Chapter 1
Popper's Theory of Deductive Logic

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Abstract We present Popper's theory of deductive logic as exhibited in his articles published between 1947 and 1949. After an introduction to Popper's inferentialist approach and his idea of "inferential definitions" of logical constants, we discuss in more formal detail Popper's general theory of the deducibility relation (which, using Gentzen's terminology, might be called his "structural" theory), as well as his special theory of logical constants. We put special emphasis on his inferential notion of duality, which includes his analysis of the "anti-conditional" (today called "co-implication"), his systematic study of various forms of negation including classical, intuitionistic and weaker negations, his system of bi-intuitionistic logic, and his theory of quantification and identity. We also touch on his treatment of modal operators, which is based on Carnap's *Meaning and Necessity*.

Key words: Karl Popper, logic, inferentialism, logical constant, deducibility, negation, duality, co-implication, intuitionistic logic, bi-intuitionistic logic, quantification, identity, modal operators

Karl Popper's published articles most relevant for his inferentialist theory of logic are listed in Table 1. These core papers are accompanied in our edition by a few other papers on deductive logic published by Popper before and after them (cf. Preface). We provide a commentary to these papers and also discuss some of the objections raised by contemporary reviewers, who were quite prominent figures in mathematical logic. Even though Popper's articles appeared in well-accessible journals and proceedings (Mind, Aristotelian Society, Royal Netherlands Academy of Sciences, 10th International Congress of Philosophy) and were reviewed in the standard periodicals (Zentralblatt für Mathematik und ihre Grenzgebiete, Mathematical Reviews, Journal of Symbolic Logic), a wider reception did not take place. One may speculate about the reasons why the academic community ignored these papers, in spite of the fact that their author was a well-known figure, although not in logic, but in both philosophy of science (*Logik der Forschung*) and social and political philosophy (*The Open Society and Its Enemies*). Why was it that the reviewers, or at least some

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Table 1 Popper's articles published between 1947 and 1949.

Title	*Original publication(s)*	*This volume*
Logic without Assumptions	Popper (1947b)	Chapter 2
New Foundations for Logic	Popper (1947c)	Chapter 3
Functional Logic without Axioms or Primitive Rules of Inference	Popper (1947d)	Chapter 4
On the Theory of Deduction, Part I. Derivation and its Generalizations	Popper (1948a,b)	Chapter 5
On the Theory of Deduction, Part II. The Definitions of Classical and Intuitionist Negation	Popper (1948c,d)	Chapter 6
The Trivialization of Mathematical Logic	Popper (1949a)	Chapter 7

of them, who had read his papers in detail, did not see the promising features of Popper's approach, which should have been visible in spite of certain shortcomings (often based on misunderstandings) that they rightly or wrongly criticized? Was it that Popper's bold claims – "new foundations", "trivialization" etc. – made them sceptical? Was it Popper's proof-theoretic framework, which he considered sufficient to lay the foundations for logic, independent of any additional sort of semantics? There were exceptions. The proof-theorist Bernays saw the merits of such an approach, which from today's perspective we would call "inferentialist",[1] as did Brouwer, Kneale and Quine (cf. § 2).

From a modern standpoint, where inferentialism and proof-theoretic semantics have become respected foundational approaches, Popper's views look quite advanced. Perhaps Popper was, with his inferentialist perspective, too much ahead of his time, and did not come back to his logical investigations, when in the spirit of a re-appraisal of Gentzen's work foundational proof-theoretic approaches started to flourish. Formal logic is not what one normally associates with Popper, and there were so many important debates in philosophy of science in the late 1960s in which Popper was involved, such that contributing to the debate on the foundational role of proof theory (Dummett, 1975; Kreisel, 1971; Prawitz, 1971) was way beyond his interests at that time.

A particular role seems to have been played by Tarski, whom Popper admired, but whose semantical approach was incompatible with the inferentialism Popper put forward in his logical papers. Popper wrote in a letter of 9 July 1982 (this volume, § 32.2): "I was discouraged at the time by the fact that Alfred Tarski, whom I admire very much, did not want to have a look at these works. I had no one else".[2]

[1] He indeed planned to write a foundational paper on logic together with Popper, which unfortunately did not materialize beyond initial stages (this volume, Chapter 14).

[2] Tarski received reprints from Popper in December 1947, and it is most likely that these were reprints of Popper (1947b,c,d). Apparently, Tarski was very busy at the time and just asked Woodger in a letter of 16 December 1947 to pass to Popper his greetings: "I got a few reprints from Popper. Would you please give him my best greetings when you see him?" (Mancosu, 2021, p. 94).

With his inferentialist approach, Popper entered uncharted logical terrain. It can well be imagined that in spite of the fact that his logical papers were written "with much enthusiasm" (Popper, 1974b, p. 1095, this volume, Chapter 12, p. 217), he felt somewhat insecure and preferred to leave a field in which he did not enjoy the authority and reputation he had acquired in the philosophy of science.

In any case, Popper's theory of deductive logic is worth reading and discussing, in particular from our modern point of view. This theory brings with it a lot of conceptual and technical insights which later became fundamental aspects of the foundational debate, such as his anticipation of Prior's operator *tonk*, the collapsing of intuitionistic into classical negation in a combined system, the inferential notion of duality and the notion of co-implication, the idea of dual-intuitionistic and bi-intuitionistic logic, the formal theory of substitution, and many more.

This chapter is structured as follows. A general introduction (§ 1) presents the fundamental ideas and concepts behind Popper's approach, in particular his theory of metalinguistic "inferential definitions" of logical constants. It proposes a reading that relates these definitions to the introduction of function symbols in first-order languages and thereby tries to justify Popper's view of inferential definitions as "explicit definitions". This reading characterizes Popper's theory as a variant of inferentialism which, however, puts no emphasis on the specific form of basic inference rules as done in modern proof-theoretic semantics. § 2 discusses the reception of Popper's logical work, in particular the above mentioned reviews, and points to some critical misunderstandings of Popper's views. In § 3 we discuss Popper's overall framework, pointing to his insistence that the whole matter is a metalinguistic approach, where, for the definition of logical constants, only restricted means – essentially positive logic with propositional quantification – are permitted. § 4 presents Popper's general theory of deduction, which is the theory of "structural" rules (to use Gentzen's terminology) without taking into account the internal deductive form or power of sentences. Popper's two approaches to characterizing this structural foundation, called "Basis I" and "Basis II" are discussed in detail. It becomes obvious that Popper's ignorance of the works of Hertz and Gentzen (Hertz, 1929a, Gentzen, 1935a,b; cf. also Schroeder-Heister, 2002) makes it more complicated than it otherwise could have been. In § 5 we discuss Popper's special theory of deduction, that is, his theory of logical constants and their inferential definability. Here we take up issues from § 1 and argue that this theory can be reconstructed in a consistent way. We discuss in particular Popper's inferential account of duality, which leads to his discussion of co-implication (called "anti-conditional" by Popper). § 6 gives a systematic account of Popper's theory of negation; more precisely, of various forms of negation including the classical and intuitionistic ones and their duals, as well as his development of a system of dual-intuitionistic logic. His discovery that certain distinct operators collapse into a single operator when combined within one system, the typical example being classical and intuitionistic negation, is particularly emphasized. § 7 is a sketch of Popper's treatment of modal logic, which is in the spirit of Carnap's *Meaning and Necessity* (Carnap, 1947). Though it is interesting to see how it fits into Popper's inferential framework, it is a bit outdated given that modal logic has made such impressive progress since Carnap's pioneering book. § 8 points to Popper's anticipation of what today is called bi-intuitionistic logic and gives a full proof of the result, only sketched by Popper, that intuitionistic negation and dual-intuitionistic negation can be combined in one system without their collapsing into one single negation. Finally, § 9 reconstructs Popper's account of quantifier logic based on a basic notion of substitution which is characterized by a list of axioms and thus represents a kind of theory of explicit substitution.

We heavily rely on our previous work on Popper's logic which resulted in four articles (Schroeder-Heister, 1984, 2006; Binder and Piecha, 2017, 2021), but deviate in several respects from the interpretation given there. The reader may well skip this chapter and study Popper's contributions directly, the most general and readable being "Logic without Assumptions" (Popper, 1947b), followed by "New Foundations for Logic" (Popper, 1947c).

1 General introduction

1.1 Deducibility: From Tarskianism to inferentialism

In his papers of 1947–1949, which are the core of the present collection, Popper claims to lay "new foundations for logic" (title of Popper, 1947c), which, at the same time, represent a "trivialization of mathematical logic" (title of Popper, 1949a). By "logic" he understands the theory of deducibility. His central notion is the deducibility of a statement a from statements $a_1, \ldots, a_n$, written as $a_1, \ldots, a_n / a$. By deducibility he does not mean the derivability in a formal system, but the semantical notion of logical consequence. For Popper, "a is a consequence of $a_1, \ldots, a_n$", "a follows from $a_1, \ldots, a_n$", "a is deducible from $a_1, \ldots, a_n$" and even "a is derivable from $a_1, \ldots, a_n$" are metalinguistic expressions which are synonymous. Providing new foundations for logic thus means developing a theory of deducibility or consequence in a novel way.

There is already the Bolzano-Tarski approach to logical consequence. Popper was well-acquainted with Tarski's version of this approach. Tarski's seminal paper on the notion of logical consequence had appeared in 1936 (Tarski, 1936b), so it was still "fresh" when Popper started working on logic.[3] Moreover, Popper was an admirer of Tarski throughout his lifetime, and it was Tarski's theory of truth that led him to give up his hesitation to use the concept of truth still prevailing in his *Logik der Forschung* (Popper, 1935, § 84; cf. Popper, 1959b, § 84, fn *1 and Popper, 1974c, § 20).

Tarski's notion of logical consequence is based on the idea of truth transmission: a follows logically from $a_1, \ldots, a_n$ if every interpretation which makes $a_1, \ldots, a_n$ true, makes a true as well. As Tarski had pointed out, this definition hinges on the definition of what a "logical constant" or "logical sign" is. An interpretation that makes $a_1, \ldots, a_n$ true and carries this truth over to a, can give all non-logical

[3] According to Popper's letter to Forder of 7 May 1943 (this volume, § 26.4) Popper even assisted Tarski in preparing the German version of this paper.

expressions in $a_1, \ldots, a_n, a$ an arbitrary meaning, but the meaning of the logical signs must be kept fixed throughout. That, for example, a follows from a-or-b and not-b, only depends on the meaning of "or" and "not", which is fixed, and not on the interpretation of "a" or "b", which is variable.

Therefore, so one can argue, by providing a satisfying definition of what a logical constant is, we obtain a satisfying theory of logical consequence and thus of deducibility, given that the notion of truth itself is not problematic and is sufficiently clarified by Tarski's theory of truth. This situation is the systematic starting point of Popper's investigations (in particular of Popper, 1947c).

There are nevertheless certain consequence laws which are independent of logical constants. These are the laws constituting a finite *consequence relation* in Tarski's sense. In a proof-theoretic setting, Gentzen (1935a,b) called them *structural rules* ("Struktur-Schlussfiguren"), where "structural" means independent of the "logical" form of the statements involved. Popper calls them "absolutely valid" since for their validity we need only be able to distinguish sentences from non-sentences, disregarding the internal structure of sentences. These rules and the various forms they take in Popper's theory will be discussed in further detail in § 4, especially in § 4.6. For our initial discussion it suffices to know that there is a core set of rules which essentially comprises reflexivity, monotonicity and transitivity, as well as permutation and contraction laws depending on whether the left side of a consequence claim is considered a list, multiset or set:

1. a/a.
2. If $a_1, \ldots, a_n/a$, then $b, a_1, \ldots, a_n/a$.
3. If $a_1, \ldots, a_n/a$ and $a, b_1, \ldots, b_m/b$, then $a_1, \ldots, a_n, b_1, \ldots, b_m/b$.

As the validity of these rules is unproblematic, we can assume that, whenever we are dealing with deducibility, it is given as a consequence relation.[4]

In what follows, "sentence" and "statement" are used synonymously. "Statement" is the term preferred by Popper. We do not use the term "proposition" since this is often used in a semantic sense as denoting the meaning of a sentence. Popper deals with syntactic entities throughout, although he emphasizes that the distinction between syntactic statements and semantic propositions does not affect his theorizing (cf. Popper, 1947b, fn 4, p. 261, this volume). His syntactic stance is quite natural, as he is throughout working in a proof-theoretic setting.

In spite of this Tarskian motivation and starting point, the idea to define the logicality of operations and thus logical consequence for complex statements leads Popper to develop a conception entirely different from Tarski's. This conception has at least three central characteristics:

1. It is not assumed that a specified formal object language is given, in which logical operations are represented by functional expressions ("sentential functions") which combine one or more sentences to a compound sentence (or, in the quantifier case, operate on open sentences). A sentence c may be a conjunction of a and b without having a special syntactic form such as $a \wedge b$ or Kab (depending on one's

[4] By that we always mean a *finite* consequence relation. Tarski's (1936b) consideration of consequences from an infinite number of assumptions goes beyond Popper's finite proof-theoretic framework. Insofar it is much more related to Gentzen (1935a,b) and Hertz (1929b).

logical notation). Tarski's approach of truth transmission under preservation of logical structure assumes that such a specification is given.

2. Logical operations are *relationally* and not *functionally* characterized. For example, that c is a conjunction of a and b is a three-place metalinguistic relation which is formally defined in the following way:

$$c \text{ is a conjunction of } a \text{ and } b \text{ if and only if } \mathcal{R}_{\mathrm{conj}}(c, a, b)$$

where $\mathcal{R}_{\mathrm{conj}}(c, a, b)$ is a certain metalinguistic condition. This relational view can leave open whether a logical operation, in our example conjunction, (1) always exists and (2) is uniquely determined. Both features would have to be presupposed if one preferred a functional characterization of logical operations as implicit in standard logical notation. Tarski, in considering a structure based on logical constants, sticks to a functional view from the very beginning.

3. The relational characterization of logical operations proceeds in terms of deducibility. For Popper, results of logical operations (conjunctions, disjunctions, implications, ...) are characterized by their deductive or inferential behaviour. In the above definition of conjunction, $\mathcal{R}_{\mathrm{conj}}(c, a, b)$ describes characteristic inferences involving conjunction in terms of $/$. There are various options for choosing $\mathcal{R}_{\mathrm{conj}}(c, a, b)$ (cf. § 5.5 below), the most straightforward of which leads to the following *relational definition of conjunction*:

(RelDef-conj) $\quad c$ is a conjunction of a and b if and only if

$$a, b / c \text{ and } c / a \text{ and } c / b$$

which means that $\mathcal{R}_{\mathrm{conj}}(c, a, b)$ is "$a, b / c$ and c / a and c / b". Thus $\mathcal{R}_{\mathrm{conj}}(c, a, b)$ formulates the standard introduction and elimination rules for conjunction in the form of consequence statements. According to this definition, we call something a conjunction if it obeys these introduction and elimination rules. For Tarski, the deductive behaviour of conjunction would not be its *definiens*, but a *consequence* of its non-deductive (e.g., truth-theoretic) characterization.

Note that we speak of the logical "operation" in contradistinction to the logical "operator" of conjunction. The operator of conjunction is the logical connective that has the form of a two-place sentential function like $\land$, whereas the logical operation of conjunction denotes the relation between a, b and one of its conjunctions c in the sense of (RelDef-conj), independent of what c looks like. This terminology is not perfect, as talking of an operation of "conjunction" might suggest that we are talking of a function. However, in our sense, an operation is not necessarily deterministic, (though in the case of conjunction it is[5]).

The third point turns the project of justifying deducibility upside down as compared to Tarski. Since we rely on a notion of consequence in the relational definition of logical operations, we can no longer use this notion of logical operation to define the

[5] At least with respect to logical equivalence. In a language with a conjunction operator $\land$, the sentences $a \land b$ and $(a \land b) \land (a \land b)$ are two syntactically different conjunctions of a and b in the sense of (RelDef-conj), but they are logically equivalent and thus have the same deductive power.

validity of consequences along Tarskian lines. In fact, such a definition of validity in terms of truth transmission becomes obsolete, as deducibility must already be available at the level of logical operations. Thus the notions of truth and truth transmission do not play a role any more in the justification of logical inference. They are discarded in favour of deducibility as a primitive notion. Logical rules are set up and explained in an inferentialist framework without any recourse to truth. All this makes Popper a logical inferentialist in the genuine sense of the term, which was only much later coined by Brandom (1994, 2000). Actually, although Popper does not use the term "inferentialism" to denote his orientation, he speaks of the "inferential characterization" of expressions and, in the case of logical constants, of their "inferential definitions". So the term "inferential" is present, and even abundant, in Popper's logical writings.

At first glance this might be surprising, given that Popper is an outspoken realist according to whom science is attempting to reach truth, and given that inferentialism is normally associated with anti-realist and sometimes even instrumentalist approaches to validity, something that Popper strongly opposes (cf. Popper, 2004). However, on a closer, second look, the inferentialist approach fits very well with Popper's conjectural approach to scientific reasoning. Logic is not only a tool used in the testing of theories, in particular in the attempt to falsify them, but is itself something that is open to criticism. The great advantage of the inferentialist framework Popper proposes is that it allows for the development of many alternative logics with a variety of logical operations. In modern terminology, he is giving a *logical framework*, that is, a framework for presenting logics. Within such a framework we can compare and challenge systems of logic, also taking into account their applicability in the empirical sciences. Viewed from this perspective, the inferentialist theory fits even better into Popper's conjecturalist approach than a monolithic theory of truth. Popper's inferentialist foundation of logic is not an approach justifying a certain logical system as the right logic. For Popper, logical laws are always relative to given definitions, and whether to pose a certain definition is not a matter of principle but a matter of whether it serves one's purposes. Thus Popper's inferentialist approach allows one to check in a perspicuous way what it means to base one's reasoning on one or other system of logic. His *new foundations* of logic is definitely not a foundationist approach in the sense strongly criticized by him at various places (e.g., Popper, 2004, Introduction).

1.2 Logical relations

As relational characterizations of logical constants, Popper gives metalinguistic definitions of the following form (which the already mentioned definition (RelDef-conj) of conjunction belongs to):

c is a conjunction of a and b if and only if ...

c is a disjunction of a and b if and only if ...

c is an implication from a to b if and only if ...

c is a negation of a if and only if ...

c is a universal quantification of a with respect to x if and only if ...

c is an existential quantification of a with respect to x if and only if ...

Besides such relational definitions for the standard operations, many more for various classical and non-classical logical operations are given (details are discussed below in §§ 5.5–5.7, § 6, § 7 and § 8). The relations defined we here call *logical relations*. The defining conditions, above represented by dots (with $\mathcal{R}_{\mathrm{conj}}(c, a, b)$ being our example for conjunction), contain as their central notion deducibility $/$. Here we call them "inferential conditions". It is inferential properties described in terms of deducibility that make an operation a *logical* operation. An operation is logical if it can be relationally defined by an inferential condition. By means of these logical relations, Popper is implicitly giving a criterion of logicality (cf. § 5.3 below). A relation is logical if the definiens (the dots above) is a deducibility condition of a certain form.

Now what does such an inferential condition look like syntactically? We are dealing with a metalanguage which is not fully formalized. However, from the many examples of relational definitions Popper discusses in his papers, it is fairly clear what is intended (cf. § 3.3). The only "material" sign of the metalanguage occurring in the defining condition of a logical relation is the deducibility operator $/$. For the treatment of quantification, we need also a substitution operation. As metalinguistic logical constants we use the constants of positive logic with propositional universal quantification, where disjunction is employed only for the relational definition of modal operations. Thus negation and existential quantification are not used, and disjunction only in a special case. For example, if c is a disjunction of a and b, an inferential condition $\mathcal{R}_{\mathrm{disj}}(c, a, b)$ characterizing disjunction would be "for all d: c/d if and only if a/d and b/d", which at the metalinguistic level uses conjunction, implication and propositional quantification. Therefore a relational definition of disjunction would be the following:

(RelDef-disj) c is a disjunction of a and b if and only if
for all d: c/d if and only if a/d and b/d.

The operators used in the metalanguage are essentially those needed to describe inference rules, namely universal propositional quantification to express the generality of rules; conjunction to express that there may be more than one rule and that a rule may have more than one premise; and implication to express the inference lines in rules leading from consequence statements to consequence statements.

This does not mean that the metalanguage does not contain further means of expression, such as negation or existential quantification. It only means that those metalinguistic means which may occur in the *defining condition* of a logical operation – the right side of its relational definition – are restricted. Popper does not give a clear specification of these restrictions, but it is clear that what he has in mind are restrictions in the spirit of inferentialism, and this is essentially a fragment of positive logic. What this exactly means can only be concluded from the inferential characterizations Popper is giving in his writings. There is never a negation occurring in inferential conditions – their positivity is essential. Further means that do not seem to be problematic at all, such as quantification over finite sets of sentences rather than only sentences, are not considered by Popper. This led to considerable difficulties in formulating some of his structural frameworks (cf. § 4.6 below).

As mentioned in the previous section, it is not presupposed at all that operators of conjunction, disjunction etc. are available in the object language considered. This means that sentential functions like $\wedge$ or $\vee$ do not need to exist. In fact, as also mentioned there, a conjunction or disjunction of a and b need not be available at all, whether expressed by a sentential function or not. It is easy to construct consequence relations without there being conjunctions or disjunctions of given sentences (cf. Koslow, 1992, p. 108f.). By the relational definitions of conjunction or disjunction, (RelDef-conj) and (RelDef-disj), we are only *characterizing* a sentence c, *whatever it may look like*, as a conjunction or disjunction of a and b, provided it exists at all. We are not expecting that we have a formal language with fully specified operators, which, by its formation rules, would make sure that any a and b can be conjunctively and disjunctively composed, and, by its inference rules, would make sure that introduction and elimination rules for these connectives are assumed to hold.[6]

This makes it possible to consider logical relations that cannot exist in consistent languages. Popper's prominent example is the logical relation "c is an opponent of a" (*opp*), which is defined in such a way that the existence of opponents for arbitrary sentences makes the underlying language inconsistent; that is, it allows for the derivation of any consequence statement. He thus antedates the discussion about Prior's (1960) operator *tonk*. However, for Popper *opp* would be a logical operator, if it existed (cf. § 5.2 below).

The relational view of logic – the definition of "c is a conjunction of a and b", "c is a negation of a" etc. rather than the consideration of logical operators $a \wedge b$, $\neg a$ from the very beginning – allows for a very general view on logical phenomena. We are not confined to formal languages of a specific format, but can just study arbitrary consequence relations and the availability of logical relations within these consequence relations. If we further restrict the inferential conditions allowed in the definiens of logical relations, we can single out logical relations of a specific kind. We might, for example, consider inferential conditions which describe introduction and elimination rules, or rules expressing that operators generate strongest or weakest statements obeying certain rules. The latter approach is followed by Koslow (1992) in his structural approach to logic, where arbitrary consequence relations and maximality and minimality principles in the definientia of logical relations are studied. A concrete logic with certain logical operators in an object language is then a kind of model of this relational approach. In this theory we are considering logical relations in a very general way, comparable to model theory dealing with the general structural features

[6] We do not always thoroughly distinguish between a language and an (axiomatized or non-axiomatized) theory within that language. That is, we often assume that with a language certain inference rules are given as well. It will always be clear from the context, in which sense the term "language" is being used.

of concrete theories. Schroeder-Heister (2006) proposed to interpret Popper's theory in this structuralist sense, reading it as a descriptive theory of consequence relations structured by logical relations. However, even though this reading is a viable option, it does not do full justice to Popper's introduction of logical operators, which is a crucial step beyond the relational definition of logical operations. While the latter can be understood purely descriptively, the former cannot.

1.3 Logical operators and inferential definitions

For a foundation of logic, we need more than just a theory of consequence relations and the definition of logical relations within such a theory. We want to introduce logical operators as specific formal signs and to formulate valid rules governing these signs.

It may well be, of course, that the language under consideration already contains such operators, but this is not a formal requirement. Popper's foundations for logic apply to any language with the only restriction that its deducibility notion is a consequence relation. The main idea of Popper's "new foundations" is that a consequence relation is given, and that a logician, as the investigator of this consequence relation, extracts a logical structure out of it by identifying sentences that might be considered results of logical operations applied to other sentences. Once such operations have been defined by the logical relations "is a conjunction of", "is a disjunction of" etc., we may, in a next step, introduce syntactical operators representing these operations. This introduction of a syntactical operator is called an "inferential definition" by Popper. This means that we must distinguish between a relation such as "is a conjunction of" and an operator such as "$\wedge$" syntactically representing the relationally defined operation of conjunction.[7]

While the relational definition of "c is a conjunction of a and b" leaves it open whether, given a and b, there is a c at all, whether there is a single c or whether there are multiple c's, a logical operator ("logical constant") must be unique. *The conjunction of a and b* is the result of a sentential function $\wedge$ generating from a and b the compound expression $a \wedge b$. As deducibility is our basic concept, by uniqueness we mean uniqueness up to interdeducibility, written by Popper as "//". That we have uniqueness in the case of conjunction is clear from its relational definition (RelDef-conj). It can easily be seen that if c and d are both conjunctions of a and b, then c and d are interdeducible: $c /\!/ d$. The fact that for an operation to be logical we do have uniqueness, must always follow from the relational definition of the operation in question. There are operations which are not unique but nevertheless play an important logical role, even though, strictly speaking, they cannot be represented by (functional) operators. A prominent example is Johansson's negation (cf. § 5.3 and § 6.4). There can be two non-equivalent Johansson-negations of the same sentence.

