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S.I.: SUBSTRUCTURAL APPROACHES TO PARADOX

How to Ekman a Crabbé-Tennant

Peter Schroeder-Heister[1] · **Luca Tranchini**[1]

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Abstract

Developing early results of Prawitz, Tennant proposed a criterion for an expression to count as a paradox in the framework of Gentzen's natural deduction: paradoxical expressions give rise to non-normalizing derivations. Two distinct kinds of cases, going back to Crabbé and Tennant, show that the criterion overgenerates, that is, there are derivations which are intuitively non-paradoxical but which fail to normalize. Tennant's proposed solution consists in reformulating natural deduction elimination rules in general (or parallelized) form. Developing intuitions of Ekman we show that the adoption of general rules has the consequence of hiding redundancies within derivations. Once reductions to get rid of the hidden redundancies are devised, it is clear that the adoption of general elimination rules offers no remedy to the overgeneration of the Prawitz–Tennant analysis. In this way, we indirectly provide further support for a solution to one of the two overgeneration cases developed in previous work.

Keywords Paradox · Natural deduction · General elimination rules · Normalization · Crabbé counterexample · Set-theory

1 The Prawitz–Tennant analysis of paradoxes

The natural deduction system for minimal implicational logic $\mathtt{NM}$ consists of the following introduction and elimination rules for implication as its only primitive rules of inference:[1]

[1] In rule schemata we indicate in square brackets the assumptions which can be discharged by rule applications. In actual derivations we indicate discharge with numbers placed above the discharged assumptions and repeated next to the rule label. In schematic derivations we use square brackets to indicate an arbitrary number of occurrences of a formula, if the formulas is in assumption position, or of the whole sub-derivation having the formula in brackets as conclusion.

✉ Peter Schroeder-Heister
psh@uni-tuebingen.de
http://www.ls.informatik.uni-tuebingen.de

✉ Luca Tranchini
luca.tranchini@gmail.com
http://www.sites.google.com/site/lucatranchini/

1 Department of Computer Science, Eberhard Karls Universität Tübingen, Tübingen, Germany

$$\dfrac{\begin{matrix}[A]\\ B\end{matrix}}{A \supset B}\supset\mathrm{I} \qquad \dfrac{A\supset B \qquad A}{B}\supset\mathrm{E}$$

Negation is defined as implication of absurdity $\perp$, i.e., $\neg A =_{\text{def}} A \supset \perp$.

In natural deduction systems, the application in a derivation of an introduction rule followed immediately by an application of the corresponding elimination rule constitutes a redundancy. Redundancies can be eliminated by rearranging the structure of derivations according to certain reductions. The reduction for implication is the following:

$$\dfrac{\dfrac{\begin{matrix}\overset{n}{[A]}\\ \mathscr{D}\\ B\end{matrix}}{A\supset B}(n) \qquad \begin{matrix}\mathscr{D}'\\ A\end{matrix}}{B} \quad \overset{\supset\text{-Red}}{\triangleright} \quad \begin{matrix}\mathscr{D}'\\ [A]\\ \mathscr{D}\\ B\end{matrix}$$

The occurrence of $A \supset B$ in the left derivation, which is removed by this reduction step, will be called an $\supset$-*redundant formula occurrence*. A derivation is normal if and only if it is redundancy-free. In his book on natural deduction, Prawitz (1965) showed that all derivations in minimal (as well as intuitionistic and classical) logic can be transformed into normal form.

In Appendix B to this book, Prawitz considered a system for naive set theory, we will refer to it as $\mathtt{NM}^{\in}$, obtained by extending the one for minimal logic with an introduction and an elimination rule for formulas of the form $t \in \{x : A\}$ to express set-theoretical comprehension:

$$\dfrac{A(t/x)}{t \in \{x : A\}}\in\mathrm{I} \qquad \dfrac{t \in \{x : A\}}{A(t/x)}\in\mathrm{E}$$

where $A(t/x)$ is the result of substituting t for x in A. Also in this case an application of the introduction rule immediately followed by an application of the corresponding elimination rules constitutes a redundancy which can be eliminated according to the following $\in$-*reduction*:

$$\dfrac{\dfrac{\begin{matrix}\mathscr{D}\\ A(t/x)\end{matrix}}{t \in \{x : A\}}}{A(t/x)} \quad \overset{\in\text{-Red}}{\triangleright} \quad \begin{matrix}\mathscr{D}\\ A(t/x)\end{matrix}$$

The occurrence of $t \in \{x : A\}$, which is removed by this reduction, will be called an $\in$-*redundant formula occurrence*.

Taking ρ to be $\mathfrak{r} \in \mathfrak{r}$, where $\mathfrak{r}$ is the Russell term $\{x : x \notin x\}$, an application of $\in$I allows one to pass over from $\neg\rho$ to ρ, and an application of $\in$E from ρ back to $\neg\rho$:

$$\dfrac{\neg\rho}{\rho}\in\mathrm{I} \qquad \dfrac{\rho}{\neg\rho}\in\mathrm{E}$$

Using these instances of $\in$I and $\in$E one can construct the following derivation of Russell's paradox in $\mathtt{NM}^{\in}$:

$$
\begin{prooftree}
\AxiomC{$\overset{1}{\rho}$}
\RightLabel{$\in$E}
\UnaryInfC{$\neg\rho$}
\AxiomC{$\overset{1}{\rho}$}
\RightLabel{$\supset$E}
\BinaryInfC{$\bot$}
\RightLabel{$\supset$I (1)}
\UnaryInfC{$\neg\rho$}
\AxiomC{$\overset{1}{\rho}$}
\RightLabel{$\in$E}
\UnaryInfC{$\neg\rho$}
\AxiomC{$\overset{1}{\rho}$}
\RightLabel{$\supset$E}
\BinaryInfC{$\bot$}
\RightLabel{$\supset$I (1)}
\UnaryInfC{$\neg\rho$}
\RightLabel{$\in$I}
\UnaryInfC{ρ}
\RightLabel{$\supset$E}
\BinaryInfC{$\bot$}
\end{prooftree}
\qquad (\mathbf{R})
$$

Since the encircled occurrence of $\neg\rho$ is an $\supset$-redundant formula occurrence, this derivation is not normal. By applying the implication reduction $\supset$-$\mathtt{Red}$ we obtain the following derivation:

$$
\begin{prooftree}
\AxiomC{$\overset{1}{\rho}$}
\RightLabel{$\in$E}
\UnaryInfC{$\neg\rho$}
\AxiomC{$\overset{1}{\rho}$}
\RightLabel{$\supset$E}
\BinaryInfC{$\bot$}
\RightLabel{$\supset$I (1)}
\UnaryInfC{$\neg\rho$}
\RightLabel{$\in$I}
\UnaryInfC{ρ}
\RightLabel{$\in$E}
\UnaryInfC{$\neg\rho$}
\AxiomC{$\overset{1}{\rho}$}
\RightLabel{$\in$E}
\UnaryInfC{$\neg\rho$}
\AxiomC{$\overset{1}{\rho}$}
\RightLabel{$\supset$E}
\BinaryInfC{$\bot$}
\RightLabel{$\supset$I (1)}
\UnaryInfC{$\neg\rho$}
\RightLabel{$\in$I}
\UnaryInfC{ρ}
\RightLabel{$\supset$E}
\BinaryInfC{$\bot$}
\end{prooftree}
\qquad (\mathbf{R}')
$$

Here the encircled occurrence of ρ is an $\in$-redundant formula occurrence. By applying the following instance of $\in$-$\mathtt{Red}$:

$$
\begin{array}{c}
\mathscr{D} \\
\neg\rho \\
\hline
\rho \\
\hline
\neg\rho
\end{array}
\quad \overset{\in\text{-}\mathtt{Red}}{\rhd} \quad
\begin{array}{c}
\mathscr{D} \\
\neg\rho
\end{array}
$$

we obtain the derivation **R** we started with. Since at each of the two steps there was only a single possibility to reduce the derivation, all possible ways of reducing the derivation (called *reduction sequences*) get stuck in a loop. Prawitz proposed this to be the distinctive feature of Russell's paradox.

