

Constructive Logics: Proof-Theoretic Semantics and Dialogue Semantics

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Preface

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1 Constructive logics

In constructive logics one investigates principles of deductive reasoning that are based on certain constructivistic approaches to mathematics. One such approach goes back to L. E. J. Brouwer (1881–1966) and A. Heyting (1898–1980), among others. This form of constructivism is called *intuitionism*. Essential aspects are:

intuitionism

- (i) Mental constructions are primary in mathematics. It is not about formal operations with symbols of a language of mathematics. The latter is just an auxiliary means to communicate our mental constructions.
- (ii) The view that mathematical statements are true or false independently of our knowledge about them is regarded as being without meaning. A mathematical statement is true, if we have a proof of it; it is false, if we can show that the assumption that there is a proof of the statement leads to a contradiction. Thus we cannot claim for arbitrary statements that they are true or false. Consequently, *tertium non datur* $A \vee \neg A$ does not hold in general; it can hold at best only for finite domains.
- (iii) Intuitionism is an opposite standpoint to platonism: In mathematics one does not discover truths about mathematical objects existing independently from us; these objects are rather created by us. It is also possible to investigate constructions that do not terminate.

Besides intuitionism there are other forms of constructivism, which we do not take into consideration here; for an overview see (Bridges and Palmgren, 2018).

In the following, we present some results on intuitionistic logic, where we presuppose that the meaning of the logical constants is given along the lines of the Brouwer–Heyting–Kolmogorov interpretation (Section 1.2). Using proof-theoretic methods (Section 1.3) we then discuss the two notions of derivable and admissible rules (Sections 1.4 and 1.5), indicate certain connections to computability (Section 1.6), and we show some results on the relation between intuitionistic and classical logic (Section 1.7). Afterwards we introduce Kripke semantics for intuitionistic logic (Section 1.8). Throughout, we will restrict ourselves to the propositional fragment of intuitionistic logic.

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1.1 Weak counterexamples

To begin with we consider two examples that illustrate why certain laws of classical logic have to be rejected when a constructivistic understanding of the logical constants is presupposed.

Example. We consider the statement

There are two irrational numbers x and y , such that x^y is rational.

It can be proved easily by arguing classically as follows.

$\sqrt{2}$ is irrational, and by *tertium non datur* we have: $\sqrt{2}^{\sqrt{2}}$ is rational or it is not rational, i.e., irrational. We consider both cases:

- (i) Assume $\sqrt{2}^{\sqrt{2}}$ is rational. We let $x = \sqrt{2}$ and $y = \sqrt{2}$, such that $x^y = \sqrt{2}^{\sqrt{2}}$, which is a rational number by assumption.
- (ii) Assume $\sqrt{2}^{\sqrt{2}}$ is irrational. We let $x = \sqrt{2}^{\sqrt{2}}$ and $y = \sqrt{2}$. Then $x^y = \left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}} = (\sqrt{2})^2 = 2$, which is rational.

However, the proof given in the example is not a *constructive* proof, since we cannot present two numbers x and y , such that x^y is rational. Under a constructivistic understanding of the considered existential statement, where the existential quantifier is interpreted as “it can be constructed”, we have thus not given a satisfying proof of that statement.

Example. We consider the conjecture p

There are infinitely many twin primes, i.e., prime numbers n , such that also $n + 2$ is a prime number.

This conjecture has not been decided yet. That is, we neither have a proof of p nor do we have a proof of $\neg p$. We therefore cannot claim that $p \vee \neg p$ holds.

This is a so-called *weak counterexample* for *tertium non datur*. From the constructivistic point of view, *tertium non datur* $A \vee \neg A$ says that for any statement A we have a proof of A or a proof of $\neg A$, i.e., a construction which transforms a hypothetical proof of A into a proof of the absurdity \perp . But then we would be in the position to decide for any statement whether it holds or not. But an example like the statement “There are infinitely many twin primes”, whose validity has not been decided yet, shows that this is not the case.

This is only a *weak* counterexample, since *tertium non datur* has not been refuted, i.e., the assumption of *tertium non datur* has not been shown to lead to absurdity. It has only been shown that *tertium non datur* is not an acceptable logical principle from the constructivistic point of view.

Moreover, it is impossible (from a constructivistic or intuitionistic point of view) to refute *tertium non datur* by finding some statement A such that $\neg(A \vee \neg A)$ holds, since $\neg\neg(A \vee \neg A)$ holds intuitionistically for all statements A .

1.2 The BHK-interpretation

The meaning of the logical constants \wedge (conjunction), \vee (disjunction), \rightarrow (implication), \perp (*falsum*, absurdity) and \neg (negation) shall be explained more precisely by the

following *proof interpretation* or *Brouwer–Heyting–Kolmogorov interpretation* (short: *BHK-interpretation*; cf. Heyting, 1971):

BHK-interpretation

- (H1) a is a proof of $A \wedge B$ iff a is a pair $\langle b, c \rangle$, such that b is a proof of A , and c is a proof of B .
- (H2) a is a proof of $A \vee B$ iff a is a pair $\langle b, c \rangle$, such that $b \in \{0, 1\}$ and c is a proof of A , if $b = 0$, and c is a proof of B , if $b = 1$.
- (H3) a is a proof of $A \rightarrow B$ iff a is a construction that transforms any proof b of A into a proof $a(b)$ of B .
- (H4) There is no proof a of \perp . A proof a of $\neg A$ is a construction that transforms any hypothetical proof b of A into a proof $a(b)$ of \perp .

- Remarks.**
- (i) The BHK-interpretation of the logical constants is not a well-founded inductive definition of “is a proof of A ”, since a base clause defining this notion for all atomic formulas is missing. The BHK-interpretation is rather an informal explication of the meaning of the logical constants.
 - (ii) The notion of construction can be understood more or less broadly. Intended is an understanding as algorithm or computable function.
 - (iii) It is usually presupposed that a is a proof of a formula A if and only if a is a proof of arbitrary instances of A . Such a requirement is stated explicitly in Heyting’s (1971, p. 103) presentation of the BHK-interpretation.
 - (iv) In clause (H4) the *falsum* \perp is used as a symbol for an arbitrary contradiction. (In the language of arithmetic this could be the statement $0 = 1$, for example.)
 - (v) For a critical discussion of the BHK clause (H3) for implication cf. (de Campos Sanz and Piecha, 2014).

Examples. The following formulas are valid under the BHK-interpretation:

- (i) $A \rightarrow (B \rightarrow A)$: We have to find a construction c that transforms a proof a of A into a proof of $B \rightarrow A$. For a given proof a of A the construction $c(b) = a$ is what we are looking for; it maps each proof b of B to the proof a of A .
- (ii) $(A \wedge B) \rightarrow A$: Let $\langle a, b \rangle$ be a proof of $A \wedge B$. Then the construction c , where $c(a, b) = a$ (i.e., the construction that projects to the first of two arguments), transforms the proof of $A \wedge B$ into a proof of A . By clause (H3), c is a proof of $(A \wedge B) \rightarrow A$.
- (iii) $\perp \rightarrow A$: Since \perp has no proof, any function (e.g., the identity $c(a) = a$) can be taken as a construction that transforms a hypothetical proof of \perp into a proof of A . (Note that the domain of such a function is always empty.)

However, $A \vee \neg A$ is not valid: By clause (H2), $A \vee \neg A$ means that we either have a proof of A or a proof of $\neg A$, for any statement A . But then for example the twin prime conjecture would be decided, which, however, is not the case. The *tertium non datur* can thus not hold in general.

Also note that we cannot validate something like $(A \wedge B) \rightarrow D$ by a construction $c(a, b) = d$, where d is supposed to be a proof of the formula D . The problem is that d would have to be a proof for each instance of D , including \perp , for example, for which there is no proof by clause (H4).

1.3 The calculus of natural deduction

The BHK-interpretation can be used to justify the inference rules of the calculus NI of natural deduction for intuitionistic logic. As examples we consider the rules for conjunction and implication:

- (i) Clause (H1) justifies the conjunction introduction rule, when read from right to left:

$$\frac{\mathcal{D}_b \quad \mathcal{D}_c}{\frac{A \quad B}{A \wedge B}}$$

When read from left to right, (H1) justifies the pair of conjunction elimination rules:

$$\frac{\mathcal{D}_a}{\frac{A \wedge B}{A}} \quad \frac{\mathcal{D}_a}{\frac{A \wedge B}{B}}$$

(where the left rule corresponds to case b , and the right rule corresponds to case c).

- (ii) Clause (H3) justifies the implication introduction rule, when read from right to left: Suppose we have shown B directly or by (possibly repeatedly) using the assumption A . Then this means that we have found a construction that transforms a (hypothetical) proof of A into a proof of B . By clause (H3) this is a proof of the implication $A \rightarrow B$, which no longer depends on assumptions A :

$$\frac{[A] \quad \mathcal{D}_{a(b)}}{\frac{B}{A \rightarrow B}}$$

When read from left to right, (H3) justifies the rule of implication elimination (i.e., *modus ponens*): Suppose we have shown $A \rightarrow B$. Then this means that we have found a construction that transforms proofs of A into a proof of B . If in addition we have shown A , then we obtain by an application of this construction to A the statement B :

$$\frac{\mathcal{D}_a \quad \mathcal{D}_b}{\frac{A \rightarrow B \quad A}{B}}$$

- (iii) We have already seen that by clause (H4) the principle *ex falso* $\perp \rightarrow A$ is valid under the BHK-interpretation. The corresponding rule is

$$\frac{\perp}{A} (\perp)$$

Note that the rule (\perp) does not allow for the discharge of assumptions $\neg A$, in contradistinction to the classical rule of *reductio ad absurdum*

$$\frac{[\neg A] \quad \frac{\perp}{A} (\perp)_c}{A}$$

The rule (\perp) is therefore weaker than the rule $(\perp)_c$.

However, the classical rule $(\perp)_c$ cannot be justified by the BHK-interpretation, since it allows to show $A \vee \neg A$, which is not valid under the BHK-interpretation.

Remarks. (i) In the following we use *proposition letters* (also called *proposition variables*) p, q, r, \dots *proposition letters*

We refer to the *set of proposition letters* as $PV := \{p, q, r, \dots\}$.

PV

(ii) As before, letters A, B, C, \dots are used as meta-variables for *formulas*, which are constructed from proposition letters with the logical constants \wedge (conjunction), \vee (disjunction), \rightarrow (implication) and \perp (*falsum*, absurdity).

formulas

Proposition letters and \perp are *atomic formulas* (short: *atoms*).

atomic formulas

(iii) As usual, we define negation \neg by implication and *falsum* as follows: $\neg A := A \rightarrow \perp$.

Note that this corresponds well with the BHK-interpretation of negation, where a proof of $\neg A$ consists in a construction which transforms any assumed proof of A into a proof of the absurdity \perp , for which there can be no proof by definition.

(iv) Moreover, we use letters Γ, Δ, \dots to refer to sets of formulas.

Definition 1.1 (i) The *calculus NI of natural deduction (for intuitionistic logic)* is given by the following rules: *calculus NI*

<i>Introduction rule</i>	<i>Elimination rule</i>
$\frac{A_1 \quad A_2}{A_1 \wedge A_2} (\wedge I)$	$\frac{A_1 \wedge A_2}{A_i} (\wedge E) (i = 1 \text{ or } 2)$
$\frac{A_i}{A_1 \vee A_2} (\vee I) (i = 1 \text{ or } 2)$	$\frac{[A_1] \quad [A_2]}{A_1 \vee A_2} \frac{C}{C} (\vee E)$
$\frac{[A]}{A \rightarrow B} (\rightarrow I)$	$\frac{A \rightarrow B \quad A}{B} (\rightarrow E)$
	<i>Ex-falso rule</i>
	$\frac{\perp}{A} (\perp)$

(ii) *Derivations in NI* are defined as usual.

(iii) If A is *derivable in NI* from a set of assumptions Γ , then we write $\Gamma \vdash_{NI} A$.

derivable in NI

(iv) If A is *provable in NI*, then we write $\vdash_{NI} A$.

provable in NI

Remarks. (i) The principle of *ex falso quodlibet sequitur* $\perp \rightarrow A$ could be rejected as being non-constructive by arguing that \perp is a statement for which we just do not know yet whether it is provable. In this case we would not exclude the possibility that there could be a proof of \perp , for example, if the currently accepted mathematics turns out to be inconsistent. Such an understanding leads to what is called *minimal logic*.

minimal logic

We obtain the *calculus NM* for minimal logic by removing the ex-falso rule from NI.

calculus NM

(ii) By replacing the ex-falso rule with the classical rule of *reductio ad absurdum* $(\perp)_c$ we obtain the *calculus NK* for classical logic.

calculus NK

- (iii) Intuitionistic and minimal logic are examples of non-classical (philosophical) logics that are weaker than classical logic; less logical laws hold in them than in classical logic. It is $\{A \mid \vdash_{\text{NM}} A\} \subset \{A \mid \vdash_{\text{NI}} A\} \subset \{A \mid \vdash_{\text{NK}} A\}$.

Example. We show $\vdash_{\text{NI}} \neg\neg(A \vee \neg A)$:

$$\frac{\frac{\frac{[\neg(A \vee \neg A)]^2}{\perp} (\perp) \quad \frac{[A]^1}{A \vee \neg A} (\vee \text{I})}{\neg A} (\rightarrow \text{I})^1}{A \vee \neg A} (\vee \text{I}) \quad \frac{[\neg(A \vee \neg A)]^2}{\perp} (\perp) \quad \frac{[A]^1}{A \vee \neg A} (\vee \text{I})}{\neg A} (\rightarrow \text{I})^1}{\neg\neg(A \vee \neg A)} (\rightarrow \text{I})^2$$

This illustrates that from a constructive point of view only *weak* counterexamples for *tertium non datur* $A \vee \neg A$ can be given. A *strong* counterexample would consist in showing that $p \vee \neg p$, for a certain statement p , leads to a contradiction; in other words, that $\neg(p \vee \neg p)$ holds. But this is not possible, since $\neg\neg(A \vee \neg A)$ holds for all statements A .

Remarks. (i) The formulas

$$A \rightarrow (B \rightarrow A) \quad (\text{ex quodlibet verum sequitur})$$

and

$$\neg A \rightarrow (A \rightarrow B) \quad (\text{ex falso quodlibet sequitur})$$

(resp. *ex contradictione quodlibet sequitur*), which are provable in NI, are sometimes called *paradoxes of implication*. If we consider their derivations

$$\frac{\frac{[A]^1}{B \rightarrow A} (\rightarrow \text{I})}{A \rightarrow (B \rightarrow A)} (\rightarrow \text{I})^1 \quad \text{and} \quad \frac{\frac{[\neg A]^2}{\perp} (\perp) \quad \frac{[A]^1}{A \rightarrow B} (\rightarrow \text{I})^1}{\neg A \rightarrow (A \rightarrow B)} (\rightarrow \text{I})^2$$

we see that in both derivations B can be chosen arbitrarily. In the first derivation the formula B in $B \rightarrow A$ is in this sense not relevant for A , and in the second derivation the formula A in $A \rightarrow B$ is not relevant for B .

A logic in which neither $A \rightarrow (B \rightarrow A)$ nor $\neg A \rightarrow (A \rightarrow B)$ holds is called *relevance logic* (or *relevant logic*). *relevance logic*

- (ii) Besides logical rules, i.e., rules which introduce or eliminate a logical constant, structural operations are of importance.

In the derivation of $A \rightarrow (B \rightarrow A)$ we went from the premiss A to the conclusion $B \rightarrow A$ without discharging an assumption B . This corresponds to the structural operation of *weakening*. *weakening*

In the derivation of $\neg\neg(A \vee \neg A)$ we discharged two occurrences of the assumption $\neg(A \vee \neg A)$ in one rule application. This corresponds to the structural operation of *contraction*. *contraction*

Another example is the proof of the *law of non-contradiction* $\neg(A \wedge \neg A)$, in which *law of non-contradiction*

contraction is essential:

$$\frac{\frac{[A \wedge \neg A]^1}{\neg A} (\wedge E) \quad \frac{[A \wedge \neg A]^1}{A} (\wedge E)}{\frac{\perp}{\neg(A \wedge \neg A)} (\rightarrow I)^1}$$

By imposing certain restrictions concerning structural operations one obtains *substructural logics* (cf. Došen & Schroeder-Heister, 1993).

substructural logics

Theorem 1.2 *The derivability relation is transitive, i.e., the following holds: If $\Gamma \vdash_{NI} A$ and $\Delta, A \vdash_{NI} B$, then $\Gamma, \Delta \vdash_{NI} B$.*

Proof. Assume $\Gamma \vdash_{NI} A$ and $\Delta, A \vdash_{NI} B$. Then there are derivations

$$\begin{array}{ccc} \Gamma & & \Delta, A \\ \mathcal{D} & \text{and} & \mathcal{D}' \\ A & & B \end{array}$$

Now we replace all assumptions A in the second derivation by the first derivation. We obtain the derivation

$$\begin{array}{c} \Gamma \\ \mathcal{D} \\ \Delta, A \\ \mathcal{D}' \\ B \end{array}$$

showing $\Gamma, \Delta \vdash_{NI} B$.

QED

Remark. In sequent calculus LI (see Gentzen, 1935) transitivity is made explicit by the rule

$$\frac{\Gamma \vdash A \quad \Delta, A \vdash B}{\Gamma, \Delta \vdash B} (\text{Cut})$$

where A is called the *cut formula*.

1.4 Derivability and admissibility of rules

Besides the derivability of formulas the two notions of derivability and admissibility of rules are important. In contradistinction to classical logic, these two notions do not have the same extension in intuitionistic logic.

Definition 1.3 A rule

$$\frac{A_1 \quad \dots \quad A_n}{B}$$

derivable rule

is called *derivable* in a calculus C, if $A_1, \dots, A_n \vdash_C B$.

Example. The rule

$$\frac{A \vee \neg A}{\neg\neg A \rightarrow A}$$

is derivable in NI. (Exercise)

Definition 1.4 A rule R is called *admissible* in a calculus C, if the following holds:

admissible rule

If $\vdash_{C+R} A$, then $\vdash_C A$.

(Here $\vdash_{C+R} A$ means that A is provable in the calculus C extended by the rule R , and $\vdash_C A$ means that A is provable in C without R .)

Remarks. Let R be an arbitrary rule of the form $\frac{A_1 \dots A_n}{B}$.

