denotes that agent i does not strictly prefer the void activity compared to any of the activities in A'. When we use this notation to define a preference order for an agent i it means that we may choose any order of these activities such that $a \succsim_i a_\emptyset$ for all $a \in A'$ and $a_\emptyset \succsim_i b$ for all $b \in A \setminus A'$.

Theorem 9 *Let $q \in \{1, 2\}$. Further, let $\ell(a) = u(a) = q$ for each activity $a \in A^*$. Then, we can compute a maximum individually rational assignment in polynomial time.*

Proof: We show that any instance of s-GASP with $\ell(a) = u(a) = q$ for each activity $a \in A^*$ can be reduced to Maximum Matching in polynomial time. Let $\mathcal{I} = (N, A, P, R)$ be such an instance of s-GASP.

Let us first consider the case $q = 1$. We construct a bipartite graph $G = (V_N \cup V_A, E)$ as follows: Let $V_N := \{v_i \mid i \in N\}$ and $V_A := \{v_a \mid a \in A^*\}$, i.e., we create a vertex v_i for each agent $i \in N$ and a vertex v_a for each activity $a \in A^*$. For each pair $(v_i, v_a) \in V_N \times V_A$ we create the edge $\{v_i, v_a\} \in E$ if and only if agent i ranks activity a higher than the void activity. It is easy to see that a maximum matching in the constructed graph corresponds to a maximum individually rational assignment for the given instance of s-GASP.

Let us now consider the case $q = 2$. We construct a graph $G = (V_N \cup V_A \cup V_{A'}, E)$ as follows: First, we create the same vertex sets V_N and V_A as in the previous case. Then, we create another vertex set $V'_A := \{v'_a \mid a \in A^*\}$ that contains a copy of each vertex in V_A, i.e., each activity $a \in A^*$ is represented by exactly two vertices. Informally, v_a and v'_a represent the two available places of each activity and each agent that is interested in the activity is therefore interested in both of these places. For each activity $a \in A^*$ and each agent $i \in N$, we create the edges $\{v_i, v_a\} \in E$ and $\{v_i, v'_a\} \in E$ if and only if agent i ranks the activity a higher than the void activity (this preference does not have to be strict). Additionally, we create the edges $\{v_a, v'_a\} \in E$ for each activity $a \in A^*$. Note that the graph is not bipartite anymore. W.l.o.g. we can assume that for each activity $a \in A^*$ a maximum matching in the constructed graph either contains $\{v_a, v'_a\}$ or two edges $\{v_i, v_a\}$, $\{v_j, v'_a\}$ for some $i, j \in N$ with $i \neq j$ (if only one of the vertices v_a, v'_a is matched, we simply replace the corresponding edge with $\{v_a, v'_a\}$). The first part represents that no agent is assigned to activity a and the second part that agents i and j are both assigned to a. It is easy to see that such a maximum matching corresponds to a maximum individually rational assignment for the given instance of s-GASP(see Figure 2 in the Appendix for an example). ■

Corollary 10 *Let $q \in \{1, 2\}$. Further, let $\ell(a) = u(a) = q$ for each activity $a \in A^*$. Then, we can decide in polynomial time if there is an individually rational assignment that assigns each agent to some $a \in A^*$.*

The above result also holds for the case that the groups sizes are all in the interval $[1, 2]$ (see Appendix; the below corollary also follows from a more general theorem for weighted many-to-one matchings by Arulselvan et al. [2]).

Corollary 11 *Let $1 \leq \ell(a) \leq u(a) \leq 2$ for each activity $a \in A^*$. Then, we can decide in polynomial time if there is an individually rational assignment that assigns each agent to some $a \in A^*$.*

Theorem 12 *Let $q \geq 3$. It is* NP*-complete to decide if there is an individually rational assignment that assigns each agent to some $a \in A^*$, even if for each activity $a \in A^*$ we have $\ell(a) = u(a) = q$.*

Proof: Membership in NP is easy to see. We show NP-hardness by reduction from PERFECT H-MATCHING, where we are given a graph $G = (V, E)$ and a second graph H. The task is to cover the graph G with vertex-disjoint copies of H, i.e., the vertex sets of the copies of H must be a partition of V. The problem to decide if such a cover exists is NP-complete for all graphs H containing a component with at least three vertices [18] (our notation is similar to Berman et al. [4]). Given an instance of PERFECT H-MATCHING, where H is the complete graph on q vertices (i.e., a *clique* of size q), we construct the instance of s-GASP as follows.

Let $V = \{v_1, v_2, \ldots, v_n\}$ be the set of vertices of G. Further, let c_q be the number of cliques of size q in G and let $\mathcal{C} := \{C_j \mid 1 \leq j \leq c_q\}$ be the corresponding subsets of vertices. Note that $\mathcal{C}$ can be computed in polynomial time since q is a constant. For each clique $C_j \in \mathcal{C}$ we create an activity a_j. The set A of activities is then $A = \{a_\emptyset\} \cup \bigcup_{j=1}^{q} \{a_j\}$. Then, we create an agent i for each vertex v_i and construct her preference as follows: Let A_i^+ be the set of activities corresponding to the cliques $\{C_j \subseteq \mathcal{C} \mid v_i \in C_j\}$, i.e., the cliques of size q that contain the vertex v_i. Then, we set $A_i^+ \succsim_i a_\emptyset \succsim_i A \setminus A_i^+$, where the ranking of the activities in A_i^+ and $A \setminus A_i^+$ can be chosen arbitrarily. The restrictions R are given by $|\pi^a| \in \{0, q\}$ for all $a \in A^*$. Then there exists a vertex-disjoint cover of G if and only if there is an individually rational assignment as required.

"$\Rightarrow$" Let $\mathcal{S} \subseteq \mathcal{C}$ cover the graph with vertex-disjoint copies of the complete graph with q vertices. For each clique $C_j \in \mathcal{S}$, we assign all agents i where $v_i \in C_j$ to the activity a_j. By construction, the restrictions are satisfied and each agent is assigned to exactly one (non-void) activity ranked higher than $a_\emptyset$. Hence, the assignment is individually rational.

"$\Leftarrow$" Let π be an individually rational assignment of the agents to non-void activities that satisfies $|\pi^a| \in \{0, q\}$ for each activity $a \in A^*$. Let $\mathcal{J}$ be the set of indices of the activities a_j with $|\pi^{a_j}| = q$. Then, $\mathcal{S} = \bigcup_{j \in \mathcal{J}} \{C_j\}$ covers the graph G with vertex-disjoint copies of the complete graph with q vertices (this follows from the fact that each agent is assigned to exactly one non-void activity). ■

Corollary 13 *Let $q \geq 3$ and $k \in \mathbb{N}$. It is* NP*-complete to decide if there is an individually rational assignment that assigns at least k agents to activities, even if for each activity $a \in A^*$ we have $\ell(a) = u(a) = q$.*

3.2.2 Stable and Envy-Free Assignments

Unfortunately, deciding whether or not there is a a virtually individually stable or strictly core stable assignment exists turns out to be compuationally hard even if the upper bound of each activity equals n.