The second requirement to be able to talk of *the conjunction of a and b* is existence:

[7] In these introductory remarks we only consider propositional operators; for the whole picture including quantification cf. § 9 below.

there must be a c in the language considered which is a conjunction of a and b in the sense of the relational definition. We can, of course, postulate its existence by means of formation rules which say that for every a and b the expression $a \wedge b$ belongs to our language and by introduction and elimination rules for conjunction assumed as (metalinguistic) axioms. We might call this way of proceeding the generation of logical constants *by postulation*. This is certainly legitimate and also done by Popper when constructing a logical language (cf. Popper, 1947c, beginning of § 4). However, we must make sure that in this way we do not generate undesired results, and in particular no inconsistency. This would actually happen in the case of the opponent $opp(a)$ of a. If we postulated the availability of $opp(a)$ for every a with the rules characteristic of opp, we would obtain a language which is inconsistent. This would not be conceptually faulty in any sense. But it would, of course, mean that the language constructed is of no use.

Popper is fully aware of this situation. According to his approach, we either consider a language given by a consequence relation, in which uniquely determined operations exist which can be relationally defined, or we construct such a language. Whether the language is consistent is a highly important but different question, which does not impair its character as a logical language.[8,9]

Now suppose we have a relational definition of a logical operation where existence and uniqueness are met. In the example of conjunction, we suppose we have (RelDef-conj) with conjunctions existing and their uniqueness following from the right side of (RelDef-conj). Defining an operator of conjunction means to define $a \wedge b$ in such a way that the relation of being a conjunction is the graph of the function $\wedge$. Due to uniqueness, $a \wedge b$ can be viewed as a special term representing any conjunction (mathematically, a representative of the equivalence class of all conjunctions). It is introduced by Popper by postulating

(InfDef-conj) $$c /\!/ a \wedge b \text{ if and only if } a, b / c \text{ and } c / a \text{ and } c / b.$$

Metalinguistic statements of this kind are called "inferential definitions" by Popper, as they uniquely define the deductive power of certain operators.

For a better understanding we consider the following parallel situation in first-order logic. Here an analogue to inferential definitions is an axiom that would be used to introduce a function sign into a first-order theory with identity, for example

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[8] Here it is important to remember that on the right side of a relational definition only positive metalinguistic operators are allowed to occur. Otherwise we could construct conditions which are metalinguistically inconsistent, as Lejewski (1974) has shown (misleadingly using them as a counterargument against the Popperian approach); cf. § 5.2 below.

[9] Although not discussed by Popper, one could discard the existence requirement by considering a logical operator to be a *partial* function, which allows that for certain arguments the result of the operation is not defined. For example, there might be an opponent of certain, but not of all sentences. This would certainly result in an elegant theory, comparable to recursive function theory in which not every partial recursive function need be total. The advantage of this partial approach would be that we do not have to make sure that an operation must deliver a result for any argument. The totality of this operation would then be something to be proved subsequently and not something to be assumed in advance.

Peano arithmetic. Suppose there is a formula $\varphi(y, x_1, \ldots, x_n)$ such that we can prove existence of uniqueness for the leftmost argument:

$$\forall x_1, \ldots, x_n \exists y \forall z (z = y \leftrightarrow \varphi(z, x_1, \ldots, x_n)).$$

Then we can conservatively introduce a function symbol f by means of the axiom

(Ax-f) $$\forall x_1, \ldots, x_n, z (z = f(x_1, \ldots, x_n) \leftrightarrow \varphi(z, x_1, \ldots, x_n)).$$

Conversely, if the theory in question contains already a function constant f for which (Ax-f) is derivable, we may call f "explicitly definable" in the theory. It is therefore not absurd to call the axiom (Ax-f) a "definition" of f, as it fixes the meaning of the newly introduced constant f, even though it is not a *fully* explicit definition, which would define f through a term τ:

$$f(x_1, \ldots, x_n) = \tau(x_1, \ldots, x_n).$$

Correspondingly, it is perfectly legitimate that Popper speaks of expressions of the form (InfDef-conj) as (inferential) *definitions* and even calls them "explicit definitions", even though they are not fully explicit definitions in the sense in which, for example, conjunction $a \wedge b$ can be defined through the Sheffer stroke[10]:

$$a \wedge b \leftrightarrow (a \mid b) \mid (a \mid b).$$

As in the example from first-order logic, the operator introduced by an inferential definition is eliminable from any context, if the inferential definition is assumed as an axiom.[11]. Thus, formally, Popper's inferential definitions are well-formed entities, which makes them a precise conceptual tool. For a further discussion cf. § 5.2 below.

1.4 Logical laws and the trivialization of logic

Given inferential definitions of certain logical constants, we immediately obtain from them basic laws for these constants, namely the right hand side of the inferential definitions with the logically composed statement inserted for the leftmost variable. From the inferential definition of conjunction we obtain $\mathcal{R}_{\text{conj}}(a \wedge b, a, b)$, from the one of disjunction $\mathcal{R}_{\text{disj}}(a \vee b, a, b)$ etc. And since these conditions uniquely (up to interdeducibility) describe what is meant by conjunction, disjunction etc., they are exhaustive in the characterization of these connectives. This means that we

[10] For Popper's discussion of the Sheffer stroke under the term "alternative denial" cf. the handwritten note Popper (n.d.[a]). Cf. also § 5.8.

[11] This cannot be spelled out in detail here. A proof would proceed along the lines of the first-order case, which is very closely related to the elimination of definite descriptions. For a semantical proof cf. Mendelson (1997, § 2.9), for syntactic proofs Hilbert and Bernays (1968, § 8) and Kleene (1952, § 74). Another analogy is the handling of set terms in set-theoretic languages (which normally do not contain explicit set terms as primitives).

obtain all logical laws involving these connectives by taking $\mathcal{R}_{\text{conj}}(a \wedge b, a, b)$ and $\mathcal{R}_{\text{disj}}(a \vee b, a, b)$ etc. as axioms or rules for the deducibility relation.

This makes sense only if $\mathcal{R}_{\text{conj}}(a \wedge b, a, b)$ and $\mathcal{R}_{\text{disj}}(a \vee b, a, b)$ etc. are formulated in such a way that they can be read as axioms or rules governing deducibility. As already mentioned above, this is always guaranteed for the inferential definitions Popper presents, even though the expressive means of the metalanguage are not formally specified. Thus a relational definition of a logical operation could always be formulated as: "... if and only if the following rules hold" rather than metalinguistically circumscribing these rules. In the case of conjunction, rather than (InfDef-conj), we would write:

(InfDef-conj′) $c /\!/ a \wedge b$ if and only if c, a, b obey the following rules:

$$\frac{a \quad b}{c} \qquad \frac{c}{a} \qquad \frac{c}{b}$$

Substituting $a \wedge b$ for c, we immediately obtain that the following rules hold:

$$\frac{a \quad b}{a \wedge b} \qquad \frac{a \wedge b}{a} \qquad \frac{a \wedge b}{b}$$

which are inference rules fully characterizing conjunction.

More generally we can say that the universal form of an inferential definition of an n-place propositional operator $S(a_1, \ldots, a_n)$ is

(InfDef-S) $c /\!/ S(a_1, \ldots, a_n)$ if and only if

$c, a_1, \ldots, a_n$ obey the rules $\mathcal{R}_S(c, a_1, \ldots, a_n)$.

By substituting $S(a_1, \ldots, a_n)$ for c we obtain the inference rules

$$\mathcal{R}_S(S(a_1, \ldots, a_n), a_1, \ldots, a_n)$$

governing S. As this instantiation is immediate and yields inference rules in a trivial way, Popper may speak of the trivialization of logic, namely the metalinguistic deduction of valid inference rules immediately from a definition. One may consider this terminology misleading and argue that the main conceptual work lies in the formulation of the rules $\mathcal{R}_S(c, a_1, \ldots, a_n)$ in the first place, and thus in the formulation of the inferential definition of S. But this is a different matter. *Given* the inferential definitions of logical constants in the form (InfDef-S) (for conjunction in the form (InfDef-conj′)), obtaining their governing rules is *trivial*.

1.5 Popper and proof-theoretic semantics

We do not find any particular reflection in Popper what the basic rules for an operator – the $\mathcal{R}_S(c, a_1, \ldots, a_n)$ in (InfDef-S) – should look like. When reading Popper's discussions, one gets the impression that there is no preference whatsoever. The only criterion seems to be that for a logical constant an inferential definition of whatever form can be given. There does not seem to be any criterion to distinguish between better and worse inferential conditions, provided the criteria of existence and uniqueness are met. From this point of view Popper's approach looks as if any kind of inferential characterization of logical operators is possible, in particular a characterization by any finite list of inference rules. This makes his inferentialism differ from proof-theoretic semantics (Schroeder-Heister, 2018; Francez, 2015; Wansing, 2000; Piecha and Schroeder-Heister, 2016), where one tries to directly justify *specific* inference rules as meaning conveying, and as appropriate to provide epistemologically convincing argument steps. To be sure, one finds all the rules considered in proof-theoretic semantics also in Popper, such as introduction and elimination rules, sequent-style right and left introduction rules etc. But Popper does not specifically argue in favour of any of these presentations.

The fact that Popper does not consider consistency or conservativeness to be an admissibility criterion when introducing logical operators confirms this view. Whereas in proof-theoretic semantics great emphasis is put on conservativity and on the fact that the introduction of new operators is non-creative, this is not an issue in Popper. His consideration of *opp* as an example of a logical operator (and not as a counterexample comparable to the discussion of *tonk* in modern proof-theoretic semantics), or his observation that different negations collapse into classical negation, if the latter is present, show that consistency and conservativeness are no big issues for Popper. This also corresponds to the fact that Popper finds the distinction between classical and intuitionistic logical systems very interesting without considering it to be his task to argue for one of these systems against the other, or at least to provide criteria along which to differentiate between alternative logical systems.

From the standpoint of proof-theoretic semantics, this looks more like delineating logics than providing a semantics of logical signs. Schroeder-Heister (1984) actually proposed to read Popper's approach as a theory of logicality rather than a semantics of logical constants, considering an inferential definition to be a logicality condition for the operator in question rather than a semantic definition of its meaning. It is certainly correct that Popper wants logicality determined that way (cf. § 5.2 below). And, as just mentioned, for a genuine proof-theoretic semantics of logical operators a more specific discussion of semantical rules should be required. However, reducing Popper's approach to a definition of logicality gives too little justice to the fact that inferential definitions extract inferential content from consequence relations and generate logical rules, beyond the pure fact that the operators involved are logical.[12,13]

Thus Popper definitely wants to give a semantics for the logical signs and not only a criterion of their logicality. But this sort of semantics is only very rudimentarily

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[12] Cf. also the letter by Popper to Schroeder-Heister of 10 July 1982 (this volume, § 32.3): "[. . .] my first goal was the characterization of formative signs. But then it seemed to me that, if one succeeds with it, one can immediately justify [German: 'begründen'] propositional logic; and somehow this is possible, although it is a separate step.".

[13] This holds, by the way, also for other proof-theoretic definitions of logicality, which can be read as semantical conditions, for example for Došen's (1989) definition, from which inference rules both in natural deduction-style and sequent-style format can be deduced.

and implicitly present in his writings. The most important point in this respect is that after Gentzen he is the first to consider the format of the sequent calculus (with "/" interpreted as the sequent arrow) for foundational considerations. With that comes the aspect that his characterization of logical constants uses a strict separativity criterion, which is inherent in the sequent calculus: namely, that for every constant there is a separate set of rules which does not involve any other constant. This means that Popper is not an inferentialist in the holistic sense that all inference rules taken together somehow simultaneously determine the meaning of all operators involved in these rules. Instead he puts great emphasis on the individual definability of logical operators and on the investigation of how operators interact when they are available in the same system, though defined independently.[14] This is a great achievement two decades before the philosophical considerations on the format of reasoning in the aftermath of Gentzen's systems took off the ground (e.g., in Prawitz, 1971). Furthermore, various forms of criteria discussed in proof-theoretic semantics can be modelled in the Popperian framework, such as Lorenzen's (1955) and Prawitz's (1965) inversion principle (cf. Schroeder-Heister, 2007; de Campos Sanz and Piecha, 2009), Dummett's (1991) and Prawitz's (2006) harmony requirements (cf. Schroeder-Heister, 2015, 2016; Tranchini, 2021), Sambin et al.'s (2000) "Basic Logic" (cf. Schroeder-Heister, 2013), Došen's (1989) approach of double-line rules, Schroeder-Heister's (1984) idea of the common content of rule systems, Tennant's (1978) maximality and minimality principles and so on. Taking this together, one might say that Popper is a full-fledged inferentialist and a rudimentary proof-theoretic semanticist.

In not insisting on a specific form of inferential conditions, that is, of the right hand side of his inferential definitions, Popper is much more liberal in admitting semantical rules than modern proof-theoretic semanticists. This re-emphasizes the view that, in accordance with Popper's general standpoint, the distinction of the proper logic is not a matter of foundationist, but rather of consequential reasoning, which would take into account all ends to be achieved and side conditions to be observed.

To summarize the overall philosophical structure of Popper's way of proceeding: We assume that a *consequence relation* is given, or we construct one by postulation. By means of *relational definitions* we characterize logical operations which may be available in this consequence relation. By means of *inferential definitions* we introduce logical operators as names for these operations. Using logically structured sentences thus introduced, we obtain the *inference laws* for these sentences immediately from the inferential definitions.

2 The reception of Popper's logical writings

Popper's articles on logic were reviewed by Ackermann (1948, 1949a,b), Beth (1948), Curry (1948a,b,c,d, 1949), Hasenjaeger (1949), Kleene (1948, 1949) and McKinsey (1948). Kemeny (1957) and Nagel (1943) are reviews of the papers Popper (1955a)

[14] Cf. § 6.3 below.

and Popper (1943), respectively, which accompany the key papers of our selection. All reviews are reprinted in Chapter 13 of this volume.

Ackermann gives a summary of Popper's views, which is in line with our interpretation. Beth stresses the relationship to the works of Gentzen (1935a,b), Jaśkowski (1934) and Ketonen (1945), without any further criticism. Curry in his five reviews emphasizes the similarity to Hertz's (1929b) and Gentzen's (1935a,b) approaches and points to the deficiencies of what Popper calls his "Basis II" of structural rules, which we discuss in detail in § 4.6. He acknowledges Popper's extensive treatment of negations (discussed in § 6) including the collapsing result when classical negation is present (cf. § 6.3). The final sentence of his last review (Curry, 1948d) summarizes his critique: "The whole program is very obscure, and has not been without serious error [. . .]; likewise it was anticipated, in many respects, by the work of Gentzen and others."

Hasenjaeger criticizes hidden existence assumptions in inferential definitions, something that is also the main point of Kleene's reviews. However, as shown above, this view is misleading and in any case cannot be taken as a critique of Popper. There are certainly existence assumptions: we cannot inferentially define an operator if the corresponding relationally defined operation yields no result for the arguments considered. But this is something of which Popper is well aware. Inferential definitions are always made under the supposition that results of logical operations exist, to which the inferential definition gives a name. This becomes absolutely clear from our analogy drawn between inferential definitions and the introduction of function symbols in a language with equality, which is a standard way of proceeding in mathematical logic (cf. § 1.4).

McKinsey criticizes that the consequence sign "/", which is introduced to denote absolutely valid inferences, is extended to cover inferences involving logical constants. As is also the case with Kleene, he misses the fact that Popper's metalinguistic theory applies to the established deductive practice in an object language which is not normally specified formally and employs from the very beginning conjunctions, disjunctions etc. in the sense of his relational definitions. This does not exclude, as a limiting case, that the object language is formally structured. However, when we define such a formal language, we do a creative job and must be prepared for unwanted results that may show up, including inconsistencies; Popper does not ignore these issues. A discussion of McKinsey's criticism by Popper himself can be found in an unpublished manuscript (cf. this volume, Chapter 17, Typescript 1, § 3, footnote 8).

An important point in several reviews is the emphasis that Gentzen's work (and also the related work of Hertz, 1929b, Jaśkowski, 1934 and Ketonen, 1945), in particular his sequent calculus, is highly relevant to Popper's undertakings. As pointed out in § 4.6, it can be asked whether or not Popper was aware of Hertz's and Gentzen's work when he developed his theory.

Bernays, Brouwer, Kneale and Quine responded quite positively, as their correspondence with Popper shows (cf. Chapters 21, 22, 29 and 30). Bernays planned to write a foundational paper on logic together with Popper, which unfortunately did not materialize beyond initial stages (this volume, Chapter 14). Brouwer presented several of Popper's papers to the Royal Netherlands Academy of Sciences.[15] For Kneale, Popper's formulation of logical rules became a key ingredient of his theory of logicality (cf. Kneale, 1956 and also Kneale and Kneale, 1962, § IX), which itself provides a background to later proof-theoretic approaches to logical constants (cf., e.g., Hacking, 1979 and Došen, 1989). Detailed investigations of Popper's theory are given in the articles of Lejewski (1974) in the Schilpp volume on Popper (cf. Schilpp, 1974)[16]; Schroeder-Heister (1984, 2006) and Binder and Piecha (2017, 2021), which were already mentioned above; Bar-Am (2009), which discusses a distinction between the notions of sound inference and proof; and Moriconi (2019), which provides a detailed overview and elucidates in particular Popper's treatment of negation and implication by making use of a sequent calculus setting. There are some unpublished theses dealing with Popper's logic, of which we know Cohen (1953b), Brooke-Wavell (1958) and Dunn (1963). Here we just sketch Cohen's thesis, as for systematic reasons it is the most interesting one.

Cohen's (1953b) Oxford B.Litt. thesis is highly relevant to matters discussed below in § 5.4 and § 6.2. Its second part concerns the development of a system of dual-intuitionistic logic.[17] Cohen starts from Gentzen's observation that intuitionistic logic can be obtained from the sequent calculus for classical logic by restricting the number of formulas occurring in the succedent of sequents to at most one. He then formulates a sequent calculus where the number of formulas occurring in the antecedent of sequents is restricted to at most one (without restricting succedents), which he calls the "dualintuitionistic restricted predicate calculus GL2". The idea of developing this system was suggested to him by Popper.[18] The inception of dual-intuitionistic logic can thus be attributed to Popper, and Cohen was the first to develop and investigate a system of dual-intuitionistic logic. Cohen's sequent calculus GL2 is, however, not exactly dual to intuitionistic logic, since it contains rules for both the anti-conditional and the conditional.[19]

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[15] These are Popper (1947d), communicated at the meeting of 25 October 1947, and Popper (1948a,c), both communicated at the meeting of 29 November 1947.

[16] Popper's (1974b) reply to Lejewski is reprinted in Chapter 12 of this volume.

[17] Popper (2004, fn 8, p. 431f.) cites this system as an answer to his question "whether we can construct a system of logic in which contradictory statements do not entail every statement." However, "[i]n [Popper's] opinion, such a system [which lacks e.g. modus ponens] is of no use for drawing inferences."

[18] As remarked by Cohen (1953b, p. 188). Popper (2004, fn 8, p. 431f.) first refers to his (1948a; 1948c), and then mentions that "[. . .] Cohen has developed the system [of dual-intuitionistic logic] in some detail." In fact, Cohen also gave a full analysis of this system, including a proof of cut elimination. Popper (ibid.) has "a simple interpretation of this calculus. All the statements may be taken to be modal statements asserting possibility." This interpretation can already be found in a letter from Popper to Cohen of 28 July 1953 (this volume, § 25.7), where he interprets every statement a as "a is possible" or, equivalently, as "a is satisfiable". In this letter Popper says that he "consider[s] publishing these results", without, however, doing so. Cohen (1953b, p. 208f.) names Popper's (1948a, p. 181) interpretation of sequents in terms of relative demonstrability (cf. § 4.4) as the most suitable one.

[19] Cohen (1953b, part II, § 3) first formulates the system GL1, which includes the rules FS and FA for the conditional. This system is then extended to his dualintuitionistic restricted predicate calculus

3 Popper's structural framework

Popper considers an object language $\mathcal{L}$ together with a deducibility relation on $\mathcal{L}$, which today we would call a finite consequence relation in Tarski's sense. As already mentioned, Popper uses the terms "deducible", "derivable", "follows from" and "is a consequence of" synonymously. Popper knew Tarski's work very well. Whether he knew any details of Hertz's and Gentzen's work, in particular of their structural framework, or whether he was aware of it at all, is questionable, as mentioned above and further discussed in § 4.6. The set of axioms characterizing a deducibility relation is called a *basis*. Popper mainly uses two alternative bases, called Basis I and Basis II, which also occur in different versions. We use a version of Basis I as the foundation for the rest of this introduction.[20]

3.1 Object languages

In contradistinction to modern approaches that proceed by giving an alphabet of the object language under consideration and either a grammar or inductive definition of those expressions that are to be counted as terms and formulas, Popper's approach does not presuppose any knowledge about the form or syntactic structure of the object language under consideration. Popper explicitly points to this difference; he gives a sketch of the classical approach at the beginning of Popper (1949a). The idea that logical analysis is not only applicable to formally specified languages is still present later, in Popper (2004, addendum 5 from 1963), where he stresses this fact in the context of discussing Tarski's theory of truth:

> It has often been said that Tarski's theory of truth is applicable only to formalized language systems. I do not believe that this is correct. Admittedly it needs a language—an object-language—with a certain degree of artificiality; and it needs a distinction between an object-language and a meta-language [. . .] [N]ot every language which is subject to some stated rules, or based on more or less clearly formulated rules [. . .] need be a fully formalized language. The recognition of the existence of a whole range of more or less artificial though not formalized languages seems to me a point of considerable importance, and specially important for the philosophical evaluation of the theory of truth.

His approach is indeed intended to be not only applicable to formally defined languages but also to natural language. For example, the conjunction of two statements a and b of the object language need not have any particular syntactic form like "$a \wedge b$" or "a and b".

GL2 by adding the rules GS and GA for the anti-conditional; cf. Cohen (1953b, part II, § 4). Cf. Kapsner, Miller, and Dyckhoff (2014) and Kapsner (2014, p. 128, fn 6).

[20] This version of Basis I can be found in Popper (1947c).

> [Forming the conjunction of a and b] is done, in English, by linking them together with the help of the word "and". But we need not suppose that any such word exists: the link may be effected in very different ways; moreover, the new statement need not even contain the old ones as recognizable separate parts (or "components"). (Popper, 1947c, p. 205)

To make this point clearer, consider as an example the propositional language containing only signs for negation ($\neg$) and disjunction ($\vee$) together with a calculus in which all classically valid formulas of this restricted language are derivable. This language nevertheless contains for any two formulas φ and ψ a conjunction of φ and ψ. One such conjunction may be $\neg(\neg\varphi \vee \neg\psi)$; but other variants are possible, and there is in general no way to specify *the* canonical form of conjunction in that language.

This does not preclude that an object language is formally specified in the common way by inductive definitions. Popper himself considers such languages. However, it is not required at all, and Popper considers any sort of language, formal or non-formal, provided we know what it means that a sentence of the language follows from other sentences.

We will write $\mathcal{L}$ for an object language and, following Popper, use small Latin letters $a, b, c, \ldots$ (also with indices) as variables ranging over $\mathcal{L}$. The members of an object language are assumed to be *statements* (the Popperian term; we also speak of "sentences"), so that it makes sense to say that some of them are deducible from others. Popper (1947c, p. 204) calls them "expressions of which we might reasonably say that they are true or that they are false".[21] We furthermore assume that any object language considered is nonempty.

3.2 The concept of deducibility

Popper's approach is based on the concept of deducibility (or "derivability"). It is the only undefined notion as far as propositional logic (including modal logic) is concerned. An operation of substitution is added for the treatment of first-order logic (discussed below in § 9.1). Deducibility is a relation, written with the solidus /, that ranges over the object language and holds between finitely many premises (including the case of no premises) $a_1, \ldots, a_n$ and exactly one conclusion b. In /-notation, Popper writes

$$a_1, \ldots, a_n / b$$

to express that the statement b can be deduced from the statements $a_1, \ldots, a_n$. The case $n = 0$, in which no premises occur, was not yet considered in Popper (1947c). It was added later in Popper (1948a), where the so-called *D-notation* is introduced. In

[21] He states explicitly (ibid.): "Nothing is presupposed of our $a, b, c, \ldots$ except that they are statements, and our theory shows, thereby, that there exists a rudimentary theory of inference for any language that contains statements, whatever their logical structure or lack of structure may be."

this notation, $D(a_1, a_2, \ldots, a_n)$ stands for $a_2, \ldots, a_n / a_1$, and the special case $D(a_1)$ corresponds to $/a_1$, meaning that a_1 is deducible without premises.

