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Chair of Statistics, Econometrics and Empirical Economics

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S414
Advanced Mathematical Methods
Exercises

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LINEAR ALGEBRA

EXERCISE 1 Eigenvalues

Derive the characteristic equations for the matrices from exercise a)-c) and determine the eigenvalues.

$$\text{a) } \mathbf{B} = \begin{pmatrix} 4 & 1 \\ -2 & -0,5 \end{pmatrix}$$

$$\text{b) } \mathbf{C} = \begin{pmatrix} 1 & 2 \\ 3 & -4 \end{pmatrix} \quad \text{c) } \mathbf{D} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & -3 \\ 0 & -1 & 3 \end{pmatrix}$$

EXERCISE 2 Eigenvalues and Eigenvectors

Given the matrix:

$$\mathbf{A} = \begin{bmatrix} -3 & 2 \\ -2 & 2 \end{bmatrix}$$

- Calculate the eigenvalues and the respective eigenvectors of \mathbf{A} .
- Use the eigenvalues to calculate the determinant of \mathbf{A} .

EXERCISE 3 Eigenvalues

A 3×3 matrix \mathbf{A} has the eigenvalues $\lambda_1 = 1$, $\lambda_2 = 3$ and $\lambda_3 = 4$. Compute the determinant of \mathbf{A} , $\text{rg}(\mathbf{A})$, the determinant of \mathbf{A}^{-1} and the eigenvalues of \mathbf{A}^{-1} . What can be said about the quadratic form $\mathbf{x}'\mathbf{A}\mathbf{x}$ of the matrix \mathbf{A} for any vectors of \mathbf{x} ?

EXERCISE 4 Eigenvalues

Find the characteristic vectors of the matrix $\begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix}$:

EXERCISE 5 Quadratic Form

Given the matrix

$$\mathbf{A} = \begin{pmatrix} 2 & -2 \\ 1 & 2 \end{pmatrix} .$$

- Determine the definiteness of the quadratic form $Q = \mathbf{x}'\mathbf{A}\mathbf{x}$.
- Explain in two sentences maximum what this means for the graph $\{(x_1, x_2, Q) | Q = (x_1; x_2)\mathbf{A}(x_1; x_2)'\}$.

EXERCISE 6 Quadratic Form

Write the quadratic form

$$Q = 4x_1^2 + 4x_1x_2 - x_2^2$$

in matrix notation and determine its definiteness.

EXERCISE 7 Sign definiteness

Express each quadratic form below as a matrix product involving a *symmetric* coefficient matrix:

- $q = 3u^2 - 4uv + 7v^2$
- $q = u^2 + 7uv + 3v^2$
- $q = 8uv - u^2 - 31v^2$
- $q = 6xy - 5y^2 - 2x^2$
- $q = 3u_1^2 - 2u_1u_2 + 4u_1u_3 + 5u_2^2 + 4u_3^2 - 2u_2u_3$
- $q = -u^2 + 4uv - 6uw - 4v^2 - 7w^2$

EXERCISE 8 Sign definiteness

Given a quadratic form $u'Du$, where D is 2×2 , the characteristic equation of D can be written as:

$$\begin{vmatrix} d_{11} - r & d_{12} \\ d_{21} & d_{22} - r \end{vmatrix} = 0 \quad (d_{12} = d_{21})$$

Expand the determinant; express the roots of this equation by use of the quadratic formula and deduce the following:

- a) No imaginary number (a number involving $\sqrt{-1}$) can occur in r_1 and r_2 .
- b) To have repeated roots, the matrix D must be in the form of $\begin{pmatrix} c & 0 \\ 0 & c \end{pmatrix}$
- c) To have either positive or negative smidefiniteness, the determinant of the matrix D must vanish, i.e. $|D| = 0$.

Solution Exercise 1:

- a) $\lambda_1 = 3.5$ and $\lambda_2 = 0$
b) $\lambda_1 = 2$ and $\lambda_2 = -5$
c) $\lambda_1 = 4.30278$; $\lambda_2 = 0.69722$; $\lambda_3 = 1$

Solution Exercise 2:

- a) Eigenvector for $\lambda_1 = 1$:
 $\Rightarrow \begin{pmatrix} a \\ 2a \end{pmatrix}$ for $a \in \mathbb{R} \setminus \{0\}$
Eigenvector for $\lambda_2 = -2$:
 $\Rightarrow \begin{pmatrix} b \\ \frac{1}{2}b \end{pmatrix}$ for $b \in \mathbb{R} \setminus \{0\}$

b) $\det(\mathbf{A}) = -2$

Solution Exercise 4:

$$v_1 = \begin{pmatrix} -\frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{pmatrix}, \quad v_2 = \begin{pmatrix} \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{pmatrix}$$

Solution Exercise 5:

- a) positive definite

Solution Exercise 6:

$$Q = \mathbf{x}'\mathbf{A}\mathbf{x} \text{ with } A = \begin{pmatrix} 4 & 2 \\ 2 & -1 \end{pmatrix}$$

\mathbf{A} is indefinite

Solution Exercise 7:Quadratic form: $q = \mathbf{x}'\mathbf{A}\mathbf{x}$

a)

$$q = \begin{pmatrix} u \\ v \end{pmatrix}' \begin{pmatrix} 3 & -2 \\ -2 & 7 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$$

b)

$$q = \begin{pmatrix} u \\ v \end{pmatrix}' \begin{pmatrix} 1 & 3.5 \\ 3.5 & 3 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$$

c)

$$q = \begin{pmatrix} u \\ v \end{pmatrix}' \begin{pmatrix} -1 & 4 \\ 4 & -31 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$$

d)

$$q = \begin{pmatrix} x \\ y \end{pmatrix}' \begin{pmatrix} -2 & 3 \\ 3 & -5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

e)

$$q = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}' \begin{pmatrix} 3 & -1 & 2 \\ -1 & 5 & -1 \\ 2 & -1 & 4 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$$

f)

$$q = \begin{pmatrix} u \\ v \\ w \end{pmatrix}' \begin{pmatrix} -1 & 2 & -3 \\ 2 & -4 & 0 \\ -3 & 0 & -7 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix}$$

Solution Exercise 8:

Refer to the video.