Model-Theoretic Inferentialism and Categoricity

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Proofs, arguments and dialogues: history, epistemology and logic of justification practices

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Outline

I. Model-Theoretic Inferentialism and Categoricity

II. The Categoricity Problem for Propositional Logic

III. The Categoricity Problem for Quantificational Logic

IV. Why the ω-logic?

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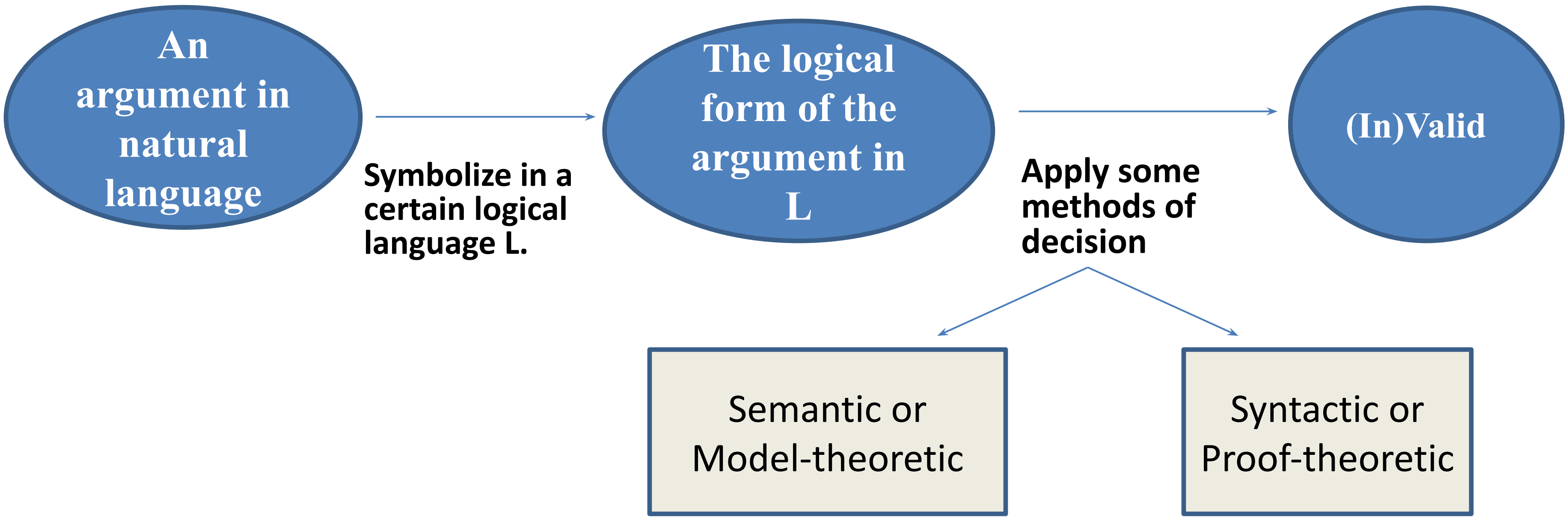
Deductive Logical Reasoning

- Logic is the study of valid arguments.



Logic:

1. A language L (well formed formula).
2. An interpretation for L/ a model-theory for L (Truth in a model and logical consequence)
3. A deductive system for L/ a proof theory for L (Sound derivation and proof)

Propositional and Quantificational Logic

Language

1. Propositional letters: A, B, C, D …
2. Logical operators: & (and), v (or), ~ (not), → (if …, then), ↔ (if, and only if, …, then ..)
3. Predicate letters: F, G, H …
4. Individual constants: a, b, c, d …
5. Individual variables: x, y, z …
6. Quantifiers: universal (∀), existential (∃)
7. Parantheses: (,)

Model Theory

- Each sentence is either true or false.
- Truth-conditions definitions for logical operators.
- Validity: there is no interpretation which makes the premises true and the conclusion false.
- ⊨ model-theoretic logical consequence.

