Probabilistic Machine Learning
Lecture 20
Latent Dirichlet Allocation

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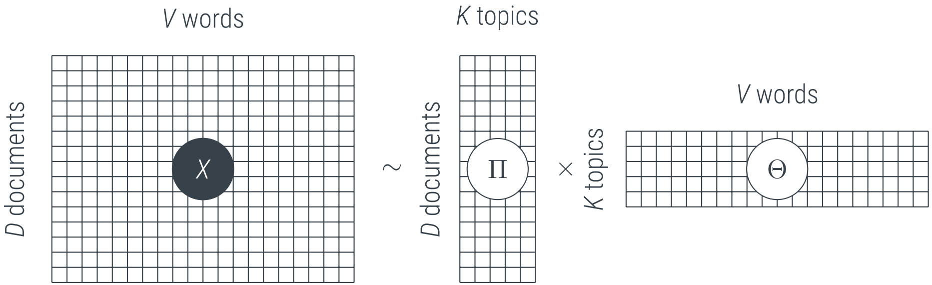
Designing a probabilistic machine learning method:

1. get the **data**
   1.1 try to collect as much meta-data as possible
2. build the **model**
   2.1 identify quantities and datastructures; assign names
   2.2 design a generative process (graphical model)
   2.3 assign (conditional) distributions to factors/arrows (use exponential families!)
3. design the **algorithm**
   3.1 consider conditional independence
   3.2 try standard methods for early experiments
   3.3 run unit-tests and sanity-checks
   3.4 identify bottlenecks, find customized approximations and refinements

A Mixture of Probabilities?

Desiderata



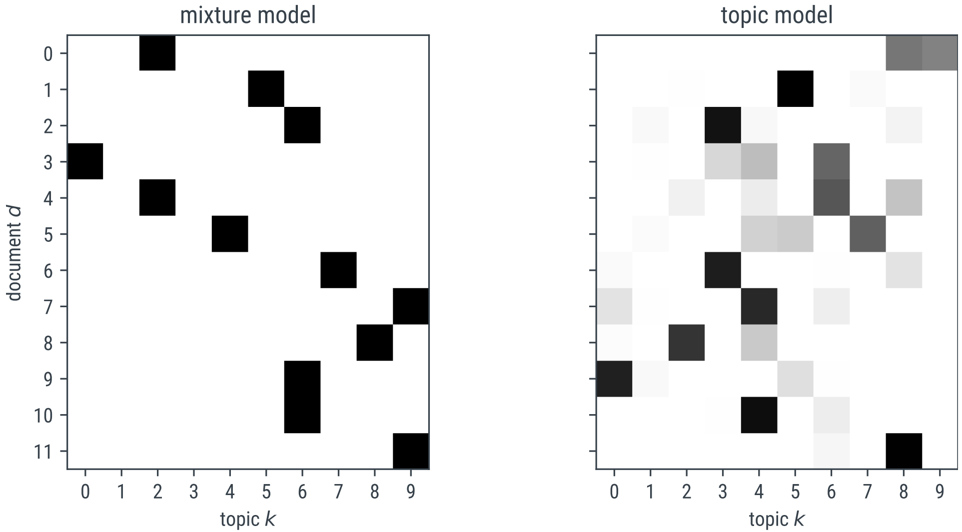
- topics should be probabilities: $p(\boldsymbol{x}_d \mid k) = \prod_{v=1}^{V} \prod_{i=1}^{x_{dv}} \theta_{k(i),v}$
- but documents should have *several* topics! Let π_{dk} be the *probability* to draw a word from topic k

doc sparsity each document d should only contain a small number of topics

word sparsity each topic k should only contain a small number of the words v in the vocabulary