Theorem 14 *It is* NP*-complete to decide if there is a virtually individually stable assignment that assigns each agent to some $a \in A^*$, even if for each activity $a \in A^*$ we have $u(a) = n$.*

Theorem 15 *It is* NP*-complete to decide if there is a strictly core stable assignment that assigns each agent to some* $a \in A^*$*, even if for each activity* $a \in A^*$ *we have* $u(a) = n$*.*

For envy-freeness, maximizing the number of "active" agents turns again out to be hard.

Theorem 16 *It is* NP*-complete to decide if there is an envy-free assignment that assigns each agent to some* $a \in A^*$*.*

However, we obtain tractability for envy-freeness if we loosen the constraints on the upper bounds of the group sizes: Clearly, if there is an activity with "unlimited" capacity (i.e., its upper bound equals n), we can assign all agents to it and obtain envy-freeness. It is not clear yet whether the problem becomes tractable if $\ell(a) = 1$ holds for all $a \in A^*$. This is the case though if all preference orders are strict (see Proposition 27 in Appendix).

3.3 Pareto Optimality

In this subsection, we consider the computational complexity involved in maximizing the number of agents assigned to non-void activities in Pareto optimal assignments. In the framework of course allocation, if all agents have strict preferences, it is known that a Pareto optimal matching—that assigns an agent to an activity (course) only if the activity is acceptable for the agent—can be found in polynomial time ([5, 17]). Since in our setting (i) the agents' preferences are represented by weak orders and (ii) Pareto optimality does not require individual rationality, these results do not immediately translate. For the latter reason, the computational intractability result by Cechlárová and Fleiner [5] (for finding a Pareto optimal matching maximizing the number of agents assigned to a non-void activity if each agent can be assigned to at most one activity) does not immediately translate to our setting either. Our first results shows that finding a Pareto optimal assignment (or of finding one that maximizes the number of agents assigned to non-void activities) in s-GASP is NP-hard even for the case $u(a) = n$ for each activity a.

Theorem 17 *It is* NP*-hard to find a Pareto optimal assignment in an instance of* s-GASP*, even when* $u(a) = n$ *for each* $a \in A^*$*.*

However, as the following theorem shows, if we relax the constraints on the lower bound of the group sizes the problem of finding a Pareto optimal assignment becomes computationally tractable.

Theorem 18 *If for each activity* $a \in A^*$ *we have* $\ell(a) = 1$*, then in polynomial time we can find a Pareto optimal assignment that maximizes the number of agents assigned to a non-void activity.*

Note that in the case $\ell(a) = 1$ for each $a \in A^*$, also any strictly core stable, core stable, or individually stable assignment is individually rational. In addition, in this case virtual (strict) core stability coincides with (strict) core stability, and virtually individual stability coincides with individual stability. Hence we can state the following corollary.

Corollary 19 *If for each activity* $a \in A^*$ *we have* $\ell(a) = 1$*, then in polynomial time we can find a maximum individually rational assignment that is Pareto optimal, (virtually) individually stable, (virtually) core stable and (virtually) strictly core stable.*

However, as the following theorem shows, in contrast to the case of the lower bound being 1 for each activity (Theorem 18), in general it turns out to be computationally hard to decide if there is a Pareto optimal assignment that assigns each agent to a non-void activity.

Theorem 20 *It is* NP*-hard to decide if there is a Pareto optimal assignment that assigns each agent to a non-void activity.*

4 Envy-Freeness versus Pareto Optimality

In many social choice settings, there is a tension between envy-freeness and Pareto optimality. This is also the case for our simplified group activity selection problem, as the following propositions show.

Proposition 21 *For any $k \geq 2$, there is an instance (N, A, P, R) of s-GASP with $|N| = k$ and $\ell(a) = 1$ for each $a \in A^*$, for which there does not exist an assignment π which is both Pareto optimal and envy-free.*

Proof: We provide a proof for $k = 2$, which easily extends to $n = k$ for any $k > 2$. Consider the instance with $N = \{1, 2\}$, $A^* = \{a\}$, with the rankings $a \succ_1 a_\emptyset$ and $a \succ_2 a_\emptyset$, and the restrictions given by $\ell(a) = u(a) = 1$. Any Pareto optimal assignment assigns exactly one agent to a, which is clearly not envy-free. ■

Interestingly, this tension also holds if the only relevant constraint is the lower bound of the activities (i.e., $u(a) = n$ for all a).

Proposition 22 *For any $k \geq 6$, there is an instance (N, A, P, R) of s-GASP with $|N| = k$ and $u(a) = k$ for each $a \in A^*$, for which there does not exist an assignment π which is both Pareto optimal and envy-free.*

Proof: We provide the idea of the proof for $k = 6$ and refer to the Appendix for details. Consider the instance of s-GASP with $N = \{1, 2, 3, 4, 5, 6\}$, $A^* = \{a, b, c\}$ and for any $x \in A^*$ we have $\ell(x) = 3, u(x) = 6$. The rankings are

$$\begin{array}{llll} \succsim_1: & a \succ_1 b \succ_1 c \succ_1 a_\emptyset & \succsim_4: & a \succ_4 b \succ_4 c \succ_4 a_\emptyset \\ \succsim_2: & b \succ_2 c \succ_2 a \succ_2 a_\emptyset & \succsim_5: & b \succ_5 c \succ_5 a \succ_5 a_\emptyset \\ \succsim_3: & c \succ_3 a \succ_3 b \succ_3 a_\emptyset & \succsim_6: & c \succ_6 a \succ_6 b \succ_6 a_\emptyset \end{array} .$$

Due to the feasibility constraints, there are only 4 types of feasible assignments, none of which is both envy-free and Pareto optimal. ■

5 Conclusion

We have formulated a simplified version of GASP where the assignment of agents to activities depends on the agents' preferences as well as on exogenous constraints. This model is powerful enough to capture many real world applications. We have made a first step by analyzing one family of constraints and have studied several solution concepts for this family.

An obvious next step is to drive a similar analysis for other interesting classes of constraints as described in Section 2. In particular, it would be interesting to characterize families of constraints guaranteeing or not guaranteeing existence of a stable solution for the different solution concepts we considered, or exploring forbidden structures that prevent stability. Also, it would be nice to provide a detailed analysis of the parameterized complexity of the hard cases, as done by Lee and Williams [19] for the stable invitation problem, and by Igarashi et al. [15, 16] and Gupta et al. [12] for GASP on social networks. Another variant would be to consider typed agents as in the work by Spradling and Goldsmith [25].