(i) To demonstrate that R is derivable in NI we have to show

$$A_1, \dots, A_n \vdash_{\text{NI}} B$$

That is, we have to present a corresponding derivation.

(ii) To demonstrate that R is admissible in NI we have to show:

$$\text{If } \vdash_{\text{NI}} A_1, \dots, \vdash_{\text{NI}} A_n, \text{ then } \vdash_{\text{NI}} B.$$

(iii) The transitivity of the derivability relation \vdash_{NI} justifies the application of rules which are derivable in NI.

(iv) Derivability of a rule implies its admissibility.

(v) In the sequent calculus LI the (Cut) rule is admissible.

Example. To illustrate the difference between the derivability and the admissibility of rules we consider the following calculus for the generation of natural numbers \mathbb{N} :

$$\frac{}{0 \in \mathbb{N}} (1) \qquad \frac{k \in \mathbb{N}}{k' \in \mathbb{N}} (2)$$

The first rule is an axiom and says that 0 is a natural number. The second rule says that if k is a natural number, then its successor k' is a natural number as well.

(i) The rule

$$\frac{k \in \mathbb{N}}{k'' \in \mathbb{N}} (3)$$

is a derivable rule in this calculus, as witnessed by the derivation

$$\frac{\frac{k \in \mathbb{N}}{k' \in \mathbb{N}} (2)}{k'' \in \mathbb{N}} (2)$$

(ii) The rule

$$\frac{k' \in \mathbb{N}}{k \in \mathbb{N}} (4)$$

is an example of an admissible rule. Assuming that the premiss $k' \in \mathbb{N}$ is derivable, we have to show that the conclusion $k \in \mathbb{N}$ is also derivable.

The premiss cannot have been derived by rule (1), since k' cannot be 0. Hence the premiss must have been derived by rule (2). An application of (2) requires that there is a derivation of $k \in \mathbb{N}$. This is the desired derivation of the conclusion of (4).

However, rule (4) is obviously not derivable.

Remark. The notion of admissible rule is central in P. Lorenzen's (1915–1994) approach to logic (see Lorenzen, 1955), where we can also find the idea that the admissibility of a rule R in a calculus C has to be shown by an *elimination procedure* that eliminates every application of R from every derivation in the calculus $C + R$ (cp. Gentzen's (1935) result on cut elimination for LI). We illustrate this idea in the proof of the following lemma.

Lemma 1.5 *Let $(\perp)^a$ be the restriction to atomic conclusions of the ex-falso rule (\perp) , and let $NI^a = NM + (\perp)^a$ (i.e., NI^a is the calculus obtained by replacing (\perp) by $(\perp)^a$ in NI). The rule (\perp) is admissible in NI^a .*

Proof. We have to show: If $\vdash_{NI} A$, then $\vdash_{NI^a} A$. Suppose $\vdash_{NI} A$ is shown by the derivation

$$\begin{array}{c} \mathcal{D} \\ \frac{\perp}{C} (\perp) \\ \mathcal{D}' \\ A \end{array}$$

in which the exposed application of the ex-falso rule (\perp) has conclusion C of arbitrary complexity, and where any other possibly occurring applications of (\perp) have conclusions of lower complexity. We consider the structure of the formula C . (If the derivation of A in NI does not contain an application of (\perp) , then $\vdash_{NI^a} A$ holds trivially.)

Induction base: C is atomic. Then the derivation has the form

$$\begin{array}{c} \mathcal{D} \\ \frac{\perp}{C} (\perp)^a \\ \mathcal{D}' \\ A \end{array}$$

(where the shown application of $(\perp)^a$ can be omitted, if C is \perp).

Induction hypothesis: The rule (\perp) is admissible in NI^a for conclusions D and E .

Induction step: We have to show that (\perp) is then also admissible for conclusions C of the form $\neg D$, $D \wedge E$, $D \vee E$ and $D \rightarrow E$.

We consider the case $D \wedge E$, i.e., the derivation

$$\begin{array}{c} \mathcal{D} \\ \frac{\perp}{D \wedge E} (\perp) \\ \mathcal{D}' \\ A \end{array}$$

Using the induction hypothesis we can transform this derivation into the derivation

$$\begin{array}{c} \mathcal{D} \qquad \mathcal{D} \\ \frac{\frac{\perp}{D} (\perp) \quad \frac{\perp}{E} (\perp)}{D \wedge E} (\wedge I) \\ \mathcal{D}' \\ A \end{array}$$

Hence the rule (\perp) is also admissible for conclusions of the form $D \wedge E$.

Remaining cases as exercise.

QED

Remark. The proof by induction shows how the complexity of the conclusion C of (\perp) can be reduced step by step until C is finally atomic. Instead of (\perp) we can thus use $(\perp)^a$ without reducing the strength of NI.

For NI one can show that there are admissible rules which are not derivable. Here we can make use of the fact that every derivation can be transformed into a normal form with certain useful properties.

1.5 Normalisability and properties of NI

Definition 1.6 (i) A formula occurrence in a derivation is called *maximal*, if it is the conclusion of an introduction rule and at the same time the major premiss of an elimination rule. The corresponding formula is called *maximal formula*.

maximal

maximal formula

(ii) Maximal formula occurrences can be eliminated by *reductions* which transform a derivation with a maximal formula occurrence into a derivation without that occurrence.

reductions

For implicational maximal formulas $A \rightarrow B$ the \rightarrow -reduction ($\triangleright_{\rightarrow}$) is defined as follows:

\rightarrow -reduction

$$\frac{\frac{[A]}{\mathcal{D}} \quad \frac{B}{A \rightarrow B} (\rightarrow I) \quad \mathcal{D}' \quad A}{A} (\rightarrow E)}{B} \quad \triangleright_{\rightarrow} \quad \frac{\mathcal{D}' \quad A}{\mathcal{D}} \quad B$$

If \mathcal{D} does not contain assumptions A , then the result of the \rightarrow -reduction is the derivation \mathcal{D} .

Similarly, one can define \wedge - and \vee -reductions for maximal formulas of the form $A \wedge B$ and $A \vee B$. (Exercise)

(iii) A derivation is called *normal*, if it does not contain a maximal formula occurrence. In this case the derivation is in *normal form*.

normal

normal form

Remark. Besides reductions one has to consider certain permutations in addition; this is because of the form of the disjunction elimination rule, where the minor premisses and the conclusion are the same formula. However, we do not have to go into that here.

Theorem 1.7 (Normalisability)

If $\Gamma \vdash_{\text{NI}} A$ holds, then there exists a normal derivation in NI of A from Γ .

Proof. See Prawitz (1965, Ch. IV § 1).

QED

Corollary 1.8 The following properties hold:

(i) *Subformula property:* In a normal derivation of A from assumptions Γ every occurring formula is a subformula of A or a subformula of formulas in Γ .

subformula property

(ii) *Separation property:* In a normal derivation of A from assumptions Γ there occur only rules dealing with the logical constants occurring in A and Γ .

separation property

(iii) *Derivations in normal form which do not contain open assumptions always end with an introduction rule.*

(iv) Since \perp cannot be derived by an introduction rule we have $\not\vdash_{\text{NI}} \perp$, i.e., NI is consistent. (One can argue likewise for every proposition letter.)

(v) Disjunction property: If $\vdash_{\text{NI}} A \vee B$, then $\vdash_{\text{NI}} A$ or $\vdash_{\text{NI}} B$.

disjunction property

(vi) Generalised disjunction property: Let Γ be a \vee -free set of formulas. Then the following holds: If $\Gamma \vdash_{\text{NI}} A \vee B$, then $\Gamma \vdash_{\text{NI}} A$ or $\Gamma \vdash_{\text{NI}} B$.

generalised disjunction property

Remarks. (i) Normalisability holds as well for NM and NK.

(ii) Every property mentioned in the corollary holds for NM, too. Note that in minimal logic \perp is not a logical constant but is treated like a proposition letter.

(iii) Except for consistency (iv) none of the properties mentioned in the corollary holds for NK.

At least the subformula property holds in a restricted form; excluded are assumptions discharged in applications of *reductio ad absurdum*

$$\frac{[\neg A]}{\frac{\perp}{A} (\perp)_c}$$

and occurrences of \perp immediately below such assumptions.

A counterexample for the separation property is given by Peirce's law

$$((A \rightarrow B) \rightarrow A) \rightarrow A$$

Any proof of this purely implicative formula requires besides $(\rightarrow \text{I})$ and $(\rightarrow \text{E})$ also *reductio ad absurdum* $(\perp)_c$.

(iv) For the generalised disjunction property (vi) it is sufficient to make the weaker presupposition that no formula in Γ contains a strictly positive subformula with main connective \vee . (See Prawitz, 1965, p. 43, 55.)

Definition 1.9 Bi-implication (or equivalence) \leftrightarrow is defined as usual:

$$A \leftrightarrow B := (A \rightarrow B) \wedge (B \rightarrow A)$$

Bi-implication \leftrightarrow shall bind as strongly as \rightarrow .

Theorem 1.10 There are admissible but non-derivable rules in NI.

Proof. Harrop's rule

Harrop's rule

$$\frac{\neg A \rightarrow (B \vee C)}{(\neg A \rightarrow B) \vee (\neg A \rightarrow C)}$$

is admissible in NI, but not derivable.

Admissibility follows from the generalised disjunction property and the fact that a negated formula $\neg A$ can always be transformed into a \vee -free formula A' and vice versa, such that $\vdash_{\text{NI}} \neg A \leftrightarrow A'$ holds.

Suppose $\vdash_{\text{NI}} \neg A \rightarrow (B \vee C)$ holds due to a derivation \mathcal{D} in normal form. By Corollary 1.8 (iii) this derivation \mathcal{D} ends with $(\rightarrow \text{I})$, i.e., $\neg A \vdash_{\text{NI}} B \vee C$ holds, too. Now we transform $\neg A$ using

$$(\vee\text{-removal}) \left\{ \begin{array}{l} \vdash_{\text{NI}} \neg(A \vee B) \leftrightarrow (\neg A \wedge \neg B) \\ \vdash_{\text{NI}} \neg(A \wedge B) \leftrightarrow \neg(\neg A \wedge \neg B) \\ \vdash_{\text{NI}} \neg(A \rightarrow B) \leftrightarrow (\neg \neg A \wedge \neg B) \end{array} \right.$$

into a \vee -free formula A' , such that $\vdash_{\text{NI}} \neg A \leftrightarrow A'$. We thus obtain $A' \vdash_{\text{NI}} B \vee C$, to which we can apply the generalised disjunction property in order to obtain $A' \vdash_{\text{NI}} B$ or $A' \vdash_{\text{NI}} C$. Using (*\vee -removal*) again yields $\neg A \vdash_{\text{NI}} B$ and $\neg A \vdash_{\text{NI}} C$, respectively. In both cases there must be corresponding derivations \mathcal{D}_1 and \mathcal{D}_2 , respectively, which we can extend to a proof of the conclusion of Harrop's rule:

$$\frac{\frac{[\neg A]^1}{\mathcal{D}_1} \quad \frac{B}{\neg A \rightarrow B} (\rightarrow \text{I})^1}{(\neg A \rightarrow B) \vee (\neg A \rightarrow C)} (\vee \text{I}) \quad \text{respectively} \quad \frac{\frac{[\neg A]^1}{\mathcal{D}_2} \quad \frac{C}{\neg A \rightarrow C} (\rightarrow \text{I})^1}{(\neg A \rightarrow B) \vee (\neg A \rightarrow C)} (\vee \text{I})$$

We have thus shown: If $\vdash_{\text{NI}} \neg A \rightarrow (B \vee C)$, then $\vdash_{\text{NI}} (\neg A \rightarrow B) \vee (\neg A \rightarrow C)$. That is, Harrop's rule is admissible in NI.

We do not prove non-derivability of Harrop's rule here. A proof can be given by a counterexample in Kripke semantics (see Section 1.8), for which NI is sound (and complete). QED

Remarks. (i) An admissible but non-derivable rule may only be used in a derivation if none of its premisses depends on open assumptions. Otherwise one could derive formulas in NI which are not provable:

$$\frac{\frac{[\neg A \rightarrow (B \vee C)]^1}{(\neg A \rightarrow B) \vee (\neg A \rightarrow C)} (\text{Harrop's rule}) \not\vdash}{(\neg A \rightarrow (B \vee C)) \rightarrow ((\neg A \rightarrow B) \vee (\neg A \rightarrow C))} (\rightarrow \text{I})^1$$

In this (incorrect) application of Harrop's rule its premiss still depends on itself as an open assumption; one would obtain the *Kreisel–Putnam formula*

$$(\neg A \rightarrow (B \vee C)) \rightarrow ((\neg A \rightarrow B) \vee (\neg A \rightarrow C)),$$

*Kreisel–Putnam
formula*

which is, however, not provable in NI.

(ii) If one extends NI by adding admissible but non-derivable rules, then one obtains so-called *superintuitionistic* or *intermediate logics*, which are located between intuitionistic and classical logic in strength. *intermediate logics*

For example, if one extends NI by adding Harrop's rule, then one obtains the so-called *Kreisel–Putnam logic*. Another example is the so-called *Gödel–Dummett logic*, which one obtains by adding

$$(A \rightarrow B) \vee (B \rightarrow A)$$

as an axiom.

There are infinitely many intermediate logics. (See Gödel, 1932.)

Examples. Further examples of rules that are admissible but not derivable in NI are:

(i) The rule $\frac{\neg\neg A \rightarrow A}{A \vee \neg A}$

(ii) *Mints's rule*: $\frac{(A \rightarrow B) \rightarrow (A \vee C)}{((A \rightarrow B) \rightarrow A) \vee ((A \rightarrow B) \rightarrow C)}$

Mints's rule

- Remarks.** (i) Logics in which not every admissible rule is derivable are also called *structurally incomplete*.
(ii) In contradistinction to NI every rule admissible in NK is also derivable, i.e., classical logic is *structurally complete*.

It is easy to show this (classically) by contraposition: Suppose the rule

$$\frac{A_1 \quad \dots \quad A_n}{B}$$

is not derivable in NK, i.e., $A_1, \dots, A_n \not\vdash_{\text{NK}} B$. Then by completeness $A_1, \dots, A_n \not\models B$ (for the semantic consequence relation \models of classical logic). Hence there must exist a valuation v , such that $\llbracket A_1 \rrbracket^v = \dots = \llbracket A_n \rrbracket^v = \text{true}$, but $\llbracket B \rrbracket^v = \text{false}$. Now we replace all proposition letters $A \in \text{PV}$ in A_1, \dots, A_n, B either by $\top := p \rightarrow p$, if $v(A) = \text{true}$, or by \perp , if $v(A) = \text{false}$. Then $\models A_1, \dots, \models A_n$ and $\models \neg B$, and hence especially $\not\models B$. By completeness and soundness of NK therefore $\vdash_{\text{NK}} A_1, \dots, \vdash_{\text{NK}} A_n$, but $\not\vdash_{\text{NK}} B$, i.e., the rule cannot be admissible.

Definition 1.11 We say that a logical constant $*$ $\in \{\neg, \wedge, \vee, \rightarrow\}$ can be *expressed* by a formula F in a calculus C , if $\vdash_C *A \leftrightarrow F$ or $\vdash_C (A * B) \leftrightarrow F$ for a formula F , in which $*$ itself does not occur. If no such formula F exists, then $*$ is called *independent*.

Theorem 1.12 In NI each of the logical constants \neg, \wedge, \vee and \rightarrow is independent.

Proof. See Wajsberg (1938) or McKinsey (1939). QED

Remarks. (i) No proper subset of $\{\neg, \wedge, \vee, \rightarrow\}$ can be functionally complete for intuitionistic logic. This is another essential difference w.r.t. classical logic, in which for example $\{\neg, \wedge\}$, $\{\neg, \vee\}$ and $\{\neg, \rightarrow\}$ are functionally complete sets.

In this sense each of the logical constants \neg, \wedge, \vee and \rightarrow has its distinct meaning in intuitionistic logic, since none can be expressed by the respective others.

(ii) A ternary Sheffer function t for the set $\{\neg, \wedge, \vee, \rightarrow\}$ is presented in Došen (1985):

$$t(A, B, C) := (A \vee B) \leftrightarrow (C \leftrightarrow \neg B)$$

Note that, since $A \leftrightarrow B := (A \rightarrow B) \wedge (B \rightarrow A)$, the definition of t depends on the whole set of constants $\{\neg, \wedge, \vee, \rightarrow\}$.

(iii) In Gödel–Dummett logic GD (also referred to as G or LC in the literature) at least \vee can be expressed using $\{\wedge, \rightarrow\}$. We have (exercise):

$$\vdash_{\text{GD}} (A \vee B) \leftrightarrow (((A \rightarrow B) \rightarrow B) \wedge ((B \rightarrow A) \rightarrow A))$$

This intermediate logic thus also lies w.r.t. the independence of logical constants between intuitionistic and classical logic.

Theorem 1.13 (Deduction theorem)

$A_1, \dots, A_n \vdash_{\text{NI}} B \iff \vdash_{\text{NI}} A_1 \wedge \dots \wedge A_n \rightarrow B$. (Likewise for NM and NK.)

Proof. Exercise. QED

1.6 The Curry-Howard correspondence

We consider the simply typed λ -calculus $\lambda \rightarrow$ (cf. [Barendregt, 1992](#)) and relate it to positive implication logic $P \rightarrow$.

Definition 1.14 The set of *types* of $\lambda \rightarrow$ is defined as follows:

- (i) Type variables $\alpha, \beta, \gamma, \delta, \alpha_1, \alpha_2, \dots$ are types. *types*
type variables
- (ii) If σ and τ are types, then $(\sigma \rightarrow \tau)$ is a type (also called *function type*). *function type*

It is $\sigma_1 \rightarrow \sigma_2 \rightarrow \dots \rightarrow \sigma_{n-1} \rightarrow \sigma_n$ an abbreviation of $\sigma_1 \rightarrow (\sigma_2 \rightarrow (\dots (\sigma_{n-1} \rightarrow \sigma_n) \dots))$; that is, we use *association to the right* for function types.