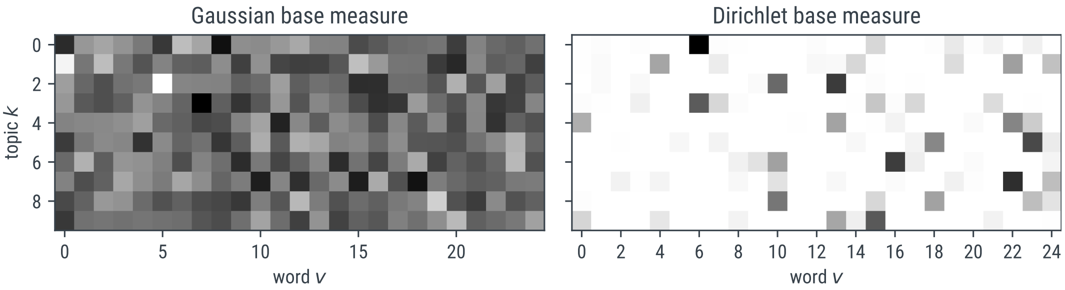
Each document discusses *a few* topics

Dirichlet sparsity in the document-topic distribution



Each topic contains *a few* words

Dirichlet sparsity in the topic-word distribution



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A Discrete Topic Model

almost there

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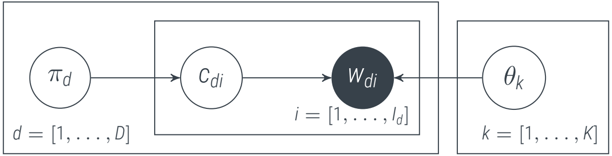


To draw I_d words $w_{di} \in [1, \dots, V]$ of document $d \in [1, \dots, D]$:

- Draw topic assignments $c_{di} = [c_{di1}, \dots, c_{diK}] \in \{0; 1\}^K, \sum_k c_{dik} = 1$ of word w_{id} from

$$\Rightarrow C \in \{0; 1\}^{D \times I_d \times K} \qquad \text{with} \qquad p(C \mid \Pi) = \prod_{d=1}^{D} \prod_{i=1}^{I_d} \prod_{k=1}^{K} \pi_{dk}^{c_{dik}}$$

- Draw word w_{di} from

$$p(w_{di} = v \mid c_{di}, \Theta) = \prod_k \theta_{kv}^{c_{dik}}$$

But we need priors for Π, Θ. And we'd like them to be sparse!

Reminder: The Dirichlet Distribution

a sparsity prior for probability distribution

$$p(\boldsymbol{x} \mid \boldsymbol{\pi}) = \prod_{i=1}^{n} \pi_{x_i} \qquad x \in \{1;\dots,K\}$$

$$= \prod_{k=1}^{K} \pi_k^{n_k} \qquad n_k := |\{x_i \mid x_i = k\}|$$

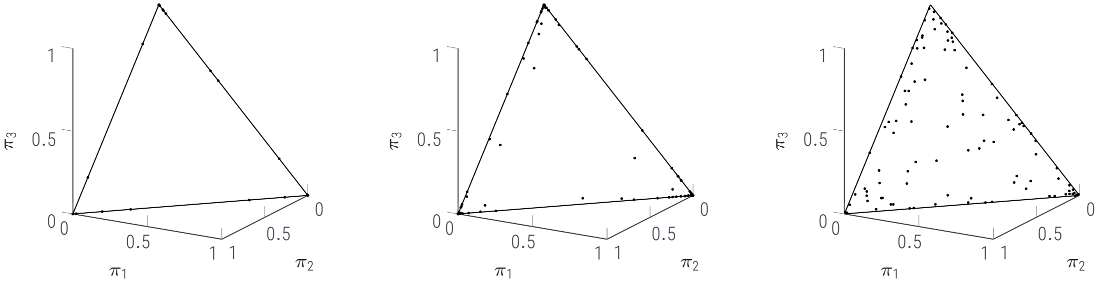
$$p(\boldsymbol{\pi} \mid \boldsymbol{\alpha}) = \mathcal{D}(\boldsymbol{\alpha}) = \frac{\Gamma(\sum_k \alpha_k)}{\prod_k \Gamma(\alpha_k)} \prod_{k=1}^{K} \pi_k^{\alpha_k - 1} = \frac{1}{B(\boldsymbol{\alpha})} \prod_{k=1}^{K} \pi_k^{\alpha_k - 1}$$

$$p(\boldsymbol{\pi} \mid \boldsymbol{x}) = \mathcal{D}(\boldsymbol{\alpha} + \boldsymbol{n})$$

Peter Gustav Lejeune Dirichlet
(1805–1859)