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Appendix

Proofs of Section 1

Theorem 1 There are s-GASP-instances with assignments which are

1. (virtually) core stable but neither (virtually) individually stable nor (virtually) strictly core stable,
2. virtually strictly core stable but not Pareto optimal,
3. Pareto optimal but not virtually strictly core stable,
4. individually stable but not individually rational,
5. individually stable and individually rational but not virtually individually stable.

Proof: Below, we give three examples which show that Statements 2-5 hold. Example 23 shows Statement 2. Example 24 shows Statement 3 and Statement 4. Finally, Example 25 shows that Statement 5 holds. ■

Example 23 *Let $N = \{1, 2\}$, $A^* = \{a, b\}$ with $\forall x \in A^* \; : \ell(x) = 1 = u(x)$. Consider the following preference profile:*

$$\begin{array}{ll} \succsim_1: & a \succ_1 \mathbf{b} \succ_1 a_\emptyset \\ \succsim_2: & b \succ_2 \mathbf{a} \succ_2 a_\emptyset \end{array}$$

Then, the assignment π with $\pi(1) = b$ and $\pi(2) = a$ is virtually strictly core stable but not Pareto optimal.

Example 24 *Let $N = \{1, 2\}$, $A^* = \{a, b\}$ with $\forall x \in A^* \; : \ell(x) = 2 = u(x)$ and the following preference profile:*

$$\begin{array}{ll} \succsim_1: & \mathbf{a} \succ_1 a_\emptyset \succ_1 b \\ \succsim_2: & b \succ_2 a_\emptyset \succ_2 \mathbf{a} \end{array}$$

Assignment π with $\pi^a = \{1, 2\}$ is Pareto optimal and individually stable, but neither virtually strictly core stable nor individually rational.

Example 25 *Let $N = \{1, 2, 3\}$, $A^* = \{a, b\}$ with $\ell(a) = 1$, $u(a) = 2$ and $\ell(b) = 2 = u(b)$, with the following preference profile:*

$$\begin{array}{ll} \succsim_1: & \mathbf{a} \succ_1 b \succ_1 a_\emptyset \\ \succsim_2: & a \succ_2 \mathbf{b} \succ_2 a_\emptyset \\ \succsim_3: & \mathbf{b} \succ_3 a \succ_3 a_\emptyset \end{array}$$

Assignment π with $\pi(1) = a$ and $\pi^b = \{2, 3\}$ is individually stable and individually rational but not virtually individually stable.

Proofs of Section 3

Proofs of Section 3.1

Theorem 2 A strictly core stable assignment always exists and can be found in polynomial time.

Proof: The basic idea behind algorithm 1 is as follows. Starting with a feasible assignment π, for each agent i and each activity b which i prefers to $\pi(i)$ we check whether there is a subset of agents including agent i that want to deviate to b such that the resulting assignment is feasible. That is, we check whether there is a subset $E \supset \pi^b$ such that (i) for all $j \in E$ we have that $b \succsim_j \pi(j)$ holds (recall that for agent i $b \succ_i \pi(i)$ holds) and (ii) π' with $\pi'(i) = b$ for $i \in E$ and $\pi'(j) = \pi(j)$ for $j \in N \setminus E$ is a feasible assignment. In order to do so, for each activity $c \in A \setminus \{b\}$, we compute the possible numbers of agents in the set π^c that agree with joining b and can be removed from π^c while still enabling a feasible assignment—these numbers are stored in the set R^c. Finally, given these numbers, we need to verify if—including i and the agents in π^b—these add up to an integer contained in $[\ell(b), u(b)]$ by taking exactly one number from each activity (note that 0 must be removed from R^a since we need to include agent i; also, note that for activity b we need to include the whole amount $|\pi^b|$ of agents assigned to b under π). The latter problem reduces to the Multiple-Choice Subset-Sum problem (see [23]), which, in our case, allows for an overall polynomial time algorithm for finding a strictly core stable assignment.

Algorithm 1 Algorithm for finding a strictly core stable assignment in an instance (N, A, P, R) of s-GASP.

Let π be a maximum individually rational assignment.
$R^{a'} := \emptyset$ for all $a' \in A$, $N' := \emptyset$, $B := \emptyset$, $i := 0$, $\pi'(j) = a_\emptyset$ for all $j \in N$
while $N \setminus N' \neq \emptyset$ **do**
  $i := \min N \setminus N'$
  $a := \pi(i)$
  $D := \{b \in A : b \succ_i a\}$
  while $D \neq \emptyset$ **do**
    take $b \in D$
    $B := \{j \in N : b \succsim_j \pi(j)\}$
    for $c \in A \setminus \{b\}$ **do**
      $R^c := \{0\}$
      for $1 \leq h \leq |\pi^c \cap B|$ **do**
        if $|\pi^c| - h \in [\ell(c), u(c)]$ **then**
          $R^c := R^c \cup \{h\}$
        end if
      end for
    end for
    $R^a := R^a \setminus \{0\}$
    if $R^a \neq \emptyset$ **then**
      if $\exists S$ with $(|\pi^b| + \sum_{j \in S} j) \in [\ell(b), u(b)]$ such that (i) $|S| = |A| - 1$ and (ii) for all $a' \in A \setminus \{b\}$ we have $|S \cap R^{a'}| = 1$ **then**
        take such a set S
        for $a' \in A \setminus \{b\}$, let $h_{a'}$ denote the unique element in $S \cap R_{a'}$
        $\pi'(i) := b$
        set $\pi'(j) := b$ for $(h_a - 1)$ arbitrarily chosen $j \in (\pi^a \cap B) \setminus \{i\}$
        for $a' \in A \setminus \{a, b\}$ **do**
          set $\pi'(j) := b$ for $h_{a'}$ arbitrarily chosen $j \in \pi^{a'} \cap B$
        end for
        for each of the remaining agents g set $\pi'(g) := \pi(g)$;
        $\pi := \pi'$.
      end if
    end if
    $D := D \setminus \{b\}$
  end while
  $N' := N' \cup \{i\}$
end while

As far as the running time of algorithm 1 is concerned, its bottleneck is to decide whether we can add up the above-mentioned numbers to be in the interval $[\ell(b), u(b)]$. In the Multiple-Choice Subset-Sum problem, we ask if taking exactly one number of each member of a given family of subsets of non-negative integers adds up to a given number. Applying for instance Pisinger's algorithm [23] for that problem[3] requires a running time of $\mathcal{O}(m^2)$ per execution. Clearly, for each interval, we need to execute the algorithm at most $(m+1)$ times. Thus, the overall running time of our algorithm can roughly be bounded by $\mathcal{O}(nm^4)$, because in the worst case we solve an instance of the Multiple-Choice Subset-Sum problem at most once for each agent and activity, i.e., nm times. ■

Example 26 *Let $N = \{1, 2, 3\}$ and $A^* = \{a, b, c\}$, with $a \succ_1 b \succ_1 c \succ a_\emptyset$, $b \succ_2 c \succ_2 a \succ a_\emptyset$, and $c \succ_3 a \succ_3 b \succ a_\emptyset$. The restrictions on the activities are given by $|\pi^x| \in \{0\} \cup [2, 3]$, for each $x \in A^*$. By the restrictions given, there is at most one non-void activity to which agents can be assigned. Clearly, for any activity $z \in A$ there is a $y \in A^*$ such that two agents prefer y to z. As a consequence, there can be no virtually core stable assignment.*

Theorem 4 It is NP-complete to decide if there is a virtually strictly core stable assignment, even when for each activity $a \in A^*$ we have $u(a) = n$.