The /-notation is introduced as a horizontal variant of the notation

$$\frac{\begin{array}{c} a_1 \\ \vdots \\ a_n \end{array}}{b}$$

that is often used to state rules of inference like, for example,

$$\frac{\begin{array}{c} \text{If } A\text{, then } B \\ A \end{array}}{B}$$

which says that from the premise "If A, then B" together with the premise A the conclusion B can be deduced (cf. Popper, 1947c, p. 194). The /-notation is also used to express rules that lead from deducibility statements to deducibility statements, for example, the rule called *Thinning* by Gentzen:

$$\text{If } a_1, \ldots, a_n / b \text{ then } a_1, \ldots, a_{n+1} / b \quad \text{(ibid. p. 196).}$$

In the context of deducibility neither the order of premises nor the multiplicity of identical premises is relevant, since deducibility enjoys the following structural properties, which we state as a lemma:

Lemma 3.1

1. *Exchange of premises:*

$$\textit{If } a_1, \ldots, a_i, a_{i+1}, \ldots, a_n / b, \textit{ then } a_1, \ldots, a_{i+1}, a_i, \ldots, a_n / b.$$

2. *Contraction of premises:*

$$\textit{If } a_1, \ldots, a_i, a_i, \ldots, a_n / b, \textit{ then } a_1, \ldots, a_i, \ldots, a_n / b.$$

These properties are consequences of Popper's characterization of deducibility by his Basis I.[22] Premises $a_1, \ldots, a_n$ can thus be understood as a set $\{a_1, \ldots, a_n\}$.

3.3 The metalanguage

When Popper defines logical operations inferentially, this is carried out in a metalanguage, whose basic relation is deducibility /, and not in a syntactically specified object language. If we consider, for example, a two-place logical operation $\circ$, which we call "connection" and which can stand for any operation such as conjunction or implication, or mutatis mutandis for operations of other arities such as negation, Popper relies on relational definitions of the form

[22] Cf. also Popper (1947c).

$$c \text{ is a connection of } a \text{ and } b \text{ if and only if } \mathcal{R}(c, a, b)$$

and on inferential definitions of the form

$$c // a \circ b \text{ if and only if } \mathcal{R}(c, a, b).$$

As already mentioned above (cf. § 2), in most cases the defining condition $\mathcal{R}(c, a, b)$ has the form of a rule, which is described in terms of deducibility /, metalinguistic conjunction, metalinguistic implication and metalinguistic universal quantification. Popper does not specify exactly the means of expression allowed in a defining condition, but from all contexts it is clear that positive logic is sufficient.

To improve readability of metalinguistic expressions, we use the following symbolic notation for it, which is similar to Popper's:

Symbol	$\rightarrow$	$\leftrightarrow$	&	$\bigvee$	(a)
Meaning	if-then	if and only if	and	or	for all a

The universal quantifier (a) ranges over statements a of the object language. Popper notes that as long as one does not want to introduce modal connectives, one can do without disjunction.[23] Therefore, given the special status of disjunction (discussed below in § 7) and the definability of equivalence in terms of conjunction and implication, the fundamental logical operations used in the metalanguage to define arbitrary logical operations of the object language are conjunction, implication and universal quantification (over sentences). We have therefore omitted negation and existential quantification from our list of symbols (though we use, of course, these operations in our metalanguage, but in the usual informal way, not within the defining condition of a logical operation).[24]

Due to the fact that only positive logic is permitted in the defining condition of an operation, the existence of this operation can never imply that the metalanguage is inconsistent. The trivial deducibility relation, which holds for all arguments, validates any defining condition $\mathcal{R}$ and therefore falsifies the negation of it, which means that not every metalinguistic statement is true. Trivialization of the object language does not imply trivialization of the metalanguage. We state this as a lemma:

Lemma 3.2 *In any nonempty object language with a trivial deducibility relation (i.e., $a_1, \ldots, a_n / b$ holds for any $a_1, \ldots, a_n, b$), every defining condition $\mathcal{R}$ for a logical operation is satisfied.*

[23] Cf. Popper (1947d, p. 1216): "[. . .] it may be mentioned that the only logical rules needed in the metalanguage (except where we treat modalities) are those of the positive part of the Hilbert–Bernays calculus of propositions as far as they pertain to 'if-then', 'if, and only if', and to 'and' [. . .], and the rules for identity. The rules for negation need not be assumed [. . .]; but we need rules for universal quantification, especially the rule of specification [. . .]."

[24] In his letter of 10 July 1982 to Schroeder-Heister (cf. this volume, § 32.3), Popper compares this restriction of the metalinguistic means of expression to Hilbert's programme: "[. . .] you will find that the metalinguistic use of ∀ and → needed for their full object-linguistic use is *greatly* restricted. [. . .] Here is indeed something similar to Hilbert's programme of finding a finite justification for a non-finite calculus.".

This will become relevant in the discussion of the logical constant *opp* later on (cf. § 5.2).[25]

For certain expressions of the metalanguage Popper uses a special vocabulary. Statements of the form

$$a_1, \ldots, a_n/b$$

are also called *absolute rules of derivation*, and statements of the form

$$a_1, \ldots, a_n/b \rightarrow c_1, \ldots, c_m/d$$

and iterated versions thereof are also called *conditional rules of derivation* or just *rules of derivation*. Note that rules of derivation are not rules in a calculus understood as a proof system. Popper does not develop a calculus in this sense, and what he calls rules of derivation are metalinguistically formulated statements about the deducibility relation.[26] However, due to their specific form they can be read as descriptions of rules, as they tell us that from certain deducibility statements we may pass over to another deducibility statement. Thus we follow Popper in using the term *rule* to speak about such metalinguistic expressions. This means in particular that we will often speak of $\mathcal{R}(c, a, b)$ as "defining rules" instead of "defining conditions", even if they cannot be translated immediately into rules of some object language.

3.4 The characterization of deducibility by a basis

So far, the deducibility relation / has only been defined by saying that it ranges over an object language $\mathcal{L}$. The next step consists in providing what Popper calls a *basis* for this relation.[27]

A basis is a complete and independent set of rules, formulated in the metalanguage, that axiomatizes the deducibility relation /. Completeness is here defined with respect to Popper's notion of *absolute validity*, which is similar to the notion of validity obtained by allowing only structural rules of inference.[28] Popper's idea seems to be

[25] Another simple example is the metalinguistic statement "there exists a, such that $\vdash a \mathbin{\&} \urcorner a$", where $\vdash a$ stands for the demonstrability of a and $\urcorner a$ for the refutability of a (cf. § 4.2 and § 4.3). This statement is also only true for object languages with a trivial deducibility relation, although it is a satisfiable expression of the metalanguage.

[26] Popper (1948c, p. 327) once uses the term "metalinguistic calculus".

[27] Popper (1947d) is an exception to this approach: the requirement to specify a basis for the deducibility relation is dropped, and the burden of making sure that the logic has certain structural properties is shifted to the definitions of the logical constants. For this reason he uses an extended definition of conjunction, called basic definition (DB2), to ensure reflexivity and transitivity of / as well as exchangeability of premises. Unfortunately, (DB2) corresponds to the defective Basis II of Popper (1947c), and is therefore just as problematic (cf. § 4.6).

[28] Popper gives his analysis of absolute validity in terms of so-called *statement preserving interpretations* in Popper (1947b), where he writes about absolutely valid inferences (ibid, p. 274): "There are inferences which are valid according to all our definitions, in spite of the fact that the logical form of the statements involved is irrelevant."

that even if we abstract away from any concrete logical system (containing a specific set of logical constants) under consideration, we still have a rudimentary residuum of deduction consisting of structural[29] inferences, like the inference from a statement a to the statement a.

Popper's Basis I is given by a *generalized reflexivity principle*, called (Rg), together with a *generalized transitivity principle*, called (Tg):[30]

$$\text{(Rg)} \qquad a_1, \ldots, a_n / a_i \qquad (1 \leq i \leq n)$$

$$\text{(Tg)} \qquad \left\{\begin{array}{l} \quad a_1, \ldots, a_n / b_1 \\ \& \ a_1, \ldots, a_n / b_2 \\ \vdots \qquad \quad \vdots \\ \& \ a_1, \ldots, a_n / b_m \end{array}\right\} \rightarrow (b_1, \ldots, b_m / c \rightarrow a_1, \ldots, a_n / c)$$

The principle (Tg) is not a rule in the strict sense, but rather a schematic rule. This means that (Tg) has, depending on the value of m, many rules as instances. There is therefore no way to directly express the content of (Tg) by means of a single formula of the metalanguage. This fact was very unsatisfactory for Popper. He searched for a replacement of (Tg) that could be expressed directly in his metalanguage. We discuss his proposals and their problems in § 4.6.

The fact that (Tg) is parametrized by a natural number actually applies not only to the multiplicity of premises by m, but also to the multiplicity n of sentences on the left side of / within both (Rg) and (Tg). This could have been overcome easily by adopting a notation for finite sets of statements corresponding to contexts Γ, Δ etc. in Gentzen sequents, where the multiplicity m of premises can be modelled by conjunctively understood sets of sentences on the right hand side of /, as in the multiple premises the left hand side of / is always the same. Obviously, Popper did not want to enlarge the formalized part of his metalanguage by such additional means of expression, insisting on the fact that deducibility / is the only primitive concept of the metalanguage (for propositional logic). Instead of talking metalinguistically about finite sets of formulas, Popper preferred, under the heading "Basis II", to incorporate objectlinguistic conjunction into his structural rules. Finite conjunctions yield, of course, a substitute for talking about finite sets, and technically this has the effect of getting rid of the parametrization of structural rules with natural numbers. Conceptually, this is not unproblematic, as it blurs the distinction between the metalinguistic association of sentences by means of a comma and their objectlinguistic association by means of a conjunction operator.

[29] Although Popper does not use the term "structural".

[30] Cf. Popper (1947c).

4 The general theory of derivation

Popper's general theory of derivation does not refer to any logical signs of the object language. It studies properties of statements and relations on statements that can be defined using only the deducibility relation. Examples of such properties are being a theorem in the deducibility structure, that is, being a demonstrable statement, or being a statement that is refutable.

We discuss the following properties and relations: mutual deducibility, complementarity, demonstrability, contradictoriness, refutability and relative demonstrability. Our focus is on mutual deducibility and relative demonstrability. Relative demonstrability in particular contains the notions of complementarity, demonstrability, contradictoriness and refutability as special cases.

4.1 Mutual deducibility

The relation of mutual deducibility // is explicitly defined as follows:

(D//) $$a//b \leftrightarrow (a/b \;\&\; b/a)$$

We also speak of "interdeducibility". It is an equivalence relation, as one can see by checking the rules of Basis I. Two mutually deducible statements *a* and *b* are said to have the same *logical force*.[31] The equivalence classes induced by // are thus logical forces.

Popper (1947c, p. 203) also calls two mutually deducible statements *logically equivalent*, and then calls (D//) the *substitutivity principle for logical equivalence*. The following substitution lemma holds:

Lemma 4.1 *If a and b are mutually deducible, then we may substitute b for a in every deducibility relation, that is, the following two statements are true:*
1. $a//b \rightarrow (a_1, \ldots, a_n/a \rightarrow a_1, \ldots, a_n/b)$.
2. $a//b \rightarrow (a_1, \ldots, a_n, a, a_{n+1}, \ldots, a_m/c \rightarrow a_1, \ldots, a_n, b, a_{n+1}, \ldots, a_m/c)$.

Proof (1) follows directly from (Tg), and (2) follows from (Tg) and Lemma 3.1. □

Popper also considers the following definition of //:[32]

(D//′) $$a//b \leftrightarrow (c)(a/c \leftrightarrow b/c)$$

For this alternative definition we can prove the following lemma:

Lemma 4.2 *In the presence of* (Tg) *and* (Rg), (D//) *is equivalent to* (D//′).

Proof We have to show that $(a/b \;\&\; b/a) \leftrightarrow (c)(a/c \leftrightarrow b/c)$ is true. The proof from left to right uses (Tg), and the proof from right to left uses (Rg). □

[31] Cf. Popper (1947b, p. 282).

[32] Cf. Popper (1947d, def. (DB1)) or Popper (1947b, def. (7.1)).

According to (D//), $a//b$ means that a and b are interdeducible; according to (D//′) it means that a and b have the same deductive content, that is, that their respective sets of consequences are the same.

4.2 Complementarity and demonstrability

The notion of complementarity for statements $a_1, \ldots, a_n$, written $\vdash a_1, \ldots, a_n$, is defined as follows:

(D⊢1) $$\vdash a_1, \ldots, a_n \leftrightarrow (b)(c)((a_1/c \;\&\; \ldots \;\&\; a_n/c) \rightarrow b/c)$$

For $n = 1$, we obtain the definition of a self-complementary or demonstrable statement, written $\vdash a$:

(D⊢1′) $$\vdash a \leftrightarrow (b)(c)(a/c \rightarrow b/c)$$

Intuitively, what is expressed by the complementarity of the statements $a_1, \ldots, a_n$ is that at least one of them has to be true, that is, that taken together they exhaust all possible states of affairs. This idea is captured by saying that if the statements $a_1, \ldots, a_n$ are complementary, then any statement c that follows from each of the statements $a_1, \ldots, a_n$ individually follows from any statement b. From (D⊢1′) one can thus obtain

$$\vdash a \leftrightarrow (b)(b/a)$$

by instantiating c by a and by using basic rules. A demonstrable statement is thus a statement that follows from any statement whatsoever.[33]

4.3 Contradictoriness and refutability

The notion of contradictoriness for statements $a_1, \ldots, a_n$, written $\urcorner\, a_1, \ldots, a_n$, is defined as follows:

(D$\urcorner$) $$\urcorner\, a_1, \ldots, a_n \leftrightarrow (b)(c)((b/a_1 \;\&\; \ldots \;\&\; b/a_n) \rightarrow b/c)$$

For $n = 1$, we get the notion of refutability of a statement a, written $\urcorner\, a$:

(D$\urcorner$′) $$\urcorner\, a \leftrightarrow (b)(c)(b/a \rightarrow b/c)$$

The intuition lying behind these notions is the following: a statement is refutable if it is false no matter the state of affairs, and a sequence of statements $a_1, \ldots, a_n$ is contradictory if its members cannot be true together. Contradictoriness of the

[33] For the definition of demonstrability cf. Popper (1947c, def. (D8.2+)) or Popper (1947d, def. (D11)). For the definition of complementarity cf. Popper (1948a, def. (D3.1)).

statements $a_1, \ldots, a_n$ is therefore defined by saying that any statement b that implies each of the statements $a_1, \ldots, a_n$ individually implies any statement c.[34] From (D$\urcorner'$) one can thus obtain

$$\urcorner a \leftrightarrow (c)(a/c)$$

by substituting a for b and by using basic rules. This is the definition of a self-contradictory statement, that is, of a refutable statement. From such a statement any other statement follows.

4.4 Relative demonstrability

Complementarity and contradictoriness can be combined in one relation that holds between statements $a_1, \ldots, a_n$ and $b_1, \ldots, b_m$. This relation is called *relative demonstrability* (or *relative refutability*), written $a_1, \ldots, a_n \vdash b_1, \ldots, b_m$. One of the definitions given by Popper is:[35]

$$\text{(D}\vdash\text{2)} \quad a_1, \ldots, a_n \vdash b_1, \ldots, b_m \leftrightarrow (c)((b_1/c \;\&\; \ldots \;\&\; b_m/c) \rightarrow a_1, \ldots, a_n/c)$$

For reasons explained in § 4.5, we will use the following slightly modified definition of relative demonstrability in the remainder of this paper:

Definition 4.3 *Relative demonstrability* $a_1, \ldots, a_n \vdash b_1, \ldots, b_m$ is defined by

$$\begin{aligned}\text{(D}\vdash\text{3)} \quad & a_1, \ldots, a_n \vdash b_1, \ldots, b_m \leftrightarrow \\ & (c)(d_1) \ldots (d_k)((b_1, d_1, \ldots, d_k/c \;\&\; \ldots \;\&\; b_m, d_1, \ldots, d_k/c) \rightarrow \\ & \qquad a_1, \ldots, a_n, d_1, \ldots, d_k/c)\end{aligned}$$

Note that in contradistinction to Popper's (D$\vdash$2) the definiens in (D$\vdash$3) is now formulated with context statements $d_1, \ldots, d_k$ for $0 \leq k$, which occur as additional premises in each of the deducibility relations. Definition (D$\vdash$3) is thus more general than (D$\vdash$2).

Lemma 4.4 *The concept of relative demonstrability contains, as special cases, the concepts of complementarity, demonstrability, contradictoriness and refutability.*

Proof Let $k = 0$. For complementarity, let $n = 0$. For demonstrability, let $n = 0$ and $m = 1$. For contradictoriness, let $m = 0$. For refutability, let $n = 1$ and $m = 0$. □

Lemma 4.5 *For all* $a_1, \ldots, a_n, b$: $a_1, \ldots, a_n/b \leftrightarrow a_1, \ldots, a_n \vdash b$.

Proof For $m = 1$ in (D$\vdash$3) we have to show that $(c)(d_1) \ldots (d_k)(b_1, d_1, \ldots, d_k/c \rightarrow a_1, \ldots, a_n, d_1, \ldots, d_k/c)$ is equivalent to $a_1, \ldots, a_n/b_1$. Let $k = 0$. Instantiating c

[34] For the definition of refutability cf. Popper (1947c, def. (D8.3)) or Popper (1947d, def. (D12)). For the definition of contradictoriness cf. Popper (1948a, def. (D3.2) and (D3.2′)).

[35] Cf. Popper (1948a, def. (D3.3′)).

by b_1 yields $b_1/b_1 \rightarrow a_1, \ldots, a_n/b_1$, and by (Rg) we get $a_1, \ldots, a_n/b_1$. Likewise for the other direction. □

This lemma allows us to replace / by $\vdash$ in all formulas of the metalanguage. This will be necessary further on to bring some of Popper's definitions into a form that makes them dualizable. Note that, conversely, $\vdash$ can only be replaced by / if $\vdash$ has exactly one succedent b. For mutual deducibility we have $a//b \leftrightarrow (a \vdash b \;\&\; b \vdash a)$.

Lemma 4.6 *The following structural rules hold for* $a_1, \ldots, a_n \vdash b_1, \ldots, b_m$*:*
1. Weakening on the left (LW) *and weakening on the right* (RW)*:*

(LW) $\qquad a_1, \ldots, a_n \vdash b_1, \ldots, b_m \rightarrow a_1, \ldots, a_n, a_{n+1} \vdash b_1, \ldots, b_m$

(RW) $\qquad a_1, \ldots, a_n \vdash b_1, \ldots, b_m \rightarrow a_1, \ldots, a_n \vdash b_1, \ldots, b_m, b_{m+1}$

2. Exchange on the left (LE) *and exchange on the right* (RE)*:*

(LE)
$$a_1, \ldots, a_i, a_{i+1}, \ldots, a_n \vdash b_1, \ldots, b_m \rightarrow a_1, \ldots, a_{i+1}, a_i, \ldots, a_n \vdash b_1, \ldots, b_m$$
(RE)
$$a_1, \ldots, a_n \vdash b_1, \ldots, b_j, b_{j+1}, \ldots, b_m \rightarrow a_1, \ldots, a_n \vdash b_1, \ldots, b_{j+1}, b_j, \ldots, b_m$$

3. Contraction on the left (LC) *and contraction on the right* (RC)*:*

(LC) $\quad a_1, \ldots, a_i, a_i, \ldots, a_n \vdash b_1, \ldots, b_m \rightarrow a_1, \ldots, a_i, \ldots, a_n \vdash b_1, \ldots, b_m$

(RC) $\quad a_1, \ldots, a_n \vdash b_1, \ldots, b_j, b_j, \ldots, b_m \rightarrow a_1, \ldots, a_n \vdash b_1, \ldots, b_j, \ldots, b_m$

4. If there are i, j *(for* $1 \leq i \leq n$ *and* $1 \leq j \leq m$*) such that* $a_i = b_j$*, then* $a_1, \ldots, a_n \vdash b_1, \ldots, b_m$.

Proof Consider the definiens of $a_1, \ldots, a_n \vdash b_1, \ldots, b_m$:

$$(c)(d_1)\ldots(d_k)((b_1, d_1, \ldots, d_k/c \;\&\; \ldots \;\&\; b_m, d_1, \ldots, d_k/c) \rightarrow a_1, \ldots, a_n, d_1, \ldots, d_k/c).$$

1. We can always strengthen the antecedent of the implication or weaken its succedent. Left weakening of / follows from Basis I. We thus get (LW) by weakening the succedent $a_1, \ldots, a_n, d_1, \ldots, d_k/c$ to $a_1, \ldots, a_n, a_{n+1}, d_1, \ldots, d_k/c$. The antecedent can be strengthened by adding the conjunct $b_{m+1}, d_1, \ldots, d_k/c$, which gives us (RW).

2. Exchange on the left (LE) is due to Lemma 3.1(1). Exchange on the right (RE) follows from the commutativity of metalinguistic conjunction.

3. Contraction on the left (LC) is due to Lemma 3.1(2). Consider $a, a/b$; by (Rg) we have a/a, and (Tg) yields a/b. By replacing / by $\vdash$ and subsequent applications of weakening and exchange we obtain (LC). Contraction on the right (RC) follows from the idempotence of metalinguistic conjunction.

4. For $a_i = b_j$ we have a_i/b_j by (Rg). Lemma 4.5 gives $a_i \vdash b_j$. By applications of (LW), (RW), (LE) and (RE) we get $a_1, \ldots, a_n \vdash b_1, \ldots, b_m$. □

Relative demonstrability $a_1, \ldots, a_n \vdash b_1, \ldots, b_m$ can be interpreted as derivability of the disjunction of $b_1, \ldots, b_m$ from the conjunction of $a_1, \ldots, a_n$. This interpretation is justified by the fact that for object languages containing conjunction $\wedge$ and disjunction $\vee$ one can show the following:

$$a_1, \ldots, a_n \vdash b_1, \ldots, b_m \;\leftrightarrow\; a_1 \wedge \ldots \wedge a_n \vdash b_1 \vee \ldots \vee b_m.$$

This is a consequence of Lemma 5.5, proved below. The concept of relative demonstrability gives thus an interpretation of Gentzen's (1935a; 1935b) sequents.[36]

4.5 Relative demonstrability and cut

Popper's definition (D⊢2) of relative demonstrability

(D⊢2) $a_1, \ldots, a_n \vdash b_1, \ldots, b_m \;\leftrightarrow\; (c)((b_1/c \;\&\; \ldots \;\&\; b_m/c) \rightarrow a_1, \ldots, a_n/c)$

is not wholly satisfactory, since it does not allow to show that

(Cut) $a_1, \ldots, a_n \vdash b_1, \ldots, b_m, c \rightarrow$
$(c, a_1, \ldots, a_n \vdash b_1, \ldots, b_m \rightarrow a_1, \ldots, a_n \vdash b_1, \ldots, b_m)$

holds for any object language.

Theorem 4.7 *The rule* (Cut) *does not follow from* (D⊢2) *and Basis I.*

Proof We show this by means of a counterexample for the instance

$$(a \vdash b, c \;\&\; c, a \vdash b) \rightarrow a \vdash b.$$

The negation of this claim is metalinguistically equivalent to

$$(d)((b/d \;\&\; c/d) \rightarrow a/d) \;\&\; c, a/b \;\&\; \neg(a/b).$$

Now consider an object language that contains only the three statements a, b and c. Then this is equivalent (also using the rules (Rg) and (Tg) of the Basis I) to the statement

$$(c/b \rightarrow a/b) \;\&\; (b/c \rightarrow a/c) \;\&\; c, a/b \;\&\; \neg(a/b).$$

The model under which the deducibility relation is only true for $c, a/b$ (and the instances required by (Rg)) satisfies (Tg) and is the desired countermodel. □

However, if one presupposes that the object language contains conjunction, disjunction and implication, then (Cut) can be shown to hold.

[36] Cf. Popper (1948a, p. 181); cf. Cohen (1953b, p. 69f. and p. 208f.).

Theorem 4.8 *In the presence of conjunction, disjunction and implication,* (Cut) *does follow from* (D$\vdash$2) *and Basis I.*

Proof In the presence of conjunction and disjunction, and in view of Lemma 5.5, it is sufficient to show that $(a \vdash b, c \;\&\; c, a \vdash b) \rightarrow a \vdash b$ holds, which is done as follows:

1. From the instance $b, a/b$ of (Rg) and (C>) we get $b \vdash a > b$.
2. From the assumption $c, a \vdash b$ and (C>) we get $c \vdash a > b$.
3. From the assumption $a \vdash b, c$ and (D$\vdash$2) we then get $a \vdash a > b$.
4. From (C>) and (LC) we get $a \vdash b$. □

Since we do not want to presuppose the existence of any specific logical constants in object languages, we use (D$\vdash$3) as a more general definition of relative demonstrability, which is formulated with additional context statements $d_1, \ldots, d_k$ (for $0 \leq k$) occurring as additional premises in each of the deducibility relations in the definiens (cf. § 4.4), which we reproduce here for easier reference:

$$\text{(D}\vdash\text{3)} \quad a_1, \ldots, a_n \vdash b_1, \ldots, b_m \leftrightarrow (c)(d_1)\ldots(d_k)((b_1, d_1, \ldots, d_k/c \;\&\; \ldots \;\&\; b_m, d_1, \ldots, d_k/c) \rightarrow a_1, \ldots, a_n, d_1, \ldots, d_k/c)$$

Given this definition, (Cut) follows from the rules of Basis I alone, without presupposing the existence of any logical constants.

Theorem 4.9 *If relative demonstrability is defined by* (D$\vdash$3) *instead of* (D$\vdash$2)*, then* (Cut) *follows from the rules of Basis I.*

Proof We have to show

$$\underbrace{a_1, \ldots, a_n \vdash b_1, \ldots, b_m, e}_{A} \rightarrow (\underbrace{e, a_1, \ldots, a_n \vdash b_1, \ldots, b_m}_{B} \rightarrow \underbrace{a_1, \ldots, a_n \vdash b_1, \ldots, b_m}_{C})$$

(where e is the cut formula). We assume A and B, and have to show C. In order to show C, we further assume $b_1, d_1, \ldots, d_k/c \;\&\; \ldots \;\&\; b_m, d_1, \ldots, d_k/c$, and have to show $a_1, \ldots, a_n, d_1, \ldots, d_k/c$.