Dirichlets can encode sparsity

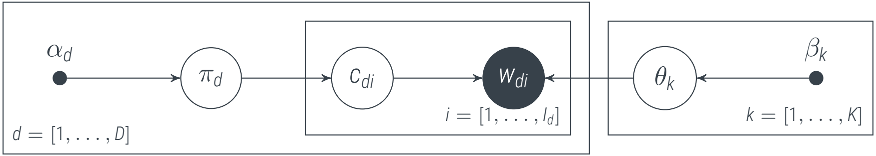
for $\alpha \sim 0.01, 0.1, 0.5$ (100 samples each)



Latent Dirichlet Allocation

our generative model

[Blei, D. M., Ng, A. Y. & Jordan, M. I. (2003) JMLR 3, 993–1022]



To draw I_d words $w_{di} \in [1, \dots, V]$ of document $d \in [1, \dots, D]$:

- Draw K topic distributions θ_k over V words from $\quad p(\Theta \mid \boldsymbol{\beta}) = \prod_{k=1}^{K} \mathcal{D}(\theta_k; \beta_k)$
- Draw D document distributions over K topics from $\quad p(\Pi \mid \boldsymbol{\alpha}) = \prod_{d=1}^{D} \mathcal{D}(\pi_d; \alpha_d)$
- Draw topic assignments c_{dik} of word w_{di} from $\quad p(C \mid \Pi) = \prod_{i,d,k} \pi_{dk}^{c_{dik}}$
- Draw word w_{di} from $\quad p(w_{di} = v \mid c_{di}, \Theta) = \prod_k \theta_{kv}^{c_{dik}}$

Useful notation: $n_{dkv} = \#\{i : w_{di} = v, c_{dik} = 1\}$. Write $n_{dk:} := [n_{dk1}, \dots, n_{dkV}]$ and $n_{dk\cdot} = \sum_v n_{dkv}$, etc.

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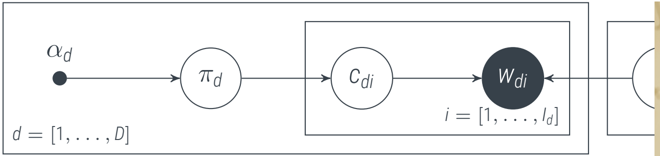

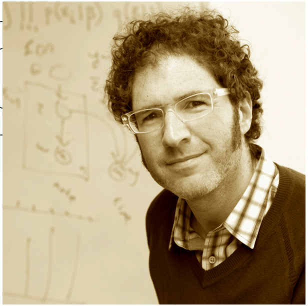

David M. Blei

image: Columbia U

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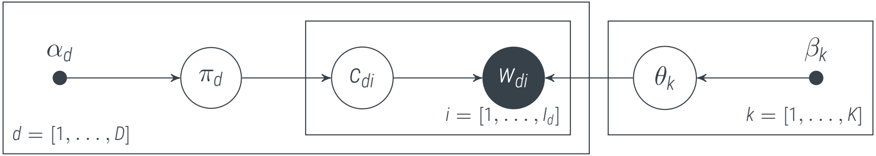
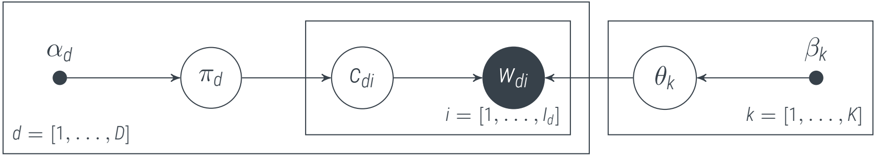


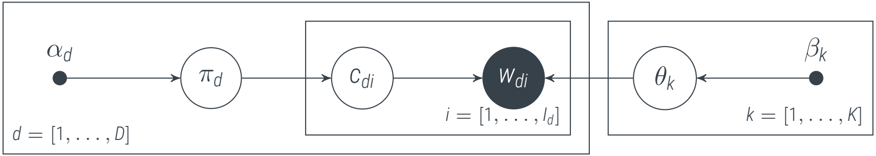
$$
\begin{aligned}
p(C, \Pi, \Theta \mid W) &= \frac{p(C, \Pi, \Theta, W)}{\iiint p(C, \Pi, \Theta, W)\, dC\, d\Pi, d\Theta} \\
&= \frac{p(W \mid C, \Pi, \Theta) \cdot p(C, \Pi, \Theta)}{\iiint p(C, \Pi, \Theta, W)\, dC\, d\Pi, d\Theta}
\end{aligned}
$$