Tennant (1982) considered a wide range of examples and claimed that all prominent mathematical and logical paradoxes follow this pattern. The steps playing the role of $\in$I and $\in$E are called *id est inferences*, as they result from extra-logical principles: In the case of Russell's paradox, from set-theoretic comprehension. In the case of the liar paradox, to take another example, analogous *id est* inferences would be based on the observation that a certain sentence says of itself that it is not true. Here, "observation" is not necessarily empirical inspection, but may result from some arithmetical referencing mechanism.[2]

[2] Instead of looping reduction sequences one can, more generally, consider non-terminating reduction sequences, which covers paradoxes such as Yablo's (see Tennant 1995). In the following, we will throughout speak of the *looping feature* of paradoxical derivations, keeping in mind that "non-termination" of reduction sequences is the appropriate more general term.

Schroeder-Heister and Tranchini (2017) dubbed the 'Prawitz–Tennant analysis of paradox' the thesis that a paradoxical derivation is a derivation of $\bot$ or of any other "unwanted" sentence (as in the case of Curry's paradox) that fails to normalize.

The Prawitz–Tennant analysis of paradoxes is a way to characterize paradoxes by their proof-theoretic behavior, looking at the derivation of absurdity generated. Although this is not *per se* a solution to the paradoxes and Tennant stresses it should not be meant as such (see, e.g., Tennant 1982, p. 268), it can be naturally turned into a solution (as implicitly suggested by both Prawitz and Tennant and, in a more explicit manner, by Schroeder-Heister 2012; Tranchini 2016, 2018). Derivations in natural deduction aim at representing proofs. According to Prawitz and Tennant, however, only normalisable derivations 'really' represent proofs. Tennant's moral is thus the following:

> The general loss of normalisability [...] is a small price to pay for the protection it gives against paradox itself. Logic plays its role as an instrument of knowledge only insofar as it keeps proofs in sharp focus, through the lens of normality. (Tennant 1982, p. 284)

As the paradoxical derivations of absurdity do not normalize, they are no 'real' proofs.

Although we are strongly sympathetic to the Prawitz–Tennant analysis, in a previous article (see Schroeder-Heister and Tranchini 2017) we suggested that certain results by Ekman (1994) can be naturally seen as showing that the proposed criterion for paradoxicality overgenerates. To solve the overgeneration problem, we argued that the notion of reduction underlying the criterion must be appropriately qualified, by requiring the reductions to preserve the identity of the proofs represented by derivations (or more philosophically, by requiring the reductions to be meaning-theoretically justified).

Tennant (2016) discards our solution as too baroque, and attempts to untrigger the overgeneration we observed (as well as another one Tennant observed in 1982) without having to assume any fundamental difference between "good" and "bad" reductions. His attempted solution, corresponding to observations by von Plato (2000), consists in rejecting *modus ponens* in favor of the so-called general (or parallel) implication elimination rule.

In this paper, we show that Tennant's attempted solution is prone to the difficulties of the original proposal, too. That is, without a criterion for the acceptability of reduction procedures, the Prawitz–Tennant analysis overgenerates even when reformulated using general elimination rules. The results presented thus aim at providing further evidence in favor of our previously advocated solution.

2 Two cases of overgeneration

In this section we discuss two distinct cases in which the Prawitz–Tennant analysis overgenerates, i.e. in which it ascribes paradoxality to derivations of $\bot$ that fail to normalize, although they belong to deductive settings that are too weak to allow for the fomulation of paradoxes.

2.1 From naive comprehension to separation

The first case of overgeneration arises in a consistent set theory in which Zermelo's separation axiom is formulated in rule form:

$$\dfrac{t \in s \qquad A(t/x)}{t \in \{x \in s : A\}}\;{\in^z}\mathrm{I} \qquad\qquad \dfrac{t \in \{x \in s : A\}}{A(t/x)}\;{\in^z}\mathrm{E} \qquad \dfrac{t \in \{x \in s : A\}}{t \in s}\;{\in^z}\mathrm{E}_2$$

(the second elimination rule will actually play no role in what follows). We call the resulting system $\mathtt{NM}^{\in^z}$.

Again an application of $\in^z$I followed by $\in^z$E constitutes a redundancy that can be eliminated according to the following reduction (we call the formula eliminated a *Zermelo-redundant formula*) :

$$\dfrac{\dfrac{\begin{matrix}\mathscr{D}_1\\ t \in s\end{matrix} \qquad \begin{matrix}\mathscr{D}_2\\ A(t/x)\end{matrix}}{t \in \{x \in s : A\}}}{A(t/x)} \quad \overset{\in^z\text{-Red}}{\rhd} \quad \begin{matrix}\mathscr{D}_2\\ A(t/x)\end{matrix}$$

What if we now try to reconstruct Russell's reasoning in this setting? For any set y, we can construct a term denoting the Russell subset of y, i.e. the set of all elements of y which do not belong to themselves, $\mathfrak{r}_y =_{def} \{x \in y : x \notin x\}$. Taking now ρ_y to be $\mathfrak{r}_y \in \mathfrak{r}_y$, an application of $\in^z$E allows one to pass over from ρ_y to $\neg\rho_y$, but in order to pass over from $\neg\rho_y$ to ρ_y using an application of $\in^z$I one needs a further premise, namely $\mathfrak{r}_y \in y$:

$$\dfrac{\mathfrak{r}_y \in y \qquad \neg\rho_y}{\rho_y}\;{\in^z}\mathrm{I} \qquad \dfrac{\rho_y}{\neg\rho_y}\;{\in^z}\mathrm{E}$$

Thus, by following Russell's reasoning in $\mathtt{NM}^{\in^z}$ one obtains a derivation of absurdity $\bot$ in $\mathtt{NM}^{\in^z}$ that, contrary to **R**, depends on an assumption, namely the assumption $\mathfrak{r}_y \in y$ that is needed for the application of $\in^z$I (for visibility this assumption is boxed):

$$\dfrac{\dfrac{\dfrac{\dfrac{\overset{1}{\rho_y}}{\neg\rho_y}\;{\in^z}\mathrm{E} \qquad \overset{1}{\rho_y}}{\bot}\;{\supset}\mathrm{E}}{\neg\rho_y}\;{\supset}\mathrm{I}\,(1) \qquad \dfrac{\boxed{\mathfrak{r}_y \in y} \qquad \dfrac{\dfrac{\dfrac{\overset{1}{\rho_y}}{\neg\rho_y}\;{\in^z}\mathrm{E} \qquad \overset{1}{\rho_y}}{\bot}\;{\supset}\mathrm{E}}{\neg\rho_y}\;{\supset}\mathrm{I}\,(1)}{\rho_y}\;{\in^z}\mathrm{I}}{\bot}\;{\supset}\mathrm{E} \qquad (\mathbf{R^z})$$

Now assume that existential quantification with its standard rules is available. As y does not occur free in the conclusion nor in any undischarged assumption other than $\mathfrak{r}_y \in y$, by assuming $\exists y(\mathfrak{r}_y \in y)$ we can obtain by $\exists E$ a derivation of $\bot$ from $\exists y(\mathfrak{r}_y \in y)$ and by $\supset$I we can thereby establish $\neg\exists y(\mathfrak{r}_y \in y)$, that is that no set contains its own Russell subset:

$$
\begin{prooftree}
\AxiomC{$\overset{3}{\exists y(\mathfrak{r}_y \in y)}$}
\AxiomC{$\overset{1}{\rho_y}$}
\RightLabel{$\in^z$E}
\UnaryInfC{$\neg\rho_y$}
\AxiomC{$\overset{1}{\rho_y}$}
\RightLabel{$\supset$E}
\BinaryInfC{$\bot$}
\RightLabel{$\supset$I (1)}
\UnaryInfC{$\neg\rho_y$}
\AxiomC{$\overset{2}{\mathfrak{r}_y \in y}$}
\AxiomC{$\overset{1}{\rho_y}$}
\RightLabel{$\in^z$E}
\UnaryInfC{$\neg\rho_y$}
\AxiomC{$\overset{1}{\rho_y}$}
\RightLabel{$\supset$E}
\BinaryInfC{$\bot$}
\RightLabel{$\supset$I (1)}
\UnaryInfC{$\neg\rho_y$}
\RightLabel{$\in^z$I}
\BinaryInfC{ρ_y}
\RightLabel{$\supset$E}
\BinaryInfC{$\bot$}
\RightLabel{$\exists$E (2)}
\BinaryInfC{$\bot$}
\RightLabel{$\supset$I (3)}
\UnaryInfC{$\neg\exists y(\mathfrak{r}_y \in y)$}
\end{prooftree}
$$

