The concept of ergodcity restricts the memory of a stationary stochastic process

Stationary process

Intuitively : for two random variables in the stochastic process Z_t and Z_{t+j} far apart

$$f(z_t, z_{t-j}) \underset{j\to\infty}{\to} f(z_t)\cdot f(z_{t-j})$$

Ergodicity when process is Gaussian (each Z_t normally distributed)

Autocovariances $Cov(Z_t, Z_{t-j}) = \gamma_j$ go to 0 sufficiently quickly as j becomes large

Important Result: A stationary Gaussian process is ergodic if

$$|\gamma_1| + |\gamma_2| + |\gamma_3| + \ldots\ldots = \sum_{j=1}^{\infty} |\gamma_j| < \infty$$

The need for ergodic stationarity

Ergodic theorem

If a stochastic process is stationary and ergodic with $E(Z_t) = \mu$

and we observe a realisation $z_1, z_2, z_3, \ldots\ldots z_T$

then $\frac{1}{T}\sum_{t=1}^{T} z_T \underset{T\to\infty}{\to} \mu$

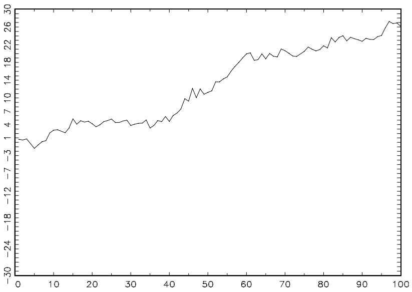
when stochastic process generating the Z_t is stationary and ergodic then the sequence of random variables $f(Z_1), f(Z_2), f(Z_3), \ldots\ldots\ldots$ also generates a stationary and ergodic process

$\Rightarrow$ moments, $E(Z_t^2), Var(Z_t), Cov(Z_t, Z_{t-j})$ consistently estimated by sample means if the stochastic process stationary and ergodic

It is important to distinguish the realisation from the process

stochastic process

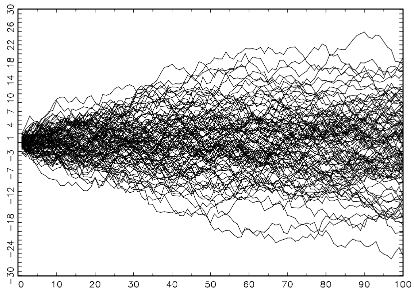
$$Y_t = Y_{t-1} + \varepsilon_t, \quad \varepsilon_t \sim N(0,1)$$
$$Y_0 = 0$$



Estimate by taking sample averages

$$\hat{\mu} = \frac{1}{T}\sum_{t=1}^{100} Y_t = 6.377$$

$$\hat{\sigma}^2 = \frac{1}{T}\sum_{t=1}^{100} (Y_t - \hat{\mu})^2 = 25.130$$



Estimate by taking ensemble averages at each point

$$\hat{\mu}_1 = \frac{1}{10000}\sum_{s=1}^{10000} Y_1^s = \text{-}0.004$$

$$\hat{\sigma}_1^2 = \frac{1}{10000}\sum_{s=1}^{10000} \left(Y_1^s - \hat{\mu}_1\right)^2 = 0.991$$

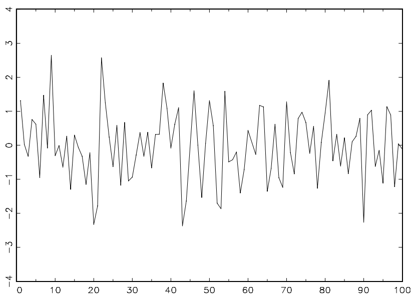
$$\hat{\mu}_{100} = \frac{1}{10000}\sum_{s=1}^{10000} Y_{100}^s = 0.023$$

$$\hat{\sigma}_{100}^2 = \frac{1}{10000}\sum_{s=1}^{10000} \left(Y_{100}^s - \hat{\mu}_{100}\right)^2 = 99.028$$

It is important to distinguish the realisation from the process

stochastic process

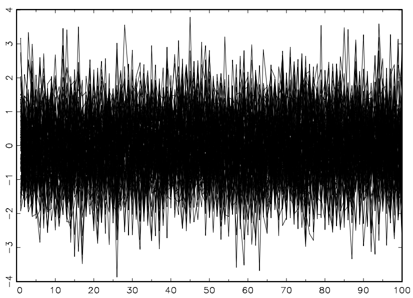
$$Y_t = \varepsilon_t, \quad \varepsilon_t \sim N(0,1)$$
$$Y_0 = 0$$