- A *judgement* has the form $M : \sigma$ for a λ -term M and a type σ . *judgement*
- The term M is called the *subject* of the judgement. (The type σ is then also called the *predicate*. One says “ M has type σ ” or something like that.) *subject*
- A *declaration* is a judgement whose subject is a term variable. *declaration*
- A *basis* Γ is a finite set of declarations whose subjects are pairwise distinct. *basis*
- A *sequent* has the form $\Gamma \vdash M : \sigma$ for a basis Γ and a judgement $M : \sigma$. *sequent*

Definition 1.15 Sequents $\Gamma \vdash M : \sigma$ expressing that the judgement $M : \sigma$ holds in basis Γ can be derived in the *calculus* $\lambda \rightarrow$, which is given by the axiom scheme

$$\text{(Id)} \quad \Gamma, x : \sigma \vdash x : \sigma$$

together with the following \rightarrow -introduction and \rightarrow -elimination rules:

$$\begin{array}{c} \text{(\(\rightarrow\)I)} \quad \frac{\Gamma, x : \sigma \vdash M : \tau}{\Gamma \vdash (\lambda x.M) : \sigma \rightarrow \tau} \end{array} \quad \begin{array}{c} \text{(\(\rightarrow\)E)} \quad \frac{\Gamma \vdash M : \sigma \rightarrow \tau \quad \Gamma \vdash N : \sigma}{\Gamma \vdash (MN) : \tau} \end{array}$$

If $\Gamma \vdash M : \sigma$ is *derivable* in $\lambda \rightarrow$, then we write $\Gamma \vdash_{\lambda \rightarrow} M : \sigma$ or $\Gamma \vdash M : \sigma$. (Thus we often identify sequents with the assertion of their derivability. What is meant in each case should be clear from the context.) *derivable*

There is a certain correspondence between typed λ -calculus and logic, which can be roughly described as follows:

<i>Typed λ-calculus</i>	<i>Logic</i>
type	formula, proposition
typable open term	derivation with assumptions
typable closed term	proof (derivation without assumptions)
β -contraction	contraction of derivation
typable term in β -normal form	derivation in normal form
β -equality	equality of derivations

Definition 1.16

- Type variables are also called *propositional variables*, types also (*implicational*) *formulas*. *formulas*
 - A finite set of formulas is called *context*. *context*
- Metalinguistic variables for contexts are Δ, Δ', \dots

– Positive implication logic $P \rightarrow$ is given by the axiom scheme

$$(Id) \Delta, \sigma \vdash \sigma$$

positive implication
logic $P \rightarrow$

and the two rules:

$$(\rightarrow I) \frac{\Delta, \sigma \vdash \tau}{\Delta \vdash \sigma \rightarrow \tau} \quad (\rightarrow E) \frac{\Delta \vdash \sigma \rightarrow \tau \quad \Delta \vdash \sigma}{\Delta \vdash \tau}$$

($P \rightarrow$ is called *positive* implication logic, since negation does not occur.)

– $\Delta \vdash_{P \rightarrow} \sigma$ means that $\Delta \vdash \sigma$ is *derivable* in $P \rightarrow$.

derivable

– For a judgement $M : \sigma$ let $(M : \sigma)^\circ \simeq \sigma$.

– For a basis $\Gamma = \{x_1 : \sigma_1, \dots, x_n : \sigma_n\}$ let Γ° be the context $\{\sigma_1, \dots, \sigma_n\}$.

Lemma 1.17 *If $\Gamma \vdash_{\lambda \rightarrow} M : \sigma$, then $\Gamma^\circ \vdash_{P \rightarrow} \sigma$.*

Proof. By application of $^\circ$ to every judgement in the $\lambda \rightarrow$ -derivation of $\Gamma \vdash M : \sigma$ one obtains a $P \rightarrow$ -derivation of $\Gamma^\circ \vdash \sigma$. QED

Lemma 1.18 *There exists an algorithm which yields for every typable term M a derivation of $\Delta \vdash \sigma$ in $P \rightarrow$ such that $\Delta = \Gamma^\circ$ and $\Gamma \vdash_{\lambda \rightarrow} M : \sigma$.*

Proof. There exists an algorithm that can generate for every typable term M its principal pair (Γ, σ) ; this can then be transformed directly into a $P \rightarrow$ -sequent $\Delta \vdash \sigma$ with $\Delta = \Gamma^\circ$. The $P \rightarrow$ -rule application necessary to derive this sequent in the last step is always determined by the form of M . QED

This means: Every typable term M encodes a derivation in $P \rightarrow$. From this derivation one can obtain by substitution all derivations of $\Gamma^\circ \vdash \sigma$ in $P \rightarrow$ which correspond to derivations of $\Gamma \vdash M : \sigma$ in $\lambda \rightarrow$.

Lemma 1.19 *For every derivation of $\Delta \vdash \sigma$ in $P \rightarrow$ one can construct a term M and a derivation of $\Gamma \vdash M : \sigma$ in $\lambda \rightarrow$ such that $\Gamma^\circ = \Delta$.*

Proof. Induction on the structure of the derivation of $\Delta \vdash \sigma$ in $P \rightarrow$ (where $\Delta = \{\sigma_1, \dots, \sigma_n\}$):

Case (Id): All formulas σ occurring in instances of (Id) of $P \rightarrow$ are replaced by type declarations $x : \sigma$. The variable x is chosen in such a way that

- all occurrences of a formula σ have the *same* corresponding declaration $x : \sigma$;
- *different* formulas σ and τ have corresponding declarations $x : \sigma$ and $y : \tau$ with *different* variables x and y .

Case ($\rightarrow I$): The derivation in $P \rightarrow$ ends with

$$(\rightarrow I) \frac{\sigma_1, \dots, \sigma_n, \sigma \vdash \tau}{\sigma_1, \dots, \sigma_n \vdash \sigma \rightarrow \tau}$$

For the premiss $\sigma_1, \dots, \sigma_n, \sigma \vdash \tau$ there is by the induction hypothesis a derivation in $\lambda \rightarrow$ of $x_1 : \sigma_1, \dots, x_n : \sigma_n, x : \sigma \vdash M : \tau$. We extend this derivation by an application of ($\rightarrow I$) in $\lambda \rightarrow$ to obtain $x_1 : \sigma_1, \dots, x_n : \sigma_n \vdash \lambda x. M : \sigma \rightarrow \tau$.

Case (\rightarrow E): The derivation in $P\rightarrow$ ends with

$$(\rightarrow\text{E}) \frac{\sigma_1, \dots, \sigma_n \vdash \sigma \rightarrow \tau \quad \sigma_1, \dots, \sigma_n \vdash \sigma}{\sigma_1, \dots, \sigma_n \vdash \tau}$$

For the premisses $\sigma_1, \dots, \sigma_n \vdash \sigma \rightarrow \tau$ and $\sigma_1, \dots, \sigma_n \vdash \sigma$ there are by the induction hypothesis derivations in $\lambda\rightarrow$ of

$$x_1 : \sigma_1, \dots, x_n : \sigma_n \vdash M : \sigma \rightarrow \tau \quad \text{and} \quad x_1 : \sigma_1, \dots, x_n : \sigma_n \vdash N : \sigma$$

Note that each type σ_i is assigned to exactly one variable x_i . By an application of (\rightarrow E) we thus obtain a derivation in $\lambda\rightarrow$ of $x_1 : \sigma_1, \dots, x_n : \sigma_n \vdash MN : \tau$. QED

Theorem 1.20 (Curry-Howard isomorphism)

Let M_P be the derivation in $P\rightarrow$ which corresponds to a term M typable in $\lambda\rightarrow$, as given by Lemma 1.18. Let Π_λ be the $\lambda\rightarrow$ -term which corresponds to a derivation in $P\rightarrow$, as given by Lemma 1.19. Then the following holds:

- (i) $(\Pi_\lambda)_P$ is a derivation in $P\rightarrow$ from which we can obtain Π by substitution of formulas for propositional variables.
- (ii) $(M_P)_\lambda$ is (modulo the renaming of free and/or bound variables) a term which results from M by identification of free or bound variables.

Proof. By Lemmas 1.18 and 1.19. QED

An example for (ii) is the λ -term

$$u(zx)(zy)$$

for which the type assignment algorithm yields

$$u : \alpha \rightarrow \alpha \rightarrow \beta, z : \gamma \rightarrow \alpha, x : \gamma, y : \gamma \vdash_{\lambda\rightarrow} u(zx)(zy) : \beta$$

For $\Gamma = \{u : \alpha \rightarrow \alpha \rightarrow \beta, z : \gamma \rightarrow \alpha, x : \gamma, y : \gamma\}$ we have $\Gamma^\circ = \{\alpha \rightarrow \alpha \rightarrow \beta, \gamma \rightarrow \alpha, \gamma\}$. The corresponding derivation $(u(zx)(zy))_P$ in $P\rightarrow$

$$\begin{array}{c} \text{(Id)} \frac{}{\Gamma^\circ \vdash \gamma \rightarrow \alpha} \quad \text{(Id)} \frac{}{\Gamma^\circ \vdash \gamma} \\ \text{(Id)} \frac{}{\Gamma^\circ \vdash \alpha \rightarrow (\alpha \rightarrow \beta)} \quad \text{(\rightarrow E)} \frac{\text{(Id)} \frac{}{\Gamma^\circ \vdash \gamma \rightarrow \alpha} \quad \text{(Id)} \frac{}{\Gamma^\circ \vdash \gamma}}{\Gamma^\circ \vdash \alpha} \\ \text{(\rightarrow E)} \frac{\text{(Id)} \frac{}{\Gamma^\circ \vdash \alpha \rightarrow (\alpha \rightarrow \beta)} \quad \text{(\rightarrow E)} \frac{\text{(Id)} \frac{}{\Gamma^\circ \vdash \gamma \rightarrow \alpha} \quad \text{(Id)} \frac{}{\Gamma^\circ \vdash \gamma}}{\Gamma^\circ \vdash \alpha}}{\text{(\rightarrow E)} \frac{\Gamma^\circ \vdash \alpha \rightarrow \beta}{\Gamma^\circ \vdash \beta}} \end{array}$$

yields

$$\alpha \rightarrow \alpha \rightarrow \beta, \gamma \rightarrow \alpha, \gamma \vdash_{P\rightarrow} \beta.$$

According to Lemma 1.19, a λ -term of the form $u(zx)(zx)$, where x and y are identified, corresponds to this derivation.

The reason for this identification of variables is that information is lost in going from $\lambda\rightarrow$ to $P\rightarrow$, which cannot be regained by going from $P\rightarrow$ to $\lambda\rightarrow$. Note that by Lemma 1.19 the mapping of variables to formulas (= types) in instances of (Id) of $\lambda\rightarrow$ can never result in two variables having the same type. That is, by going from $\lambda\rightarrow$ to $P\rightarrow$ a sequent of the form $\Gamma, x : \sigma, y : \sigma \vdash M : \tau$ cannot occur.

In view of this loss of information one might prefer the weaker term *correspondence* instead

of isomorphism. However, for a variant of natural deduction this loss of information can be avoided (see Troelstra & Schwichtenberg, 2001, Ch. 6; cp. also Sørensen & Urzyczyn, 2006, Ch. 4).

The Curry-Howard isomorphism induces reducibility and equality relations for derivations which correspond to β -reduction \triangleright_β and β -equality $=_\beta$ in $\lambda \rightarrow$. These relations for derivations are investigated in proof theory, most prominently on the basis of the calculus of natural deduction (see Prawitz 2006).

Consider the β -redex $(\lambda x.M)N$ with type τ :

$$\frac{\begin{array}{c} \mathcal{D}_1 \\ (\rightarrow\text{I}) \frac{\Gamma, x:\sigma \vdash M:\tau}{\Gamma \vdash \lambda x.M:\sigma \rightarrow \tau} \end{array} \quad \begin{array}{c} \mathcal{D}_2 \\ (\rightarrow\text{E}) \frac{\Gamma \vdash \lambda x.M:\sigma \rightarrow \tau \quad \Gamma \vdash N:\sigma}{\Gamma \vdash (\lambda x.M)N:\tau} \end{array}}{\Gamma \vdash (\lambda x.M)N:\tau}$$

It is

$$(\lambda x.M)N \triangleright_{1\beta} M[N/x]$$

To this there corresponds a contraction $\triangleright_{1\beta}^\circ$ for derivations in natural deduction:

$$\frac{\begin{array}{c} [\sigma]^n \\ \mathcal{D}_1^\circ \\ \frac{\tau}{\sigma \rightarrow \tau} (\rightarrow\text{I})^n \end{array} \quad \begin{array}{c} \mathcal{D}_2^\circ \\ \sigma \\ \frac{\sigma}{\tau} (\rightarrow\text{E}) \end{array}}{\tau} \triangleright_{1\beta}^\circ \begin{array}{c} \mathcal{D}_2^\circ \\ \sigma \\ \mathcal{D}_1^\circ \\ \tau \end{array}$$

where all occurrences of the assumption σ which are discharged by $(\rightarrow\text{I})$ are replaced by copies of the derivation ending with σ :

$$\begin{array}{c} \mathcal{D}_2^\circ \\ \sigma \end{array}$$

If there are no such occurrences, then the derivation is transformed into

$$\begin{array}{c} \mathcal{D}_1^\circ \\ \tau \end{array}$$

The replacement of all occurrences of the assumption σ corresponds to the replacement of all occurrences of x in M by N , i.e., to the substitution $M[N/x]$.

Normalisability of λ -terms corresponds to normalisability of derivations in natural deduction and vice versa. If two derivations have the same normal form, then they are equal in the sense of β -equality.

The left derivation represents an argumentation in which a lemma of the form $\sigma \rightarrow \tau$ is used. This lemma does no longer occur in the right, contracted derivation. Normalisability of derivations ensures that by using a lemma only those things can be shown that could also be shown directly, i.e., without the lemma. This justifies the use of lemmas, which allows in general for shorter derivations.

1.7 On the relation between classical and minimal/intuitionistic logic

In the following we investigate the relation between classical logic (NK) and minimal logic (NM) as well as intuitionistic logic (NI) in more detail. As a technical means one uses certain translations of formulas. As a further means we use a result on so-called negative formulas, which constitute a fragment of our language of propositional logic.

Definition 1.21 A formula A is called *negative*, if it is \vee -free, and if every occurrence of proposition letters is negated. *negative formula*

Lemma 1.22 For negative formulas A we have: $\vdash_{\text{NM}} \neg\neg A \leftrightarrow A$.

Proof. We show this by an induction on the structure of A , where we make use of the fact that for arbitrary formulas A, B the following holds:

$$\vdash_{\text{NM}} A \rightarrow \neg\neg A \quad (1)$$

$$\vdash_{\text{NM}} \neg\neg\neg A \leftrightarrow \neg A \quad (2)$$

$$\vdash_{\text{NM}} \neg\neg(A \wedge B) \rightarrow (\neg\neg A \wedge \neg\neg B) \quad (3)$$

$$\vdash_{\text{NM}} \neg\neg(A \rightarrow B) \rightarrow (A \rightarrow \neg\neg B) \quad (4)$$

(In NI the two formulas $(\neg\neg A \wedge \neg\neg B) \rightarrow \neg\neg(A \wedge B)$ and $(A \rightarrow \neg\neg B) \rightarrow \neg\neg(A \rightarrow B)$ hold as well; however, we do not need this here.)

Induction base: If A is a proposition letter, then A cannot be a negative formula; the assertion thus follows trivially.

In case $A \equiv \perp$ (i.e., A is syntactically identical to \perp) the assertion follows from

$$\frac{(\perp \rightarrow \perp) \rightarrow \perp \quad \frac{[\perp]^1}{\perp \rightarrow \perp} (\rightarrow \text{I})^1}{\perp} (\rightarrow \text{E})$$

and (1).

Induction hypothesis: The assertion holds for B and C .

Induction step: We have to show that the assertion then holds for $\neg B$, $B \wedge C$, $B \vee C$ and $B \rightarrow C$, too.

In case $A \equiv B \vee C$ the assertion follows trivially, since A is not a negative formula.

Now we consider the case $A \equiv (B \rightarrow C)$: It is $\neg\neg A \equiv \neg\neg(B \rightarrow C)$. By (4) we have

$$\vdash_{\text{NM}} \neg\neg(B \rightarrow C) \rightarrow (B \rightarrow \neg\neg C)$$

and the deduction theorem yields

$$\neg\neg(B \rightarrow C) \vdash_{\text{NM}} B \rightarrow \neg\neg C \quad (\text{A})$$

By induction hypothesis we have as a special case $\vdash_{\text{NM}} \neg\neg C \rightarrow C$, i.e. the rule

$$\frac{\neg\neg C}{C} (*)$$

is derivable in NM as long as C is a negative formula. With

$$\frac{B \rightarrow \neg\neg C \quad [B]^1 (\rightarrow \text{E})}{\frac{\neg\neg C}{C} (*)} (\rightarrow \text{I})^1$$

one obtains

$$B \rightarrow \neg\neg C \vdash_{\text{NM}} B \rightarrow C \quad (\text{B})$$

Transitivity of \vdash_{NM} applied to (A) and (B) yields $\neg\neg(B \rightarrow C) \vdash_{\text{NM}} B \rightarrow C$, from which we get $\vdash_{\text{NM}} \neg\neg(B \rightarrow C) \rightarrow (B \rightarrow C)$ by using the deduction theorem. The assertion follows with (1).

Remaining cases as exercise.

QED

Next we consider a translation of formulas in which proposition letters are negated twice, and in which disjunctive formulas obtain a weaker meaning by expressing them using negation and conjunction.

Remark. It is $A \vee B \vdash_{\text{NM}} \neg(\neg A \wedge \neg B)$, but $\neg(\neg A \wedge \neg B) \not\vdash_{\text{NI}} A \vee B$.

Definition 1.23 The *translation* $^{\text{g}}$ is defined as follows:

translation $^{\text{g}}$

- (i) $\perp^{\text{g}} := \perp$,
- (ii) $A^{\text{g}} := \neg\neg A$, if A is a proposition letter,
- (iii) $(A \wedge B)^{\text{g}} := A^{\text{g}} \wedge B^{\text{g}}$,
- (iv) $(A \vee B)^{\text{g}} := \neg(\neg A^{\text{g}} \wedge \neg B^{\text{g}})$,
- (v) $(A \rightarrow B)^{\text{g}} := A^{\text{g}} \rightarrow B^{\text{g}}$.

For sets of formulas Γ let $\Gamma^{\text{g}} := \{B^{\text{g}} \mid B \in \Gamma\}$.