Building an algorithm

A more careful look at the model



$$
\begin{aligned}
p(C, \Pi, \Theta, W) &= p(\Pi \mid \alpha) \cdot p(C \mid \Pi) \cdot p(\Theta \mid \beta) \cdot p(W \mid C, \Theta) \\
&= \left(\prod_{d=1}^{D} p(\boldsymbol{\pi}_d \mid \boldsymbol{\alpha}_d)\right) \cdot \left(\prod_{d=1}^{D} \prod_{i=1}^{I_d} p(c_{di} \mid \pi_d)\right) \cdot \left(\prod_{d=1}^{D} \prod_{i=1}^{I_d} p(w_{di} \mid c_{di}, \Theta)\right) \cdot \left(\prod_{k=1}^{K} p(\boldsymbol{\theta}_k \mid \boldsymbol{\beta}_k)\right) \\
&= \left(\prod_{d=1}^{D} \mathcal{D}(\boldsymbol{\pi}_d; \boldsymbol{\alpha}_d)\right) \cdot \left(\prod_{d=1}^{D} \prod_{i=1}^{I_d} \left(\prod_{k=1}^{K} \pi_{dk}^{c_{dik}}\right)\right) \cdot \left(\prod_{d=1}^{D} \prod_{i=1}^{I_d} \left(\prod_{k=1}^{K} \theta_{kw_{di}}^{c_{dik}}\right)\right) \cdot \left(\prod_{k=1}^{K} \mathcal{D}(\boldsymbol{\theta}_k; \boldsymbol{\beta}_k)\right) \\
&= \left(\prod_{d=1}^{D} \left(\frac{\Gamma(\sum_k \alpha_{dk})}{\prod_k \Gamma(\alpha_{dk})} \prod_{k=1}^{K} \pi_{dk}^{\alpha_{dk}-1}\right)\right) \cdot \left(\prod_{d=1}^{D} \prod_{i=1}^{I_d} \left(\prod_{k=1}^{K} \pi_{dk}^{c_{dik}}\right)\right) \cdot \dots
\end{aligned}
$$



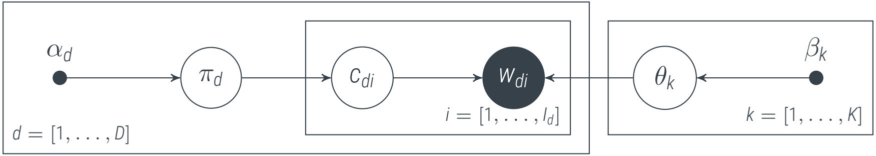
$$p(\Pi \mid \alpha) \cdot p(C \mid \Pi) = \left(\prod_{d=1}^{D} \left(\frac{\Gamma(\sum_k \alpha_{dk})}{\prod_k \Gamma(\alpha_{dk})} \prod_{k=1}^{K} \pi_{dk}^{\alpha_{dk}-1} \right) \right) \cdot \left(\prod_{d=1}^{D} \prod_{i=1}^{I_d} \left(\prod_{k=1}^{K} \pi_{dk}^{c_{dik}} \right) \right)$$

Useful notation: $n_{dkv} = \#\{i : w_{di} = v, c_{dik} = 1\}$. Write $n_{dk:} := [n_{dk1}, \dots, n_{dkV}]$ and $n_{dk\cdot} = \sum_v n_{dkv}$, etc.

$$= \left(\prod_{d=1}^{D} \frac{\Gamma(\sum_k \alpha_{dk})}{\prod_k \Gamma(\alpha_{dk})} \prod_{k=1}^{K} \pi_{dk}^{\alpha_{dk}-1+n_{dk\cdot}} \right)$$