Proof: Membership in NP is not difficult to verify. The proof proceeds by a reduction from Exact Cover by 3-Sets (X3C). The input of an instance of X3C consists of a pair $\langle X, \mathcal{Z} \rangle$, where $X = \{1, \ldots, 3q\}$ and $\mathcal{Z} = \{Z_1, \ldots, Z_p\}$ is a collection of 3-element subsets of X; the question is whether we can cover X with exactly q sets of $\mathcal{Z}$. X3C is known to be NP-complete even when each element of X is contained in exactly three sets of $\mathcal{Z}$ (see [13, 14]); note that in such a case $p = 3q$ holds. For each $i \in X$, let the sets containing i be denoted by $Z_{i_1}, Z_{i_2}, Z_{i_3}$ with $i_1 < i_2 < i_3$.
Define instance $\mathcal{I} = (N, A, P, R)$ of s-GASP as follows. Let $N = \{V_{i,1}, V_{i,2}, V_{i,3} \mid 1 \le i \le p\}$ and $A^* = \{y_i, a_i, b_i, c_i \mid 1 \le i \le p\}$. For $1 \le i \le p$, let $\ell(a_i) = \ell(b_i) = \ell(c_i) = 2$, and $\ell(y_i) = 9$. For each $a \in A^*$, let $u(a) = |N|$. Since any virtually strictly core stable assignment is individually rational, in the profile P we omit the activities ranked below $a_\emptyset$; for each $i \in \{1, \ldots, p\}$, let the ranking of the agents $V_{i,1}, V_{i,2}, V_{i,3}$ (each of which represents element $i \in X$) be given as follows:

$$\begin{aligned}
V_{i,1}:&\ y_{i_1} \succ_{i,1} y_{i_2} \succ_{i,1} y_{i_3} \succ_{i,1} a_i \succ_{i,1} b_i \succ_{i,1} c_i \succ_{i,1} a_\emptyset \\
V_{i,2}:&\ y_{i_2} \succ_{i,2} y_{i_3} \succ_{i,2} y_{i_1} \succ_{i,2} b_i \succ_{i,2} c_i \succ_{i,2} a_i \succ_{i,2} a_\emptyset \\
V_{i,3}:&\ y_{i_3} \succ_{i,3} y_{i_1} \succ_{i,3} y_{i_2} \succ_{i,3} c_i \succ_{i,3} a_i \succ_{i,3} b_i \succ_{i,3} a_\emptyset
\end{aligned}$$

Note that each set Z contains three elements, and hence each y_i, $1 \le 1 \le p$, is preferred to $a_\emptyset$ by exactly 9 agents. We show that there is an exact cover in instance $\langle X, \mathcal{Z} \rangle$ if and only if there is a virtually strictly core stable assignment in instance $\mathcal{I}$.

Assume there is an exact cover C. Consider the assignment π defined by $\pi(V_{i,h}) = y_j$ if $i \in Z_j$ and $Z_j \in C$, for $i \in \{1, \ldots, p\}$ and $h \in \{1, 2, 3\}$. Since C is an exact cover, assignment π is well-defined and feasible; note that each agent is assigned to an activity she ranks first, second or third. In addition, note that for $Z_j \in C$, each agent that prefers y_j to $a_\emptyset$ is assigned to y_j. Assume a set of agents E wishes to deviate to another activity d, such that at least one member $i \in E$ prefers d over $\pi(i)$ while there is no $j \in E$ with $\pi(j) \succ_j d$. By the definition of π, $d \in \{y_i \mid 1 \le i \le p\}$ holds. Observe that $\pi^d = \emptyset$ holds because C is an exact cover. Due to $\ell(d) = 9$, it hence follows that each agent of those who prefer d to $a_\emptyset$ must prefer d to the assigned activity, which is impossible since, by construction of the instance, for at least one of these agents j the assigned activity is top-ranked, i.e., $\pi(j) \succ_j d$ holds. Therewith, π is virtually strictly core stable.

[3] In Pisinger's work [23], the algorithm is formulated for positive weights only but extends to non-negative integers.

Conversely, assume there is a virtually strictly core stable assignment π. Assume that there is an agent $V_{i,h}$ who is not assigned to one of the activities $y_{i_1}, y_{i_2}, y_{i_3}$. Then, by $\ell(y_i) = 9$ and the fact that exactly 9 agents prefer y_i to $a_\emptyset$ for each $i \in \{1, \ldots, p\}$, it follows that no agent is assigned to one of $y_{i_1}, y_{i_2}, y_{i_3}$; in particular none of $V_{i,1}, V_{i,2}, V_{i,3}$ is assigned to one of these activities. Analogously to Example 26 it then follows that there is no virtually strictly core stable assignment, in contradiction with our assumption.
Thus, π assigns each agent $V_{i,h}$ to one of the activities $y_{i_1}, y_{i_2}, y_{i_3}$. For each $i \in \{1, \ldots, p\}$, by $\ell(y_i) = 9$ and the fact that exactly 9 agents prefer y_i to $a_\emptyset$ it follows that to exactly one of $y_{i_1}, y_{i_2}, y_{i_3}$ exactly 9 agents are assigned, while no agent is assigned to the remaining two activities. As a consequence, the set $C = \{Z_i \mid |\pi^{y_i}| = 9, 1 \leq i \leq p\}$ is an exact cover in instance $\langle X, \mathcal{Z} \rangle$.■

Proofs of Section 3.2

Proposition 6 In polynomial time we can find a feasible assignment that maximizes the number of agents assigned to a non-void activity.

Proof: We need to find the maximum number k such that taking, for each $a \in A^*$, exactly one number of $\{0\} \cup [\ell(a), u(a)]$ adds up to k. This problem corresponds to the MULTIPLE-CHOICE SUBSET-SUM problem (see [23]); in our case, the latter allows for an overall polynomial time algorithm since $u(a)$ in bounded by n. ■

Theorem 7 It is NP-complete to decide if there is an individually rational assignment that assigns each agent to some $a \in A^*$, even if for each activity $a \in A^*$ we have $u(a) = n$.