Remark. The translation $^{\text{g}}$ goes in this form back to G. Gentzen (1909–1945). Alternative translations were used by Kolmogorov, Gödel, Kuroda and Krivine. In general such translations are called *negative translations*.

negative translations

- Examples.** (i) $(p \vee \neg p)^{\text{g}} \equiv \neg(\neg p^{\text{g}} \wedge \neg\neg p^{\text{g}}) \equiv \neg(\neg\neg p \wedge \neg\neg\neg p)$.
(ii) $(\neg\neg p \rightarrow p)^{\text{g}} \equiv \neg\neg p^{\text{g}} \rightarrow p^{\text{g}} \equiv \neg\neg\neg\neg p \rightarrow \neg\neg p$.

By using $\vdash_{\text{NM}} \neg\neg\neg A \leftrightarrow \neg A$ (sub)formulas $\neg\neg\neg A$ can be further simplified into $\neg A$.

Theorem 1.24 $\Gamma \vdash_{\text{NK}} A \iff \Gamma^{\text{g}} \vdash_{\text{NM}} A^{\text{g}}$.

Proof. To prove the direction from right to left one first shows $\vdash_{\text{NK}} A \leftrightarrow A^{\text{g}}$, and then uses: $\Gamma \vdash_{\text{NM}} A \implies \Gamma \vdash_{\text{NK}} A$. (Exercise)

We prove the direction from left to right by an induction on the structure of derivations \mathcal{D} of the formula A from the set of assumptions Γ .

Induction base: Let $A \in \Gamma$; then also $A^{\text{g}} \in \Gamma^{\text{g}}$. Hence $\Gamma^{\text{g}} \vdash_{\text{NM}} A^{\text{g}}$. (This includes the case $\{A\} = \Gamma$, where the derivation \mathcal{D} is just the node A .)

Induction hypothesis: The assertion holds for the derivation(s) of the premiss(es) of the last rule application in \mathcal{D} .

Induction step: We have to consider all rules which are applicable in the last step. As examples we treat the cases $(\rightarrow \text{I})$, $(\vee \text{E})$ and $(\perp)_c$.

- (i) \mathcal{D} ends with $(\rightarrow \text{I})$:

$$\frac{\Gamma, [A]^n \quad \mathcal{D} \quad B}{A \rightarrow B} (\rightarrow \text{I})^n$$

By the induction hypothesis it holds: $\Gamma^g, A^g \vdash_{\text{NM}} B^g$. With

$$\frac{\Gamma^g, [A^g]^n \quad \mathcal{D}^g}{A^g \rightarrow B^g} (\rightarrow \text{I})^n$$

we have $\Gamma^g \vdash_{\text{NM}} A^g \rightarrow B^g$, and by definition of g also $\Gamma^g \vdash_{\text{NM}} (A \rightarrow B)^g$.

(ii) \mathcal{D} ends with $(\vee \text{E})$:

$$\frac{\frac{\Gamma \quad \Gamma, [A]^n \quad \Gamma, [B]^n}{A \vee B} \quad \frac{\Gamma, [A]^n \quad \mathcal{D}_1}{C} \quad \frac{\Gamma, [B]^n \quad \mathcal{D}_2}{C}}{C} (\vee \text{E})^n$$

By the induction hypothesis:

$$(1) \Gamma^g \vdash_{\text{NM}} (A \vee B)^g, \quad (2) \Gamma^g, A^g \vdash_{\text{NM}} C^g, \quad (3) \Gamma^g, B^g \vdash_{\text{NM}} C^g.$$

By definition of g and with (1) it then holds: $\Gamma^g \vdash_{\text{NM}} \neg(\neg A^g \wedge \neg B^g)$, by a derivation \mathcal{D}' . Moreover, due to (2) and (3) there exist derivations

$$\mathcal{D}'_1 \left\{ \begin{array}{l} \Gamma^g, [A^g]^n \\ \mathcal{D}'_1 \\ \hline C^g \\ A^g \rightarrow C^g \end{array} \right. (\rightarrow \text{I})^n \quad \text{and} \quad \mathcal{D}'_2 \left\{ \begin{array}{l} \Gamma^g, [B^g]^n \\ \mathcal{D}'_2 \\ \hline C^g \\ B^g \rightarrow C^g \end{array} \right. (\rightarrow \text{I})^n$$

Then $\Gamma^g \vdash_{\text{NM}} C^g$ holds by the following derivation:

$$\frac{\frac{\frac{\frac{\Gamma^g \quad \mathcal{D}'}{\neg(\neg A^g \wedge \neg B^g)} \quad \frac{\frac{\frac{\mathcal{D}'_1}{A^g \rightarrow C^g} \quad \frac{[A^g]^1}{C^g} (\rightarrow \text{E})}{\neg A^g} (\rightarrow \text{I})^1} \quad \frac{\frac{\mathcal{D}'_2}{B^g \rightarrow C^g} \quad \frac{[B^g]^2}{C^g} (\rightarrow \text{E})}{\neg B^g} (\rightarrow \text{I})^2}}{\neg A^g \wedge \neg B^g} (\wedge \text{I})}{\neg(\neg A^g \wedge \neg B^g)} (\rightarrow \text{E})}{\frac{\frac{\perp}{\neg\neg C^g} (\rightarrow \text{I})^3}{C^g} (\text{Lemma 1.22})} (\rightarrow \text{E})$$

In the last step we were able to apply Lemma 1.22, since C^g is a negative formula: it is \vee -free (Def. 1.23 (iv)), and only (double) negated proposition letters do occur (Def. 1.23 (ii)).

(iii) \mathcal{D} ends with $(\perp)_c$:

$$\frac{\Gamma, [\neg A]^n \quad \mathcal{D}}{\perp} (\perp)_c$$

By induction hypothesis: $\Gamma^g, (\neg A)^g \vdash_{\text{NM}} \perp^g$; and with

$$\perp^g \equiv \perp \quad \text{and} \quad (\neg A)^g \equiv (A \rightarrow \perp)^g \equiv A^g \rightarrow \perp^g \equiv A^g \rightarrow \perp \equiv \neg A^g$$

also $\Gamma^g, \neg A^g \vdash_{\text{NM}} \perp$ holds, by a derivation \mathcal{D}^g . By Lemma 1.22 we have in

addition $\vdash_{\text{NM}} \neg\neg A^g \rightarrow A^g$, since A^g is a negative formula. Then $\Gamma^g \vdash_{\text{NM}} A^g$ holds due to the derivation

$$\frac{\frac{\Gamma, [\neg A^g]^n}{\mathcal{D}^g} \quad \frac{\perp}{\neg\neg A^g} (\rightarrow \text{I})^n}{\neg\neg A^g \rightarrow A^g} (\text{Lemma 1.22}) \quad \frac{\perp}{\neg\neg A^g} (\rightarrow \text{E})}{A^g}$$

Remaining cases as exercise. QED

Corollary 1.25 For negative formulas A we have: $\vdash_{\text{NK}} A \iff \vdash_{\text{NM}} A$.

Proof. For negative formulas A also A^g is a negative formula, in which every proposition letter is prefixed by two additional negations. Therefore by Lemma 1.22 (and (1)-(4) in its proof) it holds: $\vdash_{\text{NM}} A \leftrightarrow A^g$. QED

Remarks. (i) Classical logic is thus a conservative extension of minimal logic w.r.t. negative formulas.

(ii) In classical logic every formula is equivalent to a negative formula: replace each proposition letter $A \in \text{PV}$ by $\neg\neg A$, and remove disjunctions with De Morgan. Hence classical logic is in a certain sense contained in minimal logic, although $\vdash_{\text{NK}} A \implies \vdash_{\text{NM}} A$ does not hold for arbitrary formulas A .

(iii) Since \perp is a negative formula, Corollary 1.25 yields: $\not\vdash_{\text{NK}} \perp \iff \not\vdash_{\text{NM}} \perp$. Thus NK is consistent iff NM is consistent.

Consistency of NK follows directly from $\not\vdash \perp$ and soundness. Hence NM is consistent as well. The latter already follows from the consistency of NI (Corollary 1.8 (iv)), since NM is contained in NI.

Theorem 1.26 (Glivenko) It holds:

(i) $\vdash_{\text{NK}} A \iff \vdash_{\text{NI}} \neg\neg A$.

(ii) $\vdash_{\text{NK}} \neg A \iff \vdash_{\text{NI}} \neg A$.

Proof. (i) One shows by an induction on the structure of A that $\vdash_{\text{NI}} A^g \leftrightarrow \neg\neg A$ (exercise). The assertion follows with Theorem 1.24.

(ii) Suppose $\vdash_{\text{NK}} \neg A$. Then it follows with (i) that $\vdash_{\text{NI}} \neg\neg\neg A$. With $\vdash_{\text{NI}} \neg\neg\neg A \leftrightarrow \neg A$ we can conclude $\vdash_{\text{NI}} \neg A$.

(The direction from right to left follows for both assertions already from the fact that NK is an extension of NI, and $\vdash_{\text{NK}} \neg\neg A \rightarrow A$.) QED

Remarks. (i) The following generalisation holds as well:

$$A_1, \dots, A_n \vdash_{\text{NK}} A \iff \neg\neg A_1, \dots, \neg\neg A_n \vdash_{\text{NI}} \neg\neg A$$

(ii) Glivenko's Theorem does *not* hold for first-order logic. For calculi NK and NI that are extended to first-order logic we have e.g. $\vdash_{\text{NK}} \forall x(A(x) \vee \neg A(x))$, but $\not\vdash_{\text{NI}} \neg\neg \forall x(A(x) \vee \neg A(x))$.

1.8 Kripke semantics for intuitionistic logic

Gödel (1932) showed that there cannot be a finite-valued truth-conditional semantics for intuitionistic logic.

In the following we consider Kripke semantics for intuitionistic logic. Kripke semantics is a so-called *possible-worlds semantics*, which is not truth-conditional. The calculus NI is sound and complete for this semantics.

*possible-worlds
semantics*

In order to motivate this semantics we first consider the actions of an *idealised mathematician*, who is in intuitionism also referred to as a *creating subject*; we then observe the situation in the presence of a weak counterexample for $A \vee \neg A$.

At a given moment the idealised mathematician has a certain knowledge; that is, there is a set of assertions which are accepted as being valid at that moment. Over time, our mathematician can extend this knowledge in different directions. The possible states of the mathematician that correspond to different extensions of knowledge should thus not be understood as being linearly ordered; instead these states form a *partial order*. A partial order is reflexive (each state is related to itself), antisymmetric (two different states cannot at the same time occur before the respective other) and transitive; the order is partial, since two different states need not necessarily be related to each other (there exist alternative extensions of knowledge). The knowledge of the idealised mathematician is assumed to be *monotone* w.r.t. later moments in time. This property is also called *persistence*, i.e. knowledge cannot get lost.

partial order

monotonicity

The idea of a temporal order of states serves only as an illustration here. What is essential is the idea that states which represent knowledge suggest a partial order with monotonicity.

The logical constants are now understood in such a way that the interpretation of a complex statement depends on the interpretations of its sub-statements:

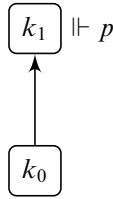
- For example, if the idealised mathematician has in state k accepted p as valid and has accepted also q as valid, then $p \wedge q$ is valid in k as well.
- Correspondingly, the statement $p \vee q$ is valid in k iff the statement p is valid in p or the statement q is valid in k .

An implication $p \rightarrow q$ might also be seen to be valid in a state k if in k it is neither known whether p is valid nor whether q is valid. For example, let p be the statement “a series of 1000 ones occurs in the decimal expansion of π ”, and let q be the statement “a series of 999 ones occurs in the decimal expansion of π ”. Even if we do neither know p nor q in state k , we nevertheless know that $p \rightarrow q$ must hold in k . Now we consider a state k' that extends our knowledge by p . Due to monotonicity $p \rightarrow q$ holds in k' as well; consequently, also q must hold in k' . On the other hand we have that an implication $p \rightarrow q$ holds in a state k , if in every extension of k , in which p holds, also q holds. By reflexivity this includes k as a trivial extension.

- Thus in a state k an implication $p \rightarrow q$ holds iff in every extension k' of k (including k) we have: If p holds, then q holds as well.
- There is no state k in which \perp holds. This follows in the described setting from the fact that our idealised mathematician has knowledge at any given moment in time.

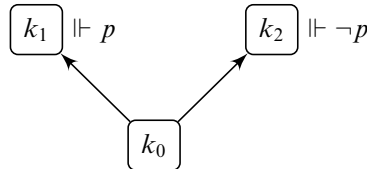
In order to illustrate this we now consider the situation for a weak counterexample for $A \vee \neg A$. We presuppose that in the present state k_0 the statement p is still undecided. It is, however, not impossible that a proof of p is found at a later moment (state k_1). This

situation can be depicted as follows (where we write $\Vdash p$ for “ p holds”, and where we omit arrows representing reflexivity):



In state k_0 we neither know whether p holds nor whether p does not hold. In k_0 we are also unable to assert $\neg p$, since there can be a state in which p holds (namely k_1). Thus also $p \vee \neg p$ cannot hold in state k_0 . However, $\neg\neg p$ holds in k_0 , since there is no state after k_0 in which $\neg p$ would hold.

Since p is still undecided in k_0 , it could happen, however, that at a later moment k_2 we find a proof of $\neg p$. We therefore have the following situation:



We still do not know in k_0 whether $\neg p$ holds. Moreover, $\neg\neg p$ cannot hold in k_0 . In this case, \perp would also have to hold in every state after k_2 (including k_2), which cannot be the case. Hence, also $\neg\neg p \vee \neg p$ does not hold in k_0 .

We now give a formal definition of Kripke semantics. We use the notion of a *model* in the neutral sense of *structure*; that is, a model describes a certain situation in which a formula A can either be valid or invalid. In the first case we will say that A is valid in the considered model, and in the second case that the model is a countermodel for A .

Definition 1.27 A *Kripke-model* is a triple $\mathcal{K} := \langle K, \leq, \Vdash \rangle$, consisting of a *frame* $\langle K, \leq \rangle$ and a *valuation* \Vdash . *Kripke-model*

– The *frame* $\langle K, \leq \rangle$ comprises a non-empty *set of states* K and a *partial order relation* \leq on $K \times K$. *frame*

(Hence the relation \leq is reflexive, antisymmetric and transitive; it is partial, since not all elements of K need to relate to each other.)

We call the elements of K *states* $k_0, k_1, \dots, k, k', k'', \dots$ *states*

Frames $\langle K, \leq \rangle$ are thus non-empty partially ordered sets of states.

(States are also called *nodes* or *possible worlds*. We also write $k' \geq k$ instead of $k \leq k'$; one says e.g. “ k' extends k ” or “ k sees k' ”.)

– A *valuation* \Vdash (read: *forces*; *forcing relation*) is a relation on $K \times \text{PV}$, i.e. between states $k \in K$ and proposition letters $A \in \text{PV}$, which obeys the following *monotonicity condition*: *valuation monotonicity condition*

$$\text{If } k \Vdash A \text{ and } k' \geq k, \text{ then } k' \Vdash A.$$

For formulas that are not proposition letters we extend the forcing relation \Vdash by the following clauses:

$$\begin{aligned} k \Vdash A \wedge B & : \iff k \Vdash A \text{ and } k \Vdash B \\ k \Vdash A \vee B & : \iff k \Vdash A \text{ or } k \Vdash B \\ k \Vdash A \rightarrow B & : \iff \text{For all } k' \geq k: \text{if } k' \Vdash A, \text{ then } k' \Vdash B \\ \text{not } k \Vdash \perp & \end{aligned}$$

(The last clause is equivalent to $k \not\Vdash \perp$; that is, there is no element k in K , such that $k \Vdash \perp$.)

A formula A is called *valid in k* iff $k \Vdash A$.

valid in k

Lemma 1.28 (i) $k \Vdash \neg A$ iff for all states $k' \geq k$: $k' \not\Vdash A$.

(ii) $k \Vdash \neg\neg A$ iff there exists for all states $k' \geq k$ a state $k'' \geq k'$, such that $k'' \Vdash A$.

Proof. (i) It is

$$\begin{aligned} k \Vdash \neg A & \iff k \Vdash A \rightarrow \perp \\ & \iff \text{for all } k' \geq k: \text{if } k' \Vdash A, \text{ then } k' \Vdash \perp & (*) \\ & \iff \text{for all } k' \geq k: k' \not\Vdash A \end{aligned}$$

In the last step, “ \implies ” holds, since in every model for each k : $k \not\Vdash \perp$. (From $(*)$ follows

$$\text{for all } k' \geq k: \text{if } k' \not\Vdash \perp, \text{ then } k' \not\Vdash A$$

from which we obtain with $k \not\Vdash \perp$ (for all k): $k' \not\Vdash A$, for all $k' \geq k$.) The opposite direction is obtained with *ex falso*.

(ii) It is

$$\begin{aligned} k \Vdash \neg\neg A & \iff \text{for all } k' \geq k: k' \not\Vdash \neg A \\ & \iff \text{for all } k' \geq k \text{ it does not hold that for all } k'' \geq k': k'' \not\Vdash A \\ & \iff \text{for all } k' \geq k \text{ there exists a } k'' \geq k', \text{ such that } k'' \Vdash A \end{aligned}$$

In the last step, “ \implies ” holds only classically (as long as we do not restrict ourselves to finite models). QED

Remark. The comment at the end of the proof of (ii) indicates that it can make a difference whether Kripke semantics is treated from a classical or from a constructivistic point of view.

Lemma 1.29 (Monotonicity) For all $k, k' \in K$ we have: If $k \Vdash A$ and $k' \geq k$, then $k' \Vdash A$. (That is, the monotonicity condition that we have imposed on proposition letters holds as well for arbitrary formulas A .)

Proof. By induction on the structure of A .

Induction base: Let A be atomic. If $A \equiv \perp$, then the assertion holds trivially, since $k \not\Vdash \perp$. If A is a proposition letter, then the assertion holds due to the monotonicity condition in Definition 1.27.

Induction hypothesis: The assertion holds for formulas B and C .

Induction step: Case $A \equiv B \wedge C$: Assume $k \Vdash B \wedge C$ and $k' \geq k$. We have $k \Vdash B \wedge C$ iff $k \Vdash B$ and $k \Vdash C$. By the induction hypothesis we have then also $k' \Vdash B$ and $k' \Vdash C$, hence $k' \Vdash B \wedge C$.

Case $A \equiv B \vee C$: Analogously to the former case.