That no set contains its own Russell subset is a perfectly acceptable conclusion in a consistent set theory like Zermelo's. It shows in particular that there is no set of all sets, which is something that any reasonable consistent set theory should be able to prove. However, and here is the problem, the derivation $\mathbf{R}^{\mathbf{z}}$ of $\bot$ from $\mathfrak{r}_y \in y$ (and likewise the one of $\bot$ from $\exists y(\mathfrak{r}_y \in y)$) fails to normalize, for the same reason as Russell's original **R**. By removing the encircled $\supset$-redundant occurrence of $\neg\rho_y$, a Zermelo-redundant formula is introduced, and by removing it, one gets back to $\mathbf{R}^{\mathbf{z}}$. So, on the Prawitz–Tennant analysis the derivation does not represent a real proof, and (as in the case of the derivation of $\bot$ in naive set theory) no other derivation fares better. That is, on the Prawitz–Tennant analysis, though we have derivations showing that there is no set of all sets in Zermelo set theory based on separation, these derivations are unacceptable as they qualify as paradoxical.

These facts were first observed by Marcel Crabbé (n.d.) in 1974 at the Logic Colloquium in Kiel (Müller et al. 1975) and have been largely neglected in the philosophical literature (in particular by Tennant), except for a short reference to them in Sundholm (1979). However, they represent the starting point of modern proof-theoretic investigations of set theory (see Hallnäs 1988; Ekman 1994).

2.2 Ekman's paradox

The other kind of overgeneration arises in an even weaker setting: pure propositional logic. Suppose we have derived A by means of a derivation $\mathscr{D}$. By assuming $A \supset B$, $\mathscr{D}$ can be extended by $\supset$E to a derivation of conclusion B. By further assuming $B \supset A$ one can conclude A, but this had already been established by $\mathscr{D}$. The two applications of $\supset E$ just make one jump back and forth between A and B:

$$
\begin{prooftree}
\AxiomC{$B \supset A$}
\AxiomC{$A \supset B$}
\AxiomC{$\mathscr{D}$}
\noLine
\UnaryInfC{A}
\RightLabel{$\supset$E}
\BinaryInfC{B}
\RightLabel{$\supset$E}
\BinaryInfC{A}
\end{prooftree}
$$

Ekman (1998) observed that although the official reductions of `NM` do not allow to get rid of patterns of this kind, such patters certainly constitute redundancies which can be easily removed by identifying the top and bottom occurrences of A and removing the two applications of $\supset$E between them. We refer to this conversion as `Ekman` and we will call an *`Ekman`*-redundant formula occurrence the occurrence of B acting as conclusion of the first application of $\supset$E and as minor premise of the second application of $\supset$E in the schema below:[3]

$$\dfrac{B \supset A \qquad \dfrac{A \supset B \qquad \begin{matrix}\mathscr{D}\\ A\end{matrix}}{B}\supset\mathrm{E}}{A}\supset\mathrm{E} \qquad \overset{\texttt{Ekman}}{\rhd} \qquad \begin{matrix}\mathscr{D}\\ A\end{matrix}$$

Observe now that $\neg A$ follows from $A \supset \neg A$:

$$\dfrac{\dfrac{\dfrac{A \supset \neg A \qquad \overset{1}{A}}{\neg A}\supset\mathrm{E} \qquad \overset{1}{A}}{\bot}\supset\mathrm{E}}{\neg A}\supset\mathrm{I}\ (1)$$

By further assuming $\neg A \supset A$, the previous derivation can be extended by $\supset$E to a derivation of A from $A \supset \neg A$ and $\neg A \supset A$:

$$\dfrac{\neg A \supset A \qquad \dfrac{\dfrac{\dfrac{A \supset \neg A \qquad \overset{1}{A}}{\neg A}\supset\mathrm{E} \qquad \overset{1}{A}}{\bot}\supset\mathrm{E}}{\neg A}\supset\mathrm{I}\ (1)}{A}\supset\mathrm{E}$$

The two derivations can be joined together by an application of $\supset$E and the result is the following derivation of $\bot$ from $A \supset \neg A$ and $\neg A \supset A$ (here and below some of the rules labels will be omitted for readability):

$$\dfrac{\dfrac{\dfrac{\dfrac{A \supset \neg A \qquad \overset{1}{A}}{\neg A}\supset\mathrm{E} \qquad \overset{1}{A}}{\bot}\supset\mathrm{E}}{\neg A}\supset\mathrm{I}\ (1) \qquad \dfrac{\neg A \supset A \qquad \dfrac{\dfrac{\dfrac{A \supset \neg A \qquad \overset{1}{A}}{\neg A} \qquad \overset{1}{A}}{\bot}}{\neg A}\ (1)}{A}\supset\mathrm{E}}{\bot}\supset\mathrm{E} \qquad \textbf{(E)}$$

The derivation **E** is not normal, since the encircled occurrence of $\neg A$ is an implication-redundant formula occurrence. By applying $\supset$-`Red` one introduces a redundancy of the kind observed by Ekman (we encircle in the derivation the `Ekman`-redundant formula occurrence):

[3] We observe that there is a fundamental distinction between `Ekman` and, say, $\supset$-`Red` in that the latter is a means of getting rid of an application of an introduction rule followed by an application of the corresponding elimination and is thus an immediate consequence of the "harmony" governing the two rules. Not so for `Ekman`, that may thus be seen as lacking a *prima facie* plausible meaning-theoretical justification. This remark has been fully developed in Schroeder-Heister and Tranchini (2017) to untrigger the kind of overgeneration discussed in this section.

$$\dfrac{\dfrac{A\supset\neg A \qquad \dfrac{\neg A\supset A \qquad \dfrac{\dfrac{\dfrac{A\supset\neg A \quad \overset{1}{A}}{\neg A} \quad \overset{1}{A}}{\perp}}{\neg A}\,{\supset}\mathrm{I}\,(1)}{A}\,{\supset}\mathrm{E}}{\neg A}\,{\supset}\mathrm{E} \qquad \dfrac{\neg A\supset A \qquad \dfrac{\dfrac{\dfrac{A\supset\neg A \quad \overset{1}{A}}{\neg A} \quad \overset{1}{A}}{\perp}}{\neg A}\,(1)}{A}\,{\supset}\mathrm{E}}{\perp} \qquad (\mathbf{E}')$$

By applying the relevant instance of `Ekman`:

$$\dfrac{A\supset\neg A \qquad \dfrac{\neg A\supset A \qquad \begin{matrix}\mathscr{D}\\ \neg A\end{matrix}}{A}\,{\supset}\mathrm{E}}{\neg A}\,{\supset}\mathrm{E} \quad \overset{\texttt{Ekman}}{\triangleright} \quad \begin{matrix}\mathscr{D}\\ \neg A\end{matrix}$$

we get back the derivation **E**.

Thus, on the natural extension of the set of conversions suggested by Ekman, we have a counterexample to normalization in NM: **E** is not normal and does not normalize, since its reduction process enters a loop. Given the Prawitz–Tennant analysis of paradoxes in term of non-normalizability, the phenomenon observed by Ekman should show that paradoxes already appear at the level of propositional logic.

In fact, Ekman's paradox can be taken to show that the logical component of Russell's paradox can be fully described using propositional logic. The derivations of Russell's paradox **R** and **R**′ can be obtained from Ekman's derivations **E** and **E**′ by suppressing all occurrences of $A \supset \neg A$ and $\neg A \supset A$ and by replacing all occurrences of the schematic letter A with ρ: In this way, the applications of $\supset$E with major premise $\neg A \supset A$ and $A \supset \neg A$ become applications of $\in$I and $\in$E respectively. In other words, the *id est* inferences involved in the derivation of Russell's paradox are simulated by applications of $\supset$E in **E** and **E**′, and the instance of $\in$-`Red` used to transform **R**′ into **R** is simulated by the instance of `Ekman` used to transform **E**′ into **E**.