1. From B we know that $a_1, \ldots, a_n, d_1, \ldots, d_k, e/c$ holds.
2. For each i with $1 \leq i \leq m$ we get from $b_i, d_1, \ldots, d_k/c$ by weakening on the left the corresponding relation $a_1, \ldots, a_n, b_i, d_1, \ldots, d_k/c$.
3. Using (D$\vdash$3), the following is an instance of A:

$$(b_1, a_1, \ldots, a_n, d_1, \ldots, d_k/c \;\&\; \ldots \;\&\; b_m, a_1, \ldots, a_n, d_1, \ldots, d_k/c \;\&\; e, a_1, \ldots, a_n, d_1, \ldots, d_k/c) \rightarrow a_1, \ldots, a_n, a_1, \ldots, a_n, d_1, \ldots, d_k/c.$$

(1), (2) and (3) together imply $a_1, \ldots, a_n, a_1, \ldots, a_n, d_1, \ldots, d_k / c$, and by contracting the premises $a_1, \ldots, a_n, a_1, \ldots, a_n$ we get $a_1, \ldots, a_n, d_1, \ldots, d_k / c$. □

A disadvantage of definition (D⊢3) is that it is not an explicit definition. As in Popper's framework no mechanism is available to handle contexts consisting of finite sets or of finite lists of statements, $a_1, \ldots, a_n \vdash b_1, \ldots, b_m$ is defined through an unspecified number k of context statements $d_1, \ldots, d_k$. The relation $\vdash$ is thus not eliminable in general. Nonetheless, it is always eliminable in a given logical argument, because the number k can then be specified.

4.6 The development of Popper's formulation of a basis

Popper presented several formulations to capture what he understands by a basis. We trace this development following the order of his publications. Note, however, that the exact order of Popper (1947b) and Popper (1947c) is not completely clear; both reference the other as forthcoming and were probably written at around the same time. We restrict ourselves again to propositional logic, and do not comment on additional constraints regarding substitution that have to be made for the treatment of quantification (cf. § 9).

There are two decisive points in this development. The first is the impasse that Popper found himself in while trying to get rid of the generalized transitivity principle (Tg), which led him to develop Basis II. But Curry (1948a, this volume, § 13.5) showed that Basis II is not, contrary to what Popper claims, equivalent to Basis I. We will point out Popper's error and will show how it may be corrected. The second decisive point was when Popper rediscovered a solution of how to replace (Tg) by a simpler rule, something which had been done similarly before by Gentzen for a system of Hertz.

In Popper (1947b) the notion of *absolute validity* is developed, which justifies the rules of a basis:

> There are inferences [. . .] which can be shown, on our definition of validity, to be valid *whatever the logical form of the statements involved*. [. . .] We shall say of these inferences that they are *absolutely valid*. (Ibid., p. 274)

The basis that he considers consists of the generalized reflexivity principle (Rg) and the rule (Tg),[37] which he claims, without giving a proof, to be complete with respect to his notion of absolute validity:

> It can be shown that all absolutely valid rules of inference [. . .] can be reduced to two [namely (Rg) and (Tg)]. By "reduced", I mean here: every inference which is an observance of some of the rules in question can be shown to be an observance of these two rules [. . .]. (Ibid., p. 277)

This claim is plausible, especially in view of Lejewski's (1974) proof of the equivalence of Popper's Basis I and the systems developed in Tarski (1930a,b, 1935b, 1936a).

[37] Cf. § 3.4. In Popper (1947b) these rules are labeled (6.1g) and (6.2g), respectively.

In Popper (1947c) two different approaches to axiomatizing the deducibility relation are developed, which are summed up ibid., p. 211. Approach I (developed ibid., § 2) consists of one of several possible variants of rules equivalent to (Rg) and (Tg). Approach II (developed ibid., § 3) takes a simpler transitivity principle than (Tg) and postulates the availability of conjunction in the object language. These two approaches are exemplified by Basis I and Basis II, respectively.

Popper considers each of the combinations (Tg) + (Rg), (Tg) + (2.1) + (2.2) + (2.3) and (Tg) + (2.41) + (2.7) as a possible basis. The components are:

$$a/a \tag{2.1}$$

$$a_1, \dots, a_n/b \to a_1, \dots, a_n, a_{n+1}/b \tag{2.2}$$

$$a_1, \dots, a_n/b \to a_n, \dots, a_1/b \tag{2.3}$$

$$a_1, \dots, a_n/a_i \quad (\text{for } 1 \leq i \leq n) \tag{(Rg) = (2.4)}$$

$$a_1, \dots, a_n/a_1 \tag{2.41}$$

$$a_1, \dots, a_{n+m}/b \to a_n, \dots, a_1, a_{n+1}, \dots, a_{n+m}/b \tag{2.7}$$

$$\left\{\begin{matrix} a_1, \dots, a_n/b_1 \\ \&\ a_1, \dots, a_n/b_2 \\ \vdots \qquad \vdots \\ \&\ a_1, \dots, a_n/b_m \end{matrix}\right\} \to (b_1, \dots, b_m/c \to a_1, \dots, a_n/c) \tag{(Tg) = (2.5g)}$$

The combination (Tg) + (Rg) is called Basis I by Popper.

All these combinations have the downside that they have to include the complicated transitivity principle (Tg) instead of a simpler one. Popper considers the rule (Tg) to be problematic and wants to get rid of it. He explains the exact problem later, in Popper (1948a, p. 177, fn 6):

> The objection against [(Tg)] is that it makes use of an unspecified number of conjunctive components in its antecedent; this may be considered as introducing a new metalinguistic concept – something like an infinite product.

This leads him to Approach II, which contains Basis II (cf. Popper, 1947c, § 3). It is supposed to axiomatize the deducibility relation by using the simpler transitivity principle (Te)

$$a_1, \dots, a_n/b \to (b/c \to a_1, \dots, a_n/c) \tag{(Te) = (2.5e)}$$

and by additionally requiring that for any two statements a and b of the object language $\mathcal{L}$ their conjunction $a \wedge b$ is also contained in $\mathcal{L}$. Popper (1947c, p. 206) expresses the motivating idea behind Basis II as follows:

> But the main point of using conjunction is that it should permit us to *link up any number of statements into one*. If it does this, then we can always replace the m conclusions of m different inferences from the same premises by one inference.

One problem with this approach is that in the absence of (Tg), the rules for conjunction have to be sufficiently strengthened in order to work. Popper also noticed this and proposed the following slightly tweaked rules for conjunction:[38]

$$(\mathrm{Cg}_1) \qquad a_1, \ldots, a_n / b \wedge c \rightarrow (a_1, \ldots, a_n / b \;\&\; a_1, \ldots, a_n / c)$$

$$(\mathrm{Cg}_2) \qquad (a_1, \ldots, a_n / b \;\&\; a_1, \ldots, a_n / c) \rightarrow a_1, \ldots, a_n / b \wedge c$$

Basis II thus consists of (Rg), (Te), (Cg_1), (Cg_2) and the existence postulate for conjunctions.

Popper uses these additional rules for conjunction to prove that (Tg) can be obtained from Basis II. That his proof contains an error was pointed out by Curry (1948a), who provided the following arithmetical counterexample: Let the object language consist of the natural numbers. Let $a \wedge b$ be the minimum of a and b. Let $a_1, \ldots, a_n / b$ hold if, and only if, $\min(a_1, \ldots, a_n) \leq b + n - 1$. Under this interpretation (Te), (Rg) and (Cg) of Basis II are satisfied but the rule (Tg) of Basis I is not. To see this, let, in (Tg), $n = 1$, $m = 2$, $a = b_1 = b_2 = 1$ and $c = 0$. A different but analogous arithmetical countermodel is given by Bernays in a letter to Popper of 12 May 1948 (cf. this volume, § 21.7) by interpreting $a \wedge b$ as the maximum of a and b, and $a_1, \ldots, a_n / b$ as $a_1 + \ldots + a_n \geq b$.

The crucial step in the proof that Basis I and Basis II are equivalent is to show that (Tg) follows from the rules of Basis II. Popper's attempted proof (ibid, p. 210f.) proceeds as follows. He assumes $a_1, \ldots, a_n / b_i$, for $1 \leq i \leq m$, as well as $b_1, \ldots, b_m / c$, and has to show that $a_1, \ldots, a_n / c$ follows. He attempts to do this in the following way:

1. From $a_1, \ldots, a_n / b_i$ (for all $1 \leq i \leq m$) obtain $a_1, \ldots, a_n / b_1 \wedge \ldots \wedge b_m$.
2. From $b_1, \ldots, b_m / c$ obtain $b_1 \wedge \ldots \wedge b_m / c$.
3. From $a_1, \ldots, a_n / b_1 \wedge \ldots \wedge b_m$ and $b_1 \wedge \ldots \wedge b_m / c$ obtain $a_1, \ldots, a_n / c$.

Step (1) can be proved by induction on m and iterated applications of (Cg_2); Popper gives this part of the proof explicitly. Step (3) is just an instance of his transitivity principle (Te). It is therefore step (2) which must be responsible for the failure of Popper's attempted proof, and it is indeed this inference that fails to be satisfied by Curry's counterexample: if $m = 3$, $b_1 = b_2 = b_3 = 1$ and $c = 0$, then $b_1 \wedge \ldots \wedge b_m / c$ does not follow from $b_1, \ldots, b_m / c$; an analogous counterexample can be given for Bernays's arithmetical interpretation. This derivation would be possible if Popper's rule (3.4g), that is,

$$(3.4\mathrm{g}) \qquad a_1, \ldots, a_n, b \wedge c / d \;\leftrightarrow\; a_1, \ldots, a_n, b, c / d$$

were secondary to (i.e., derivable from) Basis II.[39] However, both Curry's and Bernays's interpretations invalidate (3.4g)[40]. Popper (1947c, p. 210) erroneously thinks he has already proven (3.4g):

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[38] In Popper's formulation, (Cg_1) and (Cg_2) appear as one rule (Cg), which is here split up into two parts.

[39] Cf. footnote 61 for Popper's use of the term "secondary".

[40] Bernays explicitly mentions this point in his letter, and Popper explicitly acknowledges Bernays's critique of Basis II in his reply of 13 June 1948 (cf. § 21.8).

> But this situation changes completely if we drop the generalised transitivity principle (Tg) and replace it by the simpler form (Te). In this case, all the rules 3.1 to 3.5 and 3.1g to 3.4g still follow from 3.5g [= (Cg)]; but the opposite is not the case.

If rule (3.4g) were indeed secondary to Basis II, then step (2) could be proved by a simple induction on m. Basis II might therefore be salvageable if we added to the rules (Cg_1) and (Cg_2) the rule (3.4g).[41]

Popper (1947d) starts with a characterization of "[t]he customary systems of modern lower functional logic, such as *Principia Mathematica*, or the systems of Hilbert–Ackermann, Hilbert–Bernays, or Heyting, etc." (ibid., p. 1214). The fourth point of this characterization is:

> (d) Some further very general primitive rules of inference (such as some principles stating that the inference relation is transitive and reflexive) which do not refer to formative signs are assumed, either explicitly or, more often, tacitly.

This corresponds to what Popper uses his basis for. But in this article he proposes to also get rid of the basis:

> The inferential definitions of the conjunction [...] can be reformulated in such a way as to incorporate *all* the rules of inference mentioned. In this way, we can get rid of even the few trivial primitive rules (d) which were left in the previous approach; in other words, we obtain the whole formal structure of logic from metalinguistic inferential definitions alone. (Ibid., p. 1215)

Popper provides the following inferential definition (DB2) for conjunction, which contains (Cg), (Te), (2.1) and (2.2):

$$\text{(DB2)}\quad \begin{aligned} a//b \wedge c \leftrightarrow (a_1)\ldots(a_n)((a_1,\ldots,a_n/a \leftrightarrow (a_1,\ldots,a_n/b \;\&\; a_1,\ldots,a_n/c)) \\ \&\; (b/c \rightarrow (a_n,\ldots,a_1/b \rightarrow a_1,\ldots,a_n/c)) \\ \&\; (a_1,\ldots,a_n/c \rightarrow a_1,\ldots,a_n,b/c) \;\&\; a_1/a_1)) \end{aligned}$$

As this approach is based on the rules of Basis II, it suffers from the same problems. Hence (Tg) does in general not hold for the deducibility relation.

In Popper (1948a) a new notation for the deducibility relation is introduced: $D(a_1, a_2, \ldots, a_n)$ stands for $a_2, \ldots, a_n/a_1$, and Popper from now on explicitly allows the number of premises to be 0, that is, he allows $D(a)$, which stands for $/a$. The new basis that Popper considers consists of the following two rules (translated from D-notation into /-notation):

$$\text{(BI. 1)}\quad a_2/a_1 \leftrightarrow a_2, a_2/a_1$$

$$\text{(BI. 2)}\quad a_2,\ldots,a_n/a_1 \leftrightarrow (a_{n+1})\ldots(a_{n+r})(a_1,\ldots,a_n/a_{n+r} \rightarrow a_{n+r-1},\ldots,a_{n+1},a_n,\ldots,a_2/a_{n+r})$$

[41] What is actually only needed of (3.4g) is the implication from right to left.

From these two rules Popper obtains (Rg) and the rules

$$(1.44') \qquad a_1, \ldots, a_n/b \rightarrow a_n, \ldots, a_1/b$$

$$(1.45') \qquad a_1, \ldots, a_n/b \rightarrow a_1, \ldots, a_{n+r}/b$$

$$(1.46') \qquad a_1, \ldots, a_n/b \rightarrow (b, a_1, \ldots, a_n/c \rightarrow a_1, \ldots, a_n/c)$$

Popper then shows how to derive (Tg) from (1.44′), (1.45′) and (1.46′) by an induction on the number *m* in (Tg). He thus achieves the replacement of (Tg) by a rule which can, in contrast to (Tg), be stated in the metalanguage. He comments:

> The problem of avoiding [(Tg)] was discussed, but not solved, in [Popper (1947c)]. The lack of a solution led me there to construct Basis II, the need for which, as it were, has now disappeared. (Popper, 1948a, p. 177, fn 6)

This result also allows him to modify the implementation of the programme of Popper (1947d). Instead of using his definition (DB2), he can use this new formulation of Basis I. He mentions this possibility in Popper (1948a, p. 177, fn 6).

Already Curry (1948c) pointed out that this part of the proof is similar to what Gentzen (1932, esp. § 3) had shown, namely that the rule of "Syllogismus" of the system of Hertz (1929b) can be replaced by his cut rule, while Popper showed that his rule (Tg) can be replaced by the abovementioned rule (1.46′). The "Syllogismus" rule is practically identical to Popper's rule (Tg), and Gentzen's cut rule is identical to (1.46′). Gentzen's and Popper's proofs proceed in a very similar fashion. The question is how much Popper knew of the systems of Hertz and Gentzen. Popper (1947c, p. 204) remarks in a footnote that Bernays pointed him to the articles of Hertz (1923, 1931), and he mentions Gentzen (1935a,b) in order to compare his concept of relative demonstrability with Gentzen's sequent arrow. There are no further references to either of these authors in Popper's articles. In his autobiography, Popper (1974c, § 25) writes about his logic work in New Zealand: "I invented for myself something now called 'natural deduction'". Obviously, "natural deduction" is here understood not in the specific sense of the calculus of natural deduction, but in a generic sense covering Gentzen-style systems, in particular the sequent calculus.[42] This remark suggests that he had not been aware of Gentzen's systems and their foundational significance until the basic framework of his inferential logic was conceived.[43]

[42] Popper's (n.d.[b]) unpublished and undated handwritten notes entitled "Natural Deduction" and "Natural Deduction with Sheffer Strokes" confirm that Popper used the term "natural deduction" in this wider sense, which includes sequents formulated with relative demonstrability $\vdash$.

[43] In Popper (1948a, fn 7) we find a discussion of Gentzen sequents in relation to his own turnstile operation (which denotes relative demonstrability; cf. § 4.4). This footnote reads as a later comparison of his own conception with others that had come to his attention after their completion. In a letter to Paul Bernays of 22 December 1946 (this volume, § 21.1) he lists major papers of Hertz and the first paper of Gentzen, which is a direct discussion of Hertz's work, as items borrowed from Bernays, which indicates that Popper received knowledge of the development from Hertz to Gentzen not before 1947. However, Popper had always been aware of Gentzen's consistency proofs for arithmetic, as remarks in his correspondence with Forder show, but without any detailed knowledge of Gentzen's proofs, wrongly claiming that Tarski anticipated Gentzen's proof (cf., e.g., this volume, § 26.2 and § 26.4). Popper's unpublished notes on derivation and demonstration (cf. this volume, Chapter 18)

At the very end of Popper (1948a) a new Basis III is introduced. Popper takes two-place deducibility (or "two-termed derivability", as he says), that is, a/b, characterized by transitivity and reflexivity as a starting point, together with one of the two rules (the second is here translated into /-notation)

$$\text{[BIII]} \qquad b/a \leftrightarrow (c)(c/b \rightarrow c/a)$$

$$\text{(2.1)} \qquad b/a \leftrightarrow (c)(a/c \rightarrow b/c)$$

and the following definition of relative demonstrability:

$$\text{(D3.3)} \quad a_1, \ldots, a_n \vdash b_1, \ldots, b_m \leftrightarrow (c)(d)((b_1/d \;\&\; \ldots \;\&\; b_m/d) \rightarrow ((c/a_1 \;\&\; \ldots \;\&\; c/a_n) \rightarrow c/d))$$

The resulting basis is equivalent to Basis I.

Popper (1949a), the last of Popper's core articles on logic, does not contain a lot of technical development but gives an overview of his general intentions instead. Popper now seems to have adopted the approach with Basis III. He writes:

> As the sole definiens of our definitions, the idea of deducibility or derivability will be used. We can restrict ourselves to using deducibility from *one* premise. We write "$D(a, b)$" for "a is deducible from b". [...] We do not need, for the derivation of mathematical logic, to assume more about deducibility than that it is transitive and reflexive. [...] With the help of "$D(a, b)$" it is easy to define deducibility from n premises. (Ibid., p. 723)

He (ibid., p. 723) also considers to define a generalized deducibility relation $a_1, \ldots, a_n/b_1, \ldots, b_m$ in terms of $D(a, b)$ alone, with the intended meaning that if each of the premises $a_1, \ldots, a_n$ is true, then at least one of the statements $b_1, \ldots, b_m$ must be true. This corresponds to relative demonstrability.

While most of the technical development in Popper's articles is done using Basis I, a tendency can be observed towards his using two-place deducibility a/b as the underlying concept, together with derivative n-place notions like relative demonstrability that are defined in terms of two-place deducibility (which, incidentally, is the approach followed by standard categorial logic as based on two-place arrows $a \rightarrow b$; cf. Lambek and Scott, 1986).

Approach II seems ill-chosen to us, even if the suggested corrections were to be carried out. It requires conjunction in the object language in its formulation of the basis, and it confounds structural properties of the deducibility relation with the definition of a logical constant.

Our choice of (a version of) Basis I, on the other hand, allows us to stay as close as possible to Popper's technical development, which proceeds mostly by using Basis I with $n + 1$-place deducibility $a_1, \ldots, a_n/b$. This choice also makes it possible to

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refer to Gentzen's system of natural deduction without discussing it in detail. Popper (1947c, § 8), which partially overlaps with Chapter 18 of this volume, does not contain any reference to Gentzen. In his preface to Stegmüller and von Kibéd (1984, p. VII), Stegmüller mentions that he first learned of the existence of Gentzen's theory of natural deduction through Popper in 1949.

avoid a certain awkwardness in the formulation of proofs, which is already present in several proofs involving relative demonstrability; in order to show that some statement involving relative demonstrability holds, one often has to unfold the definition of relative demonstrability first, then work with the definiens, and finally reintroduce relative demonstrability. Besides its conceptual superiority, Basis I has thus also practical advantages.

5 The special theory of derivation

The general theory of derivation was purely structural (in Gentzen's terminology) and based solely on the deducibility of statements without regard for their individual form and their individual deductive power. The subject of Popper's special theory of derivation are relations between statements, which are logically complex or have a specific deductive power, and their components[44], and deals with the logical laws emerging therefrom. It is based on the relational definitions of logical operations and the inferential definitions of logical operators.

5.1 Definitions of logical constants

As discussed in § 1.3, logical constants are characterized in terms of the role they play with respect to the deducibility relation $/$. Such characterizations proceed by what Popper calls *inferential definitions*. A sign of an object language $\mathcal{L}$ is a *logical constant* if, and only if, it can be defined by an inferential definition.[45] In Popper's terminology, logical constants are called *formative signs*, in contradistinction to what he calls *descriptive signs*, such as "mountain" or "elderly disgruntled newspaper reader"; cf. Popper (1947b, p. 257). According to Popper (1947b, p. 286), "*inferential definitions* [. . .] are characterized by the fact that they define a formative sign by its logical force which is defined, in turn, by a *definition in terms of inference* (i.e., of '$/$')." Popper (1947c, p. 220) summarizes his inferential approach as follows:

> The upshot of all these considerations is this: If we have an artificial model language with signs for conjunction, the conditional . . . etc. (we have called them "formative signs" of the language in question) then the meaning of these formative signs can be exhaustively determined by the rules of inference in which the signs occur; this fact is established by defining our definitions of these formative signs explicitly in terms of rules of inference.

[44] Note that "components" are not necessarily subsentences, but sentences that are deductively related to the original sentence in a certain way, such as sentences a and b which are deductively related to a conjunction c of a and b, even if c does not syntactically contain a and b. Cf. § 1.2 above.

[45] Cf., e.g., Popper (1947b, p. 286).

As before (cf. § 3.3) we use $\circ$ as representing an arbitrary logical operation called "connection" with $\mathcal{R}$ as its defining condition ("rule"), which is used in the relational definition

$$c \text{ is a connection of } a \text{ and } b \text{ if and only if } \mathcal{R}(c, a, b)$$

and the inferential definition

$$\text{(D}\circ\text{)} \qquad c /\!/ a \circ b \;\leftrightarrow\; \mathcal{R}(c, a, b).$$

As already mentioned, $\mathcal{R}(c, a, b)$ is an expression of the metalanguage containing as relations the deducibility relation $/$ as well as the defined relations $\vdash$ and $\not\!/$. In the course of a logical argument, definitions of logical constants containing $\vdash$ and $\not\!/$ can always be replaced by definitions that contain only $/$.

The relational definition makes no special assumption about the object language – it just singles out connections c of a and b if they are available in the object language (otherwise there simply is no connection of a and b). The inferential definition, however, requires existence and uniqueness, which means that it is based on the presupposition that there is *exactly one* connection c of a and b in the object language considered. Here "uniqueness requirement" and "exactly one" is understood *modulo interdeducibility*. That is, there is a connection c of a and b and all other connections c' of a and b are interdeducible with c: $c /\!/ c'$. Note that in any case, every c' interdeducible with a connection c of a and b is itself a connection of a and b. This follows from the fact that according to the general theory of derivation, we have substitutivity of interdeducibles (Lemma 4.1), which means that all our inferential concepts and results are invariant with respect to interdeducibility. Popper is fully aware of these existence and uniqueness requirements, which have not been appreciated by some of his reviewers (cf. § 2)

As explained in § 1.3, (D$\circ$) can be viewed as an *explicit definition* of a connective $\circ$, and Popper is fully entitled to call it that way. (D$\circ$) conservatively introduces into a language a sign for a new operator $\circ$, which is eliminable following the standard procedures used for the introduction and elimination of function symbols or definite descriptions (cf. footnote 11 in § 1.3). It should be noted, however, that (D$\circ$) is a definition of a *metalinguistic* function which associates with any a and b their connection $a \circ b$, which is unique up to interdeducibility, so $a \circ b$ essentially denotes an equivalence class of objectlinguistic sentences, none of which must have a special form. However, once we have reached that stage, we can, of course, introduce into our object language sentences of the form "$a \circ b$", where $\circ$ is now an objectlinguistic operator in the usual sense, and take it to be the objectlinguistic representative of $a \circ b$ (understood metalinguistically). Viewed that way, (D$\circ$) can be used to introduce a logical operator into a suitable object language, provided the connection operation as a relation between (not further specified) sentences is available.

In fact, we can even devise a formal object language in the usual way and lay down the rules $\mathcal{R}(a \circ b, a, b)$ for all a and b. In that case a connection $a \circ b$ of a and b with certain inferential properties always exists by stipulation. However, when proceeding in that manner, we must be aware (and Popper is) that such a stipulation may have undesired consequences up to the generation of inconsistencies. In any case, for (D$\circ$)

to hold, we still must make sure that uniqueness is satisfied. If this is the case, we can proceed with the rule

$$(C\circ) \qquad \mathcal{R}(a \circ b, a, b)$$

as the *characterizing rule* (C∘), which corresponds to the definition (D∘). If uniqueness is satisfied, (D∘) and (C∘) are obviously equivalent. So we will often work with (C∘) rather than with (D∘), as it is easier to handle and as it incorporates the rules of inference in which one is interested. Philosophically, however, it is the relationship between (D∘) and (C∘), more precisely the derivation of (C∘) from (D∘), what constitutes Popper's "trivialization" of logic, that is the derivation of logical laws from inferential definitions (cf. § 1.4).

