Part 4

Adding universes

A (predicative) universe is a class that is closed under elementary comprehension and join and that consists of names only.

Universe

We write $Univ[S]$ for the conjunction of the following formulas:

- $(\forall x \in S)(x \in \Re)$.
- $\mathsf{nat} \in S$, where nat is a name of the class N.
- For every term t_φ associated to the elementary formula $\varphi[x, \vec{y}, \vec{Z}]$,

$$\forall \vec{y} (\forall \vec{z} \in S)(t_\varphi(\vec{y}, \vec{z}) \in S).$$

- $\forall f (\forall a \in S)((\forall x \dot{\in} a)(fx \in S) \rightarrow \mathsf{j}(a, f) \in S)$.

In addition,

$$\mathcal{U}[t] := \exists X(\Re(t, X) \wedge Univ[X]).$$

Universes can be regarded as the explicit analogies of

- regular sets or regular ordinals if the operations are interpreted as set-theoretic functions,
- admissible sets if the operations are interpreted as partial recursive functions.

Basic ontological properties of universes

Universes do not contain their names

- $Univ[S] \land \Re(a, S) \;\rightarrow\; a \notin S$.
- $\mathcal{U}[a] \;\rightarrow\; a \mathrel{\dot{\notin}} a$.

The names of a class cannot be in a single universe

$$Univ[S] \;\rightarrow\; \exists x(\Re(x, T) \land x \notin S).$$

EC + (J) does not prove the existence of universes

$$\mathsf{EC} + (\mathsf{J}) \;\nvdash \exists X\, Univ[X].$$

The limit axioms (Lim)

The limit axiom (Lim). For a fresh constant ℓ:

(L1) $a \in \Re \rightarrow \ell a \in \Re$,

(L2) $\ell a \in \Re \rightarrow \mathcal{U}[\ell a] \wedge a \dot{\in} \ell a$.

Attention: Non-extensionality of ℓ

$$\mathsf{EC} + (\mathsf{J}) \vdash (\exists x, y \in \Re)(x \doteq y \wedge \ell x \dot{\neq} \ell y).$$

Theorem

1. $|\mathsf{EC} + (\mathsf{J}) + (\mathsf{Lim})| + (\mathsf{C}\text{-}\mathsf{I_N}) = \Gamma_0$.
2. $|\mathsf{EC} + (\mathsf{J}) + (\mathsf{Lim})| + (\mathbb{L}\text{-}\mathsf{I_N}) = \varphi(1, \varepsilon_0, 0)$.

Sets in explicit mathematics: a sketch

We fix some universe U and introduce a further class constant S for the class of sets (with respect to U) and an individual constant σ. For better intuitive reading, write

$$\{fx : x \,\dot{\in}\, a\} \quad \text{and} \quad \sigma(a, f).$$

Closure rules for S: For all a and f:

$$a \in U \;\wedge\; (\forall x \,\dot{\in}\, a)(fx \in S) \;\rightarrow\; \{fx : x \,\dot{\in}\, a\} \in S.$$

Induction for S: For all formulas $\varphi[x]$:

$$(\forall a \in U)(\forall f \in (a \rightarrow S))\,(\,(\forall x \,\dot{\in}\, a)\varphi[fx] \;\rightarrow\; \varphi[\{fx : x \,\dot{\in}\, a\}]\,) \\ \rightarrow\; (\forall x \in S)\varphi[x].$$

Example:

- Let e be some name of the empty class and let id be the term $\lambda x.x$. Then we have

$$e \in U \quad \text{and} \quad (\forall x \,\dot{\in}\, e)(id(x) \in S).$$

  Thus $\{x : x \,\dot{\in}\, e\} \in S$, and this codes the empty set. Write emp for $\{x : x \,\dot{\in}\, e\} \in S$.

- Now consider the terms $t := \lambda x.\mathrm{emp}$. Then we have

$$\mathrm{nat} \in U \quad \text{and} \quad (\forall x \in \mathrm{nat})(tx \in S).$$

  Hence $\{tx : x \in \mathrm{nat}\} \in S$. Since all tx are equal to emp this codes the set whose only element is *emp*.