However, it is not its logical component what makes Russell's reasoning paradoxical, but the *id est* rules encoding naive comprehension. Propositional logic alone is too weak to allow for the formulation of paradoxical expression and thereby there cannot be anything paradoxical about a derivation in NM.[4]

3 A solution to overgeneration

The phenomena presented in the previous section throw a shadow on the analysis of paradoxes put forward by Prawitz and Tennant. In another article (Schroeder-Heister and Tranchini 2017) we showed how to save the Prawitz–Tennant analysis of paradoxes from the threat of Ekman by imposing some restrictions on what is to count as an appropriate reduction. If one requires that reductions have to preserve proof identity

[4] Elia Zardini observes that the derivations **E** and **E**′ are paradoxical because there are instances of them which are paradoxical. Observe however, that **R** and **R**′ are not simply instances of **E** and **E**′, as they do not arise by simply instantiating A with ρ, but moreover by replacing the assumptions $\neg A \supset A$ and $A \supset \neg A$ with genuine inferential steps, and it is to these steps that the source of paradoxicality is—in Tennant's intentions—to be ascribed.

in the sense of categorial or computational approaches to natural deduction, there are strong reasons to reject Ekman's reduction (see also footnote 3 above). There is however no immediate way of applying this strategy to solve the other kind of overgeneration cases observed by Crabbé. From the perspective of identity of proof, there is a strong asymmetry between the two cases, and the overgeneration cases observed by Crabbé show a particular resilience.

A different kind of solution to the overgeneration phenomenon observed by Ekman was put forward by von Plato (2000) and recently reinstated by Tennant (2016), who showed how it could be used to overcome also the other kind of overgeneration cases. For Tennant, this alternative solution is preferable not only because it allows one to solve both issues at once, but also because it does not require to introduce criteria to select what counts as an appropriate reduction. The aim of what follows is a critical discussion of this alternative solution, which shows, at least, that the question of what is to count as an appropriate reduction cannot be evaded so quickly as Tennant apparently supposes.

To clarify our position, we are strongly sympathetic to the Prawitz–Tennant analysis of paradoxes, and we do not take the kind of overgeneration observed by Ekman as being a real threat, provided the criterion for paradoxality is based on a qualified notion of reduction procedure. On the other hand, we do regard the kind of overgeneration observed by Crabbé as problematic (even on our refined formulation of the Prawitz–Tennant criterion) and as calling for further investigations.

What we are not at all sympathetic with is the "solution" to both kinds of overgeneration proposed by von Plato and Tennant, which will be shown in the remaining part of the present article to be, in fact, no solution at all, being flawed by the same problems of Tennant's original proposal. The line of argument developed in the present paper is thereby meant as a further—though indirect—reason to adopt our solution to the Ekman kind of overgeneration, and to further investigate the exact nature of the Crabbé kind of overgeneration.[5]

3.1 Von Plato's solution to Ekman

According to von Plato (2000), the source of Ekman's problem[6] is the form of the elimination rule for implication, and he suggested that the problem could be solved by replacing $\supset$E with its general (or parallelized) version (von Plato 2001; Tennant 2002):

$$\dfrac{A \supset B \qquad A \qquad \begin{matrix}[B] \\ C\end{matrix}}{C}\ \supset\mathrm{E}_g$$

We call $\mathtt{NM}_g$ the system obtained from $\mathtt{NM}$ by replacing $\supset$E with $\supset\mathrm{E}_g$.

[5] In the mentioned article, we also discuss and refute an attempted dismissal of the Prawitz–Tennant analysis based on a supposed case of undergeneration due to Rogerson (2007). A general investigation of possibly systematic undergeneration cases is the object of current research by the authors.

[6] It should be observed that von Plato (2000) is not in the least interested in the issue of paradoxes, and regards Ekman's phenomenon as a problem for normalization in minimal propositional logic.

Consecutive applications of the introduction and of the general elimination rule for implication also constitute a redundancy that can be eliminated according to the following reduction (we call the formula of the form $A \supset B$ eliminated by the reduction an $\supset_g$*-redundant formula occurrence*):

$$\dfrac{\dfrac{\begin{matrix}\overset{n}{[A]}\\ \mathscr{D}\\ B\end{matrix}}{A \supset B}\,(n) \qquad \begin{matrix}\mathscr{D}'\\ A\end{matrix} \qquad \begin{matrix}\overset{m}{[B]}\\ \mathscr{D}''\\ C\end{matrix}}{C}\,(m) \qquad \overset{\supset_g\texttt{-Red}}{\rhd} \qquad \begin{matrix}\mathscr{D}'\\ [A]\\ \mathscr{D}\\ [B]\\ \mathscr{D}''\\ \mathrm{C}\end{matrix}$$

In $\texttt{NM}_g$ the derivation of Ekman's paradox can be recast as follows:

$$\dfrac{\neg A \supset A \qquad \dfrac{\dfrac{A \supset \neg A \qquad \overset{3}{A} \qquad \dfrac{\overset{2}{\neg A} \qquad \overset{3}{A} \qquad \overset{1}{\bot}}{\bot}\,(1)}{\bot}\,(2)}{\neg A}\,(3) \qquad \dfrac{A \supset \neg A \qquad \overset{6}{A} \qquad \dfrac{\overset{5}{\neg A} \qquad \overset{6}{A} \qquad \overset{4}{\bot}}{\bot}\,(4)}{\bot}\,(5)}{\bot}\,(6) \qquad (\mathbf{E}'_{\mathbf{g}})$$

The reduction $\supset_g$-$\texttt{Red}$ does not apply to $\mathbf{E}'_{\mathbf{g}}$. Moreover, neither does $\texttt{Ekman}$ (obviously, since $\mathbf{E}'_{\mathbf{g}}$ is formulated with the general elimination rule and not with *modus ponens*) nor any generalization thereof,

> which have the general form of 'shrinking' to a single occurrence of A, any logically circular segments of branches (within the proof) of the form shown below to the left
>
> $$\begin{matrix}\dfrac{A}{B_1}\\ \vdots\\ \dfrac{B_n}{A}\end{matrix} \;\rhd A$$
>
> (Tennant 1995, pp. 199–200)

which Tennant (2016) calls *subproof compactification*. Note that, as Ekman (1994) already observed, $\in$-$\texttt{Red}$ and $\in^z$-$\texttt{Red}$ are also instances of subproof compactification (and so are the standard reductions for conjunction of Prawitz 1965), though neither $\supset$-$\texttt{Red}$ nor $\supset_g$-$\texttt{Red}$ are.

On these grounds, von Plato concludes that "the problem about normal form in Ekman (1998) is solved by a derivation using the general $\supset$E rule" (2000, p. 123).

3.2 Another "safe version" of Russell's paradox

Independently of Crabbé, Tennant (1978, 1982) proposed a weakening of naive comprehension, but in the context of a free logic. By free logic, one means a logic which is free from the assumption that singular terms denote. Using Zermelo's comprehension one wishes to neutralize Russell's paradox by recasting Russell's reasoning as showing that no set contains its Russell subset as element. Similarly, Tennant wishes to recast Russell's reasoning as showing that the Russell term lacks a denotation.

That a term t does possess a denotation is expressed by the formula $\exists! t =_{\text{def}} \exists x(t = x)$, and accordingly in free logic the introduction rule for identity is weakened to the effect that $t = t$ can be derived only if one has previously shown that t denotes:[7]

$$\frac{\exists! t}{t = t}$$

In this setting, Tennant proposes to replace the rules for naive comprehension with rules to introduce and eliminate set-terms in the context of identity statements:[8]

$$\dfrac{\begin{array}{cc} [A(y/x)] & [y \in s] \\ y \in s & A(y/x) \end{array}}{\{x : A\} = s}\,\{\}^{=}\mathrm{I} \qquad \dfrac{\{x : A\} = s \qquad A(t/x)}{t \in s}\,\{\}^{=}\mathrm{E}_1$$

$$\text{with } y \textit{ eigenvariable} \qquad \dfrac{\{x : A\} = s \qquad t \in s}{A(t/x)}\,\{\}^{=}\mathrm{E}_2$$

We call $\mathtt{NM}^{\in^{=}}$ the system that results by adding these rules to $\mathtt{NM}$.