Estimate by taking sample averages

$$\hat{\mu} = \frac{1}{T}\sum_{t=1}^{100} Y_t = -0.011$$

$$\hat{\sigma}^2 = \frac{1}{T}\sum_{t=1}^{100} (Y_t - \hat{\mu})^2 = 1.065$$



Estimate by taking ensemble averages at each point

$$\hat{\mu}_1 = \frac{1}{10000}\sum_{s=1}^{10000} Y_1^s = \text{-}0.004$$

$$\hat{\sigma}_1^2 = \frac{1}{10000}\sum_{s=1}^{10000} \left(Y_1^s - \hat{\mu}_1\right)^2 = 1.001$$

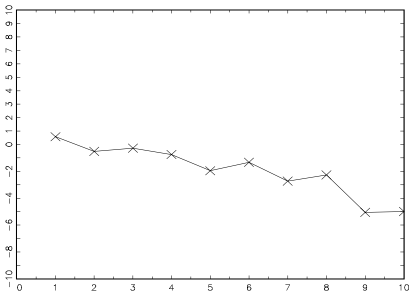
$$\hat{\mu}_{100} = \frac{1}{10000}\sum_{s=1}^{10000} Y_{100}^s = 0.000$$

$$\hat{\sigma}_{100}^2 = \frac{1}{10000}\sum_{s=1}^{10000} \left(Y_{100}^s - \hat{\mu}_{100}\right)^2 = 0.996$$

It is important to distinguish the realisation from the process

stochastic process

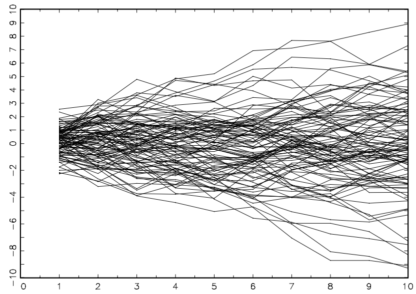
$$Y_t = Y_{t-1} + \varepsilon_t, \quad \varepsilon_t \sim N(0,1)$$
$$Y_0 = 0$$



Estimate by taking sample averages

$$\hat{\mu} = \frac{1}{T}\sum_{t=1}^{10} Y_t = \text{-}1.929$$

$$\hat{\sigma}^2 = \frac{1}{T}\sum_{t=1}^{10} (Y_t - \hat{\mu})^2 = 3.631$$



Estimate by taking ensemble averages at each point

$$\hat{\mu}_1 = \frac{1}{10000}\sum_{s=1}^{10000} Y_1^s = \text{-}0.008$$

$$\hat{\sigma}_1^2 = \frac{1}{10000}\sum_{s=1}^{10000} \left(Y_1^s - \hat{\mu}_1\right)^2 = 1.015$$

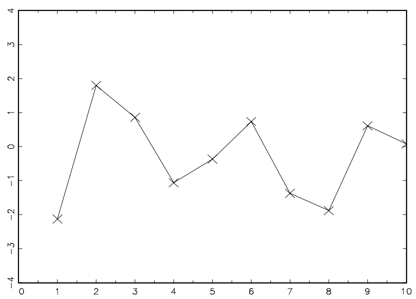
$$\hat{\mu}_{10} = \frac{1}{10000}\sum_{s=1}^{10000} Y_{10}^s = \text{-}0.036$$

$$\hat{\sigma}_{10}^2 = \frac{1}{10000}\sum_{s=1}^{10000} \left(Y_{10}^s - \hat{\mu}_{10}\right)^2 = 10.097$$

It is important to distinguish the realisation from the process

stochastic process

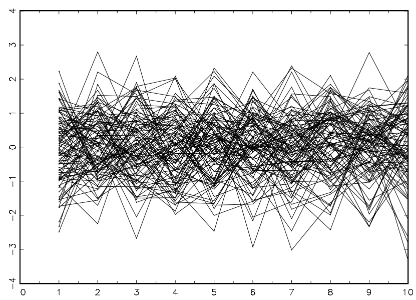
$$Y_t = \varepsilon_t, \quad \varepsilon_t \sim N(0,1)$$
$$Y_0 = 0$$



Estimate by taking sample averages

$$\hat{\mu} = \frac{1}{T}\sum_{t=1}^{10} Y_t = \text{-}0.009$$

$$\hat{\sigma}^2 = \frac{1}{T}\sum_{t=1}^{10} (Y_t - \hat{\mu})^2 = 1.914$$



Estimate by taking ensemble averages at each point

$$\hat{\mu}_1 = \frac{1}{10000}\sum_{s=1}^{10000} Y_1^s = \text{-}0.000$$

$$\hat{\sigma}_1^2 = \frac{1}{10000}\sum_{s=1}^{10000} \left(Y_1^s - \hat{\mu}_1\right)^2 = 0.993$$

$$\hat{\mu}_{10} = \frac{1}{10000}\sum_{s=1}^{10000} Y_{10}^s = 0.002$$

$$\hat{\sigma}_{10}^2 = \frac{1}{10000}\sum_{s=1}^{10000} \left(Y_{10}^s - \hat{\mu}_{10}\right)^2 = 1.002$$