Proof-theory

- A system of axioms or rules of inference that implicitly define the logical operators.
- Validity: the conclusion can be logically derived from the premises by using the axioms or rules of inference.
- ⊢ proof-theoretic logical consequence

Soundness: If P1, P2, …Pn ⊢ C, then P1, P2, …Pn ⊨ C
Completeness: If P1, P2, …Pn ⊨ C, then P1, P2, …Pn ⊢ C

Logical Inferentialism

The conception based on the idea that the meanings of the logical terms are *uniquely* determined by the formal axioms or rules of inference that govern their use in a logical calculus.

Model-theoretic Inferentialism

1) The meanings of the logical terms are determined by the rules of inference that govern their use in a logical calculus and
2) These meanings are to be defined in model-theoretic terms (truth-conditions, extension, reference, validity).

Proof-theoretic Inferentialism

1) The meanings of the logical terms are determined by the rules of inference that govern their use in a logical calculus and
2) These meanings are to be defined in proof-theoretic terms (proof-conditions, inference, derivability).

Model Theoretic Inferentialism and Categoricity

Model-theoretic Inferentialism

1) The meanings of the logical terms are uniquely determined by the rules of inference that govern their use in a logical calculus and
2) These meanings are to be defined in model-theoretic terms (truth-conditions, extension, reference).

Categoricity

A system of rules or axioms S is categorical iff all its models are standard.

or

A calculus in categorical iff it uniquely determines the intended model-theoretic meanings of the logical terms.

Carnap's Logical Inferentialism

Let any postulates and any rules of inference be chosen arbitrarily; then this choice, whatever it may be, will determine what meaning is to be assigned to the fundamental logical symbols. *[...]* The standpoint which we have suggested -we will call it the *Principle of Tolerance*- relates not only to mathematics, but to all questions of logic. (Carnap 1934/1937: xv)

While in constructing a calculus we may choose the rules arbitrarily, in constructing a calculus K in accordance with a given semantical system S we are not entirely free. In some essential respects the features of S determine those of K, although, on the other hand, there is still a freedom of choice left with respect to some features. Thus logic -if taken as a system of formal deduction, in other words, a calculus –is in one way conventional, in another not. (Carnap 1942: 218-19)

Carnap's Inferentialism in FoL

(Carnap 1943) introduces the concept of *full formalization*:

To *fully formalize* a logical theory, that is already defined on the basis of a semantical system, by a formal calculus means to show that

i. every logical truth is a theorem in the calculus;
ii. every relation of logical consequence is represented by a relation of logical derivability in the calculus, and
iii. *all logical terms have the intended semantical meanings in all the permissible interpretations of the calculus.*

(Carnap1943) discovered that both the standard (*single conclusion*, *finite*) propositional and quantificational calculi are not full formalizations of propositional and quantificational logics.

Model Theoretic Inferentialism and Categoricity

Categoricity

A calculus in categorical iff it uniquely determines the intended model-theoretic meanings of the logical terms.

Carnap's Non-Categoricity Results

Propositional Logic

v+ (φ) = true, for all wff φ of L. Thus, v+(A)= v+(~A) =true.

v⊢(φ) = true, when φ is a theorem

v⊢(φ)=false, when φ is not a theorem.

Thus, v⊢(A)= v⊢(~A)=false, but v⊢(A v ~A)= true.

Quantificational Logic

v*(∀xPx)= v*(Pa & Pb & Pc & …& Qb)=true

v*(∃xPx) = v*(Pa v Pb v Pc v…..v ~Qb)=true

Thus, we have models of propositional and quantificational calculi in which the logical terms have non-standard meanings.

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Model Theoretic Inferentialism and Categoricity

Categorical MTI

(Garson 2013) showed that MTI depends both on the format of the logical calculus and on the way in which the models for a logical calculus are defined.

Logical calculi	**Models**
Axiomatic	*Deductive*
Natural Deduction	Local
Sequent Calculus	Global

Model Theoretic Inferentialism and Categoricity

Hilbert and Bernays 1934 Axiomatic Calculus for PL

Axioms for implication

$\vdash A \to (B \to A)$

$\vdash (A \to (A \to B)) \to (A \to B)$

$\vdash (A \to B) \to ((B \to C) \to (A \to C))$

Axioms for conjunction

$\vdash (A \& B) \to A$

$\vdash (A \& B) \to B$

$\vdash (A \to B) \to ((A \to C) \to (A \to (B \& C)))$

Axioms for disjunction

$\vdash A \to A \vee B$

$\vdash B \to A \vee B$

$\vdash (A \to C) \to ((B \to C) \to ((A \vee B) \to C))$.