It is important to observe that in Tennant's reformulation we have two elimination rules for set terms, and the two elimination rules of Tennant correspond respectively to Prawitz's $\in$I and $\in$E rules. By taking s to be $\{x : A\}$ we have that Tennant's $\{\}^{=}\mathrm{E}_1$ allows one to infer $t \in \{x : A\}$ from $A(t/x)$ *together with* the premise $\{x : A\} = \{x : A\}$, and that $\{\}^{=}\mathrm{E}_2$ allows one to infer $A(t/x)$ from $t \in \{x : A\}$ *together with* the premise $\{x : A\} = \{x : A\}$ (the extra premises expressing the requirement that $\{x : A\}$ is a denoting set-term).

Redundancies constituted by consecutive applications of the introduction rule followed immediately by one of the corresponding elimination rules can be eliminated using the obvious reductions. Moreover, consecutive applications of the two elimination rules give rise to Ekmanesque redundancies of which one can get rid using the following reduction (we call respectively $\mathtt{Ekman}^{=}$ and $\mathtt{Ekman}^{=}$-*redundant formula* this transformation and the occurrence of $t \in s$ in the schematic derivation on the left-hand side):

$$\dfrac{\{x : A\} = s \qquad \dfrac{\{x : A\} = s \qquad \begin{array}{c}\mathscr{D}\\ A(t/x)\end{array}}{t \in s}\,\{\}^{=}\mathrm{E}_1}{A(t/x)}\,\{\}^{=}\mathrm{E}_2 \quad \overset{\mathtt{Ekman}^{=}}{\rhd} \quad \begin{array}{c}\mathscr{D}\\ A(t/x)\end{array}$$

To reconstruct Russell's reasoning in this further setting Tennant suggests to choose both t and s to be some variable y and to take A to be the formula $\neg(x \in x)$. One thereby obtains the following instances of $\{\}^{=}\mathrm{E}_1$, $\{\}^{=}\mathrm{E}_2$ (as before we abbreviate the Russell term $\{x : \neg(x \in x)\}$ with $\mathfrak{r}$):

---

[7] Similar modifications of the rules of the quantifiers are required as well, see, e.g., Tennant (1978, §7.10).

[8] The following rules are in fact a simplification of Tennant's rules obtained by omitting some premises and dischargeable assumptions of the form $\exists! t$. The reason for the simplification is that they help making the presentation and the discussion more concise. Each derivation using these rules should be understood as an abbreviation of a derivation using Tennant's original rules. The interested reader can easily reconstruct full derivations by adding the (in most cases trivial) sub-derivations of each of the missing premises of each rule application.

$$\dfrac{\mathfrak{t} = y \qquad \neg(y \in y)}{y \in y}\,\{\}^{=}\mathrm{E}_1 \qquad\qquad \dfrac{\mathfrak{t} = y \qquad y \in y}{\neg(y \in y)}\,\{\}^{=}\mathrm{E}_2$$

By abbreviating $y \in y$ with υ, we can reason as in Ekman's derivation **E** and thereby construct a derivation of $\bot$ depending on the assumption $\mathfrak{t} = y$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\mathfrak{t} = y \qquad \overset{1}{\upsilon}}{\neg\upsilon}\,\{\}^{=}\mathrm{E}_2 \qquad \overset{1}{\upsilon}}{\bot}\,\supset\mathrm{E}}{\neg\upsilon}\,\supset\mathrm{I}\,(1) \qquad \dfrac{\mathfrak{t} = y \qquad \dfrac{\dfrac{\dfrac{\mathfrak{t} = y \qquad \overset{1}{\upsilon}}{\neg\upsilon}\,\{\}^{=}\mathrm{E}_2 \qquad \overset{1}{\upsilon}}{\bot}\,\supset\mathrm{E}}{\neg\upsilon}\,\supset\mathrm{I}\,(1)}{\upsilon}\,\{\}^{=}\mathrm{E}_1}{\bot}\,\supset\mathrm{E} \qquad (\mathbf{R}^{=})$$

Like in $\mathbf{R^z}$, the variable y in $\mathbf{R}^{=}$ occurs free neither in the conclusion nor in any undischarged assumption other than $\mathfrak{t} = y$. The derivation $\mathbf{R}^{=}$ can thus be extended by $\exists$E and $\supset$I to a closed derivation of $\neg\exists y(\mathfrak{t} = y)$ that establishes that $\mathfrak{t}$ has no denotation.

However, as in Ekman's **E**, the encircled occurrence of $\neg\upsilon$ is an $\supset$-redundant formula occurrence. The reader can easily check that by getting rid of it using $\supset$-`Red`, an `Ekman`$^{=}$-redundant formula occurrence is introduced. By getting rid of it using the following instance of `Ekman`$^{=}$:

$$\dfrac{\mathfrak{t} = y \qquad \dfrac{\mathfrak{t} = y \qquad \overset{\mathscr{D}}{\neg\upsilon}}{\upsilon}\,\{\}^{=}\mathrm{E}_1}{\neg\upsilon}\,\{\}^{=}\mathrm{E}_2 \qquad \overset{\mathtt{Ekman}^{=}}{\rhd} \qquad \overset{\mathscr{D}}{\neg\upsilon}$$

one gets back to $\mathbf{R}^{=}$. As in the previous cases, in spite of its innocuous character the derivation fails to normalize. This overgeneration case seems a perfect blend of the two previously discussed, and Tennant (2016) showed how the (purported) solution of von Plato to Ekman's case can be applied also to this one: as soon as one replaces the elimination rules $\{\}^{=}\mathrm{E}_1$ and $\{\}^{=}\mathrm{E}_2$ with their general versions (we call the resulting system $\mathtt{NM}_g^{\in^{=}}$):

$$\dfrac{\{x : A\} = u \qquad A(t/x) \qquad \begin{matrix}[t \in u]\\ C\end{matrix}}{C}\,\{\}^{=}\mathrm{E}_{1g}$$

$$\dfrac{\{x : A\} = u \qquad t \in u \qquad \begin{matrix}[A(t/x)]\\ C\end{matrix}}{C}\,\{\}^{=}\mathrm{E}_{2g}$$

one can give the following (apparently) redundancy-free derivation of $\bot$ from $\mathfrak{t} = y$:

$$\dfrac{\mathfrak{t} = y \qquad \dfrac{\dfrac{\mathfrak{t} = y \qquad \overset{3}{\upsilon} \qquad \dfrac{\overset{2}{\neg\upsilon} \qquad \overset{3}{\upsilon} \qquad \overset{1}{\bot}}{\bot}\,(1)}{\bot}\,(2)}{\neg\upsilon}\,(3) \qquad \dfrac{\mathfrak{t} = y \qquad \overset{6}{\upsilon} \qquad \dfrac{\overset{5}{\neg\upsilon} \qquad \overset{6}{\upsilon} \qquad \overset{4}{\bot}}{\bot}\,(4)}{\bot}\,(5)}{\bot}\,(6) \qquad (\mathbf{R}_{\mathbf{g}}^{=\prime})$$

4 General `Ekman`-reductions

Although we believe that the Prawitz–Tennant analysis undoubtedly provides the basis for a proof-theoretic clarification of the phenomenon of paradoxes, we do not find the way out of the overgeneration cases proposed by von Plato and Tennant satisfactory.

It is true that in the derivations $\mathbf{E}'_{\mathbf{g}}$ and $\mathbf{R}^{=\prime}_{\mathbf{g}}$ no subproof compactification is possible. However, as we will now show, it is still possible to detect some redundancies which are *hidden* by the more involved shape of derivations constructed with general elimination rules. By defining procedures to get rid of these hidden redundancies, Ekmanesque loops will crop up again. In the remaining part of the paper this suggestion will be made precise.