Building an algorithm

A more careful look at the model



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\begin{aligned}
p(C, \Pi, \Theta, W) &= p(\Pi \mid \alpha) \cdot p(C \mid \Pi) \cdot p(\Theta \mid \beta) \cdot p(W \mid C, \Theta) \\
&= \left(\prod_{d=1}^{D} p(\boldsymbol{\pi}_d \mid \boldsymbol{\alpha}_d)\right) \cdot \left(\prod_{d=1}^{D}\prod_{i=1}^{I_d} p(c_{di} \mid \pi_d)\right) \cdot \left(\prod_{d=1}^{D}\prod_{i=1}^{I_d} p(w_{di} \mid c_{di}, \Theta)\right) \cdot \left(\prod_{k=1}^{K} p(\boldsymbol{\theta}_k \mid \boldsymbol{\beta}_k)\right) \\
&= \left(\prod_{d=1}^{D} \mathcal{D}(\boldsymbol{\pi}_d; \boldsymbol{\alpha}_d)\right) \cdot \left(\prod_{d=1}^{D}\prod_{i=1}^{I_d} \left(\prod_{k=1}^{K} \pi_{dk}^{c_{dik}}\right)\right) \cdot \left(\prod_{d=1}^{D}\prod_{i=1}^{I_d} \left(\prod_{k=1}^{K} \theta_{kw_{di}}^{c_{dik}}\right)\right) \cdot \left(\prod_{k=1}^{K} \mathcal{D}(\boldsymbol{\theta}_k; \boldsymbol{\beta}_k)\right) \\
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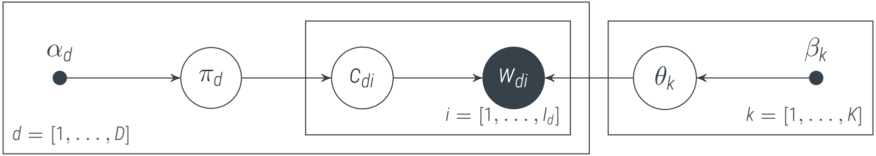


$$p(C, \Pi, \Theta, W) = \left(\prod_{d=1}^{D} \frac{\Gamma(\sum_k \alpha_{dk})}{\prod_k \Gamma(\alpha_{dk})} \prod_{k=1}^{K} \pi_{dk}^{\alpha_{dk}-1+n_{dk\cdot}} \right) \cdot \left(\prod_{k=1}^{K} \frac{\Gamma(\sum_v \beta_{kv})}{\prod_v \Gamma(\beta_{kv})} \prod_{v=1}^{V} \theta_{kv}^{\beta_{kv}-1+n_{\cdot kv}} \right)$$

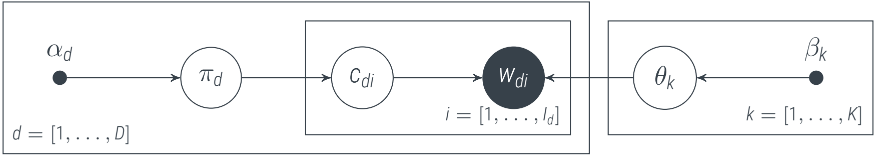
- If we had Π, Θ (which we don't), then the posterior $p(C \mid \Theta, \Pi, W)$ would be easy:

$$p(C \mid \Theta, \Pi, W) = \frac{p(W, C, \Theta, \Pi)}{\sum_C p(W, C, \Theta, \Pi)} = \prod_{d=1}^{D} \prod_{i=1}^{I_d} \frac{\prod_{k=1}^{K} (\pi_{dk} \theta_{kw_{di}})^{c_{dik}}}{\sum_{k'} (\pi_{dk'} \theta_{k' w_{di}})}$$

- note that this conditional independence can easily be read off from the above graph!

Building an algorithm

A more careful look at the model



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- If we had Π, Θ (which we don't), then the posterior $p(C \mid \Theta, \Pi, W)$ would be easy:

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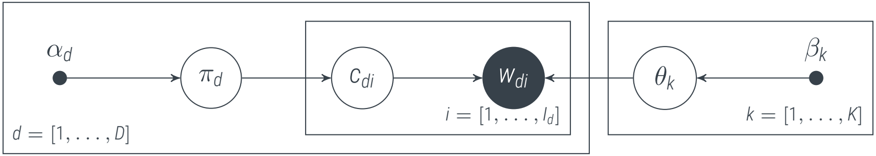