Axioms for negation

$\vdash (A \to B) \to (\sim B \to \sim A)$,

$\vdash A \to \sim\sim A$,

$\vdash \sim\sim A \to A$.

Rules of inference

1) Substitution
2) Modus Ponens

Model Theoretic Inferentialism and Categoricity

Gentzen Natural Deduction Rules 1934

STRUCTURAL RULES:

Hypothesis:

$\Gamma \vdash C$, provided that C is in Γ

Weakening:

$$\frac{\Gamma \vdash C}{\Gamma, \Gamma' \vdash C}$$

Cut:

$$\frac{\Gamma \vdash A \qquad \Gamma', A \vdash C}{\Gamma, \Gamma' \vdash C}$$

OPERATIONAL RULES (Introduction, Elimination)

Rules for conjunction

$$\frac{\Gamma \vdash A \qquad \Gamma \vdash B}{\Gamma \vdash A \,\&\, B} \qquad \frac{\Gamma \vdash A \& B}{\Gamma \vdash A} \qquad \frac{\Gamma \vdash A \,\&\, B}{\Gamma \vdash B}$$

Rules for Implication

$$\frac{\Gamma, [A] \vdash B}{\Gamma \vdash A \longrightarrow B} \qquad \frac{\Gamma \vdash A \qquad \Gamma \vdash A \longrightarrow B}{\Gamma \vdash B}$$

Rules for Negation

$$\frac{\Gamma, [A] \vdash \text{⅄}}{\Gamma \vdash \sim A} \qquad \frac{\Gamma \vdash A \qquad \Gamma \vdash \sim A}{\Gamma \vdash \text{⅄}}$$

Rules for disjunction

$$\frac{\Gamma \vdash A}{\Gamma \vdash A \vee B} \qquad \frac{\Gamma \vdash B}{\Gamma \vdash A \vee B} \qquad \frac{\Gamma \vdash A \vee B \qquad \Gamma, [A] \vdash C \qquad \Gamma, [B] \vdash C}{\Gamma \vdash C}$$

Model Theoretic Inferentialism and Categoricity

Gentzen Sequent Calculus Rules 1934

& Left

$$\frac{\Gamma, A \vdash \Delta}{\Gamma, A\&B \vdash \Delta}$$

&Left

$$\frac{\Gamma, B \vdash \Delta}{\Gamma, A\&B \vdash \Delta}$$

& Right

$$\frac{\Gamma \vdash A, \Delta \qquad \Sigma \vdash B, \Pi}{\Pi, \Sigma \vdash A \,\&\, B, \Delta, \Pi}$$

⟶ Left

$$\frac{\Gamma \vdash A, \Delta \qquad \Sigma, B \vdash \Pi}{\Gamma, \Sigma, A \longrightarrow B \vdash \Delta, \Pi}$$

⟶Right

$$\frac{\Gamma, A \vdash B, \Delta}{\Gamma \vdash A \longrightarrow B, \Delta}$$

~Left

$$\frac{\Gamma \vdash A, \Delta}{\Gamma, \sim A \vdash \Delta}$$

~Right

$$\frac{\Gamma, A \vdash \Delta}{\Gamma \vdash \sim A, \Delta}$$

v Left

$$\frac{\Gamma, A \vdash \Delta \qquad \Sigma, B \vdash \Pi}{\Gamma, \Sigma, A v B \vdash \Delta, \Pi}$$

v Right

$$\frac{\Gamma, A \vdash \Delta}{\Gamma \vdash A v B, \Delta}$$

v Right

$$\frac{\Gamma, B \vdash \Delta}{\Gamma, A v B \vdash \Delta}$$

Model Theoretic Inferentialism and Categoricity

Categorical MTI

(Garson 2013) explicitly showed that MTI depends both on the format of the logical calculus and on the way the models for the a logical calculus are defined.