4.1 Ekman's decomposing inferences

The possibility of reformulating his "paradox" using general elimination rules was clearly envisaged by Ekman in his doctoral thesis, where he introduces the notion of 'decomposing inference':

> Let Π and A designate the premise deductions and conclusion of a rule R respectively. That is, R is the inference schema:
>
> $$\frac{\Pi}{A}\,R$$
>
> We obtain the corresponding decomposing inference schema R_D as follows:
>
> $$\frac{\Pi \qquad \begin{matrix}[A]\\ \mathcal{E}\\ C\end{matrix}}{C}\,R_D$$
>
> We obtain the premise deductions of the inference schema R_D by adding one deduction $\mathcal{E}$ to the premise deductions of the R schema, where $\mathcal{E}$ designates a deduction in which occurrences of the conclusion A of the R schema, as open assumptions in $\mathcal{E}$ may be cancelled at the R_D inference. If, in the R schema, B designates an open assumption in any of the premise deductions Π and B may be cancelled at the R inference, then in the R_D schema, B also designates an open assumption of the same premise deduction and B may be cancelled at the R_D inference. (Ekman 1994, pp. 9–10)

Obviously, in the case of $\supset$E, the decomposing inference $\supset E_D$ is just the general rule $\supset E_g$.[9]

[9] The two notions of *general rule* and *decomposing inference* do not in general coincide, since according to the schema given by Ekman the decomposing inferences associated with the conjunction elimination rules of Gentzen (1935) and Prawitz (1965) differ from the (more commonly adopted) single elimination rule considered by Schroeder-Heister (1981) following Prawitz (1979):

On p. 10 Ekman introduces the notion of a *simple deduction corresponding to one with decomposing inferences* by giving an informal, though precise description of a procedure for translating derivations with decomposing inferences into derivations with the corresponding "simple" inferences. When restricted to the systems $\mathtt{NM}$ and $\mathtt{NM}_g$, Ekman's translation amounts to the following (the definition is by induction on the number of inference rules applied in a derivation):

$1^\flat$. If $\mathscr{D}$ is an assumption, then $\mathscr{D}^\flat = \mathscr{D}$
$2^\flat$. If $\mathscr{D}$ ends with an application of $\supset\mathrm{E}_g$, i.e. it is of the following form:

$$\dfrac{\begin{matrix}\mathscr{D}_1\\ A\supset B\end{matrix}\qquad \begin{matrix}\mathscr{D}_2\\ A\end{matrix}\qquad \begin{matrix}\overset{n}{[B]}\\ \mathscr{D}_3\\ C\end{matrix}}{C}\supset\mathrm{E}_g\ (n)$$

then $\mathscr{D}^\flat$ has the following form:

$$\begin{matrix}\dfrac{\begin{matrix}\mathscr{D}_1^\flat\\ A\supset B\end{matrix}\qquad \begin{matrix}\mathscr{D}_2^\flat\\ A\end{matrix}}{[B]}\supset\mathrm{E}\\ \mathscr{D}_3^\flat\\ C\end{matrix}$$

$3^\flat$. If $\mathscr{D}$ ends with an application of $\supset$I, then $\mathscr{D}^\flat$ is obtained by applying $\supset$I to the translation $\mathscr{D}_1^\flat$ of the immediate subderivation $\mathscr{D}_1$ of $\mathscr{D}$.

At this point Ekman writes:

> Let $\mathcal{H}$ and $\mathcal{H}'$ be a deduction with decomposing inferences and its corresponding simple deduction, respectively. Then indeed, $\mathcal{H}$ and $\mathcal{H}'$ both represent the same informal argument. The difference is only a matter of the display of the inferences. Therefore it ought to be the case that $\mathcal{H}$ is normal if and only if $\mathcal{H}'$ is normal. (1994, p. 13)

The translation $(\mathbf{E}'_\mathbf{g})^\flat$ of von Plato's derivation $\mathbf{E}'_\mathbf{g}$ into $\mathtt{NM}$ is indeed $\mathbf{E}'$. It is beyond doubt that the quoted passage hints at the possibility to extend the set of conversions of $\mathtt{NM}_g$ in such a way that on the extended set of conversions von Plato's derivation $\mathbf{E}'_\mathbf{g}$ fails to normalize as well.

To this we now turn.

$$\dfrac{A\wedge B\qquad \begin{matrix}[A]\\ C\end{matrix}}{C}\wedge\mathrm{E1}_D\qquad\qquad \dfrac{A\wedge B\qquad \begin{matrix}[B]\\ C\end{matrix}}{C}\wedge\mathrm{E2}_D\qquad\qquad \dfrac{A\wedge B\qquad \begin{matrix}[A][B]\\ C\end{matrix}}{C}\wedge\mathrm{E}_g$$

It is finally worth observing that the notion of decomposing inference is not restricted to elimination rules only. When applied to introduction rules, it yields what Negri and von Plato (2001, p. 213ff.) called *general introduction rules*. More on this in Sect. 5 below.

4.2 Implication-as-rule vs implication-as-link

As a starting point, we recall Schroeder-Heister's (2011) proposal to distinguish between two ways in which the assumption of an implication can be interpreted: Implication-as-rule and implication-as-link. In natural deduction the two interpretations correspond to the two distinct forms that the rule of implication elimination may take (see also Schroeder-Heister 2014).

The adoption of $\supset$E yields the implication-as-rule interpretation. Suppose we have a derivation $\mathscr{D}$ of conclusion A. By assuming the implication $A \supset B$ we can extend $\mathscr{D}$ as if we had at our disposal a rule R allowing to pass over from A to B:

$$\dfrac{A \supset B \qquad \begin{array}{c}\mathscr{D}\\ A\end{array}}{B}\supset\mathrm{E} \qquad \dfrac{\begin{array}{c}\mathscr{D}\\ A\end{array}}{B}\ R : A \Rightarrow B$$

On the other hand, the adoption of $\supset\mathrm{E}_g$ does not amount to assume only the rule to pass over from A to B, but rather to assume also the existence of a *link* connecting two distinct derivations:

$$\dfrac{A \supset B \qquad \begin{array}{c}\mathscr{D}\\ A\end{array} \qquad \begin{array}{c}\overset{n}{[B]}\\ \mathscr{D}'\\ C\end{array}}{C}\supset\mathrm{E}_g\ (n) \qquad \begin{array}{c}\mathscr{D}\\ \dfrac{A}{B}\ R : A \Rightarrow B\\ \vdots\\ [B]\\ \mathscr{D}'\\ C\end{array}$$

Applications of the rule R correspond—even graphically—to the application of $\supset$E. This is not so in $\supset\mathrm{E}_g$, where there is nothing in the structure of the rule which can be said to correspond to the application of the rule to pass from A to B. The transition from A to B remains implicit.

The implicit link in $\supset\mathrm{E}_g$ between the two sub-derivations $\mathscr{D}$ and $\mathscr{D}'$ is a form of transitivity: if B can be derived by means of $\mathscr{D}$ from a set of assumptions Γ (among which the rule R allowing to pass over from A to B), and C can be derived by means of $\mathscr{D}'$ from some other set of assumptions Δ together with (a certain number of copies of) B, then C can be derived from Γ and Δ alone.

We wish to defend the claim that the transitivity principle encoded by $\supset\mathrm{E}_g$ *hides* a redundancy in the derivation $\mathbf{E}'_\mathbf{g}$. In fact, Ekman (1994) himself refers to decomposing rules as 'cut-hiding'.

4.3 Implication-as-link and Ekman's paradox

To state this intuition in a more explicit manner we take seriously the idea that in $\supset\mathrm{E}_g$ the minor premise A is linked with the assumptions of form B which are discharged by the application of the rule.

Certain configurations of two consecutive applications of $\supset\mathrm{E}_g$ may thus be viewed as constituting a redundancy. Consider for instance situations of the following kind:

$$\dfrac{A \supset B \qquad \begin{matrix}\mathscr{D}\\ A\end{matrix} \qquad \dfrac{B \supset A \qquad \overset{m}{B} \qquad \begin{matrix}\overset{n}{[A]}\\ \mathscr{D}'\\ C\end{matrix}}{C}\,(n)}{C}\,(m)$$

The formula A which is the conclusion of $\mathscr{D}$ is linked by $A \supset B$ to the discharged occurrence of B marked with m. This in turn is linked by $B \supset A$ to the discharged assumptions A marked by n. In other words, the two applications of the general elimination rule make one jump from A to B and back in a quite unnecessary way. This intuition, which is essentially Ekman's, can be spelled out by defining a new conversion to get rid of redundancies of this kind.