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- If we had C (which we don't), then the posterior $p(\Theta, \Pi \mid C, W)$ would be easy:

$$p(\Theta, \Pi \mid C, W) = \frac{p(C, W, \Pi, \Theta)}{\int p(\Theta, \Pi, C, W) \, d\Theta \, d\Pi} = \frac{\left(\prod_d \mathcal{D}(\pi_d; \alpha_d) \left(\prod_k \pi_{dk}^{n_{dk\cdot}}\right)\right) \left(\prod_k \mathcal{D}(\theta_k; \beta_k) \left(\prod_v \theta_{kv}^{n_{\cdot kv}}\right)\right)}{p(C, W)}$$

$$= \left(\prod_d \mathcal{D}(\pi_d; \alpha_{d:} + n_{d:\cdot}) \right) \left(\prod_k \mathcal{D}(\theta_k; \beta_{k:} + n_{\cdot k:}) \right)$$

- note that this conditional independence **can not** easily be read off from the above graph!

The Toolbox

Framework:

$$\int p(x_1, x_2)\, dx_2 = p(x_1) \qquad p(x_1, x_2) = p(x_1 \mid x_2)p(x_2) \qquad p(x \mid y) = \frac{p(y \mid x)p(x)}{p(y)}$$

Modelling:

- graphical models
- Gaussian distributions
- (deep) learnt representations
- Kernels
- Markov Chains
- Exponential Families / Conjugate Priors
- Factor Graphs & Message Passing

Computation:

- Monte Carlo
- Linear algebra / Gaussian inference
- maximum likelihood / MAP
- Laplace approximations
-

Gibbs sampling:

- $x_t \leftarrow x_{t-1}; x_{ti} \sim p(x_{ti} \mid x_{t1}, x_{t2}, \dots, x_{t(i-1)}, x_{t(i+1)}, \dots)$
- a special case of Metropolis-Hastings:
  - $q(x' \mid x_t) = \delta(x'_{\setminus i} - x_{t,\setminus i}) p(x'_i \mid x_{t,\setminus i})$
  - $p(x') = p(x'_i \mid x'_{\setminus i}) p(x'_{\setminus i}) = p(x'_i \mid x_{t,\setminus i}) p(x_{t,\setminus i})$
  - acceptance rate: $a = \dfrac{p(x')}{p(x_t)} \cdot \dfrac{q(x_t \mid x')}{q(x' \mid x_t)} = \dfrac{p(x'_i \mid x_{t,\setminus i})}{p(x_{ti} \mid x_{t,\setminus i})} \cdot \dfrac{p(x_{ti} \mid x_{t,\setminus i})}{p(x'_i \mid x_{t,\setminus i})} = 1$

Markov Chain Monte Carlo Methods provide a relatively simple way to construct approximate posterior distributions. They are asymptotically exact. Compared to other approximate inference methods, like variational inference, they *tend* to be *easier to implement* but *harder to monitor*, and *may* also be more computationally expensive

$$\int f(x)p(x)\,dx \approx \frac{1}{S}\sum_{s=1}^{S} f(x_s) \quad \text{if} \quad x_s \sim p$$

Iterate between (recall $n_{dkv} = \#\{i : w_{di} = v, c_{dik} = 1\}$)

$$\Theta \sim p(\Theta \mid C, W) = \prod_k \mathcal{D}(\theta_k; \beta_{k:} + n_{\cdot k:})$$

$$\Pi \sim p(\Pi \mid C, W) = \prod_d \mathcal{D}(\pi_d; \alpha_{d:} + n_{d:\cdot})$$

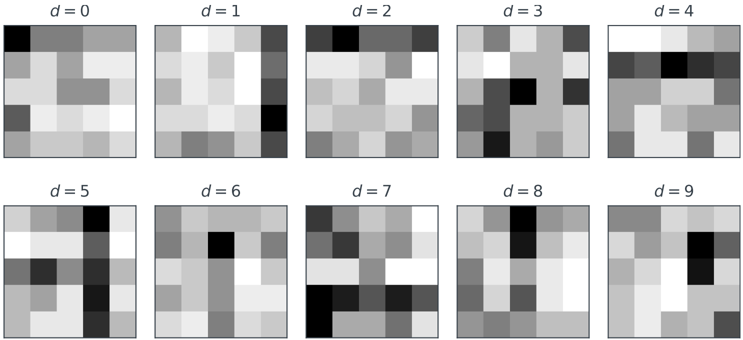
$$C \sim p(C \mid \Theta, \Pi, W) = \prod_{d=1}^{D} \prod_{i=1}^{I_d} \frac{\prod_{k=1}^{K} (\pi_{dk}\theta_{kw_{di}})^{c_{dik}}}{\sum_{k'} (\pi_{dk'}\theta_{k'w_{di}})}$$