Case $A \equiv B \rightarrow C$: Assume $k \Vdash B \rightarrow C$ and $k' \geq k$. Consider an arbitrary state k'' , such that $k'' \geq k'$ and $k'' \Vdash B$. By transitivity of \leq we have $k'' \geq k$. Since $k \Vdash B \rightarrow C$ also $k'' \Vdash C$ must hold. Therefore for all states $k'' \geq k'$: If $k'' \Vdash B$, then $k'' \Vdash C$, i.e. $k' \Vdash B \rightarrow C$. QED

Definition 1.30 Let $\mathcal{K} := \langle K, \leq, \Vdash \rangle$ be a Kripke-model. We define

Validity in a model:

$$\mathcal{K} \Vdash A \iff \text{For all } k \in K: k \Vdash A$$

validity in a model

Kripke-validity:

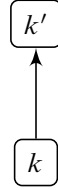
$$\Vdash A \iff \text{For all models } \mathcal{K}: \mathcal{K} \Vdash A$$

Kripke-validity

Remark. If k_0 is the smallest state in $\langle K, \leq \rangle$, then by monotonicity (Lemma 1.29): A is valid in \mathcal{K} iff A is valid in k_0 .

Kripke-models $\mathcal{K} = \langle K, \leq, \Vdash \rangle$ can also be presented diagrammatically:

- We write states $k \in K$ as boxes \boxed{k} .
- If $k \leq k'$ holds for different states k, k' , then we write



We do not use arrows to indicate reflexivity or transitivity of \leq ; however, these properties are always presupposed.

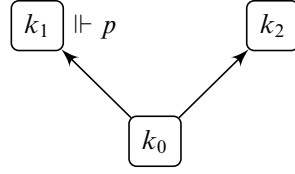
- Proposition letters that are valid in states $k \in K$ according to the valuation \Vdash are written next to the boxes for the respective states; for example, if $k \Vdash p$ holds, then we write $\boxed{k} \Vdash p$.

Examples. (i) We show $\not\Vdash \neg\neg p \vee \neg p$. That is, we have to present a Kripke-model \mathcal{K} , such that $\mathcal{K} \not\Vdash \neg\neg p \vee \neg p$.

We consider the model $\mathcal{K}_1 = \langle K, \leq, \Vdash \rangle$ with

- $K = \{k_0, k_1, k_2\}$;
- $k_0 \leq k_1$ and $k_0 \leq k_2$ (besides $k_i \leq k_i$, which we do not note explicitly);
- $k_1 \Vdash p$.

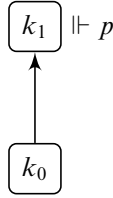
Presented as a diagram, this model \mathcal{K}_1 looks as follows:



Since $k_1 \Vdash p$, we have $k_0 \not\Vdash \neg p$. For k_2 we have only $k_2 \geq k_2$, and $k_2 \not\Vdash p$; by Lemma 1.28 (i) it holds: $k_2 \Vdash \neg p$. Furthermore, $k_0 \not\Vdash \neg \neg p$.

Thus $k_0 \not\Vdash \neg \neg p \vee \neg p$ holds, and therefore $\mathcal{K}_1 \not\Vdash \neg \neg p \vee \neg p$. That is, \mathcal{K}_1 is a countermodel for $\neg \neg p \vee \neg p$; hence $\not\Vdash \neg \neg p \vee \neg p$.

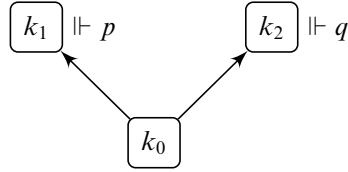
(ii) We show $\not\Vdash \neg \neg p \rightarrow p$. In the Kripke-model \mathcal{K}_2



we have that $k_0 \not\Vdash p$, and since $k_1 \Vdash p$ we have $k_0 \Vdash \neg \neg p$ (cp. Lemma 1.28 (ii)).

Hence $k_0 \not\Vdash \neg \neg p \rightarrow p$ holds, and therefore $\mathcal{K}_2 \not\Vdash \neg \neg p \rightarrow p$.

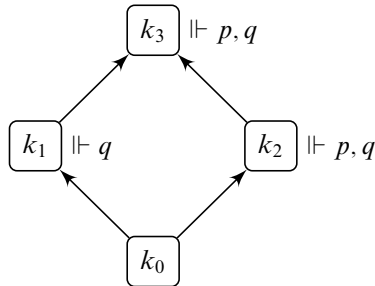
(iii) We show $\not\Vdash (p \rightarrow q) \vee (q \rightarrow p)$. A countermodel is \mathcal{K}_3



Since $k_1 \not\Vdash q$, we have $k_0 \not\Vdash p \rightarrow q$, and since $k_2 \not\Vdash p$, we have $k_0 \not\Vdash q \rightarrow p$.

Consequently, $k_0 \not\Vdash (p \rightarrow q) \vee (q \rightarrow p)$ holds, and thus $\mathcal{K}_3 \not\Vdash (p \rightarrow q) \vee (q \rightarrow p)$.

(iv) We show $\not\Vdash (p \rightarrow q) \rightarrow (\neg p \vee q)$. In the Kripke-model \mathcal{K}_4

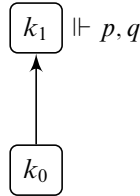


we have $k_3 \Vdash p \rightarrow q$, and $k_1 \Vdash p \rightarrow q$ holds because of $k_1 \Vdash q$ and $k_3 \Vdash q$. Hence,

with $k_0 \not\models p$ and $k_2 \models q$ also $k_0 \models p \rightarrow q$ holds. However, $k_0 \not\models \neg p$ holds due to $k_2 \models p$, and since $k_0 \not\models q$, we have $k_0 \not\models \neg p \vee q$. Therefore $k_0 \not\models (p \rightarrow q) \rightarrow (\neg p \vee q)$ holds, and thus $\mathcal{K}_4 \not\models (p \rightarrow q) \rightarrow (\neg p \vee q)$.

The Kripke-model \mathcal{K}_4 illustrates that Kripke-models need not have the form of trees.

A smaller countermodel is \mathcal{K}_5 :



Although $k_1 \models (p \rightarrow q) \rightarrow (\neg p \vee q)$, we have $k_0 \models p \rightarrow q$ but $k_0 \not\models \neg p \vee q$. Thus $k_0 \not\models (p \rightarrow q) \rightarrow (\neg p \vee q)$, and hence $\mathcal{K}_5 \not\models (p \rightarrow q) \rightarrow (\neg p \vee q)$.

Definition 1.31 A formula A is a *logical consequence* of Γ (formally: $\Gamma \Vdash A$), if for every Kripke-model $\mathcal{K} = \langle K, \leq, \Vdash \rangle$ in every state $k \in K$, in which Γ holds, also A holds. That is: *logical consequence*

$$\Gamma \Vdash A \iff \text{If } k \Vdash \Gamma, \text{ then } k \Vdash A, \text{ for every } \mathcal{K}.$$

(Where $k \Vdash \Gamma$ iff for all $B \in \Gamma$: $k \Vdash B$.)

Theorem 1.32 (Soundness and completeness) $\Gamma \vdash_{\text{NI}} A \iff \Gamma \Vdash A$.

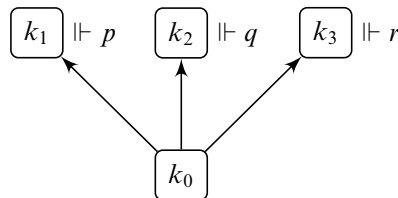
Proof. See van Dalen (2013, Ch. 5). (The result goes back to Kripke (1965), who proved soundness and completeness for an alternative calculus for intuitionistic logic.) QED

That a formula A is not derivable in NI ($\not\vdash_{\text{NI}} A$) can thus be shown by presenting a Kripke-countermodel for the formula (or for an instance of the formula). In this case $\not\models A$ holds, and with soundness follows $\not\vdash_{\text{NI}} A$.

Example. We consider the claim

$$\neg p \rightarrow (q \vee r) \Vdash (\neg p \rightarrow q) \vee (\neg p \rightarrow r)$$

where the premiss $\neg p \rightarrow (q \vee r)$ is an instance of the premiss of Harrop's rule, and where the conclusion $(\neg p \rightarrow q) \vee (\neg p \rightarrow r)$ is an instance of the conclusion of this rule. The Kripke-model \mathcal{K}



refutes this claim.

- In \mathcal{K} the premiss holds: Since $k_1 \Vdash p$, also $k_0 \not\Vdash \neg p$, and thus $k_0 \Vdash \neg p \rightarrow (q \vee r)$. Hence by monotonicity $\neg p \rightarrow (q \vee r)$ holds in every state, i.e. $\mathcal{K} \Vdash \neg p \rightarrow (q \vee r)$.
(For k_1, k_2 and k_3 one can also argue as follows: Since $k_1 \not\Vdash \neg p$, also $k_1 \Vdash \neg p \rightarrow (q \vee r)$. Since $k_2 \Vdash q$, we have $k_2 \Vdash q \vee r$, and therefore $k_2 \Vdash \neg p \rightarrow (q \vee r)$. Correspondingly for k_3 , since $k_3 \Vdash r$.)
- But \mathcal{K} is a countermodel of the conclusion: It is $k_3 \not\Vdash \neg p \rightarrow q$ and $k_2 \not\Vdash \neg p \rightarrow r$. Thus in k_0 neither $\neg p \rightarrow q$ nor $\neg p \rightarrow r$ holds, i.e. $k_0 \not\Vdash (\neg p \rightarrow q) \vee (\neg p \rightarrow r)$. Therefore $\mathcal{K} \not\Vdash (\neg p \rightarrow q) \vee (\neg p \rightarrow r)$.

Consequently

$$\neg A \rightarrow (B \vee C) \not\Vdash (\neg A \rightarrow B) \vee (\neg A \rightarrow C)$$

By soundness of NI we get

$$\neg A \rightarrow (B \vee C) \not\vdash_{\text{NI}} (\neg A \rightarrow B) \vee (\neg A \rightarrow C)$$

That is, Harrop's rule is not derivable in NI. This completes the proof of Theorem 1.10.

2 Proof-theoretic semantics

2.1 Introduction

The idea of proof-theoretic semantics goes back to G. Gentzen, D. Prawitz (born 1936) and M. Dummett (1925–2011). The meaning of logical constants $\wedge, \vee, \rightarrow, \neg, \perp, \forall, \exists, \dots$ is given in terms of the notion of *proof* or *derivation structure*, where the notion of proof is formally analyzed by derivations in natural deduction.

In one prominent approach in proof-theoretic semantics the introduction rules determine the meaning of the logical constants. A formula is *valid*, if it is provable with introduction rules alone. Elimination rules are then justified on the basis of such proofs.

Example. Consider the introduction rule for conjunction

$$\frac{A_1 \quad A_2}{A_1 \wedge A_2}$$

It tells us that $A_1 \wedge A_2$ is valid, if there is a proof of A_1 and a proof of A_2 .

Suppose there is a proof of $A_1 \wedge A_2$. By the validity of $A_1 \wedge A_2$ there must be proofs of both A_1 and A_2 . Hence the elimination rule for conjunction

$$\frac{A_1 \wedge A_2}{A_i}$$

is justified.

A difficulty arises with implication. Consider the introduction rule

$$\frac{[A]}{A \rightarrow B}$$

It says that $A \rightarrow B$ is valid, if there exists a derivation of B from assumptions A .

In order to obtain B , elimination rules may have to be applied to A , including the elimination rule for implication that we want to justify.

A solution consists in considering a proof of $A \rightarrow B$ as a derivation \mathcal{D} of B from A , which *together with a proof \mathcal{P} of A* yields a proof (without elimination rules) of B . “Yields” here means that there are reductions; for example:

$$\frac{\frac{A}{\mathcal{D}} \quad \frac{B}{A \rightarrow B}}{B} \quad \mathcal{P} \quad \text{reduces to} \quad \frac{A}{\mathcal{D}} \quad B$$

Assumptions A are thus conceived as *place holders* for proofs of A . (As we saw in Section 1.6, reductions of derivations can be compared to λ -calculus, where $(\lambda x.M)N$ β -reduces to $M[N/x]$.)

The validity of atomic formulas is given by proofs in atomic systems S , which are, for example, sets of production rules

$$\frac{a_1 \quad \dots \quad a_n}{b}$$

for atomic formulas a_i, b , where $0 \leq i \leq n$.

Example. Consider the atomic system

$$S = \left\{ \frac{}{a}, \frac{}{b}, \frac{a \quad b}{c}, \frac{b \quad c}{d} \right\}.$$

A proof of d in S is

$$\frac{\frac{}{b} \quad \frac{\frac{}{a} \quad \frac{}{b}}{c}}{d}}$$

Atomic systems S thus play the role of *structures* in this kind of semantics.

Definition 2.1 In Prawitz's semantics, the S -validity of a derivation structure is defined as follows:

S-validity of a derivation structure

- (0) Every closed (i.e., without assumptions) derivation in S is S -valid.
- (1) A closed canonical derivation structure (i.e., ending with an introduction rule) is S -valid, if its immediate substructures are S -valid.
- (2) A closed non-canonical derivation structure is S -valid, if it reduces to an S -valid canonical derivation structure.
- (3) An open derivation structure

$$\frac{A_1 \dots A_n \quad \mathcal{D}}{B}$$

from assumptions A_1, \dots, A_n is S -valid, if for every extension $S' \supseteq S$ and for every list of closed S' -valid derivation structures

$$\frac{\mathcal{P}_i}{A_i}$$

the derivation structure

$$\frac{\mathcal{P}_1 \quad \mathcal{P}_n \quad A_1 \dots A_n \quad \mathcal{D}}{B}$$

is S' -valid.

Prawitz (1971, 2014) made the following completeness conjecture:

Conjecture 2.2 *Intuitionistic logic NI is sound and complete for S-validity:*

$$A_1, \dots, A_n \vdash_{\text{NI}} B \iff \text{for every } S \text{ there is an } S\text{-valid derivation structure from } A_1, \dots, A_n \text{ to } B.$$

According to Prawitz, there is a strong intuition behind this idea: One cannot imagine stronger elimination rules than the standard ones, which are valid. However, this does not exclude the validity of other rules that are not derivable in NI. In the following, we will investigate the completeness conjecture.

Literature

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2.2 Incompleteness of intuitionistic logic with respect to proof-theoretic semantics

In (Piecha, de Campos Sanz and Schroeder-Heister, 2015) it was shown that intuitionistic propositional logic is semantically incomplete for certain notions of proof-theoretic validity (see also Piecha, 2016). This questioned a claim by Prawitz, who was the first to propose a proof-theoretic notion of validity, and claimed completeness for it (Prawitz, 1973, 2014). In (Piecha and Schroeder-Heister, 2019), which we replicate in the following, we put these and related results into a more general context.

We consider the calculus of intuitionistic propositional logic (IPC) and formulate, in Section 2.3, abstract semantic conditions for proof-theoretic validity which every proof-theoretic semantics is supposed to satisfy. They are so general that they cover most semantic approaches, even classical truth-theoretic semantics. In Section 2.4 we show that if in addition certain more special conditions are assumed, IPC fails to be complete. In Section 2.5 we study several concrete notions of proof-theoretic validity and investigate which of the conditions rendering IPC incomplete they meet. In Section 2.6 we consider Goldfarb’s (2016) semantic approach for which IPC is complete, but which is not a ‘standard’ notion of proof-theoretic validity compared to those proposed by Prawitz.

2.3 Proof-theoretic validity in an abstract setting

We consider the intuitionistic propositional calculus (IPC) with the standard constants \wedge , \vee , \rightarrow and \neg .

In validity-based proof-theoretic semantics, one normally considers the validity of atomic

formulas to be determined by an atomic system S . This atomic system corresponds to what in truth-theoretic semantics is a structure \mathfrak{A} (in propositional logic \mathfrak{A} reduces to a truth-valuation of propositional variables). Via semantical clauses for the connectives, an atomic base then inductively determines the validity with respect to S , in short: S -validity $\models_S A$ of a formula A , as well as S -consequence $\Gamma \models_S A$ between a set of formulas Γ and a formula A . In our abstract setting we completely leave open the nature of S and just assume that a nonempty finite or infinite set \mathcal{S} of entities called *bases* is given. We furthermore assume that for each base $S \in \mathcal{S}$ a consequence relation \models_S is given, that is, a relation $\Gamma \models_S A$ between a set Γ of formulas and a formula A , such that the following conditions are satisfied:

(Reflexivity) $A \models_S A$.

(Monotonicity) If $\Gamma \models_S A$, then $\Gamma, B \models_S A$.

(Transitivity) If $\Gamma \models_S A$ and $\Gamma, A \models_S B$, then $\Gamma \models_S B$.

By $\Gamma \models_S \Delta$ we mean that for all $A \in \Delta$: $\Gamma \models_S A$.

The S -validity of A (i.e., $\models_S A$) is identified with the fact that A is an S -consequence of the empty set ($\emptyset \models_S A$). We expect that S -validity respects the intended meaning of the logical connectives, where we take only the positive connectives of conjunction, disjunction and implication into account (see, however, the remark on negation at the end of Section 2.4):

(\wedge) $\models_S A \wedge B \iff \models_S A$ and $\models_S B$.

(\vee) $\models_S A \vee B \iff \models_S A$ or $\models_S B$.

(\rightarrow) $\models_S A \rightarrow B \iff A \models_S B$.

The relation of *universal* or *logical consequence* is, as usual, understood as transmitting S -validity from the antecedents to the consequent. In our abstract setting, this is achieved by assuming that besides \models_S , there is a consequence relation \models available, such that the following two conditions are satisfied:

(\models) $\Gamma \models A \iff$ For all $S \in \mathcal{S}$: ($\models_S \Gamma \implies \models_S A$).

(\models') If $\Gamma \models A$, then $\Gamma \models_S A$ for any S .

Condition (\models') expresses that \models is a generalisation of \models_S . It follows from condition (\models), if we assume that ($\models_S \Gamma \implies \models_S A$) implies $\Gamma \models_S A$, which we do not, however, want to presuppose as a necessary condition.

The five conditions (\wedge), (\vee), (\rightarrow), (\models), (\models') constitute our abstract notion of a semantics. That is, if a non-empty set \mathcal{S} of bases, and consequence relations \models_S (for each $S \in \mathcal{S}$) and \models are given such that these five conditions are met, we speak of a validity-based proof-theoretic semantics in the abstract sense, in short: a *semantics*.

Note that these conditions are also satisfied by classical truth-theoretic (or model-theoretic) semantics, if one defines $\Gamma \models_{\mathfrak{A}} A$ to mean: If $\mathfrak{A} \models \Gamma$, then $\mathfrak{A} \models A$.