Logical calculi	**Models**
Axiomatic	Deductive
Natural Deduction	Local
Sequent Calculi	Global

Model Theoretic Inferentialism and Categoricity

Deductive Model: V is a *deductive model* of S iff all provable sequents of S are V-valid.

Global Model: V is a *global model* of S iff each rule of S preserves V-validity.

Local Model: V is a *local model* of rule R iff R preserves S-satisfaction.

The difference between being a local model of a rule R and being a global model of R amounts to a difference in the scope of quantification**:**

Global Model: $(\forall v \in V)$(v sat. the inputs of R)$\rightarrow(\forall v \in V)$(v sat. the outputs of R)

Local Model: $(\forall v \in V)$(v sat. the inputs of R $\rightarrow$ v sat. the outputs of R)

Model Theoretic Inferentialism and Categoricity

- (Carnap 1943) worked with deductive models and axiomatic systems.
- He was interested in the relation between logical semantics (L-semantics) and logical syntax (C-syntax).
- He defined the logical concepts semantically (L-implication, L-true, L-disjunct, L-exclusive) and wanted to find out whether there is a symmetry between then and the corresponding syntactic concepts (C-implication, C-true, C-disjunct, C-exclusive).
- Since PL and QL are sound and complete, then:

  L-implication ↔ C-implication L-true ↔ C-true.
- The problem is with C-disjunct and C-exclusive; they are not defined by C-implication.
- L-exclusive: a sentence and its negation cannot both be true (LNC).
- L-disjunct: a sentence and its negation cannot both be false (LEM).

- Thus, we may have interpretations in which LNC and LEM do not hold.

Conjunction: PC & NTT

(&I)

$$\frac{p \quad q}{p \,\&\, q}$$

(&E)’

$$\frac{p \,\&\, q}{p}$$

(&E)’’

$$\frac{p \,\&\, q}{q}$$

	p	q	p & q
C1	⊤	⊤	⊤
C2	⊤	⊥	⊥
C3	⊥	⊤	⊥
C4	⊥	⊥	⊥

(&I) : C1
(&E)’ : C3, C4
(&E)’’ : C2, C4

Disjunction: PC & NTT

(vI)’

$$\frac{p}{p \vee q}$$

(vI)’’

$$\frac{q}{p \vee q}$$

(vE)

$$\frac{p \vee q \quad \begin{matrix}[p] \\ r\end{matrix} \quad \begin{matrix}[q] \\ r\end{matrix}}{r}$$

	p	q	p v q
D1	⊤	⊤	⊤
D2	⊤	⊥	⊤
D3	⊥	⊤	⊤
D4	⊥	⊥	**?**

(vI) : D1, D2

(vI)’ : D1, D3

(vE) : -------

Material Implication: PC & NTT

(→I)

$$\frac{\begin{array}{c}[p] \\ q\end{array}}{p \rightarrow q}$$

(→E)

$$\frac{p \quad p \rightarrow q}{q}$$

	p	q	p → q
I1	⊤	⊤	⊤
I2	⊤	⊥	⊥
I3	⊥	⊤	⊤
I4	⊥	⊥	?

(→ I) : I1, I3

(→ E) : I2

Negation: PC & NTT

(~I)

$$\dfrac{\begin{matrix}[p] \\ \curlywedge\end{matrix}}{\sim p}$$

(~E)

$$\dfrac{p \qquad \sim p}{\curlywedge}$$

	p	~p
N1	⊤	?
N2	⊥	?

(~I) : .....

(~E): .....