By directly linking together $\mathscr{D}$ and $\mathscr{D}'$, both applications of $\supset E_g$ could be eliminated as follows:

$$\begin{matrix}\mathscr{D}\\ [A]\\ \mathscr{D}'\\ C\end{matrix}$$

However, this is only possible if in the original derivation no other occurrence of B is discharged in $\mathscr{D}'$ by the application of $\supset E_g$ marked with m.

If such occurrences of B are present, than the lower application of $\supset E_g$ is still needed in order to discharge them. This is perfectly reasonable, since these occurrences of B do not belong to the detour generated by the links of the two applications of $\supset E_g$. We take the following reduction to be what in $\mathtt{NM}_g$ corresponds to $\mathtt{Ekman}$ (the occurrence of B in the leftmost derivation constituting the redundancy is encircled and will be called an $\mathtt{Ekman}_g$-*redundant formula occurrence*):

$$\dfrac{A \supset B \qquad \begin{matrix}\mathscr{D}\\ A\end{matrix} \qquad \dfrac{B \supset A \qquad \overset{n}{B} \qquad \begin{matrix}\overset{m}{[A]}\ \overset{n}{[B]}\\ \mathscr{D}'\\ C\end{matrix}}{C}\supset E_g\ (m)}{C}\supset E_g\ (n) \qquad \overset{\mathtt{Ekman}_g}{\rhd}$$

$$\overset{\mathtt{Ekman}_g}{\rhd} \qquad \dfrac{A \supset B \qquad \begin{matrix}\mathscr{D}\\ A\end{matrix} \qquad \begin{matrix}\mathscr{D}\\ [A]\ \ \overset{n}{[B]}\\ \mathscr{D}'\\ C\end{matrix}}{C}\supset E_g\ (n)$$

Observe now that von Plato's $\mathbf{E_g'}$ contains an $\mathtt{Ekman}_g$-redundant formula occurrence (encircled):

$$\dfrac{\neg A \supset A \qquad \dfrac{\dfrac{A \supset \neg A \qquad \overset{3}{A} \qquad \dfrac{\overset{2}{\neg A} \qquad \overset{3}{A} \qquad \overset{1}{\bot}}{\bot}\,(1)}{\bot}\,(2)}{\neg A}\,(3) \qquad \dfrac{A \supset \neg A \qquad \overset{6}{A} \qquad \dfrac{\overset{5}{\neg A} \qquad \overset{6}{A} \qquad \overset{4}{\bot}}{\bot}\,(4)}{\bot}\,(5)}{\bot}\,(6) \qquad (\mathbf{E_g'})$$

The redundancy can be eliminated using the following instance of $\mathtt{Ekman}_g$:

$$\dfrac{\neg A \supset A \qquad \overset{\mathscr{D}}{\neg A} \qquad \dfrac{A \supset \neg A \qquad \overset{n}{A} \qquad \begin{matrix}\overset{m}{[\neg A]}\;\;\overset{n}{[A]}\\ \mathscr{D}'\\ \bot\end{matrix}}{\bot}\,(m)}{\bot}\,(n) \qquad \overset{\mathtt{Ekman}_g}{\rhd} \qquad \dfrac{\neg A \supset A \qquad \overset{\mathscr{D}}{\neg A} \qquad \begin{matrix}\mathscr{D}\\ [\neg A]\;\;\overset{n}{[A]}\\ \mathscr{D}'\\ \bot\end{matrix}}{\bot}\,(n)$$

By applying this instance of $\mathtt{Ekman}_g$ to $\mathbf{E}'_{\mathbf{g}}$ one obtains the following derivation:

$$\dfrac{\neg A \supset A \qquad \dfrac{\dfrac{A \supset \neg A \qquad \overset{3}{A} \qquad \dfrac{\overset{2}{\neg A} \quad \overset{3}{A} \quad \overset{1}{\bot}}{\bot}\,(1)}{\bot}\,(2)}{\neg A}\,(3) \qquad \dfrac{\dfrac{\dfrac{A \supset \neg A \qquad \overset{6}{A} \qquad \dfrac{\overset{5}{\neg A} \quad \overset{6}{A} \quad \overset{4}{\bot}}{\bot}\,(4)}{\bot}\,(5)}{\neg A}\,(6) \qquad \overset{8}{A} \quad \overset{7}{\bot}}{\bot}\,(7)}{\bot}\,(8)$$

($\mathbf{E_g}$)

The encirled occurrence of $\neg A$ is the conclusion of an application of $\supset$I and the major premise of an application of $\supset E_g$ and thus it is an $\supset_g$-redundant formula occurrence. By applying $\supset_g$-$\mathtt{Red}$ to this derivation, one gets back to $\mathbf{E}'_{\mathbf{g}}$. That is, by enriching the set of conversions with $\mathtt{Ekman}_g$, the process of normalizing the derivations $\mathbf{E_g}$ and $\mathbf{E}'_{\mathbf{g}}$ gets stuck in a loop in the same way as that of the derivations $\mathbf{E}$ and $\mathbf{E}'$. As already observed in Sec. 4.1, $\mathbf{E}'$ is the image of $\mathbf{E}'_{\mathbf{g}}$ under the traslation $\flat$ from $\mathtt{NM}_g$ to $\mathtt{NM}$, and as the reader can easily check the same is true of $\mathbf{E}$ and $\mathbf{E_g}$.

It is easy to see that the foregoing line of reasoning can be extended in a straightforward manner to Tennant's derivation $\mathbf{R}^{=}_{\mathbf{g}}$ in $\mathtt{NM}^{\in=}_{g}$. In particular, the remarks on the cut-hiding nature of $\supset E_g$ can be applied to Tennant's $\{\}^{=}E_1$ and $\{\}^{=}E_2$ as well. In particular, we can define the following general version of the $\mathtt{Ekman}^{=}$-reduction:

$$\dfrac{\{x : A\} = s \qquad \overset{\mathscr{D}}{A(t/x)} \qquad \dfrac{\{x : A\} = s \qquad \overset{n}{t \in s} \qquad \begin{matrix}\overset{m}{[A(t/x)]}\;\overset{n}{[t \in s]}\\ \mathscr{D}'\\ C\end{matrix}}{C}\,\{\}^{=}E_{2g}\,(m)}{C}\,\{\}^{=}E_{1g}\,(n) \qquad \overset{\mathtt{Ekman}^{=}_{g}}{\rhd}$$

$$\overset{\mathtt{Ekman}^{=}_{g}}{\rhd} \qquad \dfrac{\{x : A\} = s \qquad \overset{\mathscr{D}}{A(t/x)} \qquad \begin{matrix}\mathscr{D}\\ [A(t/x)]\;\;\overset{n}{[t \in s]}\\ \mathscr{D}'\\ C\end{matrix}}{C}\,\{\}^{=}E_{1g}\,(n)$$

The derivation $\mathbf{R}^{=\prime}_{\mathbf{g}}$, like $\mathbf{E}'_{\mathbf{g}}$, contains a hidden redundancy that can be eliminated using $\mathtt{Ekman}^{=}_{g}$. As the reader can check, by applying the reduction one obtains a derivation that, like $\mathbf{E_g}$, contains an $\supset_g$-redundant formula occurrence. By eliminating it using $\supset_g$-$\mathtt{Red}$ one gets back to Tennant's $\mathbf{R}^{=\prime}_{\mathbf{g}}$.

Moreover, the translation $\flat$, mapping $\mathtt{NM}_g$-derivations onto $\mathtt{NM}$-derivations, can be easily extended to a translation $\flat^{\in}$ mapping $\mathtt{NM}_g^{\in=}$-derivations onto $\mathtt{NM}^{\in=}$-derivations. The image of Tennant's derivation $\mathbf{R}_\mathbf{g}^{='}$ and of the derivation to which $\mathbf{R}_\mathbf{g}^{='}$ reduces via $\mathtt{Ekman}_g$ are Tennant's (1978, 1982) derivations $\mathbf{R}^{='}$ and $\mathbf{R}^{=}$ respectively.