Proof: The proof proceeds by a reduction from EXACT COVER BY 3-SETS (X3C). The input of an instance of X3C consists of a pair $\langle X, \mathcal{Z} \rangle$, where $X = \{1, \ldots, 3q\}$, and $\mathcal{Z} = \{Z_1, \ldots, Z_p\}$ is a collection of 3-element subsets of X; the question is whether we can cover X with exactly q sets of $\mathcal{Z}$. X3C is known to be NP-complete even when each element of X is contained in exactly three sets of $\mathcal{Z}$ (see [13, 14]); note that in such a case $p = 3q$ holds. For each $i \in X$ let the sets containing i be denoted by $Z_{i_1}, Z_{i_2}, Z_{i_3}$ with $i_1 < i_2 < i_3$.
Given such an instance $\langle X, \mathcal{Z} \rangle$ of X3C, we construct an instance $\mathcal{I} = (N, A, P, R)$ of s-GASP as follows. Set $N := X$, $A := \{a_1, \ldots, a_p\}$ and let P be an arbitrary profile such that for each $i \in N$ the ranking restricted to the four top-ranked activities is given by $a_{i_1} \succ_i a_{i_2} \succ_i a_{i_3} \succ_i a_\emptyset$. The restrictions are given by $|\pi^a| \in \{0\} \cup [3, n]$, for all $a \in A^*$.
If there is an exact cover C, consider the assignment π given by $\pi(i) = a_j$, with $i \in Z_j$ and $Z_j \in C$, for $i \in N$. Clearly, since C is an exact cover, π is well-defined since for any $i \in X$, $i \in Z_j$ holds for exactly one set $Z_j \in C$. In addition, it is not hard to verify that π is feasible (since each Z_j contains exactly three elements and hence either no agent or exactly three agents are assigned to a_j), individually rational, and assigns each agent to a non-void activity.
On the other hand, if an assignment π satisfies these properties, then to any non-void activity either zero or exactly three agents are assigned (recall that in the original X3C-instance, each set contains exactly three elements). I.e., for exactly q activities $a \in A^*$ we have $|\pi^a| > 0$. Since all agents are assigned to a non-void activity, with individual rationality we get that $D = \{Z_j \mid |\pi^{a_j}| > 0,\ 1 \leq j \leq p\}$ is an exact cover for X. ■

Corollary 11 Let $1 \leq \ell(a) \leq u(a) \leq 2$ for each activity $a \in A^*$. Then, we can decide in polynomial time if there is an individually rational assignment that assigns each agent to some $a \in A^*$.

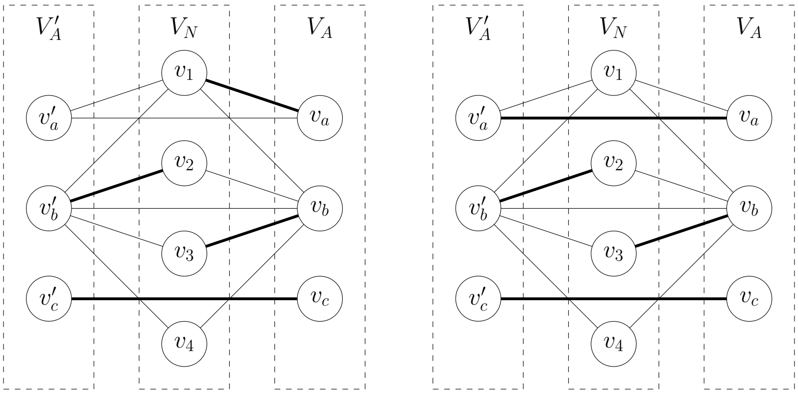

Figure 2: Reducing an instance $\mathcal{I}$ of s-GASP with $\ell(a) = u(a) = 2$ for each activity $a \in A^*$ to Maximum Matching (see Theorem 9). In this example, the set of agents is $N := \{1, 2, 3, 4\}$, the set of non-void activities is $A^* = \{a, b, c\}$ and the relevant preferences are $\{a, b\} \succsim_1 a_\emptyset$, $\{b\} \succsim_2 a_\emptyset$, $\{b\} \succsim_3 a_\emptyset$ and $\{b\} \succsim_4 a_\emptyset$. A maximum matching is indicated by bold edges. The normalized version (as described in Theorem 9) of the maximum matching on the left is depicted on the right. The matching on the right corresponds to a maximum individually rational assignment for $\mathcal{I}$ (i.e., $\pi(1) = \pi(4) = a_\emptyset$ and $\pi(2) = \pi(3) = b$).

Proof: Let $\mathcal{I} = (N, A, P, R)$ be such an instance of s-GASP. We adapt the previous proof by constructing a graph $G = (V, E)$ as follows: Let $V_N := \{v_i \mid i \in N\} \subseteq V$ and $V_A := \{v_a \mid a \in A^*\} \subseteq V$. For each pair $(v_i, v_a) \in V_N \times V_A$ we create the edge $\{v_i, v_a\} \in E$ if and only if $a \succsim_i a_\emptyset$. Let $A_{\leq 2} \subseteq A^*$ be the non-void activities with upper bound 2 and let $A_2 \subseteq A_{\leq 2}$ bet the non-void activities with lower and upper bound 2. Then, we create the vertex set $V'_{A_{\leq 2}} := \{v'_a \mid a \in A_{\leq 2}\} \subseteq V$. For each activity $a \in A_{\leq 2}$ and each agent $i \in N$, we create the edge $\{v_i, v'_a\} \in E$ if and only if $a \succsim_i a_\emptyset$. Additionally, we create the edges $\{v_a, v'_a\} \in E$ for each activity $a \in A_2$. ■

Theorem 14. It is NP-complete to decide if there is a virtually individually stable assignment that assigns each agent to some $a \in A^*$, even if for each activity $a \in A^*$ we have $u(a) = n$.

Proof: The proof follows analogously to the proof of Theorem 7. ■

Theorem 15. It is NP-complete to decide if there is a strictly core stable assignment that assigns each agent to some $a \in A^*$, even if for each activity $a \in A^*$ we have $u(a) = n$.

Proof: For membership in NP, assume an assignment π to be given. It is sufficient to determine, for each agent i and activity $a \succ_i \pi(i)$, whether there is a subset $E \supseteq (\pi(a) \cup \{i\})$ such that for each $e \in E$ it holds that $a \succsim_e \pi(e)$, such that the assignment resulting from the deviation is still feasible. Scanning, in the worst case, the whole profile for each agent and activity, and checking the feasibility constraints, can be done in polynomial time.