Most concrete versions of proof-theoretic semantics, including those considered by Prawitz, are semantics in this abstract sense. Deviant proof-theoretic semantics we are aware of are only those, which challenge the fact that \models_S or \models are standard consequence relations, for example, by changing the principles of monotonicity or transitivity. However, even those semantics could be discussed in a modified framework of our kind (see Piecha and Schroeder-Heister, 2016b, 2017), but this is not our topic here.

S-validity
S-consequence

logical consequence

The notions of valid rule and derivable rule will be used in what follows. It is \vdash the derivability relation of IPC.

Definition 2.3 A rule

$$\frac{A_1 \quad \dots \quad A_n}{B}$$

is called *valid* iff $A_1, \dots, A_n \models B$. It is called *derivable* iff $A_1, \dots, A_n \vdash B$.

The following standard results will play a prominent role.

Lemma 2.4 (i) *Harrop's rule* (see [Harrop, 1960](#))

Harrop's rule

$$\frac{\neg A \rightarrow (B_1 \vee B_2)}{(\neg A \rightarrow B_1) \vee (\neg A \rightarrow B_2)}$$

is not derivable in IPC (though it is admissible; see [Theorem 1.10](#), cf. [Iemhoff, 2001](#)).

(ii) For \vdash the following generalised disjunction property holds:

generalised disjunction property

GDP(\vdash) If $\Gamma \vdash A \vee B$, where \vee does not occur in Γ , then $\Gamma \vdash A$ or $\Gamma \vdash B$.

(iii) Disjunctions can always be removed from a negated formula, by the following principles:

\vee -removal

$$(\vee\text{-removal}) \left\{ \begin{array}{l} \neg(A \vee B) \dashv\vdash \neg A \wedge \neg B; \\ \neg(A \wedge B) \dashv\vdash \neg(\neg A \wedge \neg B); \\ \neg(A \rightarrow B) \dashv\vdash \neg\neg A \wedge \neg B. \end{array} \right.$$

Remark. A stronger version of GDP(\vdash), in which it is only assumed that \vee does not occur *positively* in Γ , was proven by [Harrop \(1960\)](#) and, in a natural deduction setting, by [Prawitz \(1965\)](#). We here only need the stated weaker version GDP(\vdash), where the stronger assumption is made that \vee does not occur in Γ at all.

Soundness and completeness of IPC are understood in the usual way.

Definition 2.5 *Soundness* of IPC means:

soundness

For any Γ and A : if $\Gamma \vdash A$, then $\Gamma \models A$;

and *completeness* of IPC means:

completeness

For any Γ and A : if $\Gamma \models A$, then $\Gamma \vdash A$.

Lemma 2.6 In view of (\models'), soundness implies the following: For any Γ and A , if $\Gamma \vdash A$, then $\Gamma \models_S A$ for any S .

2.4 Conditions for incompleteness of IPC

We show that IPC turns out to be incomplete, if the semantics given satisfies certain special conditions.

A crucial role is played by the *generalised disjunction property*, which was stated above for the derivability relation \vdash of IPC. We are particularly interested in its semantical version. Therefore we formulate it for an arbitrary consequence relation \Vdash in the language of IPC:

GDP(\Vdash) If $\Gamma \Vdash A \vee B$, where \vee does not occur in Γ , then $\Gamma \Vdash A$ or $\Gamma \Vdash B$.

We assume in the following that a semantics in the abstract sense of Section 2.3 is given, with respect to which IPC is sound.

Lemma 2.7 *If $\text{GDP}(\models_S)$ for every S , then Harrop's rule*

$$\frac{\neg A \rightarrow (B_1 \vee B_2)}{(\neg A \rightarrow B_1) \vee (\neg A \rightarrow B_2)}$$

is valid.

Proof.

$$\begin{aligned} \models_S \neg A \rightarrow (B_1 \vee B_2) &\implies \neg A \models_S B_1 \vee B_2; \text{ by } (\rightarrow) \\ &\implies A' \models_S B_1 \vee B_2 \text{ for some } \vee\text{-free formula } A' \text{ such} \\ &\quad \text{that } A' \Vdash \neg A; \text{ by } (\vee\text{-removal}), \text{ Lemma 2.6 and} \\ &\quad \text{transitivity of } \models_S \\ &\implies A' \models_S B_i \text{ for } i = 1 \text{ or } 2; \text{ by } \text{GDP}(\models_S) \\ &\implies \neg A \models_S B_i; \text{ by } (\vee\text{-removal}), \text{ Lemma 2.6 and transitivity of } \models_S \\ &\implies \models_S \neg A \rightarrow B_i; \text{ by } (\rightarrow) \\ &\implies \models_S (\neg A \rightarrow B_1) \vee (\neg A \rightarrow B_2); \text{ by } (\vee). \end{aligned}$$

As this holds for any S , condition (\models) gives us $\neg A \rightarrow (B_1 \vee B_2) \models (\neg A \rightarrow B_1) \vee (\neg A \rightarrow B_2)$. QED

This means that if we have $\text{GDP}(\models_S)$ for every S , then completeness fails, since Harrop's rule is not derivable in IPC.

Now consider the following property:

(Export) For every base S there is a set of \vee -free formulas S^* such that for all Γ and A :
 $\Gamma \models_S A \iff \Gamma, S^* \models A$.

This condition means that the base S of non-logical consequence \models_S can be 'exported' as a set of assumptions (S^*) of logical consequence \models .

Lemma 2.8 *Assume completeness of IPC. Then Export implies $\text{GDP}(\models_S)$ for every S .*

Proof. Suppose completeness holds, and \vee does not occur in Γ . Then we obtain $\text{GDP}(\models_S)$ as follows:

$$\begin{aligned} \Gamma \models_S A_1 \vee A_2 &\implies \Gamma, S^* \models A_1 \vee A_2; \text{ by Export} \\ &\implies \Gamma, S^* \vdash A_1 \vee A_2; \text{ by completeness} \\ &\implies \Gamma, S^* \vdash A_i \text{ for } i = 1 \text{ or } 2; \text{ by } \text{GDP}(\vdash) \\ &\implies \Gamma, S^* \models A_i \text{ for } i = 1 \text{ or } 2; \text{ by soundness} \\ &\implies \Gamma \models_S A_i \text{ for } i = 1 \text{ or } 2; \text{ by Export.} \end{aligned} \quad \text{QED}$$

This means that assuming completeness we obtain that Harrop's rule is valid. Again assuming completeness, this implies that Harrop's rule is derivable in IPC, which is not the case. Thus we have refuted completeness.

Note that we have not shown $\text{GDP}(\vDash_S)$ outright, but only under the assumption of completeness, which is, however, sufficient to refute completeness.

Now consider the condition

$$(\vDash_S) \quad \Gamma \vDash_S A \iff (\vDash_S \Gamma \implies \vDash_S A).$$

We obtain the following interesting result. (Note that the direction from left to right in (\vDash_S) follows already from the fact that \vDash_S is a consequence relation.)

Lemma 2.9 *Suppose (\vDash_S) . Then, using classical logic in the metalanguage, $\text{GDP}(\vDash_S)$ can be proved.*

Proof.

$$\begin{aligned} \Gamma \vDash_S A \vee B & \\ \implies (\vDash_S \Gamma \implies \vDash_S A \vee B); & \text{ by } (\vDash_S) \\ \implies (\vDash_S \Gamma \implies (\vDash_S A \text{ or } \vDash_S B)); & \text{ by } (\vee) \\ \implies (\vDash_S \Gamma \implies \vDash_S A) \text{ or } & (\vDash_S \Gamma \implies \vDash_S B); \text{ classical metalanguage} \\ \implies \Gamma \vDash_S A \text{ or } \Gamma \vDash_S B; & \text{ by } (\vDash_S). \end{aligned}$$

(We do not need the supposition that \vee does not occur in Γ .)

QED

However, to show $\text{GDP}(\vDash_S)$ we do not have to rely on a classical metalanguage, if we can make use of the following principle:

(Import) For every S , every \vee -free Γ and every A there is a base $S + \Gamma$ such that:
 $\Gamma \vDash_S A \iff \vDash_{S+\Gamma} A$.

This condition means that any disjunction-free set of assumptions of logical consequence \vDash can be ‘imported’ into a base S of non-logical consequence \vDash_S .

Lemma 2.10 *Import implies $\text{GDP}(\vDash_S)$.*

Proof. Suppose \vee does not occur in Γ .

$$\begin{aligned} \Gamma \vDash_S A \vee B & \implies \vDash_{S+\Gamma} A \vee B; \text{ by Import} \\ & \implies \vDash_{S+\Gamma} A \text{ or } \vDash_{S+\Gamma} B; \text{ by } (\vee) \\ & \implies \Gamma \vDash_S A \text{ or } \Gamma \vDash_S B; \text{ by Import.} \end{aligned} \quad \text{QED}$$

Import is a condition that played a crucial role in (Piecha, de Campos Sanz and Schroeder-Heister, 2015), where we considered higher-level inference rules, but it is not needed in the general setting here.

Summary. For any semantics with respect to which IPC is sound, we have shown the following.

Theorem 2.11

- (i) $\text{GDP}(\vDash_S)$ for all $S \implies$ validity of Harrop’s rule,
thus: $\text{GDP}(\vDash_S)$ for all $S \implies$ incompleteness.
- (ii) *Export + completeness $\implies \text{GDP}(\vDash_S)$ for all S ,*
thus: Export \implies incompleteness.

- (iii) $\text{Condition } (\models_S) \implies \text{GDP}(\models_S)$ for all S (using classical metalanguage),
 thus: $\text{Condition } (\models_S) \implies \text{incompleteness}$.
- (iv) $\text{Import} \implies \text{GDP}(\models_S)$ for all S ,
 thus: $\text{Import} \implies \text{incompleteness}$.

Remark. Theorem 2.11(i) and (ii) continue to hold if for $\text{GDP}(\models_S)$ and $\text{GDP}(\vdash)$ it is only assumed that \forall does not occur *positively* in Γ .

Therefore, in order to establish the incompleteness of IPC for a semantics, for which IPC is sound, we only need to establish one of the four conditions stated in the clauses of Theorem 2.11.

Remark on negation. The counterexample to completeness, which is based on Harrop's rule, relies heavily on negation being available, in addition to implication and disjunction. The reason why there is no principle explicitly required of negation in our abstract semantics in Section 2.3 nor in the conditions for incompleteness in the current section is due to the fact that for our incompleteness results we throughout assume soundness, which means that the principles governing negation with respect to \models and \models_S are inherited from those derivable in IPC. If we wanted to establish incompleteness results for semantics for which IPC is not even required to be sound, we would have to formulate explicit semantic principles for negation or absurdity.

2.5 Incompleteness results for concrete proof-theoretic semantics

By a concrete semantics we understand a semantical approach in which bases S are explicitly specified, and in which consequence relations \models and \models_S are defined in such a way that the result is a semantics in the abstract sense of Section 2.3. The specification of \models and \models_S can proceed by explicit or inductive definition. Another possibility would be to start with a different fundamental concept in terms of which \models_S is then defined. The latter is the case in Prawitz's definition of the validity of a derivation or derivation structure, on which the definition of valid consequence is based.

We consider certain types of concrete semantics. All of them are proof-theoretic semantics in the sense that bases are understood as atomic systems generating valid atomic formulas by means of inference rules.

Definition 2.12 An *atomic system* S is a deductive system with rules of the form

atomic system

$$\frac{a_1 \quad \dots \quad a_n}{b}$$

where a_1, \dots, a_n, b are *atoms*. As a limiting case, n can be 0, in which case we have a rule without premisses, that is, an *axiom*. The set of rules may be empty, in which case S takes the form \emptyset .

The atoms a_1, \dots, a_n, b can be of a specific form different from the atomic formulas (= propositional variables) of IPC. In that case, in order to interpret IPC semantically, one would have to consider valuations which interpret propositional variables by such atoms. Alternatively, one might consider the atoms of an atomic system S to be just the propositional variables. By giving rules for propositional variables and a notion of

derivability $\vdash_S p$ of propositional variables p in S one obtains a way of interpreting propositional variables in an atomic system S in analogy to truth valuations in classical logic. This is how we proceed in the following. That is, the atomic formulas derived by an atomic system are propositional variables.

The S -validity of an atomic formula a is defined as the derivability of a in S :

$$(At) \quad \vDash_S a :\iff \vdash_S a.$$

The set \mathcal{S} of bases considered is the set of all atomic systems, where atomic systems are identified with the sets of their rules. The systems within \mathcal{S} are ordered in the usual way by set inclusion \subseteq .

Remark. Different kinds of concrete semantics are obtained from different kinds of atomic systems. Besides the kind of atomic systems considered here, one may, for example, consider systems of higher-level rules (see [Piecha, de Campos Sanz and Schroeder-Heister, 2015](#)). Furthermore, for a given kind of atomic systems different kinds of derivability relations \vdash_S can be examined. For example, an interpretation of atomic systems as definitions justifies additional principles for deductions in atomic systems, which yields a derivability relation which is no longer monotone with respect to extensions of atomic systems (see [Piecha and Schroeder-Heister, 2016b, 2017](#)). We will not treat such variants here.

Concrete semantics based on (any kind of) atomic systems can be classified into so-called extension semantics and non-extension semantics, depending on how they interpret implication.

In *extension semantics*, S -consequence is defined using extensions $S' \supseteq S$, for atomic systems S and S' :

extension semantics

$$(\vDash_S^{\text{ext}}) \quad \Gamma \vDash_S A :\iff \text{For all } S' \supseteq S: (\vDash_{S'} \Gamma \implies \vDash_{S'} A).$$

In *non-extension semantics*, S -consequence is just defined by

non-extension semantics

$$(\vDash_S) \quad \Gamma \vDash_S A :\iff (\vDash_S \Gamma \implies \vDash_S A).$$

Remark. Concerning (\rightarrow) this means that with (\vDash_S^{ext}) we have

$$\vDash_S A \rightarrow B \iff \text{For all } S' \supseteq S: (\vDash_{S'} A \implies \vDash_{S'} B),$$

whereas with (\vDash_S) we have $\vDash_S A \rightarrow B \iff (\vDash_S A \implies \vDash_S B)$.

Whether one should prefer one or the other kind of concrete semantics depends on how atomic systems are to be interpreted. Note that for non-extension semantics S -consequence fails to be monotone with respect to atomic systems, whereas extension semantics guarantee monotonicity (for details see [Piecha and Schroeder-Heister, 2016b, 2017](#)).

Remark. Obviously, in extension semantics \vDash_S is monotone with respect to atomic systems, that is, $\Gamma \vDash_S A \implies \Gamma \vDash_{S \cup S'} A$ for any S and S' .

Remark. In extension semantics we can strengthen (At) to

$$(At') \quad a_1, \dots, a_n \vDash_S a \iff a_1, \dots, a_n \vdash_S a.$$

The direction from right to left is trivial. For the direction from left to right we consider any extension S' of S , which has a_1, \dots, a_n as additional axioms. Then $a_1, \dots, a_n \vDash_S a$ implies $\vDash_{S'} a$, and thus $\vdash_{S'} a$ by (At). Now $\vdash_{S'} a$ means the same as $a_1, \dots, a_n \vdash_S a$.

Lemma 2.13 *For extension semantics we can establish*

(Export) *For every S there is a set of \forall -free formulas S^* such that for all Γ and A :*

$$\Gamma \vDash_S A \iff \Gamma, S^* \vDash A.$$

Proof. First note that every atomic system S can be represented by a set of \forall -free formulas S^* :

- (i) Axioms \overline{a} are represented by the atom a .
- (ii) Rules $\frac{a_1 \dots a_n}{b}$ are represented by formulas $a_1 \wedge \dots \wedge a_n \rightarrow b$.

Obviously $\vDash_S S^*$ holds true.

To show $\Gamma, S^* \vDash A \implies \Gamma \vDash_S A$ for sets S and S^* as described, assume $\Gamma, S^* \vDash A$. By monotonicity with respect to atomic systems (Remark 13), this implies $\Gamma, S^* \vDash_S A$ for any S . Since $\vDash_S S^*$, we get $\Gamma \vDash_S A$ by transitivity of \vDash_S .

To show $\Gamma \vDash_S A \implies \Gamma, S^* \vDash A$, assume $\Gamma \vDash_S A$. By (\vDash_S^{ext}) it holds that

$$\Gamma \vDash_S A \iff \text{For all } S_1: (\vDash_{S \cup S_1} \Gamma \implies \vDash_{S \cup S_1} A).$$

Now assume $\vDash_{S_2} \Gamma$, for an arbitrary S_2 . Thus $\vDash_{S \cup S_2} \Gamma$ by monotonicity with respect to atomic systems (Remark 13). Hence $\vDash_{S \cup S_2} A$, and $\Gamma \vDash_{S \cup S_2} A$ by (\vDash_S^{ext}).

Assuming $\vDash_{S_2} S^*$, we can infer $\Gamma \vDash_{S_2} A$ by using

$$(\vDash_{S_2} S^* \text{ and } \Gamma \vDash_{S \cup S_2} A) \implies \Gamma \vDash_{S_2} A.$$

We prove this implication by induction on the joint complexity of Γ and A .

For atomic formulas we have $\vDash_{S \cup S_2} a \iff \vdash_{S \cup S_2} a$ by (At), and $a_1, \dots, a_n \vDash_{S_2} b \iff a_1, \dots, a_n \vdash_{S_2} b$ by (At'). The latter implies for $\vDash_{S_2} S^*$ that all rules of S are derivable in S_2 . Hence $\vdash_{S_2} a$, and thus $\vDash_{S_2} a$ by (At).