- Now suppose that we have the sets $\{fx : x \dot{\in} a\}$ and $\{gy : y \dot{\in} b\}$ with $a, b \in U$. Then we first define the class

  $$\{\langle 0, x\rangle : x \dot{\in} a\} \cup \{\langle 1, y\rangle : y \dot{\in} b\}$$

  and let c be one of its names that belongs to U. We alss let h be an operation, i.e. a first-order term that satisfies

  $$h(\langle i, x\rangle) = \begin{cases} gx & \text{if } i = 0, \\ hx & \text{if } i = 1, \end{cases}$$

  Then $(\forall x \dot{\in} c)(hx \in S)$. Hence $\{hx \dot{\in} c\} \in S$ and codes the union of $\{fx : x \dot{\in} a\}$ and $\{gy : y \dot{\in} b\}$.

Recursive definition of the extensional equality of sets

$$\{fx : x \mathrel{\dot{\in}} a\} \equiv \{gy : y \mathrel{\dot{\in}} b\} \quad :\Leftrightarrow$$

$$(\forall x \mathrel{\dot{\in}} a)(\exists y \mathrel{\dot{\in}} b)(fx \equiv gy) \ \wedge \ (\forall y \mathrel{\dot{\in}} b)(\exists x \mathrel{\dot{\in}} a)(fx \equiv gy).$$

Definition of the elementhood of sets: For all objects $r \in S$:

$$r \mathrel{\tilde{\in}} \{gy : y \mathrel{\dot{\in}} b\} \quad \Leftrightarrow \quad (\exists y \mathrel{\dot{\in}} b)(r \equiv gy).$$

A short interlude: least universes

Replace the constant ℓ by $\hat{\ell}$ and the axioms (Lim) by $(\widehat{\mathsf{Lim}})$:

$(\hat{\mathrm{L}}1)$ $a \in \Re \;\to\; \hat{\ell}a \in \Re$,

$(\hat{\mathrm{L}}2)$ $\hat{\ell}a \in \Re \;\to\; \mathcal{U}[\hat{\ell}a] \wedge a \mathrel{\dot{\in}} \hat{\ell}a \wedge (\forall b \in \Re)(\mathcal{U}[b] \wedge a \mathrel{\dot{\in}} b \to \hat{\ell}a \mathrel{\dot{\subseteq}} b)$.

Thus:

- $\hat{\ell}a$ is the name of the intersection of all universes that contain a.
- Least universes are defined by reference to the totality of all classes.
- $\mathsf{EC} + (\mathsf{J}) + (\widehat{\mathsf{Lim}}) + (\mathbb{L}\text{-}\mathsf{I_N})$ closely related to $\mathsf{T_0}$ and thus much stronger than $\mathsf{EC} + (\mathsf{J}) + (\mathsf{Lim}) + (\mathbb{L}\text{-}\mathsf{I_N})$.

Explicit Mahlo

EC + (J) + (Lim) describes the explicit analogue of an inaccessible universe or of an recursively inaccessible universe. It is predicatively justified or reducible according the the Feferman-Schütte approach.

What if we go a step further?

An ordinal α is callled a Mahlo ordinal iff

$$(\forall f : \alpha \to \alpha)(\exists \beta < \alpha)(\beta \in \mathit{Reg} \wedge f : \beta \to \beta).$$

We live in a Mahlo world – roughly speaking – if for every class A and for every operation f that maps classes to classes there exists a universe $U(A, f)$ such that A is represented in $U(A, f)$ and f maps $U(A, f)$ to $U(A, f)$.

In the language of explicit mathematics:

$$f \in [\Re]_1 \;:=\; (\forall x \in \Re)(fx \in \Re),$$

$$f \in [a]_1 \;:=\; (\forall x \mathrel{\dot{\in}} a)(fx \mathrel{\dot{\in}} a).$$

Mahlo axioms (M)

(M1) $a \in \Re \wedge f \in [\Re]_1 \;\rightarrow\; \mathsf{m}(a, f) \in \Re,$

(M2) $\mathsf{m}(a, f) \in \Re \;\rightarrow\; \mathcal{U}[\mathsf{m}(a, f)] \wedge a \mathrel{\dot{\in}} \mathsf{m}(a, f) \wedge f \in [\mathsf{m}(a, f)]_1.$

Theorem

1. $|\mathsf{EC} + (\mathsf{J}) + (\mathsf{M}) + (\mathsf{C}\text{-}\mathsf{I}_\mathsf{N})| \;=\; \varphi(\omega, 0, 0).$
2. $|\mathsf{EC} + (\mathsf{J}) + (\mathsf{M}) + (\mathbb{L}\text{-}\mathsf{I}_\mathsf{N})| \;=\; \varphi(\varepsilon_0, 0, 0).$

Thank you for your attention!