For NP-hardness, we again reduce from Exact Cover by 3-Sets (X3C). Given an instance $\langle X, \mathcal{Z} \rangle$ of X3C, where $X = \{1, \ldots, 3q\}$ and $\mathcal{Z} = \{Z_1, \ldots, Z_p\}$ is a collection of 3-element subsets of X, we ask whether X can be covered by q sets of $\mathcal{Z}$. Again, we assume that each element of X is contained in exactly three sets of $\mathcal{Z}$ (see [13, 14]) (this implies $p = 3q$ holds). For each $i \in X$ let $Z_{i_1}, Z_{i_2}, Z_{i_3}$ denote the sets containing i.
Given $\langle X, \mathcal{Z} \rangle$ we construct instance $\mathcal{I} = (N, A, P, R)$ of s-GASP as follows. Set $N := X$, $A := \{a_1, \ldots, a_p\}$ and let P be an arbitrary profile such that for each $i \in N$ the ranking restricted

to the four top-ranked activities is given by $a_{i_1} \sim_i a_{i_2} \sim_i a_{i_3} \succ_i a_\emptyset$. The restrictions are given by $|\pi^a| \in \{0\} \cup [3, n]$, for all $a \in A^*$.

Assume there is an exact cover C. Let the assignment π be given by $\pi(i) = a_j$, with $i \in Z_j$ and $Z_j \in C$, for $i \in N$. Because C is an exact cover, π is well-defined and feasible (each set Z_j contains exactly three elements of X, therefore either exactly three agents or no agent at all is assigned to activity a_j). Also, π assigns each agent to one of the three activities she has top-ranked. As a consequence, π is strictly core stable because no agent i can prefer any activity over $\pi(i)$.

On the other hand, assume there is a strictly core stable assignment π which assigns each agent to a non-void activity. By the restrictions, to any non-void activity either zero or exactly three agents are assigned. That means, for exactly q activities $a \in A^*$ we have $|\pi^a| > 0$. Assume an agent $i$is assigned to an activity a with $a_\emptyset \succsim_i a$. Then, since each set of Z contains exactly three elements, the group E of agents with $E := \{i \in N \mid a_{i_1} \succ a_\emptyset\} \cup \pi^{a_{i_1}}$ is of size at least 3, and each $e \in E$ has $a \succsim_e \pi(e)$ with $i \in E$ strictly preferring a_{i_1} over $\pi(i)$. This, however, contradicts with strict core stability of π. Hence, each agent is assigned to one of her top-ranked activities; as a consequence, $D = \{Z_j \mid |\pi^{a_j}| > 0,\ 1 \le j \le p\}$ is an exact cover for X. ■

Theorem 16 It is NP-complete to decide if there is an envy-free assignment that assigns each agent to some $a \in A^*$.

Proof: Reduction from X3C. Let $\langle X, \mathcal{Z}\rangle$ be an instance of X3C with $X = \{1, \dots, 3q\}$ and $\mathcal{Z} = \{Z_1, \dots, Z_p\}$, where each element of X is contained in exactly three sets of $\mathcal{Z}$. Again, let the sets containing i be denoted by $Z_{i_1}, Z_{i_2}, Z_{i_3}$ with $i_1 < i_2 < i_3$, for each $i \in X$. Construct instance $\mathcal{I} = (N, A, P, R)$ of s-GASP as follows. Let $N := \{1, \dots, 6q\}$ and $A^* := \{a_1, \dots, a_p\} \cup \{b_1, \dots b_{3q}\}$. Let σ be a fixed strict ranking on $A' := \{a_1, \dots, a_p\}$. Finally, let P be a profile such that (i) for each $i \in X$ the ranking restricted to the $(2 + p)$ top-ranked activities is given by $a_{i_1} \succ_i a_{i_2} \succ_i a_{i_3} \succ_i b_i \succ_i a_\emptyset \succ_i \sigma_i$, where σ_i corresponds to the ranking σ restricted to the set $A' \setminus \{a_{i_1}, a_{i_2}, a_{i_3}\}$; and (ii) for $1 \le i \le 3q$ the ranking of agent $(3q + i)$ restricted to the $(2 + p)$ top-ranked activities is $b_i \succ_{(3q+i)} a_\emptyset \succ_{(3q+i)} \sigma$. The restrictions in R are given by $|\pi^a| \in \{0\} \cup [3, 3]$, for $a = a_j \in A^*$ with $j \in \{1, \dots, p\}$, and $|\pi^b| \in \{0\} \cup [1, 1]$, for $b = b_j \in A^*$, $j \in \{1, \dots, 3q\}$.

Given an exact cover C, consider the assignment π defined by (i) for $i \in N \setminus X$: $\pi(i) = b_i$ and (ii) for $i \in X$: $\pi(i) = a_j$ iff $i \in Z_j$ with $Z_j \in C$. It is easy to see that π is feasible and assigns each agent to a non-void activity. Obviously, none of the agents $i \in N \setminus X$ envies another agent because each of these agents is assigned to her top-ranked activity. On the other hand, each agent $i \in X$ is assigned to one of her three top-ranked activities. By the fact that C is an exact cover, it follows that from the three top-ranked activities of agent i, only $\pi(i)$ is used, i.e., a positive number of agents is assigned. Thus, also $i \in X$ does not envy another agent.

On the other hand, let assignment π satisfy the stated properties. In particular, each agent is assigned to a non-void activity. Clearly, no agent $i \in X$ can be assigned to some b_j because otherwise agent $3q + j$ would envy i. I.e.,

$$\pi(i) \in A' \text{ for all } i \in X \tag{1}$$

holds. Let $N' = \{g \in N \setminus X \mid \pi(g) \in \{a_1, \dots, a_p\}\}$.

If $N' = \emptyset$, then $\pi(3q + i) = b_i$ follows for each $i \in \{1, \dots, 3q\}$. Thus, in order to avoid envy, each agent $i \in X$ must be assigned to one of her three top-ranked activities. For any activity $a \in A'$ with $\pi^a \ne \emptyset$ the restrictions imply that exactly three agents are assigned to a. Hence, it is not difficult to verify that $D = \{Z_j \in \mathcal{Z} : |\pi^{a_j}| > 0,\ 1 \le j \le p\}$ forms an exact cover for X in instance $\langle X, \mathcal{Z}\rangle$.