5.2 Popper's definitional criterion of logicality

The question we are facing now is what form characterizing rules like $\mathcal{R}(a \circ b, a, b)$ might be allowed to take. Should certain rules be disallowed because their use in a definition of form (D∘) does not in fact define a logical constant, or does any rule $\mathcal{R}$ give rise to a definition of a logical constant?

For Popper any characterizing rule which is equivalent to an inferential definition characterizes a logical constant. He calls such rules *fully characterizing*. We state this as a definition:

Definition 5.1 A rule $\mathcal{R}$ characterizing an operation ∘ is *fully characterizing* if, and only if, it is equivalent to an inferential definition of ∘.[46]

In case that $\mathcal{R}$ is given in the form $\mathcal{R}(a \circ b, a, b)$, this means uniqueness of $\mathcal{R}$ for its first argument. For Popper uniqueness is essential for any definition of a logical constant.[47] We state this as an immediate corollary of our definition.

Corollary 5.2 *A rule of the form $\mathcal{R}(c, a_1, \ldots, a_n)$ is fully characterizing if, and only if*

$$(\mathcal{R}(a, a_1, \ldots, a_n) \;\&\; \mathcal{R}(b, a_1, \ldots, a_n)) \rightarrow a /\!/ b.$$

In other words, if, and only if, a rule $\mathcal{R}$ characterizes a statement c up to mutual deducibility, then $\mathcal{R}$ is fully characterizing c.

We distinguish between the definition and the corollary because not every characterizing rule has the form $\mathcal{R}(c, a_1, \ldots, a_n)$ and can thus be used in a relational definition. Only logical constants, which are *fully* characterized, are always definable by rules of the form $\mathcal{R}(c, a_1, \ldots, a_n)$.

What about the existence requirement? The inferential definition (D∘) is a proper definition only if, in addition to uniqueness, there exists a connection of a and b

[46] Cf., e.g., Popper (1948c, § VI); this volume, Chapter 6, p. 187f.

[47] Cf. Popper (1948c, p. 324).

in the object language considered, for which we define a name. We may, of course, stipulate that there be a connection of a and b and even denote it by $a \circ b$, but this is nothing an inferential definition can do by itself without becoming creative. Unique connections must be there before we can single them out by means of an inferential definition. This is a point Popper's reviewers have strongly emphasized (cf. § 2), but also a point of which Popper is aware.

Forcing an operation to exist in an object language can make the object language inconsistent. However, if it is uniquely characterized, it is a logical constant. Popper thus allows, for example, the following definition for "opponent" (*opp*), with its characterizing rule:

(D*opp*) $$a//opp(b) \leftrightarrow (c)(b/a \;\&\; a/c)$$

(C*opp*) $$(c)(b/opp(b) \;\&\; opp(b)/c)$$

This obviously trivializes any system, since it implies $(c)(b/c)$ for any b. But this does not lead Popper to reject (D*opp*) as a definition. In a system, where *opp* exists (thus an inconsistent system), it is a logical constant because it is unique for trivial reasons.[48] Historically, it is interesting to note that the connective *opp* is quite similar to the connective *tonk*, which was later introduced into the philosophical discussion by Prior[49], there with the intention to discredit inferentialism.

Lejewski (1974, p. 644) considers an even stronger version of *opp*, called opp^*. This supposed logical constant not only turns the object language inconsistent but also the metalanguage. Its definition and characterizing rule is supposed to be:

(Dopp^*) $$a//opp^*(b) \leftrightarrow (c)(b/a \;\&\; \overline{b/c})$$

(Copp^*) $$(c)(b/opp^*(b) \;\&\; \overline{b/c})$$

The definition (Dopp^*) is formulated using metalinguistic negation ($\overline{\quad}$). This contrasts with all definitions considered by Popper, who never uses negation in his metalinguistic definitions of logical constants. So while it is true that the definition of opp^* would turn any metalanguage inconsistent in which it is stated and where opp^* is forced to exist, this criticism cannot be applied to the system of Popper. Relying on positive logic within defining conditions, which is mandatory (cf. § 3.3), in view of Lemma 3.2 there can be no such characterization.[50]

[48] Cf. Popper (1947b, p. 284): "We need not make sure, in any other way, that our system of definitions is consistent. For example, we may define (introducing an arbitrarily chosen name 'opponent'): [see (D*opp*)]. This definition has the consequence that every language which has a sign for 'opponent of *b*' [. . .] will be inconsistent [. . .]. But this need not lead us to abandon [(D*opp*)]; it only means that no consistent language will have a sign for 'opponent of *b*'."

[49] Prior (1960) cites Popper (1947b), but only for giving an example of what he understands by an analytically valid inference. He does not mention the fact that Popper (1947b) discusses the *tonk*-like connective *opp*. Neither Belnap (1962) nor Stevenson (1961) noticed this in their responses to Prior's article, even though they both also explicitly mention Popper's article. It is noted, however, by Merrill (1962).

[50] For an extensive discussion of Popper's criterion of logicality cf. also Schroeder-Heister (1984, 2006).

Another question concerning the form of definitions of logical constants is the following: For two given alternative rules $\mathcal{R}$ and $\mathcal{R}'$, which are equivalent, is one preferable over the other in defining a logical constant? Popper often considers more than one possible definition of a logical constant, showing (or in some cases only indicating) that these alternative definitions are equivalent. There seems to be no logical or philosophical criterion that makes Popper prefer one definition rather than another, equivalent definition. Often, it seems, he just chooses the brevity of some definition, or the ease with which he is able to explain it.

This contrasts with more modern approaches concerning definitions of logical constants. Especially in proof-theoretic semantics as initiated by Gentzen, Prawitz and Dummett there is a strong sense of preferring one special kind of rules (either introduction rules or elimination rules) as being constitutive of the meaning of logical constants. Such considerations play no part in Popper's definitions. For example, Popper's (1947c, p. 228) definition of the universal quantifier resembles its elimination rule, while his definition of the existential quantifier resembles its introduction rule. For a further discussion cf. § 1.5.

However, the fact that Popper does not explicitly give a philosophical criterion for preferring one form of definition over another does not preclude the possibility that he has a logical one. In the more mature version of his theory presented in Popper (1948a,c), he sometimes states definitions of logical constants in a form that highlights the duality of certain of his definitions. We will follow this lead and give the definitions in a way that allows for the formulation of a duality function that transforms any definition of a logical constant into a definition of a dual logical constant.

5.3 Not-strictly-logical operations and the understanding of logical notation

It is the existence of fully characterizing rules that distinguishes logical constants from non-logical constants, and it is this criterion of logicality that leads Popper to reject, for example, Johansson's negation as a logical constant (cf. § 6.4).[51] As characterizing rule $\mathcal{R}_j$ for this negation Popper gives the following:

$$a, b \vdash \neg_j c \rightarrow a, c \vdash \neg_j b.$$

This actually characterizes a negation weaker than Johansson's. For example, the rule

$$a, c \vdash \neg_j c \rightarrow a \vdash \neg_j c$$

[51] In contradistinction to more modern terminology, for Popper "Johansson's negation" and "minimal negation" are different items, the latter denoting dual-intuitionistic negation. We will follow this terminology and refrain from designating Johansson's negation as "minimal".

which is valid in Johansson's system, is not derivable from it. Therefore, we here speak of "j-negation" rather than "Johansson's negation" (though our argumentation holds for Johansson's negation as well). It can be shown that the characterizing rule $\mathcal{R}_j$ is not fully characterizing (cf. Theorem 6.12), that is, it is not equivalent to an inferential definition.

Even though it cannot be inferentially defined, we nevertheless want to talk about j-negation and its laws in a symbolic way. We want, for example, to say that $a, \neg_j a/\neg_j c$ holds for any c and that $a, \neg_j a/c$ does not hold for all c. By that we mean that from the characterizing rule $\mathcal{R}_j$ we can infer these claims. However, how can the notation "$\neg_j a$" be understood if there cannot be an inferential definition of $\neg_j$?

Consider an object language, in which for every sentence a there is at least one j-negation of a. Consider the law

$$a, \neg_j a/\neg_j c.$$

It can plausibly be read as follows: Suppose we have a fixed selection function (in short: "selection") which chooses, for every sentence a, one of the j-negations of a. Let $\neg_j$ denote this selected j-negation of a. Then this law holds for the given selection of j-negations. In fact, this law not only holds for a specific selection, but for any selection.

Generalizing this idea means that laws for not-fully-characterizable operations are interpreted with respect to certain selections, and that such a law is valid, if it holds for any selection. This interpretation can be used even for unique (fully characterized) operations such as classical negation $\neg_k$, with the only difference that there is only one possible selection (up to interdeducibility), which means that, when we (metalinguistically) define the validity of an inference, we need not quantify over selections. For example

$$a, \neg_k a/\neg_k c$$

is valid as it holds for *the* unique selection of classical negations, namely for *the* classical negation of a and *the* classical negation of c, whereas

$$a, \neg_j a/\neg_j c$$

is valid as it holds for every selection of j-negations of a and of c.

Elaborating on this idea, something that cannot be done here, we propose to read a connection $a \circ b$, be it unique or not, as an ϵ-term as used in the epsilon calculus of Hilbert and Bernays (1939), that is, $a \circ b$ is understood as $\epsilon c\, \mathcal{R}(c, a, b)$. This means that $a \circ b$ just selects one element from the connections of a and b. We do not have the problem of non-denoting terms, as with logical operations we always assume that they generate a result, even if this is by stipulation. As the availability of a connection for any two statements a and b is guaranteed, the use of $a \circ b$ and the instantiation of universally quantified formulas of the metalanguage with terms of the form $a \circ b$ is no longer problematic. Reading $a \circ b$ as an ϵ-term also makes the existential presupposition explicit in cases where $a \circ b$ is the result of an instantiation of a universal quantifier of the metalanguage. If a connection is unique, that is, if $\circ$ is a logical constant, then we have a definite description as a limiting case of the epsilon operator.[52]

Our discussion serves as a philosophical explication of Popper's idea that we can define a unique logical operation $\circ$ by an inferential definition and a corresponding full characterization $\mathcal{R}(a \circ b, a, b)$, and non-unique logical operations $\circ$ or $\neg_j$ by characterizations $\mathcal{R}(a \circ b, a, b)$ or $\mathcal{R}_j$ without there being an inferential definition. For the following more technical discussions and results this philosophical distinction does not matter very much: we can just start from the rule $\mathcal{R}(a \circ b, a, b)$ as a characterization of $\circ$, or from $\mathcal{R}_j$ as a characterization of $\neg_j$, be it unique (*fully* characterizing) or not. Uniqueness and many other metalogical features are then possible results of our investigations.

This interpretation gives a clear understanding of what is meant when throughout his articles Popper uses notations like $a \wedge b$, $a \vee b$ and $a > b$ to speak about conjunctions, disjunctions and implications, respectively, as well as various negation signs to speak about certain uniquely-characterized and certain not-fully-characterizable negations of some object language $\mathcal{L}$. It can also be used to rectify certain objections by reviewers of Popper's papers, notably Kleene (1948), and by later commentators such as Lejewski (1974, §§ 5–6) and Schroeder-Heister (2006, § 3.1); cf. § 2.

5.4 Popper's notion of duality

Although Popper makes frequent use of duality,[53] he nowhere gives an explanation of his notion of duality. We think such an explanation is needed, since Popper does not only discuss duality in the context of classical logic, where the well-known duality based on truth functions can be applied, but also in the context of non-classical logics such as intuitionistic logic, where a different notion of duality is called for.

We propose to understand his notion of duality as being based on his concept of relative demonstrability ($\vdash$). More precisely, an inferential definition is said to be dual to another inferential definition, if it results from exchanging all statements on the left side of $\vdash$ with the statements on its right side, that is, by transforming $a_1, \ldots, a_n \vdash b_1, \ldots, b_m$ into $b_1, \ldots, b_m \vdash a_1, \ldots, a_n$.[54] Note that it is in general not possible to invert the direction of the sign of derivability ($/$), for it allows multiple statements on the left but only a single statement on the right (for alternative versions cf. § 4.6). As already mentioned in § 4.1, the replacement of $/$ by $\vdash$ is allowed. One can therefore formulate definitions of logical constants in terms of relative demonstrability instead of deducibility. This allows us to make the duality of logical

[52] This is more complicated in a case like j-negation, where $\mathcal{R}_j$ does not have the form $\mathcal{R}_j(\neg_j a, a)$, that is, where we do not have a relational definition in the strict sense. One would have to say that $\neg_j a$ selects one of the j-negation-functions satisfying $\mathcal{R}_j$ and applies it to a.

[53] He speaks of "the dual of", "dual rules", "dual definitions" etc.

[54] Cf. Kapsner (2014, p. 76), who also mentions this possibility. There is, however, a difference between the semantical concept $\models$ that Kapsner considers and Popper's defined concept $\vdash$.

constants obvious.[55] In the case of binary connectives we have to swap the arguments to produce its dual.[56] We make this understanding of the notion of duality precise by using a duality function, defined as follows:

Definition 5.3 Let $\star$ be a unary connective and $\circ$ a binary connective. The *duality function* δ is defined by the following clauses, where Γ and Π are lists of statements, and Γ^δ (resp. Π^δ) is the application of δ to each member of Γ (resp. Π):

$$\begin{aligned} a^\delta &=_{\text{df}} a, \text{ for non-compound statements } a; \\ (\star a)^\delta &=_{\text{df}} \star^\delta a^\delta; \\ (a \circ b)^\delta &=_{\text{df}} b^\delta \circ^\delta a^\delta; \\ (\Gamma \vdash \Pi)^\delta &=_{\text{df}} \Pi^\delta \vdash \Gamma^\delta. \end{aligned}$$

There are no clauses for / and //. This does not restrict the range of applicability of δ, since / can always be replaced by $\vdash$ (cf. Lemma 4.5). In the following we show that the function δ maps definitions of logical constants to definitions of what Popper considers to be their duals. We do this for conjunction and disjunction, conditional and anti-conditional, and for modal connectives.

5.5 Conjunction and disjunction

As was already mentioned, Popper gives several characterizing rules for conjunction ($\wedge$). We choose the following definition:

(D$\wedge$) $$a // b \wedge c \;\leftrightarrow\; (d)(a \vdash d \;\leftrightarrow\; b, c \vdash d)$$

(C$\wedge$) $$b \wedge c \vdash d \;\leftrightarrow\; b, c \vdash d$$

If we apply our duality function δ to the characterizing rule (C$\wedge$), we obtain:

(C$\wedge$)$^\delta$ $$d \vdash c \wedge^\delta b \;\leftrightarrow\; d \vdash b, c$$

which is equivalent to Popper's definition of disjunction ($\vee$):[57]

[55] To our knowledge, Popper mentions this duality based on relative demonstrability only once, in Popper (1948a, p. 181): "For certain purposes – especially if we wish to emphasize the duality or symmetry between '$\vdash$' and '$\not\nearrow$' – the use of '$(\ldots) \vdash$' turns out to be preferable to that of '$\not\nearrow(\ldots)$'."

[56] Our definition of duality resembles Cohen's (1953b, p. 82): "We define the *dual* of a postulate of any Gentzen-type system as the result of writing each sequent $p_1, \ldots, p_m \Vdash q_1, \ldots, q_n$ as $q_n, q_{n-1}, \ldots, q_1 \Vdash p_m, p_{m-1}, \ldots, p_1$, of interchanging $K\,p\,q$ with $A\,q\,p$ (hence also interchanging $A\,p\,q$ with $K\,q\,p$), of interchanging $F\,p\,q$ with $G\,q\,p$ (hence also interchanging $G\,p\,q$ with $F\,q\,p$) [. . .], and of reversing the order of premises whenever the postulate is a rule of inference having two premises." Here K stands for conjunction, A for disjunction ("alternative"), F for the conditional, and G for the anti-conditional.

[57] Our definitions (D$\wedge$) and (D$\vee$) correspond to Popper's (1948a) rules (3.71) and (3.72), respectively.

(D∨) $a//b \vee c \leftrightarrow (d)(d \vdash a \leftrightarrow d \vdash b, c)$

(C∨) $d \vdash b \vee c \leftrightarrow d \vdash b, c$

We immediately obtain the following introduction and elimination rules for conjunction and disjunction:

Lemma 5.4 *The following rules for conjunction and disjunction hold:*

(1) $a \wedge b/a$	*(4)* $a/a \vee b$
(2) $a \wedge b/b$	*(5)* $b/a \vee b$
(3) $a, b/a \wedge b$	*(6)* $(c)((a/c \ \& \ b/c) \rightarrow a \vee b/c)$

Proof For (1) consider the substitution instance $a \wedge b \vdash a \leftrightarrow a, b \vdash a$ of (C∧); $a, b \vdash a$ holds by (Rg). Thus $a \wedge b \vdash a$. Rules (2)–(5) are shown analogously. For (6) consider the substitution instance $a \vee b \vdash a \vee b \leftrightarrow a \vee b \vdash a, b$ of (C∨); $a \vee b \vdash a \vee b$ holds by (Rg). Thus $a \vee b \vdash a, b$. From definition (D⊢3) we get $(c)((a/c \ \& \ b/c) \rightarrow a \vee b/c)$. □

Lemma 5.5 *Conjunction and disjunction can be introduced and eliminated within contexts, that is, the following equivalences hold:*

$$a_1, \ldots, a_n, b, c \vdash d \leftrightarrow a_1, \ldots, a_n, b \wedge c \vdash d;$$
$$a_1, \ldots, a_n \vdash b_1, \ldots, b_m, c, d \leftrightarrow a_1, \ldots, a_n \vdash b_1, \ldots, b_m, c \vee d.$$

Proof For the first equivalence consider the following two instances of (Tg):

$$\left\{\begin{array}{l} \quad a_1, \ldots, a_n, b \wedge c \vdash a_1 \\ \& \ a_1, \ldots, a_n, b \wedge c \vdash a_2 \\ \vdots \qquad\qquad \vdots \\ \& \ a_1, \ldots, a_n, b \wedge c \vdash a_n \\ \& \ a_1, \ldots, a_n, b \wedge c \vdash b \\ \& \ a_1, \ldots, a_n, b \wedge c \vdash c \end{array}\right\} \rightarrow (a_1, \ldots, a_n, b, c \vdash d \rightarrow a_1, \ldots, a_n, b \wedge c \vdash d);$$

$$\left\{\begin{array}{l} \quad a_1, \ldots, a_n, b, c \vdash a_1 \\ \& \ a_1, \ldots, a_n, b, c \vdash a_2 \\ \vdots \qquad\qquad \vdots \\ \& \ a_1, \ldots, a_n, b, c \vdash a_n \\ \& \ a_1, \ldots, a_n, b, c \vdash b \wedge c \end{array}\right\} \rightarrow (a_1, \ldots, a_n, b \wedge c \vdash d \rightarrow a_1, \ldots, a_n, b, c \vdash d).$$

For the second equivalence we first show the direction from left to right:

1. From $a_1, \ldots, a_n \vdash b_1, \ldots, b_m, c, d$ and two applications of (Cut) with $c/c \vee d$ and $d/c \vee d$ we obtain $a_1, \ldots, a_n \vdash b_1, \ldots, b_m, c \vee d, c \vee d$.
2. By (RC) we obtain $a_1, \ldots, a_n \vdash b_1, \ldots, b_m, c \vee d$.

For the direction from right to left:

1. Assume $a_1, \ldots, a_n \vdash b_1, \ldots, b_m, c \vee d$.
2. This is equivalent to $(e)((b_1/e \ \& \ \ldots \ \& \ b_m/e \ \& \ c \vee d/e) \rightarrow a_1, \ldots, a_n/e)$.
3. Assume b_1/e to b_m/e, c/e and d/e.

4. From c/e and d/e follows $c \vee d/e$.
5. From b_1/e to b_m/e and $c \vee d/e$ we get $a_1, \ldots, a_n/e$ from (2).

Thus $a_1, \ldots, a_n \vdash b_1, \ldots, b_m, c, d \ \leftrightarrow \ a_1, \ldots, a_n \vdash b_1, \ldots, b_m, c \vee d$. □

Lemma 5.6 *Conjunction and disjunction are logical constants, that is, their rules are fully characterizing.*

Proof For conjunction we have to show that

$$((d)(a_1 \vdash d \ \leftrightarrow \ b, c \vdash d) \ \& \ (d)(a_2 \vdash d \ \leftrightarrow \ b, c \vdash d)) \rightarrow a_1 /\!/ a_2$$

is true. Assuming the antecedent, we substitute a_2 for d in both its conjuncts to obtain $a_1 \vdash a_2 \ \leftrightarrow \ b, c \vdash a_2$ and $a_2 \vdash a_2 \ \leftrightarrow \ b, c \vdash a_2$. By (Rg) we obtain $b, c \vdash a_2$ from the second conjunct, and with $b, c \vdash a_2$ we obtain $a_1 \vdash a_2$ from the first conjunct. Likewise for $a_2 \vdash a_1$. The proof for disjunction is similar. □

5.6 Conditional and anti-conditional

Again, we choose those formulations of the characterizing rules that highlight the duality of the definitions. For the conditional (>) this definition is:

(D>) $$a /\!/ b > c \ \leftrightarrow \ (d)(d \vdash a \ \leftrightarrow \ d, b \vdash c)$$

(C>) $$d \vdash b > c \ \leftrightarrow \ d, b \vdash c$$

If we dualize (C>), we obtain:

$(C>)^{\delta}$ $$c >^{\delta} b \vdash d \ \leftrightarrow \ c \vdash d, b$$

which is equivalent to Popper's definition of the anti-conditional ($\not>$):[58]

(D$\not>$) $$a /\!/ b \not> c \ \leftrightarrow \ (d)(a \vdash d \ \leftrightarrow \ c \vdash d, b)$$

(C$\not>$) $$c \not> b \vdash d \ \leftrightarrow \ c \vdash d, b$$

As an informal observation we may state that the conditional is characterized by the deduction theorem and the anti-conditional by the dual of the deduction theorem.

On the basis of these definitions we can show that modus ponens holds for the conditional, and that a dual to modus ponens holds for the anti-conditional.

Lemma 5.7 *The following rules hold:*

1. $b, b > c \vdash c$ *(modus ponens),*

2. $c \vdash c \not> b, b$ *(a dual rule to modus ponens).*

Proof (1) In (C>) let d be $b > c$ to obtain $b > c \vdash b > c \ \leftrightarrow \ b > c, b \vdash c$. By (Rg), $b > c, b \vdash c$ follows.

[58] Our definitions (D>) and (D$\not>$) correspond to Popper's (1948a) rules (3.81) and (3.82), respectively.

(2) In (C$\not>$) let d be $c \not> b$ to obtain $c \not> b \vdash c \not> b \ \leftrightarrow\ c \vdash c \not> b, b$. By (Rg), $c \vdash c \not> b, b$ follows. □

Lemma 5.8 *Conditional and anti-conditional are logical constants, that is, their rules are fully characterizing.*

Proof For the conditional we have to show that

$$((d)(d \vdash a_1 \ \leftrightarrow\ d, b \vdash c) \ \&\ (d)(d \vdash a_2 \ \leftrightarrow\ d, b \vdash c)) \to a_1 /\!/ a_2$$

is true. Assuming the antecedent, we instantiate d by a_1 in both its left and right conjunct. From the left one gets $a_1 \vdash a_1 \ \leftrightarrow\ a_1, b \vdash c$, and thus $a_1, b \vdash c$ by (Rg). From the right one gets $a_1 \vdash a_2 \ \leftrightarrow\ a_1, b \vdash c$, and thus $a_1 \vdash a_2$. Likewise for $a_2 \vdash a_1$. The proof for the anti-conditional is similar. □

Following Popper, we will use the terms "implication" and "conditional" interchangeably to refer either to the connective $>$ or to statements of the form $a > b$. In the modern discussion, the logical constant introduced and discussed by Popper as "anti-conditional" figures under the heading "co-implication" (cf. Wansing, 2008).

5.7 Characterizations of implication and Peirce's rule

As is well known, if implication (i.e., the conditional $>$) is defined solely by the usual implication introduction and elimination rules of natural deduction, then the addition of classical negation by its natural deduction rules is a non-conservative extension of the logic. For example, Peirce's law $((a > b) > a) > a$ is not derivable using only the introduction and elimination rules for implication, but becomes derivable if the rules of classical negation are added.[59]

Popper (1947c) already made this observation in the context of his characterizing rules. He calls *positive logic* that part of propositional logic obtained by the definitions of the logical constants excluding negation, and remarks in Popper (1947c, p. 215): "Positive logic as defined by these rules does not yet contain all valid rules of inferences in which no use is made of negation: there is a further region which we may call the 'extended positive logic' [. . .]." Kleene's (1948) review (cf. this volume, § 13.12) of Popper (1947d) cites Popper's observation and turns it into a critique of Popper's claim to give explicit definitions of the logical constants: "If his definitions were 'explicit,' as he claims, it should make no difference whatsoever, in the case of a formula containing only '>', whether or not the definition of classical negation has been stated." But it is clearly not enough to just "state" the definition. It is the existence postulate which tells us that for every statement there exists a classical negation of it in the object language which permits the proof of the demonstrability of, for example, Peirce's law to go through.

In order to make statements such as Peirce's law deducible without invoking negation, Popper considers the use of the following additional rule for the conditional:

[59] Cf. Dummett (1991, p. 291f.); cf. also de Campos Sanz, Piecha, and Schroeder-Heister (2014).