Carnap's Solution to the Categoricity Problem in PL

- To eliminate these non-normal interpretations, Carnap introduced new syntactical concepts based on the notion of *junctive*.
- A junctive is a potentially infinite sentential class that can be constructed either conjunctively (as it is usually done when we consider the class of premises of an argument) or disjunctively (when we consider the class of conclusions of an argument).
- For obtaining a fully formalized propositional logic, Carnap introduced two new rules of deduction:

1) $A_i \vee A_j \vdash \{ A_i, A_j\}^{v}$
2) $V^{\&} \vdash \Lambda^{v}$

- Various solutions: Alonzo Church (1944), T. Smiley (1996), Vann McGee (2000), J. Garson (2013), Bonnay & Westerståhl (2016), J. Warren (2020), Murzi & Topey (2021)

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The Categoricity Problem for Quantificational Logic

- **The objectual interpretation of ∀:**

  v(∀xFx)=true iff for all d in D, v(Fd/x)=true.

- **The substitutional interpretation of ∀:**

  v(∀xFx)=true iff for all t in L, v(Ft/x)=true.

- If every object from the domain of quantification D is named in the language, then the two interpretations are equivalent.

The Categoricity Problem for Quantificational Logic

- The possibility of non-normal interpretations in predicate logic arises because, a universally quantified sentence is not deductively equivalent with the class formed by the conjunction of all the instances of the operand.

$$\frac{\Gamma \vdash Ft}{\Gamma \vdash \forall x Fx} \qquad \frac{\Gamma \vdash \forall x Fx}{\Gamma \vdash Ft}$$

*provided that t does not occur in Γ

- $(\forall x)Fx$ is not a consequence of any proper subclass of its class of instances.
- To make $(\forall x)Fx$ a consequence of all its infinite class of instances, Carnap (1937, 1943) introduced an infinitary rule.

∀Iω –rule: $\dfrac{\Gamma \vdash Ft, \text{ for all } t \text{ of } L}{\Gamma \vdash \forall x Fx}$

$$\{A_i(i_k)\}_{\&} \vdash A_i$$

Garson's Non-Categoricity Theorem

Theorem 14.3: S$\forall$ does not express $\|s\forall\|$, nor does it express $\|d\forall\|$.

Proof sketch: The set $\{A^y/x$: y is a variable of L$\}$ $\cup$ $\{\sim\forall xA\}$ is consistent in S$\forall$, and the set *e* of all wffs B such that $\{A^y/x$: y is a variable of L$\}$ $\cup$ $\{\sim\forall xA\} \vdash$ B is deductively closed and so a member of [S$\forall$]. The set *e* however, although it contains A^y/x for each variable y, it does not contain $\forall xA$ on pain of inconsistency.

v* (A^y/x, for each variable y)=true, v*($\forall xA$)=false.

Model Theoretic Inferentialism and Categoricity

Hilbert and Ackerman/Bernays Axiomatic Calculus for QL 1927

Axiom for Universal

$$\forall xFx \longrightarrow Fy$$

Axiom for Existential

$$Fy \longrightarrow \exists xFx$$

Rules of inference

Let Fx be an arbitrary logical expression that depends on x, and A one that does not:

$$\frac{A \longrightarrow Fx}{A \longrightarrow \forall x\, Fx}$$

$$\frac{Fx \longrightarrow A}{(\exists x)Fx \longrightarrow A}$$

Model Theoretic Inferentialism and Categoricity

Gentzen Natural Deduction Rules 1934

Rules for the universal quantifier

$$\frac{\Gamma \vdash Ft}{\Gamma \vdash \forall xFx} \qquad \frac{\Gamma \vdash \forall xFx}{\Gamma \vdash Ft}$$

*provided that t does not occur in Γ

Rules for the existential quantifier

$$\frac{\Gamma \vdash Ft}{\Gamma \vdash \exists xFx} \qquad \frac{\Gamma \vdash \exists xFx \quad \Gamma, [Ft] \vdash \varphi}{\Gamma, \exists xFx \vdash \varphi}$$

*provided that t does not occur in ∃xFx, in Γ or in φ.