- This is *comparably* easy to implement because there are libraries for sampling from Dirichlet's, and discrete sampling is trivial. All we have to keep around are the counts n (which are sparse!) and Θ, Π (which are comparably small). Thanks to factorization, much can also be done in parallel!
- Unfortunately, this sampling scheme is relatively slow to move out of initialization, because z depends strongly on θ, π and vice versa.
- properly vectorizing the code is important for speed

A visual example

building a toy model

adapted from T. L. Griffiths & M. Steyvers, *Finding scientific topics*, PNAS **101**, 5228–5235



- $K = 10, V = 25, N = 100, D = 1000$
- $p(\Pi) = \prod_d \mathcal{D}(\pi_d; \alpha = 1)$

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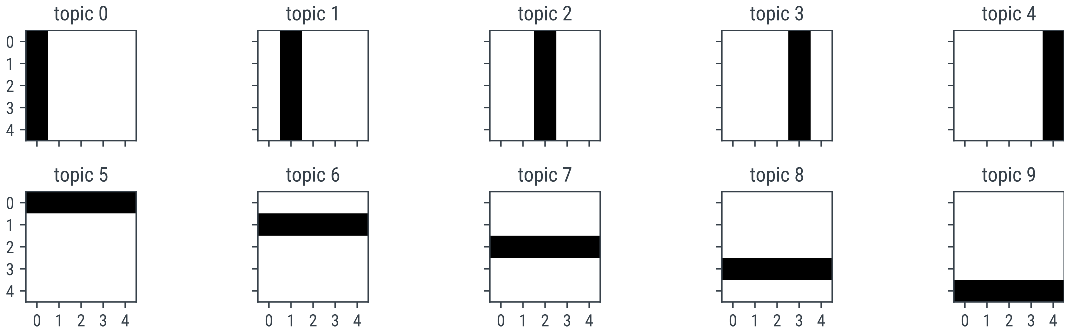
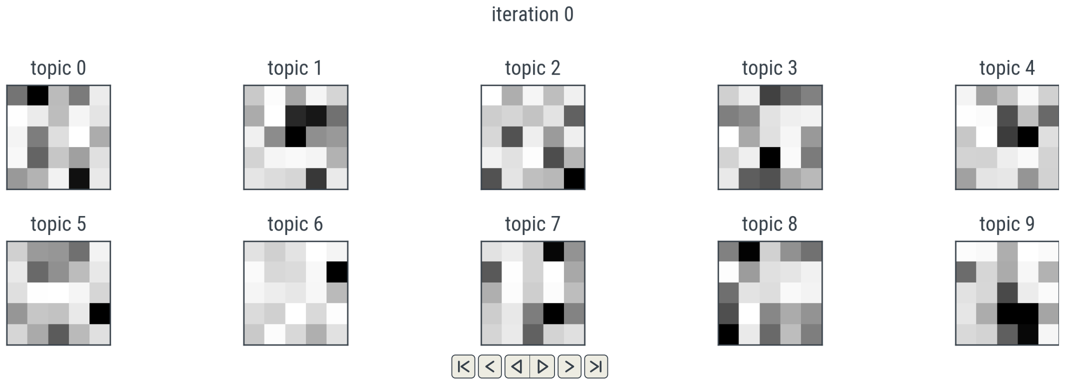


- $K = 10, V = 25, N = 100, D = 1000$
- $p(\Pi) = \prod_d \mathcal{D}(\pi_d; \alpha = 1)$

It works!

your homework this week



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Designing a Probabilistic Machine Learning Model

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   1.2 take a close look at the data
2. build the **model**
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   2.2 design a generative process (graphical model)
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4. **Test** the Setup
5. Revisit the Model and try to **improve** it, using **creativity**