(4.2e) $$a, b > c/b \leftrightarrow a/b$$

Reading (4.2e) from left to right amounts to a characterization of implication by Peirce's rule[60], that is, a rule version of Peirce's law $((b > c) > b) > b$. Popper also mentions that in the presence of (4.2e), the rules for classical negation can be obtained from the definitions of intuitionistic negation.[61]

Popper thus had the following insights, which were far from trivial at the time: The addition of classical negation to a logic containing implication defined by its usual introduction and elimination rules forms a non-conservative extension. Alternatively, implication can be characterized by a stronger set of rules containing a version of Peirce's rule. If this stronger set of rules for implication is used, then intuitionistic and classical negation coincide. Whether Popper was the first to consider Peirce's rule is not completely clear. Seldin (2008, § 3) "think[s . . .] that Popper and Curry thought of this rule independently."

Further remarks on implication

In later unpublished typescripts, entitled "A Note on the Classical Conditional" (this volume, Chapter 17), which were written at the beginning of 1952, Popper wants to show how to derive the truth table of classical implication from the generally accepted inference rules of implication (e.g., modus ponens), together with the definition of a valid inference as an inference which transmits truth from its premises to its conclusion. One can thus say that the inference rules are considered to be primary, and the truth tables are to be justified in terms of the generally accepted rules of inference, and are thus secondary. Popper furthermore discusses the difference between classical, strict and intuitionist implication, and makes the following remark on intuitionism (this volume, Chapter 17, Typescript 2, § 8):

> While contending that the truth table of the classical conditional does not, upon closer inspection, conflict with the usages of an "ordinary language", the tendencies and the consistently developed usages of an ordinary language, I am very ready to admit with the Intuitionists (Brouwer, Heyting) that "ordinary language" usages involve us into difficulties when problems of infinities are involved, and that we may have to sacrifice classical negation, with its characteristic truth table, and especially ⟨6.5⟩ [b is false if and only if $\neg b$ is true] (and the law of the excluded middle).

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[60] Popper did not call his rule "Peirce's rule".

[61] Cf. Popper (1947c, p. 216): "Also we do not need, in the presence of rule 4.2e, the whole force of our rule of negation 4.6 [i.e., classical negation], but can obtain this rule as a secondary rule from some weaker rules [. . .] (from the rules of the so-called intuitionist logic)." Popper (1947c, p. 200) calls a rule *secondary* if it is derivable from a set of given so-called primary rules, that is, if "every inference which is asserted as valid by the secondary rule could be drawn merely by force of the primary rules alone."

5.8 Conjoint denial, alternative denial and further connectives

Popper (1947c, p. 219) defines some connectives in terms of connectives that have already been defined by characterizing rules, noting that these definitions are unnecessary in the sense that one can always use the characterizing rules directly instead. We mention the conjoint denial

(D↓) $$a /\!/ a \downarrow b \;\leftrightarrow\; a /\!/ \neg(a \vee b)$$

and the alternative denial

(D⅄) $$a /\!/ a \curlywedge b \;\leftrightarrow\; a /\!/ \neg(a \wedge b)$$

In a letter to Alonzo Church of 2 February 1948 (this volume, § 24.3) Popper provides characterizing rules for these two connectives as follows and notes their duality:

(C↓) $$a \downarrow b \vdash c \;\leftrightarrow\; \vdash a, b, c$$
(C⅄) $$a \vdash b \curlywedge c \;\leftrightarrow\; a, b, c \vdash$$

In a handwritten note Popper (n.d.[a]) defines negation, conjunction, disjunction, conditional, anti-conditional and bi-conditional both in terms of alternative denial and in terms of joint denial.

After introducing the bi-conditional

(D$\widehat{\equiv}$) $$a /\!/ b \mathrel{\widehat{\equiv}} c \;\leftrightarrow\; (\text{for any } a_1\text{: } a_1/a \;\leftrightarrow\; a_1, c/b \;\&\; a_1, b/c)$$

Popper defines the exclusive disjunction as follows:

(D≠) $$a /\!/ a \neq b \;\leftrightarrow\; a /\!/ \neg(a \mathrel{\widehat{\equiv}} b)$$

Further connectives defined by using already defined logical constants are the tautology

(Dt) $$a /\!/ t(b) \;\leftrightarrow\; a /\!/ b \vee \neg b$$

and the contradiction

(Df) $$a /\!/ f(b) \;\leftrightarrow\; a /\!/ b \wedge \neg b$$

Since, for any a and b, $t(a) /\!/ t(b)$ and $f(a) /\!/ f(b)$ hold for these unary connectives, one can simply write t and f; that is, these connectives correspond to the nullary constants verum and falsum (cf. also the end of § 6.4).

6 Negations

Popper considered several different kinds of negation. We discuss their definitions and investigate how they relate to each other. These definitions are taken from Popper (1948c) unless indicated otherwise.[62] Some of the following results were only stated without proof by Popper, and for some he gave only proof sketches. We will provide more details and fill in the missing proofs.

6.1 Classical negation

For classical negation ($\neg_k$) Popper considers several definitions, among them the following two:[63]

(D$\neg_k$1) $$a /\!/ \neg_k b \;\leftrightarrow\; (a, b \vdash \;\&\; \vdash a, b)$$

(D$\neg_k$2) $$a /\!/ \neg_k b \;\leftrightarrow\; (c)(d)(d, a \vdash c \;\leftrightarrow\; d \vdash b, c)$$

with the following two characterizing rules:

(C$\neg_k$1) $$\neg_k b, b \vdash \;\&\; \vdash \neg_k b, b$$

(C$\neg_k$2) $$(c)(d)(d, \neg_k b \vdash c \;\leftrightarrow\; d \vdash b, c)$$

These two definitions reflect two different ways of characterizing classical negation. Definition (D$\neg_k$1) is based on the idea that the classical negation of a statement b is a statement which is at the same time complementary and contradictory to b, whereas (D$\neg_k$2) is very similar to the rules for negation used in classical sequent calculus.

Lemma 6.1 *The definitions* (D$\neg_k$1) *and* (D$\neg_k$2) *are equivalent.*

Proof We consider the characterizing rules, and show first that (C$\neg_k$1) follows from (C$\neg_k$2):

In $(c)(d)(d, \neg_k b \vdash c \;\leftrightarrow\; d \vdash b, c)$ let d be b to obtain $(c)(b, \neg_k b \vdash c \;\leftrightarrow\; b \vdash b, c)$. By (Rg) one obtains $(c)(b, \neg_k b \vdash c)$, that is, $b, \neg_k b \vdash$; hence $\neg_k b, b \vdash$ by (LE). In $(c)(d)(d, \neg_k b \vdash c \;\leftrightarrow\; d \vdash b, c)$ let c be $\neg_k b$ to obtain $(d)(d, \neg_k b \vdash \neg_k b \;\leftrightarrow\; d \vdash b, \neg_k b)$. By (Rg) one obtains $(d)(d \vdash b, \neg_k b)$, that is, $\vdash b, \neg_k b$; hence $\vdash \neg_k b, b$ by (RE).

To show that (C$\neg_k$2) follows from (C$\neg_k$1) it is sufficient to show that the two implications $\vdash \neg_k b, b \rightarrow (d, \neg_k b \vdash c \rightarrow d \vdash b, c)$ and $\neg_k b, b \vdash \rightarrow (d \vdash b, c \rightarrow d, \neg_k b \vdash c)$ hold. This can be done by using (Cut). To show the first implication we assume $\vdash \neg_k b, b$, from which we obtain $d \vdash \neg_k b, b$ by (LW). Assuming $d, \neg_k b \vdash c$ we get $d, d, b \vdash c$ by (Cut), from which we get $d, b \vdash c$ by (LC). The second implication is a direct instance of (Cut), where the cut formula is the statement b. □

[62] Popper's notation for negations varies; we write $\neg_k a$, for example, where Popper would, e.g., have written a^k.

[63] In Popper (1948c), (D$\neg_k$1) is (D4.3), and (D$\neg_k$2) is (4.31).

The use of (Cut) can be justified by using our definition (D$\vdash$3)[64] for relative demonstrability ($\vdash$) instead of Popper's definition (D$\vdash$2), or by assuming that the object language contains conjunction ($\wedge$), disjunction ($\vee$) and the conditional ($>$); cf. § 4.5.[65] We take the first option here, and presuppose our definition (D$\vdash$3).

Lemma 6.2 *Classical negation $\neg_k$ is self-dual, that is, $(\neg_k a)^\delta = \neg_k a$ for the duality function δ.*

Proof By applying the duality function δ to (D$\neg_k$1). □

6.2 Intuitionistic negation and dual-intuitionistic negation

The study of formalized intuitionistic logic started with Heyting's set of axioms.[66] Originally, Heyting's formalization was written in response to a prize question proposed by Mannoury in 1927, which also asked "to investigate whether from the system to be constructed [for intuitionistic logic] a dual system may be obtained by (formally) interchanging the *principium tertii exclusi* and the *principium contradictionis*."[67] Thus the idea of dualizing intuitionistic logic was present from the very beginning of its development.[68] The principium tertii exclusi is in Popper's theory mirrored by the property of a negation of b to be complementary to b, and the principium contradictionis is mirrored by the contradictoriness of b and the negation of b. Popper uses the term "minimum definable negation" when he writes about what we call dual-intuitionistic negation ($\neg_m$), but still mentions the fact that it forms some kind of dual to intuitionistic negation.[69] For intuitionistic negation ($\neg_i$), Popper gives the following definition and characterizing rule:[70]

(D$\neg_i$) $$a /\!/ \neg_i b \leftrightarrow (c)(c \vdash a \leftrightarrow c, b \vdash)$$

(C$\neg_i$) $$c \vdash \neg_i b \leftrightarrow c, b \vdash$$

If we dualize (D$\neg_i$), we get

(D$\neg_i$)$^\delta$ $$a /\!/ \neg_i{}^\delta b \leftrightarrow (c)(a \vdash c \leftrightarrow \;\vdash c, b)$$

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[64] Cf. the discussion in § 4.5.

[65] Popper states that (D$\neg_k$1) and (D$\neg_k$2) are equivalent if the existence of either conjunction or disjunction is guaranteed for arbitrary statements.

[66] Cf. Heyting (1930).

[67] Quoted from Troelstra (1990, p. 4).

[68] Cf. Kapsner (2014, p. 128).

[69] Cf. Popper (1948c, p. 323). Cohen (1953b, p. 188) remarks that "[t]he negation functor in the dualintuitionistic restricted predicate calculus GL2 [developed by Cohen] has the same properties as Popper's 'minimum definable (non-modal) negation [$\neg_m$].'" To avoid misunderstandings, it should be emphasized that Popper's usage of the term "minimum negation" has nothing to do with Johansson's (1937) "minimal calculus" and its negation.

[70] Cf. Popper (1948c, def. (D4.1)).

which is identical to Popper's definition for dual-intuitionistic negation ($\neg_m$):[71]

(D$\neg_m$) $$a /\!/ \neg_m b \leftrightarrow (c)(a \vdash c \leftrightarrow \; \vdash c, b)$$

(C$\neg_m$) $$\neg_m b \vdash c \leftrightarrow \; \vdash c, b$$

Lemma 6.3 *For intuitionistic negation* $b, \neg_i b \vdash$ *holds, and for dual-intuitionistic negation* $\vdash b, \neg_m b$ *holds.*

Proof In (C$\neg_i$) let c be $\neg_i b$; by (Rg) and (LE) we get $b, \neg_i b \vdash$. In (C$\neg_m$) let c be $\neg_m b$; by (Rg) and (RE) we get $\vdash b, \neg_m b$. □

Lemma 6.4 *Intuitionistic negation* $\neg_i$ *and dual-intuitionistic negation* $\neg_m$ *are logical constants, that is, their rules are fully characterizing.*

Proof For intuitionistic negation we have to show that

$$((c)(c \vdash a_1 \leftrightarrow c, b \vdash) \;\&\; (c)(c \vdash a_2 \leftrightarrow c, b \vdash)) \rightarrow a_1 /\!/ a_2$$

is true. In both conjuncts let c be a_1 to obtain

$$a_1 \vdash a_1 \leftrightarrow a_1, b \vdash \quad \text{and} \quad a_1 \vdash a_2 \leftrightarrow a_1, b \vdash .$$

From the first conjunct we get $a_1, b \vdash$ by (Rg), and from $a_1, b \vdash$ and the second conjunct we obtain $a_1 \vdash a_2$. The proof of $a_2 \vdash a_1$ is similar.

For dual-intuitionistic negation we have to show that

$$((c)(a_1 \vdash c \leftrightarrow \; \vdash c, b) \;\&\; (c)(a_2 \vdash c \leftrightarrow \; \vdash c, b)) \rightarrow a_1 /\!/ a_2$$

is true. In both conjuncts let c be a_1 to obtain

$$a_1 \vdash a_1 \leftrightarrow \; \vdash a_1, b \quad \text{and} \quad a_2 \vdash a_1 \leftrightarrow \; \vdash a_1, b.$$

From the first conjunct we obtain $\vdash a_1, b$ by (Rg), and from $\vdash a_1, b$ and the second conjunct we get $a_2 \vdash a_1$. The proof of $a_1 \vdash a_2$ is similar. □

6.3 Non-conservative language extensions

Popper (1948c, § V) considers non-conservative language extensions, although without using these terms. Popper (1947b, p. 282, fn 20) observes that "if, in one language, a classical as well as an intuitionistic negation exists of every statement, then the latter becomes equivalent to the former, or in other words, classical negation then absorbs or assimilates its weaker kin."[72] Moreover, he observes that this does not happen if classical negation is put together with Johansson's negation (or with yet

[71] Cf. Popper (1948c, def. (D4.2)).

[72] Cf. also Popper's remark at the end of § 3 in Chapter 16 of this volume.

another negation, "the impossibility of b", proposed by him ibid., p. 283). However, Johansson's negation $\neg_j$ is not a logical constant in Popper's sense (cf. Theorem 6.12).

An example of a non-conservative language extension is the addition of classical negation $\neg_k$ to a language containing both intuitionistic negation $\neg_i$ and dual-intuitionistic negation $\neg_m$ since this addition makes classical laws hold for the two weaker negations $\neg_i$ and $\neg_m$.

Theorem 6.5 *In the presence of* $\neg_k$ *we have* $\neg_k a /\!/ \neg_i a$, $\neg_k a /\!/ \neg_m a$ *and* $\neg_i a /\!/ \neg_m a$. *In other words, the three negations* $\neg_k$, $\neg_i$ *and* $\neg_m$ *collapse into a single one (i.e., they become synonymous).*

Proof Classical negation $\neg_k$ satisfies the rules for $\neg_i$ and $\neg_m$, that is, we have $a \vdash \neg_k b \;\leftrightarrow\; a, b \vdash$ and $\neg_k a \vdash b \;\leftrightarrow\; \vdash a, b$, respectively. Both equivalences are direct consequences of (C$\neg_k$1) and (C$\neg_k$2), by using (Cut). Since the rules for $\neg_i$ and $\neg_m$ are fully characterizing, we have $\neg_k a /\!/ \neg_i a$ and $\neg_k a /\!/ \neg_m a$. Hence also $\neg_i a /\!/ \neg_m a$, for any object language containing $\neg_k$, $\neg_i$ and $\neg_m$. □

Popper (1948c, p. 324) also considers the more general situation where two logical functions S_1 and S_2 have been introduced by sets of primitive rules R_1 and R_2, respectively, such that $R_2 \subset R_1$. If both S_1 and S_2 are definable, and S_1 is given, then one can show that S_1 and S_2 are equivalent.

This can be generalized further, since R_2 need not be a subset of R_1; it is sufficient that R_1 implies R_2. Consider the following setting with two fully characterizing rules $\mathcal{R}_1$ and $\mathcal{R}_2$ for two n-ary constants such that $\mathcal{R}_1$ implies $\mathcal{R}_2$:

$$\mathcal{R}_1(a, a_1, \ldots, a_n) \tag{1.1}$$

$$\mathcal{R}_2(a, a_1, \ldots, a_n) \tag{1.2}$$

$$\mathcal{R}_1(a, a_1, \ldots, a_n) \rightarrow \mathcal{R}_2(a, a_1, \ldots, a_n) \tag{1.3}$$

Now assume that $\mathcal{R}_2(b, a_1, \ldots, a_n)$ holds. We get $\mathcal{R}_2(a, a_1, \ldots, a_n)$ from (1.1) and (1.3), which implies $a /\!/ b$, since we have fully characterizing rules. Hence $\mathcal{R}_1(b, a_1, \ldots, a_n)$. The two characterized constants become thus synonymous; in other words, adding $\mathcal{R}_1$ yields a non-conservative extension of systems containing $\mathcal{R}_2$.

Popper's treatment of conservativeness also throws some light on his logical approach in general. Based on the fact that Popper does not use conservativeness as a criterion for accepting characterizing rules, one could argue that Popper is not aiming at a semantic justification of logical theories, since from a semantic theory we expect that the introduction of a new constant is always a conservative extension; cf. § 1.5.

6.4 Six further kinds of negation

Popper (1948c, § VI) considers three further kinds of negation explicitly, namely $\neg_j$, $\neg_l$ and $\neg_n$. He mentions their duals in a footnote[73] but does not study them. The negation $\neg_n$ coincides with what is nowadays called *subminimal negation*[74].

Definition 6.6 The six negations $\neg_j$, $\neg_{dj}$, $\neg_l$, $\neg_{dl}$, $\neg_n$ and $\neg_{dn}$ are given by the following characterizing rules. We also indicate the respective duals. (The characterizing rules for $\neg_k$, $\neg_i$ and $\neg_m$ are repeated below for comparison.)

Negation	*Characterizing rule*	*Rule name*	*Dual negation*	*Rule in Popper (1948c)*
$\neg_j$	$a, b \vdash \neg_j c \rightarrow a, c \vdash \neg_j b$	$(\mathrm{C}\neg_j)$	$\neg_{dj}$	(6.1)
$\neg_{dj}$	$\neg_{dj} c \vdash a, b \rightarrow \neg_{dj} b \vdash a, c$	$(\mathrm{C}\neg_{dj})$	$\neg_j$	–
$\neg_l$	$a, \neg_l b \vdash c \rightarrow a, \neg_l c \vdash b$	$(\mathrm{C}\neg_l)$	$\neg_{dl}$	(6.2)
$\neg_{dl}$	$c \vdash a, \neg_{dl} b \rightarrow b \vdash a, \neg_{dl} c$	$(\mathrm{C}\neg_{dl})$	$\neg_l$	–
$\neg_n$	$a, b \vdash c \rightarrow a, \neg_n c \vdash \neg_n b$	$(\mathrm{C}\neg_n)$	$\neg_{dn}$	(6.3)
$\neg_{dn}$	$c \vdash a, b \rightarrow \neg_{dn} b \vdash a, \neg_{dn} c$	$(\mathrm{C}\neg_{dn})$	$\neg_n$	–
$\neg_k$	$a, \neg_k b \vdash c \leftrightarrow a \vdash b, c$	$(\mathrm{C}\neg_k 2)$	$\neg_k$	(4.32)
$\neg_i$	$a \vdash \neg_i b \leftrightarrow a, b \vdash$	$(\mathrm{C}\neg_i)$	$\neg_m$	(4.1)
$\neg_m$	$\neg_m a \vdash b \leftrightarrow \vdash a, b$	$(\mathrm{C}\neg_m)$	$\neg_i$	(4.2)

Popper (1948c, p. 328) explains his names for these negations as follows: "In view of 6.2 [cf. Definition 6.6], we may call [$\neg_l a$] the 'left-hand side negation of a' (in contradistinction to Johansson's [$\neg_j a$] which, in view of 6.1 [cf. Definition 6.6], is a 'right hand side negation'). [$\neg_n a$] may be called the 'neutral negation'; it is neutral with respect to right-sidedness and left-sidedness [. . .]." As mentioned above (cf. § 5.3), Popper is wrong in calling the j-negation $\neg_j$ "Johansson's negation".

The two rules $(\mathrm{C}\neg_i)$ and $(\mathrm{C}\neg_m)$ differ slightly from the other rules in that they have only two instead of three statements occurring in each relation of relative demonstrability. However, they can also be given the same form with three such statements, as the following lemma shows.

Lemma 6.7 *The rule* $(\mathrm{C}\neg_i)$ *is equivalent to*

$$(\mathrm{C}\neg_i{}') \qquad a, c \vdash \neg_i b \leftrightarrow a, b, c \vdash$$

[73] Popper (1948c, p. 328, fn 16) writes: "There are, of course, dual rules of 6.1 [$\neg_j$], 6.2 [$\neg_l$] and 6.3 [$\neg_n$], two of which are satisfied by [$\neg_m$], just as 6.1 [$\neg_j$] and 6.3 [$\neg_n$] are satisfied by [$\neg_i$]." (In Curry (1949, this volume, § 13.9), $\neg_l$ is mistakenly considered to be the dual of $\neg_j$, which it is not in the sense used by Popper.)

[74] Cf., e.g., Dunn (1999) and the references therein.

and the rule (C$\neg_m$) *is equivalent to*

(C$\neg_m{}'$) $$\neg_m a \vdash b, c \;\leftrightarrow\; \vdash a, b, c.$$

Proof Rule (C$\neg_i$) follows from (C$\neg_i{}'$) by substituting a for c and applications of (LC). Rule (C$\neg_m$) follows from (C$\neg_m{}'$) by substituting b for c and applications of (RC).

Next we show that (C$\neg_m{}'$) can be obtained from (C$\neg_m$). For the direction from right to left, assume $\neg_m a \vdash b, c$ and use (Cut) with $\vdash a, \neg_m a$, which holds by Lemma 6.3.

For the direction from left to right, we first show that $\neg_m a \vdash a, b \to \neg_m a \vdash b$ holds:

1. Assume $\neg_m a \vdash a, b$.
2. Use (Cut) with $\vdash a, \neg_m a$ (Lemma 6.3) to obtain $\vdash a, a, b$, from which $\vdash a, b$ follows.
3. Applying (C $\neg_m$), $\neg_m a \vdash b$ follows.

We now show that $\vdash a, b, c$ implies $\neg_m a \vdash b, c$.

1. Assume $\vdash a, b, c$.
2. By using definition (D$\vdash$3) and by applying universal instantiation twice, we obtain $(a/d \;\&\; b/d \;\&\; c/d) \to \neg_m a/d$.
3. Assume c/d to obtain $(a/d \;\&\; b/d) \to \neg_m a/d$.
4. Using an instance of $\neg_m a \vdash a, b \to \neg_m a \vdash b$, we obtain $b/d \to \neg_m a/d$.
5. Reintroduce c/d to obtain $(c/d \;\&\; b/d) \to \neg_m a/d$.
6. By universal quantification and application of (D$\vdash$3) we get $\neg_m a \vdash b, c$.

To show that (C$\neg_i{}'$) can be obtained from (C$\neg_i$), we first apply the duality function δ to (C$\neg_i$), which yields (C$\neg_m$). An application of δ to the equivalent (C$\neg_m{}'$) yields (C$\neg_i{}'$). □

In the following, we show how all these negations relate to each other.

Theorem 6.8 *The following statements hold for the characterizing rules given in Definition 6.6:*

(1) $\neg_k$ satisfies the rule for $\neg_i$.
(2) $\neg_i$ satisfies the rule for $\neg_j$.
(3) $\neg_j$ satisfies the rule for $\neg_n$.
(4) $\neg_k$ satisfies the rule for $\neg_l$.
(5) $\neg_l$ satisfies the rule for $\neg_n$.
(6) $\neg_k$ satisfies the rule for $\neg_m$.
(7) $\neg_m$ satisfies the rule for $\neg_{dj}$.
(8) $\neg_{dj}$ satisfies the rule for $\neg_{dn}$.
(9) $\neg_k$ satisfies the rule for $\neg_{dl}$.
(10) $\neg_{dl}$ satisfies the rule for $\neg_{dn}$.

Proof We show (7), (8), (9) and (10). The structural rules (LE) and (RE) will be used tacitly.

(7) We have to show $\neg_m c \vdash a, b \;\to\; \neg_m b \vdash a, c$, presupposing (C$\neg_m$). By Lemma 6.7 we can use (C$\neg_m{}'$) equivalently.

a. Assume $\neg_m c \vdash a, b$.
b. Therefore $\vdash c, a, b$, by (C$\neg_m{}'$).
c. Therefore $\neg_m b \vdash a, c$, again by (C$\neg_m{}'$).

(8) We have to show $c \vdash a, b \to \neg_{dj} b \vdash a, \neg_{dj} c$, presupposing (C$\neg_{dj}$).

a. Assume $c \vdash a, b$.
b. We have $\neg_{dj}\, c \vdash a, \neg_{dj}\, c \rightarrow \neg_{dj} \neg_{dj}\, c \vdash a, c$, as an instance of (C$\neg_{dj}$).
c. We have $\neg_{dj}\, c \vdash a, \neg_{dj}\, c$, by (Rg) and (RW).
d. Therefore $\neg_{dj} \neg_{dj}\, c \vdash a, c$, from (b) and (c) by modus ponens.
e. Therefore $\neg_{dj} \neg_{dj}\, c \vdash a, b$, from (a) and (d) by (Cut) and (RC).
f. We have $\neg_{dj} \neg_{dj}\, c \vdash a, b \rightarrow \neg_{dj}\, b \vdash a, \neg_{dj}\, c$, as an instance of (C$\neg_{dj}$).
g. Therefore $\neg_{dj}\, b \vdash a, \neg_{dj}\, c$, from (e) and (f) by modus ponens.