Model Theoretic Inferentialism and Categoricity

Gentzen Sequent Calculus Rules 1934

∀ Left

$$\frac{\Gamma, Ft \vdash \Delta}{\Gamma, \forall x Fx \vdash \Delta}$$

∀ Right

$$\frac{\Gamma \vdash Fy, \Delta}{\Gamma \vdash \forall x Fx, \Delta}$$

*provided that y does not occur in Γ, Δ, ∀xFx

∃ Left

$$\frac{\Gamma, Fy \vdash \Delta}{\Gamma, \exists x Fx \vdash \Delta}$$

*provided that y does not occur in Γ, Δ , ∃xFx

∃ Right

$$\frac{\Gamma \vdash Ft, \Delta}{\Gamma \vdash \exists x Fx, \Delta}$$

Two Ideals of Logical Formalization

(Hintikka 1989: 71-73) distinguishes two main uses of logic in mathematics:

1. The descriptive use of logic:
   - The use of logical notions for the purpose of capturing different structures studied in mathematical theories.
   - **Ideal**: to attain a ***categorical axiomatization*** S for a mathemathical theory, i.e., any two models of S are isomorphic. (descriptive completeness)
2. The deductive use of logic:
   - The use of logical inferences for systematizing and criticizing mathematicians reasoning about the structures they are interested in.
   - **Ideal**: to attain a ***deductively complete system*** S, i.e. for each sentence C in the language of S, either $S \vdash C$ or $S \vdash \sim C$.

Remark: categoricity and deductive completeness are properties of the mathematical theory formally axiomatized in a certain logical system and not of the underlying logical system itself.

The underlying logical system L of S is ***semantically complete*** iff all valid formulas and all consequences can be derived in the system, i.e. if $\Gamma \vDash \varphi$, then $\Gamma \vdash_L \varphi$.

Two Ideals of Logical Formalization

Some connections:

- If S is deductively complete, then L is semantically complete.
- If S is deductively complete, then the models of S are elementary equivalent.
- If S is not deductively complete, then either its logic is not semantically complete, or its models are not elementary equivalent, i.e., S is non-categorical.
- In the case of first order PA, since L is semantically complete, then S is non-categorical.

Gödel's Incompleteness Theorem: If we reach a categorical mathematical theory S (containing elementary arithmetic), then our underlying logic cannot be sematically complete. Thus, we can have categoricity only at the price of semantical completeness of the underlying logic. We can have either the categoricity of S, or the semantical completeness of L, but not both.

(See Hintikka 1989, Tennant 2000, Hazen 2006, Smith 2020)

Two Ideals of Logical Formalization

Which should be the preferences of a MT-logical inferentialist? Would he abandon the semantic completeness of L for obtaining the categoricity of S?

- (Murzi and Topey 2021) argued that the open-ended rules for SOL fix the standard meanings of the second order quantifiers. Thus, they prefer the categoricity of S.
- I think that the semantic completeness of L should not be abandoned by a MT-logical inferentialist. One of his main aims, after all, is to formally capture all valid arguments.
- My view is that the MT-inferentialist should accept in his inferential framework the infinitary rules of inference. In this way he uniquely determines the *standard* meanings of the first-order quantifiers and, in addition, ω-logic is semantically complete and makes PA deductively complete.
- The price paid is the categoricity of first-order theories. Due to Löweinhem-Skolem theorem, PA closed under ω-rule is non-categorical.

Carnap on Infinitary Rules

The Principle of Conventionality of Language Forms

There are no theoretical constraints that the consequence relation of language must satisfy.

- Thus, each formal theory T determines a consequence relation.

The Determinacy Desideratum

The syntactic consequence relation of T should be such that any logico-mathematical sentence φ is either analytic, i.e. provable, or contradictory, i.e. refutable. (T⊢ φ or T⊢ ~φ)

- If T is negation-complete, then all its terms will be logical.
- If T is not negation-complete, then some of its terms will be non-logical, i.e., descriptive.

The ‘Non-normality’ of ∀ in LSL

P1. PRA_0 (BA) is negation-complete.

P2. PRA (Q) is not negation-complete.

P3. $PRA = PRA_0 + \forall$

C1. Therefore, ∀ is a logical sign in PRA_0 (BA) and a non-logical sign in PRA (Q).