Let $N' \neq \emptyset$. If N' contains two agents g, h such that $\pi(g) \neq \pi(h)$, then one of these agents must envy the other by the fact that the two agents' rankings over $\{a_1, \ldots, a_p\}$ coincide. Hence, all members of N' must be assigned to the same activity $a \in \{a_1, \ldots, a_p\}$. Let $X' = \{i \in X : \pi(i) \notin \{a_{i_1}, a_{i_2}, a_{i_3}\}\}$.
Assume $X' \neq \emptyset$, i.e., there is an agent $i \in X$ which is not assigned to one of her three top-ranked activities.
First, we show that $\pi(i) = a$ must hold for each $i \in X'$. Assume the opposite, i.e., $\pi(i) \neq a$ for some $i \in X'$. Recall that $\pi(i) \in A'$ holds (see (1)), and thus $\pi(i) \in A' \setminus \{a_{i_1}, a_{i_2}, a_{i_3}\}$ follows. Then, in the strict ranking σ we must have $a \succ_\sigma \pi(i)$ because otherwise the members in N' envy i. However, this implies that in $\succ_i$ we must have $a \succ_i \pi(i)$ due to $\pi(i) \in A' \setminus \{a_{i_1}, a_{i_2}, a_{i_3}\}$; i.e., i envies the members in N'. Therewith, $\pi(i) = a$ follows.
From $\pi^a \neq \emptyset$ (due to $N' \neq \emptyset$) and the given restrictions we can conclude that exactly three agents are assigned to a. With $|N'| \neq 0$ it follows that $|X'| \in \{1, 2\}$ holds. Recall that each agent of $X \setminus X'$ is assigned to one of her three top-ranked activities. In addition, note that by the given restrictions to each active $a' \in A' \setminus \{a\}$ exactly three agents must be assigned, all of which must be members of $X \setminus X'$ (recall that (i) all members of X' are assigned to a and (ii) all members of $N \setminus X$ are either assigned to a or to some $b \in \{b_1, \ldots, b_{3q}\}$). This, however, is impossible, since from $|X| = 3q$ and $|X'| \in \{1, 2\}$ with $X' \subset X$ it follows that $|X \setminus X'| \in \{3q-1, 3q-2\}$ holds.
Thus, $X' = \emptyset$ holds. Assume $\pi(i) = a$ for some $i \in X$. By $N' \neq \emptyset$ we know that there can be at most two such agents and we get a contradiction to the size of X analogously to above. Hence, $\pi(i) \neq a$ holds for all $i \in X$. As a consequence, for each agent $i \in X$ we have $\pi(i) \in \{a_{i_1}, a_{i_2}, a_{i_3}\}$ such that to $\pi(i)$ exactly three agents, all members of X, are assigned (again, recall that for $j \in N \setminus X$ it holds that $\pi(j) \in \{a, b_1, \ldots, b_{3q}\}$). In other words, the collection $D = \{Z_j \in \mathcal{Z} : |\pi^{a_j} \cap X| > 0,\ 1 \leq j \leq p\}$ is an exact cover for X. ■

Proposition 27 *If all agents have strict linear orders on the set of activities and $\ell(a) = 1$ for all $a \in A^*$, then it can be decided in polynomial time if there exists an envy-free assignment which assigns all agents to a non-void activity.*

Proof: A simple greedy algorithm can determine if there exists such an envy-free assignment. The first step is to assign each agent to her most preferred non-void activity; this is possible because of $\ell(a) = 1$ for all $a \in A^*$. If the resulting (provisional) assignment π_p is feasible, i.e., for all activities $a \in A*$ the statement $u(a) \geq |\pi_p^a|$ holds true, we are done. If this is not the case, we remove all activities a with $u(a) > |\pi_p^a|$ from the set of activities (envy-freeness will only be accomplished if no agent is assigned to any of these activities) and start over with the first step. This algorithm either stops because all activities have been removed, or it produces an envy-free assignments which assign all agents to a non-void activity. ■

Proofs of Section 3.3

Theorem 17 It is NP-hard to find a Pareto optimal assignment in an instance of s-GASP, even when $u(a) = n$ for each $a \in A^*$.

Proof: Again we reduce from Exact Cover by 3-Sets (X3C). An instance $\langle X, \mathcal{Z} \rangle$ of X3C consists of a set $X = \{1, \ldots, 3q\}$ and a collection $\mathcal{Z} = \{Z_1, \ldots, Z_p\}$ of 3-element subsets of X; we ask whether X can be covered by q sets of $\mathcal{Z}$. We can assume that each element of X is contained in exactly three sets of $\mathcal{Z}$ (see [13, 14]) (this implies $p = 3q$ holds). For $i \in X$ let again $Z_{i_1}, Z_{i_2}, Z_{i_3}$ denote the three sets containing element i.

Given $\langle X, \mathcal{Z}\rangle$ we construct instance $\mathcal{I} = (N, A, P, R)$ of s-GASP as follows. Let $N = X$, $A = \{a_1, \ldots, a_p\}$; let P be some profile such that for each $i \in N$, (i) the ranking restricted to the four top-ranked activities corresponds to $a_{i_1} \sim_i a_{i_2} \sim_i a_{i_3} \succ_i a_\emptyset$, and (ii) for every remaining activity c we have $a_\emptyset \succ_i c$. With $n = |N|$, the restrictions are given by $|\pi^a| \in \{0\} \cup [3, n]$ for all $a \in A^*$.

Consider instance $\mathcal{I}$ of s-GASP. Let π be a Pareto optimal assignment. If π assigns an agent $i \in N$ to an alternative c with $a_\emptyset \succsim_i c$, then by Pareto optimality of π it follows that there does not exist a Pareto optimal assignment μ with $\mu(i) \succ_i a_\emptyset$ for each agent i. I.e., either each Pareto optimal assignment or no Pareto optimal assignment assigns at least one agent i to an activity different from $a_{i_1}, a_{i_2}, a_{i_3}$. We complete the proof by showing that a Pareto optimal assignment μ with $\mu(i) \succ_i a_\emptyset$ for each $i \in N$ exists if and only if there is an exact cover in instance $\langle X, \mathcal{Z}\rangle$ of X3C.

If there is a Pareto optimal assignment for $\mu(i) \succ_i a_\emptyset$ for each $i \in N$, each agent $i \in N$ is assigned to a non-void activity. By $\ell(a) = 3$ for $a \in A^*$ and since each a_j is preferred over $a_\emptyset$ by exactly three agents, it follows that the collection $D = \{Z_j \mid |\pi^{a_j}| = 3,\ 1 \leq j \leq p\}$ is an exact cover for X. On the other hand, given an exact cover C for instance $\langle X, \mathcal{Z}\rangle$, the assignment π which assigns $i \in N$ to activity a_j for the unique set Z_j of C which contains element i is feasible and assigns each agent to one of her top-ranked activities. As a consequence, π is Pareto optimal. ■

Theorem 18 If for each activity $a \in A^*$ we have $\ell(a) = 1$, then in polynomial time we can find a Pareto optimal assignment that maximizes the number of agents assigned to a non-void activity.