(9) We have to show $c \vdash a, \neg_k\, b \rightarrow b \vdash a\, \neg_k\, c$, presupposing (C$\neg_k$).

a. Assume $c \vdash a, \neg_k\, b$.
b. We have $\neg_k\, b, b \vdash$ and $\vdash c, \neg_k\, c$, by (C$\neg_k$).
c. Therefore $b, c \vdash a$, from (a) and (b) by (Cut).
d. Therefore $\vdash c, \neg_k\, c$, from (b) and (c) by (Cut).

(10) We have to show $c \vdash a, b \rightarrow \neg_{dl}\, b \vdash a, \neg_{dl}\, c$, presupposing (C$\neg_{dl}$).

a. Assume $c \vdash a, b$.
b. We have $\neg_{dl}\, b \vdash a, \neg_{dl}\, b \rightarrow b \vdash a, \neg_{dl} \neg_{dl}\, b$, as an instance of (C$\neg_{dl}$).
c. We have $\neg_{dl}\, b \vdash a, \neg_{dl}\, b$, by (Rg) and (RW).
d. Therefore $b \vdash a, \neg_{dl} \neg_{dl}\, b$, from (b) and (c) by modus ponens.
e. Therefore $c \vdash a, \neg_{dl} \neg_{dl}\, b$, from (a) and (d) by (Cut) and (RC).
f. We have $c \vdash a, \neg_{dl} \neg_{dl}\, b \rightarrow \neg_{dl}\, b \vdash a, \neg_{dl}\, c$, as an instance of (C$\neg_{dl}$).
g. Therefore $\neg_{dl}\, b \vdash a, \neg_{dl}\, c$, from (e) and (f) by modus ponens.

Statements (2)–(5) can be shown analogously, and were already presented by Popper (1948c, p. 328) without proof. For statements (1) and (6) cf. Theorem 6.5. □

Corollary 6.9 *By transitivity we have in addition that $\neg_k$ satisfies the rules for $\neg_j$, $\neg_n$, $\neg_{dn}$ and $\neg_{dj}$; $\neg_i$ satisfies the rule for $\neg_n$, and $\neg_m$ satisfies the rule for $\neg_{dn}$.*

Theorem 6.10 *We have the following dualities: $(\neg_k, \neg_k)$, $(\neg_i, \neg_m)$, $(\neg_j, \neg_{dj})$, $(\neg_l, \neg_{dl})$ and $(\neg_n, \neg_{dn})$.*

Proof The self-duality of $\neg_k$ is given by Lemma 6.2. For the duality between $\neg_i$ and $\neg_m$ cf. § 6.2. The dualities $(\neg_j, \neg_{dj})$, $(\neg_l, \neg_{dl})$ and $(\neg_n, \neg_{dn})$ can be shown by applications of the duality function δ. □

Our results on negations are summarized in Figure 6.1.[75]

In contradistinction to classical, intuitionistic and dual-intuitionistic negation, the six negations $\neg_j$, $\neg_{dj}$, $\neg_l$, $\neg_{dl}$, $\neg_n$ and $\neg_{dn}$ are not given by fully characterizing rules. Hence, these negations cannot be considered as logical constants.[76] This is especially

[75] That $\neg_m$ does not satisfy $\neg_n$ was already observed by Popper (1948c, p. 328).

[76] The idea that certain negations could be too weak to be still considered as logical constants is already present in Popper (1943, p. 50): "Systems containing the operation of *negation* may be so much weakened that contradictoriness [i.e., the derivability of any two formulas such that one is the negation of the other] only implies n-embracingness [*i.e.* we can prove that any *negation of any formula whatsoever* can be derived (ibid., p. 49)]. It appears, however, that we cannot weaken them further without depriving *negation* of the character of a logical operation."

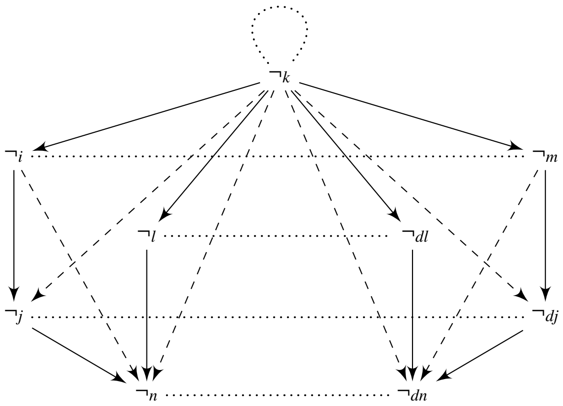


Fig. 6.1 Results on negations. The dotted lines connect those negations that are dual in the sense of Popper, with $\neg_k$ being self-dual, and the arrows show which negations satisfy the rules of which other negations; the solid arrows are due to Theorem 6.8, and the dashed arrows are due to Corollary 6.9. The diagram is complete; no further (non-trivial) relations hold.

interesting in the case of Johansson's negation $\neg_j$. In order to show this, Popper first introduces the two logical connectives t and f:

(D t) $$a/\!/t(b) \leftrightarrow (c)(b/a \leftrightarrow c/a)$$

(C t) $$(c)(b/t(b) \leftrightarrow c/t(b))$$

(D f) $$a/\!/f(b) \leftrightarrow (c)(a/b \leftrightarrow a/c)$$

(C f) $$(c)(f(b)/b \leftrightarrow f(b)/c)$$

For t and f, the following lemma holds.

Lemma 6.11 *For all statements b:* $\vdash t(b)$ *and* $f(b) \vdash$.

Proof In (C t) let c be $t(b)$ in order to obtain $(b/t(b) \leftrightarrow t(b)/t(b))$, from which one can obtain $b \vdash t(b)$. From $b/t(b)$ and (C t) one immediately obtains $(c)(c/t(b))$, and thus $\vdash t(b)$. Similarly for $f(b) \vdash$. □

This shows that t is a unary verum and that f is a unary falsum.

Theorem 6.12 *None of the rules for* $\neg_j$, $\neg_{dj}$, $\neg_l$, $\neg_{dl}$, $\neg_n$ *and* $\neg_{dn}$ *are fully characterizing.*

Proof The verum t satisfies the rules for $\neg_j$, $\neg_n$, $\neg_{dl}$ and $\neg_{dn}$. The falsum f satisfies the rules for $\neg_l$, $\neg_n$, $\neg_{dj}$ and $\neg_{dn}$. Classical negation $\neg_k$ satisfies the rules for $\neg_j$, $\neg_{dj}$, $\neg_l$, $\neg_{dl}$, $\neg_n$ and $\neg_{dn}$. If the rules for the six latter negations were each fully characterizing, then we would have for all a that $\neg_k\, a/\!/t(a)$ or $\neg_k\, a/\!/f(a)$, depending on the negation considered. But both $\neg_k\, a/\!/t(a)$ and $\neg_k\, a/\!/f(a)$ can only hold for contradictory object languages. □

7 Modal logic

Modal logic is introduced in Popper (1947d, § IX). Popper uses modal connectives in his proof of the compatibility of intuitionistic and dual-intuitionistic logic, which will be dealt with in § 8.

Popper considers the following six modal connectives: *necessary*, *impossible*, *logical*, *contingent*, *possible* and *uncertain*. They are taken from Carnap (1947, p. 175), and the definitions given for them by Popper are strongly influenced by Carnap's treatment of modality. Popper's definitions and characterizing rules for the modal connectives all have the following form:

(DM) $$a /\!/ M\, b \;\leftrightarrow\; (\vdash a \bigvee \text{⅂}\, a) \;\&\; \mathcal{R}(a, b)$$

(CM) $$(\vdash M\, b \bigvee \text{⅂}\, M\, b) \;\&\; \mathcal{R}(M\, b, b)$$

where M stands for any of the six modal connectives, and $\mathcal{R}(a, b)$ varies depending on the modal connective to be defined.

Table 1.3 combines the table given by Carnap (1947, p. 175)[77] and the list of definitions given by Popper (1947d, p. 1223). Carnap's half of the table contains

Table 1.3 The modal connectives from Carnap and Popper.

Carnap			*Popper*		
Name	$\Box$, $\Diamond$-*Definition*	*Semantic property*	*Sign*	*Name*	$\mathcal{R}(M\, b, b)$
Necessary	$\Box p$ $\neg\Diamond\neg p$	L-true	N	Necessary	$\vdash N\, b \leftrightarrow\; \vdash b$
Impossible	$\Box\neg p$ $\neg\Diamond p$	L-false	I	Impossible	$\vdash I\, b \leftrightarrow \text{⅂}\, b$
Contingent	$\neg\Box p \wedge \neg\Box\neg p$ $\Diamond\neg p \wedge \Diamond p$	factual	C	Contingent	$\text{⅂}\, C\, b \leftrightarrow (\vdash b \bigvee \text{⅂}\, b)$
Non-necessary	$\neg\Box p$ $\Diamond\neg p$	not L-true	U	Uncertain	$\text{⅂}\, U\, b \leftrightarrow\; \vdash b$
Possible	$\neg\Box\neg p$ $\Diamond p$	not L-false	P	Possible	$\text{⅂}\, P\, b \leftrightarrow \text{⅂}\, b$
Noncontingent	$\Box p \vee \Box\neg p$ $\neg\Diamond\neg p \vee \neg\Diamond p$	L-determinate	L	Logical	$\vdash L\, b \leftrightarrow (\vdash b \bigvee \text{⅂}\, b)$

his names for the modal connectives, their definition in terms of necessity $\Box$ and possibility $\Diamond$ and the semantic property to which they correspond. In Carnap's system every statement (or sentence, to use Carnap's terminology) falls into one of three disjoint categories. It can either be L-true, L-false or factual. A statement is called

[77] We have replaced N by $\Box$ here.

L-determinate, if it is either L-true or L-false. If a statement is not L-true, then it is either L-false or factual and so on. Popper's half of the table contains his names for the modal connectives, the corresponding symbol and the part $\mathcal{R}(M\,b, b)$ of the definition that varies with the modal connectives.

If L-truth is matched with demonstrability and L-falsity with refutability, then the close correspondence between the semantic properties given by Carnap's and Popper's definitions becomes obvious. For example, in the case of a noncontingent (Carnap's terminology) or logical (Popper's terminology) statement, the semantic property of being L-determinate corresponds to the property of being either demonstrable or refutable according to Popper's definition.

We observe that the part $\vdash M\,b \ \bigvee\ \urcorner\!/ M\,b$ in Popper's definitions (C*M*) of the modal connectives *M* corresponds to the fact that in Carnap's system all modal statements (i.e., statements whose outermost logical constant is a modal constant) are L-determinate.[78] To express this determinateness is the only situation where Popper uses (metalinguistic) disjunction in the defining conditions of logical operators. And even here it plays only the role of a necessary side condition rather than something that enters the content of the rule $\mathcal{R}(M\,b, b)$. A more modern treatment of modal logic within the Popperian framework but not tied to Carnap's exposition might well do without disjunction in defining conditions in line with other logical operators.

Carnap's modal logic is S5, which can be axiomatized by the axioms K, T and 5. The same is the case for Popper's modal logic, as the following theorem shows.

Theorem 7.1 *From the definitions* (D*N*) *and* (D>) *we can prove the S5 axioms K, T and 5.*

Proof We consider axiom 5, that is, $\vdash N\,b > N\,N\,b$, which is shown as follows (K and T can be shown similarly):

1. $\vdash N\,N\,b \ \leftrightarrow\ \vdash N\,b$, from (D*N*).
2. $\vdash N\,b \ \bigvee\ \urcorner\!/ N\,b$, from (D*N*).
3. If $\vdash N\,b$, then $\vdash N\,N\,b$, from (1).
4. If $\vdash N\,N\,b$, then $N\,b \vdash N\,N\,b$, by (LW).
5. If $\urcorner\!/ N\,b$, then $(c)\,N\,b \vdash c$, hence $N\,b \vdash N\,N\,b$, by instantiating c by $N\,N\,b$.
6. Therefore $N\,b \vdash N\,N\,b$, from (2), (3) and (5).
7. Therefore $\vdash N\,b > N\,N\,b$, from (6) and (D>).

The proof can be generalized for all modal connectives *M* defined by Popper, that is, $\vdash M\,b > N\,M\,b$ holds for any *M*. □

The following result will be important in § 8, where it is used to show that the interpretation of intuitionistic negation by *I* and of dual-intuitionistic negation by *U* works.

Theorem 7.2 *Uncertainty U and impossibility I are dual modal notions.*

Proof By applications of the duality function δ (Definition 5.3). □

Each of the considered modal connectives is a logical constant in Popper's sense. We show this for *N* as an example:

[78] Carnap (1947, p. 174) mentions this explicitly for necessity.

Lemma 7.3 *N is a logical constant, that is, its rule is fully characterizing.*

Proof We have to show that the following is true:

$$((\vdash a_1 \bigvee \not 7\, a_1) \,\&\, (\vdash a_1 \leftrightarrow \vdash b) \,\&\, (\vdash a_2 \bigvee \not 7\, a_2) \,\&\, (\vdash a_2 \leftrightarrow \vdash b)) \rightarrow a_1 /\!/ a_2.$$

Assume $\vdash a_1$. From $\vdash a_1 \leftrightarrow \vdash b$ we get $\vdash b$. From $\vdash b$ and $\vdash a_2 \leftrightarrow \vdash b$ we get $\vdash a_2$. From $\vdash a_2$ we get $a_1 \vdash a_2$. Assume $\not 7\, a_1$. Hence $(c)(a_1/c)$, and by instantiating c by a_2 we get a_1/a_2. Therefore $a_1 \vdash a_2$. The proof of a_2/a_1 is similar. □

8 Bi-intuitionistic logic

The logical constants of intuitionistic and dual-intuitionistic negation are compatible in the sense that there is a logic containing both, without them collapsing into classical negation. A proof of this result was sketched by Popper (1948c, § V). We give a full proof in what follows. It consists in the exposition of the logic $\mathcal{L}_1$, for which we show that it has the following, desired properties:

1. It satisfies the rules of Basis I.
2. It contains for any two statements a and b also $a \wedge b$, $a \vee b$, $a > b$, $a \not> b$, $I\,a$ and $U\,a$.
3. It contains for every statement a its intuitionistic negation $\neg_i\, a$ and its dual-intuitionistic negation $\neg_m\, a$.
4. It satisfies all the inferential definitions of the logical connectives it contains.
5. Both $\neg_i\, a /\!/ I\,a$ and $\neg_m\, a /\!/ U\,a$ holds.
6. The duals $I\,a$ and $U\,a$ (cf. Theorem 7.2) do not collapse into each other.

Definition 8.1 The *logic* $\mathcal{L}_1$ is three-valued with truth-values d, c and r.[79] It contains a statement s with constant truth-value c. The truth-values of the compound statements of $\mathcal{L}_1$ and of s are given by the truth-tables in Table 1.4; for completeness, we add the truth-tables for N, P, L and C, which are not included in Popper (1948c).

A deducibility relation $a_1, \ldots, a_n/b$ is $\mathcal{L}_1$*-valid*, if the following conditions hold:

1. If $a_1 \wedge \ldots \wedge a_n$ has truth-value d, then b must have truth-value d as well.
2. If $a_1 \wedge \ldots \wedge a_n$ has truth-value c, then b must have either truth-value c or truth-value d.
3. If $a_1 \wedge \ldots \wedge a_n$ has truth-value r, then b can have any truth-value.

The truth-values d, c and r of $\mathcal{L}_1$ reflect Carnap's tripartition into demonstrable, factual (contingent), and refutable statements, respectively. That is, if d is read as L-true, c as factual in the sense of Carnap, and r as L-false, then the given truth-tables for $\mathcal{L}_1$ are an adequate semantics for this reading.

Lemma 8.2 $\mathcal{L}_1$ *satisfies the rules of Basis I.*

Proof That $\mathcal{L}_1$ satisfies (Rg) and (Tg) is a direct consequence of the definition of $\mathcal{L}_1$-validity. □

[79] Popper (1948c) uses the truth-values 1, 2 and 3, which are here replaced by d, c and r, respectively.

Table 1.4 The truth-tables for the logic $\mathcal{L}_1$ and for N, P, L and C.

a	b	$a \wedge b$	$a \vee b$	$a > b$	$a \not> b$
d	d	d	d	d	r
d	c	c	d	c	d
d	r	r	d	r	d
c	d	c	d	d	r
c	c	c	c	d	r
c	r	r	c	r	c
r	d	r	d	d	r
r	c	r	c	d	r
r	r	r	r	d	r

s
c

a	$I\,a$	$U\,a$
d	r	r
c	r	d
r	d	d

a	$N\,a$	$P\,a$	$L\,a$	$C\,a$
d	d	d	d	r
c	r	d	r	d
r	r	r	d	r

Lemma 8.3 *The following propositions are true:*

1. *A statement of $\mathcal{L}_1$ is demonstrable if, and only if, it has the value* d *for all valuations.*
2. *A demonstrable statement exists in $\mathcal{L}_1$.*
3. *A statement is refutable if, and only if, it has the value* r *for all valuations.*
4. *A refutable statement exists in $\mathcal{L}_1$.*
5. *It is $a /\!/ b$ if, and only if, the value of a is identical with the value of b.*

Proof (1), (3) and (5) follow directly from the definitions of demonstrability and refutability (cf. § 4.2 and § 4.3), together with the interpretation of deducibility in $\mathcal{L}_1$. A witness for (2) is the demonstrable statement $a > a$. A witness for (4) is the refutable statement $a \not> a$. □

Lemma 8.4 *The logical constants in $\mathcal{L}_1$ satisfy their respective inferential definitions* (D$\wedge$)*,* (D$\vee$)*,* (D$>$)*,* (D$\not>$)*,* (DI) *and* (DU)*.*

Proof This is only shown for U, the other cases are analogous. The characterizing rule for U is:

$$(\vdash U\,b \bigvee \dashv U\,b)\ \&\ (\dashv U\,b \leftrightarrow\ \vdash b)$$

That the left conjunct is satisfied can be seen by looking at the truth-table for U, which contains only either d or r. If $U\,a$ has the truth-value d, then it is demonstrable, and if it has the truth-value r, then it is refutable.

For the right conjunct, we consider first the part $\dashv U\,b \rightarrow\ \vdash b$, which is also satisfied: If $U\,b$ is refutable, then it has the value r, and if it has the value r, then b must have the value d, thus being demonstrable. The remaining part $\vdash b \rightarrow \dashv U\,b$ is also satisfied: If b is demonstrable, then it has the value d; hence $U\,b$ has the value r, thus being refutable. □

Lemma 8.5 *$I\,a$ and $U\,a$ do not collapse in $\mathcal{L}_1$.*

Proof This is guaranteed by the existence of the statement s with truth-value c. The value of $U\,s$ is d, and the value of $I\,s$ is r. Therefore it cannot be the case that $U\,s /\!/ I\,s$, by Lemma 8.3. □

Lemma 8.6 *From the definitions* (D$\neg_i$), (D$\neg_m$), (D$>$), (D$\not>$), (DI) *and* (DU) *we can show that* $\neg_i\, a /\!/ a > I\, a$ *and* $\neg_m\, a /\!/ U\, a \not> a$.

Proof For the proof of $\neg_i\, a /\!/ a > I\, a$ we have to show that $(c)(c \vdash b > I\, b \rightarrow c, b \vdash)$ and $(c)(c, b \vdash\ \rightarrow c \vdash b > I\, b)$, which is just an instance of (D$\neg_i$). The latter is shown as follows:

1. Assume $c, b \vdash$.
2. Therefore $c, b \vdash I\, b$, by (RW).
3. Therefore $c \vdash b > I\, b$, by (C$>$).
4. Therefore $(c)(c, b \vdash\ \rightarrow c \vdash b > I\, b)$.

The proof of $(c)(c \vdash b > I\, b \rightarrow c, b \vdash)$ is:

1. Assume $c \vdash b > I\, b$.
2. Therefore $c, b \vdash I\, b$, by (C$>$).
3. $\vdash I\, b \bigvee I\, b \vdash$, by (C$I$). We argue by cases.
4. Assume $I\, b \vdash$.
5. Therefore $c, b \vdash$, by (2), (4) and (Tg).
6. Assume $\vdash I\, b$.
7. Therefore $b \vdash$, by (CI).
8. Therefore $c, b \vdash$, by (LW).
9. Therefore $c, b \vdash$, by (3), (5) and (8).
10. Therefore $(c)(c \vdash b > I\, b \rightarrow c, b \vdash)$.

Analogously for the proof of $\neg_m\, a /\!/ U\, a \not> a$. □

Lemma 8.7 *In* $\mathcal{L}_1$ *there is for every statement* a *an intuitionistic as well as a dual-intuitionistic negation of* a. *For* $\neg_i\, a$ *we have* $\neg_i\, a /\!/ I\, a$, *and for* $\neg_m\, a$ *we have* $\neg_m\, a /\!/ U\, a$.

Proof We have $a > I\, a /\!/ I\, a$ and $U\, a \not> a /\!/ U\, a$ in $\mathcal{L}_1$. This can be checked by constructing the respective truth-tables. Using Lemma 8.6, we obtain $I\, a /\!/ \neg_i\, a$ and $U\, a /\!/ \neg_m\, a$. The modal statements $I\, a$ and $U\, a$ exist for any statement in $\mathcal{L}_1$. Therefore intuitionistic and dual-intuitionistic negations exist for any statement in $\mathcal{L}_1$. □

Theorem 8.8 *If a logic contains for any statement* a *also its intuitionistic negation* $\neg_i\, a$ *and its dual-intuitionistic negation* $\neg_m\, a$, *then these two negations do not (necessarily) collapse, that is, we do not have* $\neg_i\, a /\!/ \neg_m\, a$.

Proof The logic $\mathcal{L}_1$ is such a logic. □

Popper (1948c) thus showed that there exists a bi-intuitionistic logic. The logic $\mathcal{L}_1$ might not be very interesting in itself. Nevertheless, it is at least interesting from a historical point of view, since it is perhaps the first example of a bi-intuitionistic logic to be found in the literature after Moisil (1942), which Popper almost certainly did not have access to.[80] Moreover, it shows that already Popper had the idea of combining different logics, a topic that today is receiving considerable attention (cf. Carnielli and Coniglio, 2020).

[80] The modern discussion of bi-intuitionistic logic was initiated by Rauszer (1974).

9 The theory of quantification

Popper extended his framework of inferential definitions to a theory of quantification at the beginning of 1947. In a letter to Paul Bernays of 19 October 1947 (this volume, § 21.6), he wrote:

> The first important result which I had finished about one week after I saw you, was the extension of the method of $a/b \wedge c \;\leftrightarrow\; a/b \;\&\; a/c$ to quantification.

The meeting that Popper refers to probably took place in Zürich on 11 or 12 April 1947[81], where Popper met Bernays in order to discuss the possibility of publishing a joint article on logic. The manuscript (this volume, Chapter 14) for this unpublished article does not have a title; in a letter to Bernays of 3 March 1947 (this volume, § 21.3), Popper suggests the title "On Systems of Rules of Inference", noting that "[t]he title is not very good, but so far I could not think of a better one".

Although they did not publish this manuscript, Popper's results found their way into several of his published articles. The most extensive discussion of these results can be found in Popper (1947c, § 7 and § 8). Additionally, there is an important footnote in Popper (1948c), an alternative axiomatization in Popper (1947d), and a very short but clear summary of his treatment of quantification in Popper (1949a). We follow the presentation of Popper (1947c) but refer to some modifications which can be found in his other articles. Some modifications of his view on quantification were only discussed in hitherto unpublished correspondence, which we will discuss in § 9.4.

Popper's theory of quantification underwent significant modifications over the course of his published articles, subsequent corrections to those articles, and in unpublished correspondence with other logicians. We present what we consider to be his most mature view on these matters, taking unpublished material into account.

Popper uses the terms "theory of quantification" or "quantification theory" instead of "first-order logic". At first, he extends his concept of object language to include open statements and his deducibility relation to range over open statements. He then adds a substitution operation which replaces free variables by other free variables, and gives rules and postulates which characterize this substitution operation. We discuss his definitions of the auxiliary concepts of identity and non-free-occurrence of a variable in a statement and, finally, his definitions of the quantifiers.

9.1 Formulas, name-variables and substitution

As explained in § 3, for propositional logic Popper considers pairs

[81] Bernays writes to Popper: "[. . .] nothing stands, as far as I can see, in the way of us seeing each other on the 11th of April in Zürich; I will certainly also be available in the midmorning of the 12th. I am looking forward to the receipt of the concept you promised me, – also with regards to the possible joint publication." (This volume, § 21.4.)

$$(\mathcal{L}; a_1, \ldots, a_n/b)$$

of an object language $\mathcal{L}$ and a deducibility relation $/$, axiomatized by a basis consisting of the rules (Rg) and (Tg). Each element of the object language $\mathcal{L}$ is presumed to be a statement, that is, something which has a truth value.

The first modification Popper makes in order to treat quantification is to consider quadruples

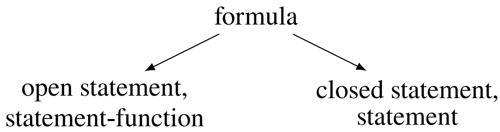
$$(\mathcal{L}; \mathcal{P}; a_1, \ldots, a_n/b; a\binom{x}{y})$$

consisting of a set $\mathcal{L}$ of *formulas*, a set $\mathcal{P}$ of *name-variables* (or pronouns), a *deducibility relation* on $\mathcal{L}$ and a *substitution operation*

$$a\binom{x}{y}$$

which substitutes the name-variable y for the name-variable x in the formula a. Variables $a, b, \ldots$ now range over *formulas* in $\mathcal{L}$, and variables $x, y, \ldots$ range over name-variables in $\mathcal{P}$.