The universal operator (𝔷) in both [Princ. Math.] and IId is not logical but descriptive. By this nothing is said against the usual translation, in which the correlate of 𝔊 is a logical sentence (for example, the identically worded sentence 𝔊 in II), and the correlate of (𝔷) is a logical expression (for example, a proper universal operator in II). The fact that 𝔊 and (𝔷) are descriptive only means that in addition to this usual translation others are possible, amongst them some in which the correlates of 𝔊 and (𝔷) are descriptive. (Carnap 1937:231)

The ‘Non-normality’ of ∀ in LSL

P1. ∀ is a proper logical sign iff a universal quantified sentence is a syntactic logical consequence of the class of all its instances.

P2. A universal quantified sentence is a syntactic logical consequence of the class of all its instances in a logical calculus iff the calculus contains infinite rules of inference.

C. ∀ is a proper logical sign in a logical calculus iff the calculus contains infinite rules on inference.

In Languages I and II the universal operators with 𝔷 are proper universal operators. For not only is every sentence—and hence every closed sentence—of the form $\text{pr}_1(\mathfrak{Z})$ a consequence of $(\mathfrak{z}_1)(\text{pr}_1(\mathfrak{z}_1))$, but, conversely, this universal sentence is also a consequence of the class of those closed sentences (by DC 2, p. 38) and therefore equipollent to it. In the other languages which we have mentioned, on the contrary, the same thing is not true for the universal operators with 𝔳 or with 𝔷 (unless Hilbert's new rule is laid down; hence these operators are improper. (Carnap 1937:197)

Carnap's Gödelian Dilemma

∀ is a proper logical sign in a logical calculus iff the calculus contains infinite rules on inference.

Because of its generality Gödel's theorem presents Carnap with a dilemma: every Carnapian language strong enough to contain quantified arithmetic must either have a non-effective consequence relation or make some arithmetical vocabulary count as descriptive. (Potter 2002: 269)

Murzi and Topey –Categoricity by convention

- Consider the $\forall$I- rule:

  $$\frac{\Gamma \vdash \varphi}{\Gamma \vdash \forall x\varphi}$$, where x does not appear free in Γ.
- This rule is locally valid with respect to a class of valuations V iff it preserves sequent satisfaction in V.
- (Standard meaning of $\forall$): For any φ with at most x free, $\forall x\varphi$ is true iff *every object in the domain is in the extension of φ in v*.

1. If the rules are locally valid, then $\forall x\varphi$ is true in v iff the range of x in v is included in the extension of φ in v. (*Weakened First Order Thesis)*
2. If the rules are locally valid and the interpretation of ∀ is permutation invariant (or the rules are open-ended), then the standard meaning of ∀ is determined by the rules. (*First Order Thesis*)

Is the local validity of ∀I-rule compatible with Carnap's v+?

- We can define a classical valuation v+ such that:
  v+(∀xFx) = true
  v+(Fa & Fb & Fc & … & Qb) =true.

- Consider the ∀I- rule:
  $\Gamma \vdash \varphi$
  $\Gamma \vdash \forall x\varphi$, where x does not appear free in Γ.
- This rule is locally valid with respect to a class of valuations V iff it preserves sequent satisfaction in V. A sequent $\Gamma \vdash \varphi$ is satisfied by a valuation v relative to a variable assignement s iff s makes φ true, or makes a member of Γ false.

Case 1: $\Gamma \neq \varnothing$. v+ satisfies $\Gamma \vdash \varphi$ since it satisfies φ, and also satisfies $\Gamma \vdash \forall x\varphi$ since it satisfies $\forall x\varphi$.
Case 2: $\Gamma = \varnothing$. Idem.

Thus, v+ ∊ V, but v+ provides ∀ with a different meaning.

Is the local validity of ∀I-rule compatible with Garson's v*?

- The set {$\varphi(t/x)$ for every term t of L, $\sim\forall x\varphi$} is consistent in the standard sense. A sentence may be provable from an infinite set of premises without being provable from a finite subset of the initial set.
- We can define a classical valuation v* such that v*($\varphi(t/x)$, for every term t of L) is true, while v*($\forall x\varphi$) is false.
- Consider the ∀I-rule:

  $\Gamma \vdash \varphi$

  $\Gamma \vdash \forall x\varphi$, where x does not appear free in Γ.