For non-atomic formulas A we consider the case where A is an implication $B \rightarrow C$ (the cases where A has the form $B \wedge C$ or $B \vee C$ are similar):

$$\begin{aligned} \vDash_{S \cup S_2} B \rightarrow C &\iff B \vDash_{S \cup S_2} C; \text{ by } (\rightarrow) \\ &\implies B \vDash_{S_2} C; \text{ by } \vDash_{S_2} S^* \text{ and induction hypothesis} \\ &\implies \vDash_{S_2} B \rightarrow C; \text{ by } (\rightarrow). \end{aligned}$$

For S -consequence we have the following:

$$\begin{aligned} \Gamma \vDash_{S \cup S_2} A &\iff \text{For all } S_3 \supseteq (S \cup S_2): (\vDash_{S_3} \Gamma \implies \vDash_{S_3} A) \\ &\iff \text{For all } S_4: (\vDash_{S_4 \cup S \cup S_2} \Gamma \implies \vDash_{S_4 \cup S \cup S_2} A) \\ &\iff \text{For all } S_4: (\vDash_{S_4 \cup S_2} \Gamma \implies \vDash_{S_4 \cup S_2} A); \text{ by } \vDash_{S_4 \cup S_2} S^*, \text{ i.h.} \\ &\iff \Gamma \vDash_{S_2} A; \text{ by } (\vDash). \end{aligned}$$

Having assumed $\vDash_{S_2} \Gamma$ and $\vDash_{S_2} S^*$, we can conclude $\vDash_{S_2} A$, and thus $\Gamma, S^* \vDash A$ by (\vDash_S^{ext}). QED

Remark. For extension semantics one can also establish Import. However, this presupposes atomic systems of higher-level rules, that is, rules which allow at least for the discharge of atomic assumptions (cf. [Piecha and Schroeder-Heister, 2016b, 2017](#)). Whereas in Export we proceed, in the first place, from rules to implicational formulas, we proceed in Import from implicational formulas to rules. To establish Import we thus have to be able to translate left-iterated implications into rules, which in general can only be done by using higher-level rules.

Remark. Suppose we formulate completeness in a stronger way as

$$\Gamma \vDash_S A \iff \Gamma, S^* \vdash A$$

where S^* is chosen as in Export, that is, as a set of \vee -free formulas. Then this strong completeness (which implies completeness in the sense of [Definition 2.5](#)) is refuted outright, since it gives us $\text{GDP}(\vDash_S)$, which can be directly inferred from $\text{GDP}(\vdash)$.

Corollary 2.14 *In view of [Theorem 2.11](#) we have the following results:*

- (i) *By [Lemma 2.9](#) we obtain by classical reasoning that IPC is incomplete with respect to non-extension semantics.*
- (ii) *By [Lemma 2.13](#) we obtain that IPC is incomplete with respect to extension semantics.*
- (iii) *By [Remark 15](#) IPC is incomplete with respect to extension semantics based on higher-level atomic systems.*

These results pertain to proof-theoretic semantics that do not directly specify \vDash_S and \vDash , but define some other basic concept which then leads to relations \vDash_S and \vDash in the sense of non-extension or extension semantics. A prominent example is Prawitz's definition of validity for derivations or derivation structures (which was adopted to some extent by Dummett). Here one defines the *S-validity of a derivation structure* (cp. [Definition 2.1](#)), that is, a tree structure of formulas which results from the application of rules in natural deduction style. These rules can be any rules, possibly discharging free assumptions, and do not necessarily have to be the introduction and elimination rules used in standard natural deduction.

*S-validity of a
derivation structure*

Once the notion of an *S*-valid derivation structure has been defined, the consequence relation $\Gamma \vDash_S A$ can be defined by requiring that there be an *S*-valid derivation structure from Γ to A . Then $\Gamma \vDash A$ means that $\Gamma \vDash_S A$ holds for every *S*. Depending on whether in the definition of an *S*-valid derivation structure we define the validity of an open derivation structure by reference to extensions $S' \supseteq S$ of atomic systems *S* or not, we obtain an extension semantics or a non-extension semantics. If we define the validity of an open derivation by using extensions, which is certainly what Prawitz intended in his first proposal ([Prawitz, 1971](#)) of proof-theoretic validity, then by [Corollary 2.14\(ii\)](#) we obtain the incompleteness of IPC for this semantics, thus refuting Prawitz's conjecture that IPC is complete for this semantics. If we understand Prawitz's semantics as a non-extension semantics (which is not without problems, see [Piecha and Schroeder-Heister, 2017](#)), then by [Corollary 2.14\(i\)](#) we again obtain incompleteness of IPC, albeit by means of classical reasoning in the metalanguage, which, as a negative result, is as devastating for the completeness conjecture as a constructive proof.

2.6 Observations on semantics for which IPC is complete

Our focus has been on incompleteness. We established semantical conditions under which IPC is incomplete and which are satisfied by basic notions of proof-theoretic validity. Applying these results to semantics for which IPC is not incomplete but complete, gives us nonetheless substantial insight, in particular for the condition of Export. Theorem 2.11 tells us that if we have a semantics for which IPC is sound and complete, then Export cannot hold. In other words, if IPC is sound and complete, the bases S of the semantics, which constitute S -consequence $\dots \vDash_S \dots$, cannot be represented by means of sets of \vee -free formulas Γ functioning as assumptions of logical consequence $\dots, \Gamma \vDash \dots$. In particular it is not necessarily true that the S -validity of a formula A (i.e., $\vDash_S A$) can be expressed as the universal validity of A with respect to some set of \vee -free assumptions Γ (i.e., $\Gamma \vDash A$). This is a significant result, for example for Kripke semantics of IPC.

It can easily be seen that Kripke semantics for IPC is a semantics in the abstract sense of Section 2.3. A base is an entity $\langle \mathfrak{W}, \geq, v, w \rangle$, where $\langle \mathfrak{W}, \geq \rangle$ is a Kripke frame for intuitionistic propositional logic, v is a valuation assigning a truth value to every propositional variable in any reference point, and w is a reference point. All five conditions for a semantics are satisfied if we define logical consequence as follows:

$$\Gamma \vDash_{\langle \mathfrak{W}, \geq, v, w \rangle} A \iff \text{For all } w' \geq w: (\vDash_{\langle \mathfrak{W}, \geq, v, w' \rangle} \Gamma \implies \vDash_{\langle \mathfrak{W}, \geq, v, w' \rangle} A).$$

The non-validity of Export for Kripke semantics means in particular that we cannot internalise, that is, code the validity of A in a reference point w (i.e., $\vDash_{\langle \mathfrak{W}, \geq, v, w \rangle} A$) as the derivability from a suitably chosen set of \vee -free assumptions Γ .

Within the realm of proof-theoretic semantics, Goldfarb (2016) has given a semantics for which IPC is complete. It can be reconstructed in our framework as follows. Take a base S to be a pair $\langle \mathcal{R}, \alpha \rangle$, where α is a set of propositional variables, and \mathcal{R} is an atomic system (in the sense of Definition 2.12), such that α is closed under the rules of \mathcal{R} . Then we obtain a semantics in the sense of Section 2.3, if we define

$$\vDash_{\langle \mathcal{R}, \alpha \rangle} p \iff p \in \alpha, \text{ for propositional variables } p$$

and

$$\Gamma \vDash_{\langle \mathcal{R}, \alpha \rangle} A \iff \text{For all } \beta \supseteq \alpha: (\vDash_{\langle \mathcal{R}, \beta \rangle} \Gamma \implies \vDash_{\langle \mathcal{R}, \beta \rangle} A).$$

Goldfarb (2016) was able to show that IPC is complete for this semantics, by interpreting standard Kripke semantics in it. Our results then show that Export cannot hold for this semantics, that is, we cannot code $\langle \mathcal{R}, \alpha \rangle$ -validity as universal validity with respect to a set of \vee -free assumptions.

Final remark. From the point of view of proof-theoretic semantics, intuitionistic logic has always been considered the main alternative to classical logic. However, in view of the results discussed here, intuitionistic logic does not capture basic ideas of proof-theoretic semantics. Given the fact that a semantics should be primary over a syntactic specification of a logic, we observe that intuitionistic logic falls short of what is valid according to proof-theoretic semantics. The incompleteness of intuitionistic logic with respect to such a semantics therefore raises the question of whether there is an intermediate logic between intuitionistic and classical logic which is complete with respect to it.

3 Dialogue semantics

Dialogues were proposed first by Lorenzen (1960, 1961) as an alternative foundation for constructive or intuitionistic logic. The general idea is that the logical constants are given an interpretation in certain game-theoretical terms. Dialogues are two-player games between a proponent and an opponent, where each of the two players can either attack claims made by the other player or defend their own claims. For example, an implication $A \rightarrow B$ is attacked by claiming A and defended by claiming B . This means that in order to have a winning strategy for $A \rightarrow B$, the proponent must be able to generate an argument for B depending on what the opponent can put forward in defense of A . The logical constant of implication has thus been given a certain game-theoretical or argumentative interpretation, and corresponding argumentative interpretations can be given for the other logical constants as well.

Literature

- Keiff, L. (2011), *Dialogical logic*, The Stanford Encyclopedia of Philosophy (Summer 2011 Edition), Edward N. Zalta (ed.), <https://plato.stanford.edu/archives/sum2011/entries/logic-dialogical/>.
- Piecha, T. (2015), *Dialogical logic*, The Internet Encyclopedia of Philosophy, <https://www.iep.utm.edu/dial-log/>.

3.1 Dialogues and strategies

We define the concepts of argumentation form, dialogue and strategy, following the presentation of Felscher (1985, 2002) with slight deviations. We focus on dialogues for intuitionistic propositional logic.

3.1.1 Dialogues

We define our language, argumentation forms for logical constants and dialogues.

- Definition 3.1**
- (i) The *language* consists of propositional *formulas* A, B, C, \dots that are constructed from *atomic formulas (atoms)* q, r, s, \dots with the *logical constants* \wedge (conjunction), \vee (disjunction), \rightarrow (implication) and \neg (negation). *language*
 - (ii) Furthermore, \wedge_1, \wedge_2 and \vee are used as *special symbols*. *special symbols*
 - (iii) In addition, the *signatures* P (‘proponent’) and O (‘opponent’) are used. *signatures*
 - (iv) An *expression* e is either a formula or a special symbol. For each expression e there is a *P-signed expression* $P e$ and an *O-signed expression* $O e$. *expression*
 - (v) A signed expression is called *assertion* if the expression is a formula; it is called *symbolic attack* if the expression is a special symbol. X and Y , where $X \neq Y$, are used as variables for P and O . *assertion*
symbolic attack

Definition 3.2 For each logical constant an *argumentation form* determines how a complex formula (having the respective constant in outermost position) that is asserted by X can be attacked by Y and how this attack can be defended (if possible) by X . The argumentation forms are as follows: *argumentation form*

conjunction \wedge :	assertion:	$X A_1 \wedge A_2$	
	attack:	$Y \wedge_i$	(Y chooses $i = 1$ or $i = 2$)
	defense:	$X A_i$	
disjunction \vee :	assertion:	$X A_1 \vee A_2$	
	attack:	$Y \vee$	
	defense:	$X A_i$	(X chooses $i = 1$ or $i = 2$)
implication \rightarrow :	assertion:	$X A \rightarrow B$	
	attack:	$Y A$	
	defense:	$X B$	
negation \neg :	assertion:	$X \neg A$	
	attack:	$Y A$	
	defense:	<i>no defense</i>	

Example. The following is a concrete instance of the argumentation form for implication:

$$\begin{array}{l}
P \neg q \rightarrow (r \vee q) \\
O \neg q \\
P r \vee q
\end{array}$$

The argumentation forms provide what Felscher (2002, p. 127) calls an *argumentative interpretation* of the logical constants in the following sense:

argumentative interpretation

- (i) An argument on a conjunctive assertion made by X consists in Y choosing one conjunct of the assertion, and X continuing the argument with that chosen conjunct. In other words, the argumentative interpretation of conjunction is given by the reduction of the argument on a conjunctive assertion made by X to the argument on one of the conjuncts chosen by Y in the attack.
- (ii) In an argument on a disjunctive assertion made by X , Y demands the continuation of the argument with any of the disjuncts. In other words, the argumentative interpretation of disjunction is given by the reduction of the argument on a disjunctive assertion made by X to the argument on one of the disjuncts chosen by X in the defense.
- (iii) An argument on an implicative assertion made by X consists in Y stating the antecedent of the implication (whereby the antecedent functions as an assumption), and X continuing the argument with the succedent. Alternatively, X could continue with an attack on the assumed antecedent. In other words, the argumentative interpretation of implication is given by the reduction of the argument on an implicative assertion made by X to the argument on the succedent under the assumption of the antecedent.
- (iv) An argument on a negative assertion $\neg A$ made by X consists in Y stating the assertion A , without X being able to continue the argument.

This argumentative interpretation of negation can be made clear by introducing the *falsum* \perp as a constant which signifies absurdity (which is taken as a primitive notion). We can then define negation by implication and *falsum*: $\neg A := A \rightarrow \perp$. An argument on $\neg A$ is thus an argument on $A \rightarrow \perp$. However, X asserting \perp would mean that Y could continue the argument with *any* assertion – assuming the principle of *ex falso quodlibet* to be applicable here. To avoid this, \perp must not be

asserted. Hence, an argument on $\neg A$ (i.e., on $A \rightarrow \perp$) can only continue with an argument on the assumption A , and cannot be reduced to an argument on \perp .

This is similar to the treatment of negation in constructive semantics, respectively in the BHK-interpretation of logical constants, as for example stated by Heyting (1971, p. 102): “[. . .] $\neg p$ can be asserted if and only if we possess a construction which from the supposition that a construction p were carried out, leads to a contradiction.” Where contradiction – or equivalently absurdity (here signified by \perp) – is usually considered to be a primitive notion.

- Definition 3.3** (i) Let $\delta(n)$, for $n \geq 0$, be a signed expression and $\eta(n)$ a pair $[m, Z]$, for $0 \leq m < n$, where Z is either A (for ‘attack’) or D (for ‘defense’), and where $\eta(0)$ is empty. Pairs $\langle \delta(n), \eta(n) \rangle$ are called *moves*. *move*
- (ii) A move $\langle \delta(n), \eta(n) = [m, A] \rangle$ is called *attack move*, and a move $\langle \delta(n), \eta(n) = [m, D] \rangle$ is called *defense move*. *attack move*
defense move

It is $\delta(n)$ a function mapping natural numbers $n \geq 0$ to signed expressions $X e$, and $\eta(n)$ is a function mapping natural numbers $n \geq 0$ to pairs $[m, Z]$. The numbers in the domain of $\delta(n)$ (resp. in the domain of $\eta(n)$) are called *positions*. *positions*

When talking about a move $\langle \delta(n), \eta(n) \rangle$, we write $\langle \delta(n) = X e, \eta(n) = [m, Z] \rangle$ to express that $\delta(n)$ has the value $X e$ for position n , and that $\eta(n)$ has the value $[m, Z]$ for position n . For example, $\langle \delta(n) = P A, \eta(n) = [m, D] \rangle$ denotes a defense move which is made by the proponent P at position n by asserting the formula A ; this defense move refers to a move made at position m . A concrete move like $\langle \delta(4) = P \wedge_1, \eta(4) = [3, A] \rangle$ will also be written as

$$4. \quad P \wedge_1 [3, A]$$

This is an attack move with symbolic attack $P \wedge_1$; it is made at position 4 and refers to a move made at position 3.

The notation $\langle \delta(n) = X e, \eta(n) = [m, Z] \rangle$ has the advantage that we can speak about a move $\langle X e, [m, Z] \rangle$ by including information about the position n at which this move is made.

Although moves are always pairs $\langle \delta(n), \eta(n) \rangle$, we will also refer to moves by giving only their $\delta(n)$ -component, as long as it is clear from the context which move is meant, or if it is irrelevant whether the move is an attack or a defense, or if it is irrelevant to which position the move refers to. And instead of $\langle \delta(n) = X e, \eta(n) \rangle$ we will also speak of the move $X e$ made at position n . We will also speak simply about attacks and defenses in order to refer to attack moves and defense moves, respectively.

Definition 3.4 A *dialogue* is a finite or infinite sequence of moves $\langle \delta(n), \eta(n) \rangle$ (for $n = 0, 1, 2, \dots$) satisfying the following conditions: *dialogue*

- (D00) $\delta(n)$ is a P -signed expression if n is even and an O -signed expression if n is odd. The expression in $\delta(0)$ is a complex formula.
- (D01) If $\eta(n) = [m, A]$, then the expression in $\delta(m)$ is a complex formula and $\delta(n)$ is an attack on this formula as determined by the relevant argumentation form.
- (D02) If $\eta(p) = [n, D]$, then $\eta(n) = [m, A]$ for $m < n < p$ and $\delta(p)$ is the defense of the attack $\delta(n)$ as determined by the relevant argumentation form.

Definition 3.5 An attack $\langle \delta(n), \eta(n) = [m, A] \rangle$ at position n on an assertion at position m is called *open at position k* for $n < k$ if there is no position n' such that $n < n' \leq k$ and $\langle \delta(n'), \eta(n') = [n, D] \rangle$, that is, if there is no defense at or before position k to an attack at position n . *open attack*

Remark. Since there is no defense to an attack $\langle \delta(n) = YA, \eta(n) = [m, A] \rangle$ on $\delta(m) = X \neg A$ for $m < n$, the attack at position n is open at all positions k for $n < k$.

3.1.2 DI-dialogues

We define DI-dialogues and strategies. With regard to the literature on dialogical logic, DI-dialogues can be considered to be the standard dialogues for intuitionistic propositional logic. The following definition of DI-dialogues is based on the definition of dialogues.

Definition 3.6 A *DI-dialogue* is a dialogue satisfying the following conditions (in addition to (D00), (D01) and (D02)): *DI-dialogue*

(D10) If, for an atomic formula q , $\delta(n) = Pq$, then there is an m such that $m < n$ and $\delta(m) = Oq$.

That is, P may assert an atomic formula only if it has been asserted by O before.

(D11) If $\eta(p) = [n, D]$, $n < n' < p$, $n' - n$ is even and $\eta(n') = [m, A]$, then there is a p' such that $n' < p' < p$ and $\eta(p') = [n', D]$.

That is, if at a position $p - 1$ there are more than one open attacks, then only the last of them may be defended at position p .

(D12) For every m there is at most one n such that $\eta(n) = [m, D]$.

That is, an attack may be defended at most once.

(D13) If m is even, then there is at most one n such that $\eta(n) = [m, A]$.

That is, a P -signed formula may be attacked at most once.

A DI-dialogue beginning with PA (i.e., $\delta(0) = PA$, where A is a complex formula) is called *DI-dialogue for the formula A* .

Remarks. (i) The objects defined by the conditions (D00)–(D02) alone are what Felscher (1985, 2002) calls ‘dialogues’, and the objects defined by adding (D10)–(D13) – which we call ‘DI-dialogues’ – are called ‘ D -dialogues’ by him. Since here we are concerned with the objects defined by (D00)–(D02) plus (D10)–(D13), we simply speak of ‘dialogues’, omitting the specifier ‘DI’ as long as no confusion can arise. *(dialogues)*

(ii) The conditions (D00)–(D13) are also called ‘structural rules’, ‘frame rules’ (‘Rahmenregeln’) or ‘special rules of the game’ (‘spezielle Spielregeln’) in the literature, and (D10) is sometimes called ‘formal rule’. The argumentation forms are also called ‘particle rules’ (‘Partikelregeln’), ‘logical rules’ or ‘general rules of the game’ (‘allgemeine Spielregeln’).