Formulas can either be *open statements* (also called *statement-functions*) or *closed statements* (also called *statements*):



An example of an open statement given by Popper is "He is a charming fellow", which can be turned into a closed statement by replacing the name-variable "He" with the name "Ernest's best friend". Popper explicitly remarks that open statements do not have a truth value on their own; an open statement cannot be considered to be true or false.

The deducibility relation is axiomatized by the same rules (Rg) and (Tg) as in the case of propositional logic, but it now ranges over arbitrary formulas, not just closed statements. For example, Popper says that one can validly deduce the open statement "He is an excellent physician" from the open statement "He is not only a charming fellow but an excellent physician".

The new substitution operation is characterized by the four postulates (PF1) to (PF4) and the six primitive rules of derivation (6.1) to (6.6), which we present in a slightly simplified form as follows.

(PF1) $\mathcal{L} \cap \mathcal{P} = \emptyset$

(PF2) If $a \in \mathcal{L}$ and $x, y \in \mathcal{P}$, then $a\binom{x}{y} \in \mathcal{L}$

(PF3) For all $a \in \mathcal{L}$ there exists an $x \in \mathcal{P}$ such that for all $y \in \mathcal{P}$: $a\binom{x}{y} /\!/ a$

(PF4) There exist $a \in \mathcal{L}$ and $x, y \in \mathcal{P}$: $a/a\binom{x}{y} \to t/f$

Note that two kinds of metalinguistic quantifiers are used: there are universal and existential quantifiers ranging over statements $a \in \mathcal{L}$ and universal and existential quantifiers ranging over name-variables $x \in \mathcal{P}$. We only use symbols for the respective metalinguistic universal quantifiers in the following; (a) means "for all statements a" and (x) means "for all name-variables x".

The postulates (PF1) and (PF2) are, in a way, only about the correct grammatical use of formulas and name-variables. The postulate (PF3) says that for every formula there is some name-variable not occurring in it. This is obvious if the set of name-variables is considered to be infinite, and if each formula is a finite object which can only mention a finite number of name-variables. The postulate (PF4), which Popper considers to be optional, excludes degenerate systems in which only one object exists. Take, for example, the open statement a to be "x likes the current weather". The deducibility of "y likes the current weather" from "x likes the current weather" only leads to a contradiction if there are at least two persons to whom x and y can refer. Postulate (PF4) was also discussed in correspondence between Popper and Carnap (cf. this volume, § 23.10 and § 23.11).

The six primitive rules of inference are given below. We will not discuss them in detail, but the reader may check that they are valid for a concrete formalized object language and a substitution operation for that language.

$$\text{If, for every } z,\ a /\!/ a\tbinom{y}{z} \text{ and } b /\!/ b\tbinom{y}{z}, \text{ then } a /\!/ b \rightarrow a\tbinom{x}{y} /\!/ b\tbinom{x}{y} \tag{6.1}$$

$$a /\!/ a\tbinom{x}{x} \tag{6.2}$$

$$\text{If } x \neq y, \text{ then } \left(a\tbinom{x}{y}\right)\tbinom{x}{z} /\!/ a\tbinom{x}{y} \tag{6.3}$$

$$\left(a\tbinom{x}{y}\right)\tbinom{y}{z} /\!/ \left(a\tbinom{x}{z}\right)\tbinom{y}{z} \tag{6.4}$$

$$\left(a\tbinom{x}{y}\right)\tbinom{z}{y} /\!/ \left(a\tbinom{z}{y}\right)\tbinom{x}{y} \tag{6.5}$$

$$\text{If } w \neq x,\ x \neq u \text{ and } u \neq y, \text{ then } \left(a\tbinom{x}{y}\right)\tbinom{u}{w} /\!/ \left(a\tbinom{u}{w}\right)\tbinom{x}{y} \tag{6.6}$$

The rules (6.1) to (6.6) characterize substitution as a structural operation; this is similar to how the basis characterizes commas in sequences of statements. It is remarkable that Popper here presents an algebraic treatment of substitution, which can be compared to the theory of explicit substitution developed much later (cf., e.g., Abadi et al., 1991).

As an intriguing sidenote, Popper compares the definition of substitution by the rules (6.1) to (6.6) to the definition of conjunction via the inferential definition (D∧). He writes:

> These six primitive rules determine the meaning of the symbol "$a\tbinom{x}{y}$" in a way precisely analogous to the way in which, say, [rule (D∧) determines] the meaning of conjunction [. . .] with the help of the concept of derivability "/". (Popper, 1947c, p. 226)

However, Popper's rules for substitution cannot be brought into the form of an inferential definition of an operator of the object language. Hence, substitution cannot have the status of a logical constant according to Popper's criterion for logicality; his rules for substitution do not have the form of characterizing rules (and, consequently, no fully characterizing rules can be given either). Indeed, Popper also explains substitution as follows:

> The notation
>
> $$\text{“}a\tbinom{x}{y}\text{”}$$
>
> will be used as a (variable) metalinguistic name of the statement which is the result of substituting, in the statement a (open or closed), the variable y for the variable x, wherever it occurs. $a\tbinom{x}{y}$ is identical with a if x does not occur in a. (Popper, 1947d, p. 1216)

Popper's rules for substitution may thus be viewed as an implicit characterization of a metalinguistic operation, and not as an inferential definition of a logical constant for object languages.

Next we discuss some auxiliary concepts defined with the help of both the deducibility relation and the substitution operation.

9.2 Non-dependence, identity and difference

If we work with some inductively defined formal object language, then we can easily specify the set of free variables of a formula by recursion on the structure of that formula. This possibility is excluded in Popper's approach, which is not restricted to formal languages. Popper therefore introduces the expression

$$a_{\hat{x}}$$

which can be read as "x does not occur among the free variables in a". Popper himself expresses this as "a does not depend on x", "*a-without-x*" and "x does not occur relevantly in a". The formula a does not depend on x if, and only if, substitution of some name-variable y for x does not change the logical strength of a. That is:

(D $a_{\hat{x}}$) $$a /\!/ a_{\hat{x}} \ \leftrightarrow\ \text{for every } y\text{: } a /\!/ a\tbinom{x}{y}$$

The second concept Popper defines with the help of deducibility and substitution is *identity*. As Popper (1947c, p. 227f., fn 24) notes, one first has to extend the object language $\mathcal{L}$ to incorporate formulas of the form $Idt(x, y)$; this is achieved by the postulate

(P *Idt*) If x and y are name variables, then $Idt(x, y)$ is a formula.

In addition, the characterizing rules for substitution have to be extended by rules of the form

(A) $$(Idt(x, y))\tbinom{x}{z} /\!/ Idt(z, y)$$

(B) $$(Idt(x, y))\tbinom{y}{z} /\!/ Idt(x, z)$$

(C) $$\text{If } x \neq u \neq y\text{, then } Idt(x, y)\tbinom{u}{z} /\!/ Idt(x, y)$$

With these preliminaries, Popper defines identity using the following idea:

> The identity statement "$Idt(x, y)$" can be defined as the weakest statement strong enough to satisfy the [...] formula [...]
>
> $$\text{“}Idt(x, y), a(x)/a(y)\text{”}$$
>
> that is to say, the formula corresponding to what Hilbert-Bernays call the second identity axiom. (Hilbert-Bernays's first axiom follows from the demand that the identity statement must be the *weakest* statement satisfying this formula.) (Popper, 1949a, p. 725f.)

Popper here refers to the identity axioms J_1 and J_2 of Hilbert and Bernays (1934, p. 164):

$$(J_1) \qquad a = a$$

$$(J_2) \qquad a = b \rightarrow (A(a) \rightarrow A(b))$$

This justifies the following definition of *identity* $Idt(x, y)$:

$$\text{(D } Idt\text{)} \quad a/\!/Idt(x, y) \leftrightarrow \text{(for every } b \text{ and } z\text{: } ((b/\!/b_{\grave{x}} \;\&\; b/\!/b_{\grave{y}}) \rightarrow a, b\tbinom{z}{x}/b\tbinom{z}{y}) \;\&$$
$$((\text{for every } c \text{ and } u\text{: } ((c/\!/c_{\grave{x}} \;\&\; c/\!/c_{\grave{y}}) \rightarrow b, c\tbinom{u}{x}/c\tbinom{u}{y}))) \rightarrow b/a))$$

Popper (1948c, p. 323f., fn 11) expands on the definition of identity $Idt(x, y)$ in order to illustrate his method of obtaining a relatively simple characterizing rule from an explicit definition that is the weakest (or strongest) statement satisfying a certain condition or axiom. He first introduces the following abbreviating notation:

$$a/\!/a_{\grave{x}\grave{y}} \leftrightarrow (w)(a/\!/a\tbinom{x}{w} \;\&\; a/\!/a\tbinom{y}{w}).$$

Using this abbreviation, he defines $Idt(x, y)$ as the weakest statement strong enough to imply the axiom J_2:

$$\text{(D } Idt^{\dagger}\text{)} \quad a/\!/Idt(x, y) \leftrightarrow$$
$$(b)(z)((b/\!/b_{\grave{x}\grave{y}} \rightarrow a, b\tbinom{z}{x}/b\tbinom{z}{y}) \;\&\; (((c)(u)(c/\!/c_{\grave{x}\grave{y}} \rightarrow b, c\tbinom{u}{x}/c\tbinom{u}{y}))) \rightarrow b/a))$$

This explicit definition, which is an abbreviated version of (D Idt), can be replaced by a definition that corresponds to the following characterizing rule:

$$\text{(C } Idt^{\ddagger}\text{)} \qquad a/Idt(x, y) \leftrightarrow (b)(z)(b/\!/b_{\grave{x}\grave{y}} \rightarrow a, b\tbinom{z}{x}/b\tbinom{z}{y})$$

This can be seen by instantiating a in (D $Idt^{\dagger}$) with $Idt(x, y)$ in order to obtain

$$(b)(z)((b/\!/b_{\grave{x}\grave{y}} \rightarrow Idt(x, y), b\tbinom{z}{x}/b\tbinom{z}{y}) \;\&$$
$$(((c)(u)(c/\!/c_{\grave{x}\grave{y}} \rightarrow b, c\tbinom{u}{x}/c\tbinom{u}{y}))) \rightarrow b/Idt(x, y))).$$

The left conjunct gives the direction from left to right in (C $Idt^{\ddagger}$), and the right conjunct gives the direction from right to left.

Finally, *difference* $\mathit{Dff}(x, y)$ is simply defined as the classical negation of identity:

$$\text{(D } \mathit{Dff}\text{)} \qquad a /\!/ \mathit{Dff}(x, y) \;\leftrightarrow\; a /\!/ \neg_k \mathit{Idt}(x, y)$$

It is interesting to see that Popper chose to treat occurrence of free variables and identity as defined notions, rather than to class them with substitution and deducibility among the primitive notions characterized by the basis. We will see in § 9.4 that Popper probably revised this position later.

9.3 Quantification

Inferential definitions of universal and existential quantification are introduced in Popper (1947c), to which he later published a list of corrections and additions (cf. Popper, 1948e), which we take into account here. Popper's aim is not to develop and analyze the theory of quantification, that is, first-order logic, but to show that his approach to quantification is at least on a par with other proposed treatments of quantification. He therefore restricts himself to stating his definitions of the quantifiers and to deriving some simple conclusions, but he does not formally develop a meta-theory of quantification. He does not, for example, discuss the completeness of his rules, the difference between classical and constructive interpretations of the existential quantifier, or the relation to models of his system.

Later, Popper (1949a) gives the clearest explanation of what intuition his inferential definition of *universal quantification* is supposed to capture. He writes:

> The result of universal quantification of a statement *a* can be defined as the weakest statement strong enough to satisfy the law of specification, that is to say, the law "what is valid for all instances is valid for every single one". Popper (1949a, p. 725)

Presupposing his rules of substitution, and writing Ax for the universal quantifier, Popper's inferential definition and the characterizing rule for universal quantification are the following:

$$\text{(D7.1)} \qquad a_{\grave{y}} /\!/ Axb_{\grave{y}} \;\leftrightarrow\; (\text{for every } c_{\grave{y}}\text{: } c_{\grave{y}} / a_{\grave{y}} \;\leftrightarrow\; c_{\grave{y}} / b_{\grave{y}} \tbinom{x}{y})$$

$$\text{(C7.1)} \qquad \text{For every } c_{\grave{y}}\text{: } c_{\grave{y}} / Axb_{\grave{y}} \;\leftrightarrow\; c_{\grave{y}} / b_{\grave{y}} \tbinom{x}{y}$$

In order to see how more ordinary presentations of the rules for universal quantification follow from these inferential definitions, we can compare them to the more familiar rules of the (intuitionistic) sequent calculus (writing $\varphi[x/y]$ for the result of substituting y for x in the formula φ):

$$\frac{\Gamma, \varphi[x/t] \vdash \psi}{\Gamma, \forall x \varphi \vdash \psi}\ (\forall \vdash) \qquad\qquad \frac{\Gamma \vdash \varphi[x/y]}{\Gamma \vdash \forall x \varphi}\ (\vdash \forall)$$

$$\frac{\Gamma, \varphi[x/y] \vdash \psi}{\Gamma, \exists x\varphi \vdash \psi}\ (\exists\vdash) \qquad\qquad \frac{\Gamma \vdash \varphi[x/t]}{\Gamma \vdash \exists x\varphi}\ (\vdash\exists)$$

with the variable condition that y does not occur free in the conclusion of $(\vdash\forall)$ and $(\exists\vdash)$.

For example, by instantiating (C7.1) with $Axb_{\hat{y}}$ and by using the rules (Tg) and (Rg) from the basis, we obtain the following rule

$$a, b_{\hat{y}}\tbinom{x}{y}/c \rightarrow a, Axb_{\hat{y}}/c$$

which can easily be seen to be a variant of the rule $(\forall\vdash)$ where the name-variable y takes the role of the term t. Similarly, by instantiating (C7.1) with $c_{\hat{y}}$ and reading the bi-implication from right to left we obtain the following rule, which corresponds to the rule $(\vdash\forall)$ with the variable condition that y does not occur relevantly in c:

$$c_{\hat{y}}/b_{\hat{y}}\tbinom{x}{y} \rightarrow c_{\hat{y}}/Axb_{\hat{y}}.$$

As was the case for universal quantification, Popper gives the clearest explanation of the inferential definition of *existential quantification* not in Popper (1947c), but in Popper (1949a, p. 725):

> The result of existential quantification of the statement a can be defined as the strongest statement weak enough to follow from every instance of a.

The inferential definition and the characterizing rule for the *existential quantifier Ex* are

(D7.2) $\quad a_{\hat{y}}//Exb_{\hat{y}} \leftrightarrow$ (for every $c_{\hat{y}}$: $a_{\hat{y}}/c_{\hat{y}} \leftrightarrow b_{\hat{y}}\tbinom{x}{y}/c_{\hat{y}}$)

(C7.2) $\quad$ For every $c_{\hat{y}}$: $Exb_{\hat{y}}/c_{\hat{y}} \leftrightarrow b_{\hat{y}}\tbinom{x}{y}/c_{\hat{y}}$

To elucidate what they mean, we derive some more familiar rules for the existential quantifier from its characterizing rule. Instantiating (C7.2) with $Exb_{\hat{y}}$ and using the rules of the basis we can obtain the rule

$$a/b_{\hat{y}}\tbinom{x}{y} \rightarrow a/Exb_{\hat{y}}$$

which corresponds to the sequent calculus rule $(\vdash\exists)$; and by instantiating (C7.2) with $c_{\hat{y}}$ and reading the bi-implication from right to left, we obtain the following rule, which corresponds to $(\exists\vdash)$:

$$b_{\hat{y}}\tbinom{x}{y}/c_{\hat{y}} \rightarrow Ex_{\hat{y}}/c_{\hat{y}}.$$

Popper does not consider the explicit definitions (D7.1) and (D7.2) to be improvements compared to the characterizing rules. They are given to show that universal and existential quantification can be defined using only his basis and the rules (6.1) to (6.6). He notices that these rules are not as simple as the rules of his basis, for example. But he points out that the concept of "$a_{\hat{x}}$" can be avoided in these definitions (cf. Popper, 1947c, p. 230, fn 26)[82]. Assuming $x \neq y$, one can use instead:

(7.1*) $$a\binom{y}{x}/Ax(b\binom{y}{x}) \leftrightarrow a\binom{y}{x}/b\binom{x}{y}$$

(7.2*) $$Ex(a\binom{y}{x})/b\binom{y}{x} \leftrightarrow a\binom{x}{y}/b\binom{y}{x}$$

(D7.1*) $$a\binom{y}{x}/\!/Ax(b\binom{y}{x}) \leftrightarrow (\text{for every } c\text{: } c\binom{y}{x}/a\binom{y}{x} \leftrightarrow c\binom{y}{x}/b\binom{x}{y})$$

(D7.2*) $$a\binom{y}{x}/\!/Ex(b\binom{y}{x}) \leftrightarrow (\text{for every } c\text{: } a\binom{y}{x}/c\binom{y}{x} \leftrightarrow b\binom{x}{y}/c\binom{y}{x})$$

He conceives his rules of quantification to be less complicated than those given by Hilbert and Ackermann (1928) or those given by Quine (1940, § 15), and he emphasizes that his rules in the end make use of only one logical concept, namely that of deducibility / as characterized by his basis.

9.4 An unfortunate misunderstanding

Popper (1947c, § 8) introduces a distinction between rules of derivation and rules of demonstration or proof, something considered very important by him. If he had not stopped publishing in logic, he would very likely have developed these ideas in more detail. For example, among his unpublished manuscripts there are two which are entitled "Derivation and Demonstration in Propositional and Functional Logic" (this volume, § 18.1) and "The Propositional and Functional Logic of Derivation and of Demonstration" (this volume, § 18.3), as well as another untitled manuscript (this volume, § 18.2), which also deals with this distinction. In a letter to John C. Eccles of 10 November 1946 Popper (1946b) explicitly mentions that he is "writing a paper on 'Derivation & Demonstration'".

In order to illustrate this distinction between derivation and demonstration we have to make use of the concept of relative demonstrability $\vdash$ (cf. § 4.4). If we specialize this concept to no formula on the left hand side and exactly one formula on the right hand side, we obtain the definition of a *provable formula* a: $\vdash a$. Consider now the following two formulas of the metalanguage:

(8.4) $$a/b \rightarrow (\vdash a \rightarrow \ \vdash b)$$

(8.4′) $$(\vdash a \rightarrow \ \vdash b) \rightarrow a/b.$$

Popper correctly remarks that while the first formula is valid, the second is not. This can be seen, as Popper claims, by instantiating a by a non-tautological consistent formula and b by a contradictory one. Using a more modern terminology, we may say that $\vdash a \rightarrow \ \vdash b$ expresses that the rule leading from a to b is *admissible*, whereas a/b expresses that this rule is *derivable*. The first formula (8.4) then expresses that the derivability of a rule implies its admissibility, and the second formula (8.4′) that the admissibility of a rule implies its derivability. The counterexample to the second formula says that admissibility does not in general imply derivability. However,

[82] Added in the corrections and additions (cf. Popper, 1948e).

it should be noted that the value of this counterexample is limited, as it does not apply to rules closed under substitution (that is, to rules identifiable with the set of their substitution instances), which is normally considered a requirement for rules to constitute a logic. In classical propositional logic, for rules closed under substitution, admissibility does imply derivability (cf. Belnap, Leblanc, and Thomason, 1963, Mints, 1976, and Humberstone, 2011).[83]

Rules of the form $\vdash a \rightarrow\ \vdash b$ are called *rules of proof* or *rules of demonstration*, in contrast to *rules of derivation* expressed by a/b (there can be more than one premise). Now Popper correctly observes that the rules of a system like Principia Mathematica (Whitehead and Russell, 1925–1927) are rules of proof and not rules of derivation. For example, the rule of modus ponens takes the form

$$\vdash a \rightarrow (\vdash a > b \rightarrow\ \vdash b)$$

rather than the form

$$a, a > b / b.$$

What Popper intends to formulate here, and in particular in his definition of a purely derivational system of primitive rules (cf. Popper, 1947c, definition (D8.1)), is, in our opinion, a criterion that allows to distinguish between formulations of logic based on axioms and rules of proof, such as those of Frege (1879), Hilbert and Bernays (1934, 1939) and Whitehead and Russell (1925–1927) on the one hand, and formulations of logic based on derivation alone, such as those of Gentzen (1935a,b) and his own, on the other hand.

Unfortunately, he applied this analysis of rules of derivation and rules of proof to the systems of Carnap as well as of Hilbert and Bernays in a way that does not take account of an important difference between his system and theirs. Popper (1947c, p. 232) warns that there are rules of proof such as

$$\vdash a_{\grave{y}} \ \leftrightarrow\ \vdash a_{\grave{y}}\binom{x}{y} \tag{8.5}$$

which are valid, whereas the corresponding rule of derivation

$$a_{\grave{y}} / a_{\grave{y}}\binom{x}{y}$$

is invalid. He continues:

> Now all the mistakes here warned against do actually vitiate some otherwise very excellent books on modern logic – an indication that the distinction between (conditional) rules of proof or rules of demonstration on the one side and rules of derivation on the other cannot be neglected without involving oneself in contradictions. (Popper, 1947c, p. 233)

Both Carnap and Bernays responded to Popper's criticism of their respective system in

[83] Formal systems not closed under substitution have many unexpected properties – in intuitionistic logic, for example, that the law of double negation elimination can be validated; cf. de Campos Sanz, Piecha, and Schroeder-Heister (2014) and Piecha, de Campos Sanz, and Schroeder-Heister (2015).

correspondence (cf. this volume, § 23.10 and § 23.11). Bernays (this volume, § 21.7) writes:

> Now I have to comment upon your critique of the formulation of the all-schema, as it is given in the "Grundlagen der Math." [Hilbert and Bernays, 1934]. I think of the passage p. 232–233 of your New Foundations. [. . .] The contradiction that you derive, starting with the schema $a_{\hat{x}} > b / a_{\hat{x}} > Axb$ which you criticize, does not arise in the formalism of the "Grundl. der Math.", because the implication plays another role here than the "hypothetical" in your formalism.

We note that Popper's letter to Carnap of 24 December 1947 (this volume, § 23.11) is also interesting for the fact that it contains an expansion of his theory of quantification by presenting several logical laws of classical first-order logic.

Popper later revised his understanding of the interaction of substitution and deducibility. While his definitions are formulated using the weaker notion of interdeducibility, he then considered it necessary to use the stronger notion of identity of statements (Popper, 1974c, p. 171, endnote 198):

> The mistake was connected with the rules of substitution or replacement of expressions: I had mistakenly thought that it was sufficient to formulate these rules in terms of interdeducibility, while in fact what was needed was identity (of expressions). To explain this remark: I postulated, for example, that if in a statement *a*, two (disjoint) subexpressions *x* and *y* are both, wherever they occur, replaced by an expression *z*, then the resulting expression (provided it is a statement) is interdeducible with the result of replacing first *x* wherever it occurs by *y* and then *y* wherever it occurs by *z*. What I should have postulated was that the first result is identical with the second result. I realized that this was stronger, but I mistakenly thought that the weaker rule would suffice. The interesting (and so far unpublished) conclusion to which I was led later by repairing this mistake was that there was an essential difference between propositional and functional logic: while propositional logic can be constructed as a theory of sets of statements, whose elements are partially ordered by the relation of deducibility, functional logic needs in addition a specifically morphological approach since it must refer to the subexpression of an expression, using a concept like identity (with respect to expressions). But no more is needed than the ideas of identity and subexpression; no further description especially of the shape of the expressions.[84]

Concerning possible future work on logic, Popper states in his reply of 13 June 1948 (this volume, § 21.8) to Bernays's letter of 12 May 1948 (this volume, § 21.7):

> I have also a number of new results – but I do not believe that I will ever dare again to publish something (except, maybe, an infinite sequence of corrections to my old publications)!

[84] Cf. the remarks in a letter to Schroeder-Heister of 19 August 1982 (this volume, § 32.5).

However, this modest remark should not be taken too literally in light of Popper's later contributions to logic reprinted in this volume (Chapters 9–11), and also in light of his work on Boolean algebra and probability theory not presented here (cf. Preface). Even Popper's (1949a) logical contribution to the Tenth International Congress of Philosophy shortly afterwards in Amsterdam[85] carries the boldest and most provocative of his titles: "The Trivialization of Mathematical Logic". Although he may have been embarassed by technical deficiencies as well as unfortunate and misleading formulations in his exposition[86], which naturally surfaced in discussions with outstanding mathematical logicians such as Bernays, Popper was certainly aware that he was making an original conceptual contribution towards the foundations of deductive logic, something confirmed by the later development of inferentialism and proof-theoretic semantics.

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[85] From 11 to 18 August 1948. Besides, Popper was also invited to give a plenary talk on social science, entitled "Prediction and Prophecy and their Significance for Social Theory"; cf. Popper (1949d). Bernays and Popper had long planned to meet there in person; cf. this volume, § 21.9 and § 21.11.

[86] Cf. Popper's letter to Schroeder-Heister of 19 August 1982 (this volume, § 32.3).

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