  Case 1: $\Gamma \neq \varnothing$. Then the rule is locally valid if some δ in Γ is made false by v*. Thus, if v*(Γ) = false, then $v^* \in V$.

  Case 2: $\Gamma = \varnothing$. The rule then says that if φ is a theorem, then its logical closure is also a theorem. Apparently, in this case $v^* \notin V$ ....

Is the local validity of ∀I-rule compatible with Garson's v*?

However, we do not always use ∀I-rule only in proofs, i.e., when the premises are theorems.

There are cases in which $\forall x\varphi$ follows semantically from the entire class of instances of φ, although φ is not a theorem. For instance a Gödel sentence for PA:

$PA \vdash \varphi 1$ $PA \vdash \varphi 2$ $PA \vdash \varphi 3$ ….. $PA \nvdash \forall x\varphi$

From $PA \vdash \varphi 1$ $PA \vdash \varphi 2$ $PA \vdash \varphi 3$ ….. it does not follow that $PA \vdash \varphi x$

We would need the Carnapian transfinite rule which says that a sentence A_i containing a free variable i_k is directly derivable from the infinite conjunctive set of all its instances:

$\{A_i(i_k)\}_\& \vdash A_i$

Therefore, the valuation v* is available for a person who uses ∀I- rule in an ordinary derivation.

Why the ω-logic?

The ω-rule is an infinite rule of inference:
From $\varphi(0)$, $\varphi(1)$, $\varphi(2)$ …. infer $\forall x\varphi(x)$

The ω-logic is formed by adding the ω-rule to the axioms or rules of inference of FOL and allowing infinite proofs.

If we consider the language L={+, x, S, 0}, an ω-model is a model M in which D={0, 1, 2, ….}. That is, M omits the set: $\{x\neq 0, x\neq 1, \ldots\}$ This formulation of ω-logic is intended for the study of the standard model of arithmetic.

ω-Soundness and Completeness Theorem:
A theory T in L is consistent in ω-logic iff T has and ω-model.

Why the ω-logic?

Gödel's Incompleteness Theorem: If we reach a categorical mathematical theory S (containing elementary arithmetic), then our underlying logic cannot be sematically complete. Thus, we can have categoricity only at the price of semantical completeness of the underlying logic. We can have either the categoricity of S, or the semantical completeness of L, but not both.

The ω-logic is semantically complete.

PA in ω-logic is deductively complete.

The ω-rule uniquely determines the standard meaning of the $\forall$.

The price paid is the categoricity of the theories formalized by the ω-logic.

However, if we find a good inferentialist reason to pick out only the ω-models, then categoricity of the PA could also be attained. (further work to be done …)

ω-rule Reasoning?

The Supertask Computer Verification

- Goldbach conjecture (GB) asserts that every even number greater than 2 is the sum of two prime numbers.
- (Warren 2020) invites us to consider a supertask computer (SC) that is able to perform a countably infinite number of computations is a finite time.
- The SC is set to verify GB and it checks 0 in half a minute, one in half of half a minute, and n in $1/2^{n+1}$ minutes and, thus, the computation will finish in one minute.
- The SC either sends a halt signal if a counterexample is found before one minute or no counterexample is found and, thus, we receive no signal.
- If the SC fails to halt, then we accept GB(0), GB(1), GB(2) … by using as evidence the computations. Then we conclude $(\forall x)GBx$.
- Therefore, on the basis of the computation, we accept each of the infinitely many premises and infer from them, according to the omega rule, the truth of GB:

$$\frac{GB(0),\ GB(1),\ GB(2)\ \ldots}{(\forall x)GBx}$$

- The acceptance of the premises is supposed to be justified by the result of the computation, namely, that no counterexample was to be found. The conclusion is inferred by using the omega rule. Thus, this would count as a situation in which we, human beings, perform an infinite reasoning, by using infinite inferences.

Outline

I. Model-theoretic Inferentialism and Categoricity

II. The Categoricity Problem for Propositional Logic

III. The Categoricity Problem for Quantificational Logic

IV. Why ω-logic?

Thank you!