Proof: In that case, any Pareto optimal assignment is individually rational. Let k be the maximum number of agents assigned to non-void activities by an individually rational assignment. Hence, it is sufficient to find a Pareto optimal assignment π with $\#(\pi) = k$. Given an instance $\mathcal{I} = (N, A, P, R)$ of s-GASP with $\ell(a) = 1$ for all $a \in A^*$, we construct an instance $\mathcal{F}$ of the minimum cost flow problem. Instance $\mathcal{F}$ corresponds to instance $\mathcal{M}$ of the proof of Theorem 8 except that we add the following edge costs: for each $a \in A^*$ and $i \in N$ edge (i, a) has cost $-(1 + |\{b \in A^* | a \succ_i b,\ b \succ_i a_\emptyset\}|)$, all remaining edges have zero cost. Let f be a minimum integer cost flow of size k in instance $\mathcal{F}$. Then f induces the assignment π by setting $\pi(i) = a$ iff f sends a unit of flow through edge (i, a). Clearly, π is Pareto optimal since otherwise a flow f' of lower total cost than the total cost of f could be induced. ■

Theorem 20 It is NP-hard to decide if there is a Pareto optimal assignment that assigns each agent to a non-void activity.

Proof: We provide a reduction from Exact Cover by 3-Sets (X3C). Let $\langle X, \mathcal{Z}\rangle$ be an instance of X3C consisting of a set $X = \{1, \ldots, 3q\}$ and a collection $\mathcal{Z} = \{Z_1, \ldots, Z_p\}$ of 3-element subsets of X. Again, we assume that each element of X is contained in exactly three sets of $\mathcal{Z}$ (see [13, 14]) (this implies $p = 3q$ holds). For $i \in X$ let again $Z_{i_1}, Z_{i_2}, Z_{i_3}$ denote the three sets containing element i.
Given $\langle X, \mathcal{Z}\rangle$ we construct instance $\mathcal{I} = (N, A, P, R)$ of s-GASP as follows. Let $N = X$, $A = \{a_1, \ldots, a_p\} \cup \{b\}$; let P be some profile such that for each $i \in N$, (i) the ranking restricted to the five top-ranked activities corresponds to $b \sim_i a_{i_1} \sim_i a_{i_2} \sim_i a_{i_3} \succ_i a_\emptyset$, and (ii) for every remaining activity c we have $a_\emptyset \succ_i c$. With $n = |N|$, the restrictions are given by $|\pi^b| \in \{0\} \cup [n-1, n-1]$, and $|\pi^a| \in \{0\} \cup [3, 3]$ for all $a \in A^* \setminus \{b\}$.

Consider instance $\mathcal{I}$ of s-GASP. Observe that an assignment π which assigns an agent $i \in N$ to an alternative c with $a_\emptyset \succ_i c$ cannot be Pareto optimal, since under π' – which assigns i to $a_\emptyset$ and each of the remaining agents to b – agent i is strictly better off while no agent is worse off. Also, any assignment π that assigns at most $n-2$ agents to a non-void

activity is not Pareto optimal: Under λ – which assigns all of these agents and all but one of the agents of $\pi^{a_\emptyset}$ to b and the remaining agent to $a_\emptyset$, makes at least one agent better off (an agent of $\pi^{a_\emptyset}$ who is assigned to b under λ) while no agent is worse off.
Therefore, a Pareto optimal assignment must be individually rational and assigns at least $n-1$ agents to non-void activities. Note that any assignment that assigns exactly $n-1$ agents to non-void activities must assign all of these agents to b, because n is a multiple of 3 and due to $\ell(a) = u(a) = 3$ for $a \in A^* \setminus \{b\}$. We complete the proof by showing that there is a Pareto optimal assignment μ with $\#(\mu) = n$ if and only if there is an exact cover in instance $\langle X, \mathcal{Z}\rangle$ of X3C.

If there is a Pareto optimal assignment for μ with $\#(\mu) = n$, each agent $i \in N$ is assigned to a non-void activity. By the above observation, μ is individually rational. Clearly, by the bounds $\ell(a) = u(a) = 3$ for $a \in A^* \setminus \{b\}$ no agent can be assigned to b; also, it follows that the collection $D = \{Z_j \mid |\pi^{a_j}| = 3,\ 1 \le j \le p\}$ is an exact cover for X. On the other hand, given an exact cover C for instance $\langle X, \mathcal{Z}\rangle$, the assignment π which assigns $i \in N$ to activity a_j for the unique set Z_j of C which contains element i is feasible and assigns each agent to one of her top-ranked activities. Thus, π is Pareto optimal. ■

Proofs of Section 4

Proposition 22 For any $k \ge 6$, there is an instance (N, A, P, R) of s-GASP with $|N| = k$ and $u(a) = k$ for each $a \in A^*$, for which there does not exist an assignment π which is both Pareto optimal and envy-free.

Proof: We provide the proof for $k = 6$, which easily extends to $k = n$ for any $n > 6$. Consider the instance of s-GASP with $N = \{1, 2, 3, 4, 5, 6\}$, $A^* = \{a, b, c\}$ and for any $x \in A^*$ we have $\ell(x) = 3, u(x) = 6$. The rankings are

$$\begin{array}{llll} \succsim_1: & a \succ_1 b \succ_1 c \succ_1 a_\emptyset & \succsim_4: & a \succ_4 b \succ_4 c \succ_4 a_\emptyset \\ \succsim_2: & b \succ_2 c \succ_2 a \succ_2 a_\emptyset & \succsim_5: & b \succ_5 c \succ_5 a \succ_5 a_\emptyset \\ \succsim_3: & c \succ_3 a \succ_3 b \succ_3 a_\emptyset & \succsim_6: & c \succ_6 a \succ_6 b \succ_6 a_\emptyset \end{array} .$$

Due to the feasibility constraints, there are only 4 types of feasible assignments:

(i) 3-5 agents are assigned to the same activity $x \neq a_\emptyset$, and the rest to $a_\emptyset$;
(ii) all agents are assigned to the void activity;
(iii) all agents are assigned to the same activity $x \neq a_\emptyset$;
(iv) 3 agents are assigned to the same activity $x \neq a_\emptyset$ and the other 3 agents are assigned to another activity $y \notin \{x, a_\emptyset\}$.

The assignments of type (i) and (ii) are Pareto dominated by some assignment of type (iii). An assignment π_1 of type (iii) is envy-free but not Pareto optimal. Due to the symmetrical construction of the preferences profiles, we can assume without loss of generality $\pi_1^a = N$. But then the assignment is Pareto dominated by the assignment π_2 with $\pi_2^a = \{1, 3, 4\}$ and $\pi_2^c = \{2, 5, 6\}$. An assignment of type (iv) cannot be envy-free. Without loss of generality we can assume $x = a$ and $y = b$. Assume, for the sake of contradiction, that there is an envy-free assignment. Agents 1 and 4 must be assigned to activity a and agents 2 and 5 to activity b. As the preference profiles of the remaining agents both rank a strictly better than b, the assignment cannot be an envy-free assignment. ■