We will stick to the notions ‘dialogue condition(s)’ (or just ‘condition(s)’) and ‘argumentation form(s)’. *(dialogue conditions)*

- (iii) Proponent P and opponent O are not interchangeable due to the asymmetries between P and O introduced by (D10) and (D13). For atomic formulas q , the proponent move $\langle \delta(n) = P q, \eta(n) = [m, Z] \rangle$ is possible only after an opponent move $\langle \delta(m) = O q, \eta(m) = [k, Z] \rangle$ for $k < m < n$, and O can attack a P -signed formula only once, whereas P can attack O -signed formulas repeatedly.

These asymmetries are introduced by dialogue conditions only. The argumentation forms themselves (as given in Definition 3.2) are symmetric with respect to the two players P and O . That is, they are independent of whether the assertion is made by the proponent P or by the opponent O ; they are thus player independent.

Definition 3.7 P wins a dialogue for a formula A if the dialogue is finite, begins with the move $P A$ and ends with a move of P such that O cannot make another move.

winning a dialogue

Remark. A dialogue won by P ends with a move $\langle \delta(n) = P q, \eta(n) = [m, Z] \rangle$, where q is an atomic formula.

Example. A dialogue for the formula $(q \vee r) \rightarrow \neg\neg(q \vee r)$ is the following:

0.	$P (q \vee r) \rightarrow \neg\neg(q \vee r)$	
1.	$O q \vee r$	[0, A]
2.	$P \vee$	[1, A]
3.	$O q$	[2, D]
4.	$P \neg\neg(q \vee r)$	[1, D]
5.	$O \neg(q \vee r)$	[4, A]
6.	$P q \vee r$	[5, A]
7.	$O \vee$	[6, A]
8.	$P q$	[7, D]

The dialogue starts with the assertion of the formula $(q \vee r) \rightarrow \neg\neg(q \vee r)$ by the proponent P in the initial move at position 0. This initial move is attacked ($\eta(1) = [0, A]$) by the opponent O with the assertion of the antecedent $q \vee r$ ($\delta(1) = O q \vee r$) of the implication asserted by P at position 0. The attack is thus made according to the argumentation form for implication.

At position 2, the proponent does not proceed according to the argumentation form for implication by defending O 's attack move with the assertion of the succedent $\neg\neg(q \vee r)$ of the attacked implication. Instead, the proponent makes the symbolic attack $P \vee$ on O 's assertion $q \vee r$. This move is thus made according to the argumentation form for disjunction. The attack is defended by O with the assertion of the left disjunct q (alternatively, O could also have chosen the right disjunct r). The moves at positions 1–3 are an instance of the argumentation form for disjunction.

As q is an atomic formula, it cannot be attacked. At position 4, the proponent defends O 's attack $O q \vee r$ by asserting the succedent $\neg\neg(q \vee r)$ of the attacked implication $(q \vee r) \rightarrow \neg\neg(q \vee r)$. The moves at positions 0, 1 and 4 are an instance of the argumentation form for implication.

The opponent now attacks $P \neg\neg(q \vee r)$ at position 5 by asserting $O \neg(q \vee r)$ according to the argumentation form for negation. By this argumentation form there is no defense for the attack. But the proponent can attack $O \neg(q \vee r)$ with the assertion $P q \vee r$. The moves at positions 4 and 5 are an instance of the argumentation form for negation, and the moves at positions 5 and 6 are another instance of that argumentation form.

Next O attacks $P q \vee r$ with the symbolic attack $O \vee$ according to the argumentation

form for disjunction at position 7. Finally, this attack is defended by P 's assertion of the left disjunct q . The moves at positions 6–8 are made according to the argumentation form for disjunction. Note that P cannot defend here by asserting the right disjunct r : the opponent has not asserted the atomic formula r before, hence such a move is prohibited by condition (D10).

The proponent's move at position 8 is the last one. The opponent cannot attack q , since it is an atomic formula. Each other P -signed formula has been attacked by O , thus no more attack moves can be made by O due to condition (D13), as these would be repetitions of attacks already made. And since each proponent attack that can be defended according to an argumentation form has already been defended by O , no more defense moves are possible either, due to condition (D12). The dialogue is finite, begins with the move $P(q \vee r) \rightarrow \neg\neg(q \vee r)$ and ends with a move of P such that O cannot make another move; the dialogue for the formula $(q \vee r) \rightarrow \neg\neg(q \vee r)$ is thus won by P .

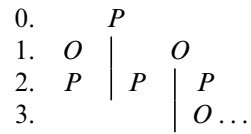
3.1.3 Strategies

We next introduce dialogue trees and define strategies. We explain first what we call a path.

Definition 3.8 A *path* in a branch of a tree with root node n_0 is a sequence n_0, n_1, \dots, n_k of nodes for $k \geq 0$ where n_i and n_{i+1} are adjacent for $0 \leq i < k$. *path*

Definition 3.9 A *dialogue tree* is a tree whose branches contain as paths all possible dialogues for a given formula. *dialogue tree*

Example. Schematic example of a dialogue tree:



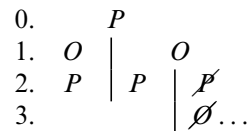
At each odd position all possible moves for O have to be considered, and at each even position all possible moves for P have to be considered.

Remark. For a given formula A there is exactly one dialogue tree, if we consider trees to be equal modulo swapping of branches.

Definition 3.10 A *strategy* for a formula A is a subtree S of the dialogue tree for A such that *strategy*

- (i) S does not branch at even positions,
- (ii) S has as many nodes at odd positions as there are possible moves for O ,
- (iii) and all branches of S are dialogues for A won by P .

Example. Schematic example of a strategy:



At each odd position all possible moves for O have to be considered (ii), but at each even position only *one* move for P has to be considered (i). The two remaining branches are dialogues won by P (iii).

Remarks. (i) In more game-theoretic terms, the strategies defined here could also be called *winning strategies for the player P* , and a corresponding definition could be given of *winning strategies for the player O* . For the dialogical treatment of logic undertaken here, only the first notion is needed, however. We can thus simply speak of *strategies*.

(ii) Strategies are finite for propositional formulas. All the branches in a strategy have finite length by definition, whereas dialogues that are not part of a strategy can be of infinite length. Dialogue trees are therefore infinite objects in general. As dialogue trees can be constructed breadth-first, of course, an existing strategy can always be found.

Formulas can have no, exactly one or more than one strategy.

Example. There is exactly one strategy for the formula $q \rightarrow \neg\neg q$:

0. $P q \rightarrow \neg\neg q$
1. $O q$ [0, A]
2. $P \neg\neg q$ [1, D]
3. $O \neg q$ [2, A]
4. $P q$ [1, A]

The strategy contains only one branch.

Example. For the formula $(q \vee r) \rightarrow \neg\neg(q \vee r)$ there are the following three strategies, among others:

- (i)
0. $P (q \vee r) \rightarrow \neg\neg(q \vee r)$
 1. $O q \vee r$ [0, A]
 2. $P \neg\neg(q \vee r)$ [1, D]
 3. $O \neg(q \vee r)$ [2, A]
 4. $P q \vee r$ [3, A]
 5. $O \vee$ [4, A]
 6. $P \vee$ [1, A]
 7. $O q$ [6, D] | $O r$ [6, D]
 8. $P q$ [5, D] | $P r$ [5, D]

- (ii)
0. $P (q \vee r) \rightarrow \neg\neg(q \vee r)$
 1. $O q \vee r$ [0, A]
 2. $P \neg\neg(q \vee r)$ [1, D]
 3. $O \neg(q \vee r)$ [2, A]
 4. $P \vee$ [1, A]
 5. $O q$ [4, D] | $O r$ [4, D]
 6. $P q \vee r$ [3, A] | $P q \vee r$ [3, A]
 7. $O \vee$ [6, A] | $O \vee$ [6, A]
 8. $P q$ [7, D] | $P r$ [7, D]

(iii)

0.	$P(q \vee r) \rightarrow \neg\neg(q \vee r)$		
1.	$O q \vee r$		[0, A]
2.	$P \vee$		[1, A]
3.	$O q$	$O r$	[2, D]
4.	$P \neg\neg(q \vee r)$	$P \neg\neg(q \vee r)$	[1, D]
5.	$O \neg(q \vee r)$	$O \neg(q \vee r)$	[4, A]
6.	$P q \vee r$	$P q \vee r$	[5, A]
7.	$O \vee$	$O \vee$	[6, A]
8.	$P q$	$P r$	[7, D]

There are more strategies for this formula than the three shown here, because the proponent can repeatedly attack formulas asserted by the opponent. For example, in strategy (iii) the proponent could at position 4 (in the left as well as in the right dialogue) repeat the attack $P \vee$ on $O q \vee r$. The subtrees below these attacks (in both dialogues) would have the same form as the subtree below position 2 in strategy (iii).

Example. There is no strategy for the formula $q \vee \neg q$, an instance of *tertium non datur*. The only possible dialogue is

0.	$P q \vee \neg q$	
1.	$O \vee$	[0, A]
2.	$P \neg q$	[1, D]
3.	$O q$	[2, A]

and P does not win.

There would be a strategy, if condition (D12) were dropped for P . Then P could defend the attack $O \vee$ a second time by stating q , thereby winning the dialogue. Condition (D11) does not have to be dropped because there are not more than one open attacks at position 3 (there is exactly one open attack at position 3; the attack $O \vee$ is not open there since it has already been defended at position 2).

3.2 Soundness and completeness

Definition 3.11 A formula A is called *dialogue-provable* (or *DI-dialogue-provable*) if there is a strategy for A . Notation: $\vdash_{DI} A$. *dialogue-provable*

Remark. We speak of *dialogue-provable* formulas here, in accordance with Felscher (2002). Contrasting Gentzen's calculi with dialogues, Felscher (2002, p. 127) remarks:

Gentzen's calculi of proofs are easily explained in that they represent the weakest consequence relation for which the provability interpretation is valid. The connection between dialogues and the argumentative interpretation of logical operations is [...] located on a different level: it is not the dialogues but the *strategies* for dialogues which will correspond to proofs. I thus formulate the *basic purpose* for the use of dialogues:

- (A₀) Logically provable assertions shall be those which, for *purely formal* reasons, can be upheld by a strategy covering every dialogue chosen by [O].

However, the fact that we speak of *provability* in the context of dialogues (thus following

Felscher) should not be misunderstood in a way that would imply that dialogues cannot be seen as a (formal) semantics (as opposed to considering dialogues only as a proof system or calculus).

Of course, such a misunderstanding could only arise if one's notion of semantics is limited to truth-conditional semantics, as opposed to proof-theoretic semantics (like the BHK-interpretation, or related justificationist, verificationist, pragmatist or falsificationist approaches in the tradition of Dummett and Prawitz) where the notion of proof or closely related notions are of central importance.

As the meaning of the logical constants is in some sense given by the argumentation forms in terms of how assertions containing the logical constants can be used in an argumentation, dialogues might very well be seen as a semantics under the heading "meaning is use", and were indeed introduced for that purpose. This aspect can be emphasised by speaking of (*logical*) *validity* instead of dialogue-provability.

validity

Theorem 3.12 (Soundness and completeness) *The dialogue-provable formulas are exactly the formulas provable in intuitionistic logic.*

This theorem has been shown (also for intuitionistic first-order logic) by Felscher (1985) by proving for Gentzen's sequent calculus LI (for intuitionistic first-order logic; see Gentzen, 1935) that every (first-order) strategy can be transformed into a proof in LI, and vice versa.

3.3 Addendum: Contraction in dialogues

In dialogues, the structural operations of thinning and contraction are only implicitly given by the dialogue conditions. This is comparable to natural deduction, where these structural operations are also only implicitly given, namely by how assumptions are discharged. Whereas in sequent calculus these operations are explicitly given as structural rules. That the structural operations are only implicitly given in dialogues can be seen as an advantage: we have argumentation forms only for the logical constants, and everything else is – in part implicitly – taken care of by the dialogue conditions.

Theorem 3.13 *In dialogues, the twofold use made by the proponent P of a formula A asserted by the opponent O corresponds to the structural operation of contraction, contracting A, A into A . The twofold use can consist either*

- (1) *in the twofold attack of a formula by the proponent P ,*
- (2) *in the twofold assertion by the proponent P of a formula asserted by the opponent O before,*

or

- (3) *in an attack of a formula A by the proponent P together with the assertion of A by P .*

That is, the twofold use can be of the following forms:

- | | |
|--|--|
| $ \begin{array}{l} (1) \quad k. \quad O A \ [k - 1, Z] \\ \qquad \qquad \vdots \\ \qquad \quad l. \quad P e \ [k, A] \\ \qquad \qquad \vdots \\ \qquad \quad m. \quad P e \ [k, A] \end{array} $ | $ \begin{array}{l} (2) \quad k. \quad O A \ [k - 1, Z] \\ \qquad \qquad \vdots \\ \qquad \quad l. \quad P A \ [i < l, Z] \\ \qquad \qquad \vdots \\ \qquad \quad m. \quad P A \ [j < m, Z] \end{array} $ |
|--|--|

$$\begin{array}{lcl}
(3) \quad k. & OA & [k-1, Z] \\
& \vdots & \\
& l. & Pe & [k, A] \\
& \vdots & \\
& m. & PA & [i < m, Z]
\end{array}
\quad \text{respectively} \quad
\begin{array}{lcl}
k. & OA & [k-1, Z] \\
& \vdots & \\
& l. & PA & [i < l, Z] \\
& \vdots & \\
& m. & Pe & [k, A]
\end{array}$$

Example. In the following two examples the twofold use made by P of an assertion made by O is of the form (1). The formulas $\neg(q \wedge \neg q)$ respectively $\neg\neg(q \vee \neg q)$ are not provable without a twofold attack on $q \wedge \neg q$ respectively $\neg(q \vee \neg q)$ by P , or without the corresponding discharge of two occurrences of the same assumption in the natural deduction derivations (where $\neg q := q \rightarrow \perp$), respectively:

$$\begin{array}{lcl}
(i) \quad 0. & P \neg(q \wedge \neg q) & \\
1. & O q \wedge \neg q & [0, A] \\
2. & P \wedge_1 & [1, A] \\
3. & O q & [2, D] \\
4. & P \wedge_2 & [1, A] \\
5. & O \neg q & [4, D] \\
6. & P q & [5, A]
\end{array}
\quad
\begin{array}{c}
\frac{[q \wedge \neg q]^1}{\neg q} (\wedge E) \quad \frac{[q \wedge \neg q]^1}{q} (\wedge E) \\
\frac{\perp}{\neg(q \wedge \neg q)} (\rightarrow I)^1
\end{array}$$

The twofold attack at positions 2 and 4 corresponds to the contraction of $q \wedge \neg q, q \wedge \neg q$ to $q \wedge \neg q$.

$$\begin{array}{lcl}
(ii) \quad 0. & P \neg\neg(q \vee \neg q) & \\
1. & O \neg(q \vee \neg q) & [0, A] \\
2. & P q \vee \neg q & [1, A] \\
3. & O \vee & [2, A] \\
4. & P \neg q & [3, D] \\
5. & O q & [4, A] \\
6. & P q \vee \neg q & [1, A] \\
7. & O \vee & [6, A] \\
8. & P q & [7, D]
\end{array}
\quad
\begin{array}{c}
\frac{[q]^1}{q \vee \neg q} (\vee I) \\
\frac{[\neg(q \vee \neg q)]^2}{\neg q} (\rightarrow I)^1 \\
\frac{[\neg(q \vee \neg q)]^2}{\neg\neg(q \vee \neg q)} (\rightarrow I)^2
\end{array}$$

The twofold attack at positions 2 and 6 corresponds to the contraction of $\neg(q \vee \neg q), \neg(q \vee \neg q)$ to $\neg(q \vee \neg q)$.

3.4 Addendum: Classical dialogues

Although we are only concerned with intuitionistic logic, we point out here how dialogues for classical (propositional) logic relate to dialogues for intuitionistic (propositional) logic.

Theorem 3.14 *If the conditions (D11) and (D12) are restricted to apply only to O (and no more to P), then the formulas provable on the basis of the thus modified dialogues are exactly the formulas provable in classical logic.*

Definition 3.15 A *classical dialogue* is a dialogue where the conditions (D11) and (D12) do hold for O but not for P , that is, where conditions (D11) and (D12) are replaced by the following conditions (D11⁺) and (D12⁺), respectively:

classical dialogue

(D11⁺) If $\eta(p) = [n, D]$ for even n , $n < n' < p$, $n' - n$ is even and $\eta(n') = [m, A]$, then there is a p' such that $n' < p' < p$ and $\eta(p') = [n', D]$.

That is, if at a position $p - 1$ there are more than one open attacks by P , then only the last of them may be defended by O at position p .

($D12^+$) For every even m there is at most one n such that $\eta(n) = [m, D]$.

That is, an attack by P may be defended by O at most once.

The notions ‘dialogue won by P ’, ‘dialogue tree’ and ‘strategy’ as defined for dialogues are directly carried over to the corresponding notions for classical dialogues.

The effects of replacing ($D11$) and ($D12$) by ($D11^+$) and ($D12^+$), respectively, are illustrated in the two following examples.

Example. There is a classical strategy for the formula $q \vee \neg q$:

0. $P q \vee \neg q$
1. $O \vee$ [0, A]
2. $P \neg q$ [1, D]
3. $O q$ [2, A]
4. $P q$ [1, D]

The last move is possible due to the replacement of condition ($D12$) by condition ($D12^+$). In the presence of ($D12$) this move is not possible, and there is thus no DI-strategy for (any instance of) *tertium non datur* (cf. the example on page 52).

Example. There is a classical strategy for the formula $\neg\neg q \rightarrow q$:

0. $P \neg\neg q \rightarrow q$
1. $O \neg\neg q$ [0, A]
2. $P \neg q$ [1, A]
3. $O q$ [2, A]
4. $P q$ [1, D]

The last move is possible due to the replacement of condition ($D11$) by condition ($D11^+$). In the presence of ($D11$) this move is not possible, and there is thus no DI-strategy for (any instance of) double negation elimination.

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