4.4 Copy-and-paste subproof compactification

As observed by Tennant, both $\mathtt{Ekman}$ and $\mathtt{Ekman}^{=}$ are instances of the general reduction pattern called by Tennant subproof compactification. Crudely put, the adoption of general elimination rules has the result of chopping up derivations and scattering around their subderivations. As a consequence, it is natural to generalize subproof compactification to a reduction pattern that could be called *copy-and-paste subproof compactification*: if a derivation $\mathscr{D}$ contains a sub-derivation $\mathscr{D}'$ of A and some assumptions of the form A are discharged in $\mathscr{D}$, the result of replacing $\mathscr{D}'$ for the discharged assumptions of A may bring to light hidden possibilities of applying subproof compactification. Although some subderivations may have to be copied in the process, the overall result will be a derivation depending on less assumptions than the original one and containing less (explicit or implicit) redundancies.

Instances of copy-and-paste sub-proof compactification are not just the conversions $\mathtt{Ekman}_g$ and $\mathtt{Ekman}_g^{=}$, but also all other known reductions, in particular $\supset$-$\mathtt{Red}$ (and $\supset_g$-$\mathtt{Red}$), that could be analysed as consisting of one (respectively two) step(s) of "copy-and-paste", where the "copy-and-paste" operation could be schematically depicted as follows:

$$\begin{array}{c} \overset{n}{[A]} \qquad \begin{array}{c}\mathscr{D}'\\ A\end{array} \\ \mathscr{D} \\ C \end{array} \quad \rhd \quad \begin{array}{c} \begin{array}{c}\mathscr{D}'\\ {[A]}\end{array} \qquad \begin{array}{c}\mathscr{D}'\\ A\end{array} \\ \mathscr{D} \\ C \end{array}$$

followed by one step of subproof compactification. In the case of $\supset$-$\mathtt{Red}$, we would have:

$$\dfrac{\dfrac{\begin{array}{c}\overset{n}{[A]}\\ \mathscr{D}\\ B\end{array}}{A\supset B}\supset\mathrm{I}\,(n) \qquad \begin{array}{c}\mathscr{D}'\\ A\end{array}}{B}\supset\mathrm{E} \quad \rhd \quad \dfrac{\dfrac{\begin{array}{c}\mathscr{D}'\\ {[A]}\\ \mathscr{D}\\ B\end{array}}{A\supset B}\supset\mathrm{I} \qquad \begin{array}{c}\mathscr{D}'\\ A\end{array}}{B}\supset\mathrm{E} \quad \rhd \quad \begin{array}{c}\mathscr{D}'\\ {[A]}\\ \mathscr{D}\\ B\end{array}$$

5 General introduction rules and $\mathtt{Ekman}_g$

It may be retorted that, compared to Ekmans's original conversion, the conversion $\mathtt{Ekman}_g$ is much less straightforward, and one may wonder whether in the end, it is not just artificial. We rebut this criticism by observing that $\mathtt{Ekman}_g$ is as much as plausible as $\mathtt{Ekman}$. Or at least, that this is the case if one (like von Plato himself) is willing to accept not only general elimination rules but general introduction rules as well.

According to Negri and von Plato (2001), not only elimination rules, but also introduction rules can be recast in general form, according to the following idea: "General introduction rules state that if a formula C follows from a formula A, then it already follows from the immediate grounds for A; general elimination rules state that if C follows from the immediate grounds for A, then it already follows from A" (*ibid.* 217).[10]

For example, the Prawitz-Gentzen introduction and elimination rules for conjunction:

$$\dfrac{A \qquad B}{A \wedge B}\wedge\mathrm{I} \qquad\qquad \dfrac{A \wedge B}{A}\wedge\mathrm{E}_1 \qquad \dfrac{A \wedge B}{B}\wedge\mathrm{E}_2$$

are recast in general form as follows:

$$\dfrac{A \qquad B \qquad \begin{matrix}[A \wedge B]\\ C\end{matrix}}{C}\wedge\mathrm{I}_g \qquad\qquad \dfrac{A \wedge B \qquad \begin{matrix}[A][B]\\ C\end{matrix}}{C}\wedge\mathrm{E}_g$$

As of today there is no systematic study of the properties of natural deduction systems with general introduction rules. In a recent paper however, Milne (2014) argued for their significance for the inferentialist project of characterizing the meaning of logical constants through the inference rules governing them. In this context he suggested reductions to eliminate consecutive applications of the general introduction and elimination rules for a connective. In the case of conjunction, Milne's proposal amounts to the following transformation:

$$\dfrac{\begin{matrix}\mathscr{D}_1\\ A\end{matrix} \qquad \begin{matrix}\mathscr{D}_2\\ B\end{matrix} \qquad \dfrac{\overset{n}{A \wedge B} \qquad \begin{matrix}\overset{m_1}{[A]}\,\overset{m_2}{[B]}\\ \mathscr{D}'\\ C\end{matrix}}{C}\wedge\mathrm{E}_g\ (m_1, m_2)}{C}\wedge\mathrm{I}_g\ (n) \qquad \rhd \qquad \begin{matrix}\mathscr{D}_1\ \mathscr{D}_2\\ [A]\,[B]\\ \mathscr{D}'\\ C\end{matrix}$$

However, one cannot exclude that the application of the general introduction rule labelled with (n) discharges some occurrences of $A \wedge B$ in $\mathscr{D}'$ as well. Such further occurrences (if any) are not part of the redundancy, and the application of $\wedge\mathrm{I}_g$ would still be needed to discharge them. The solution consists in revising Milne's proposed reduction as follows:

$$\dfrac{\begin{matrix}\mathscr{D}_1\\ A\end{matrix} \quad \begin{matrix}\mathscr{D}_2\\ B\end{matrix} \quad \dfrac{\overset{n}{A \wedge B} \qquad \begin{matrix}\overset{m_1}{[A]}\,\overset{m_2}{[B]}\,[\overset{n}{A \wedge B}]\\ \mathscr{D}'\\ C\end{matrix}}{C}\wedge\mathrm{E}_g\ (m_1, m_2)}{C}\wedge\mathrm{I}_g\ (n) \quad \overset{\wedge_G\texttt{-Red}}{\rhd} \quad \dfrac{\begin{matrix}\mathscr{D}_1\\ A\end{matrix} \quad \begin{matrix}\mathscr{D}_2\\ B\end{matrix} \quad \begin{matrix}\mathscr{D}_1\ \mathscr{D}_2\\ [A]\,[B]\,[\overset{n}{A \wedge B}]\\ \mathscr{D}'\\ C\end{matrix}}{C}\wedge\mathrm{I}_g\ (n)$$

The conversion $\wedge_G$-Red certainly has the flavor of $\texttt{Ekman}_g$. To spell out the analogy between reductions for general introduction-elimination patterns and $\texttt{Ekman}_g$ in full,

[10] In fact, general introduction rules are nothing but the decomposing inference corresponding to the usual introduction rules according to the pattern proposed by Ekman given in Sect. 4.1 above.

we consider the general version of the introduction and elimination rules for naive set theory of Prawitz:

$$\dfrac{A(t/x) \qquad \begin{matrix}[t \in \{x : A\}]\\ C\end{matrix}}{C}\ \in\mathrm{I}_g \qquad\qquad \dfrac{t \in \{x : A\} \qquad \begin{matrix}[A(t/x)]\\ C\end{matrix}}{C}\ \in\mathrm{E}_g$$

and the reduction $\in_G$-Red associated with $\in\mathrm{I}_g$ and $\in\mathrm{E}_g$ (we call the encircled formula occurrence $\in_G$*-redundant formula occurrence*):

$$\dfrac{\begin{matrix}\mathscr{D}\\ A(t/x)\end{matrix} \qquad \dfrac{\overset{n}{t \in \{x : A\}} \qquad \begin{matrix}\overset{m}{[A(t/x)]}\ \ \overset{n}{[t \in \{x : A\}]}\\ \mathscr{D}'\\ C\end{matrix}}{C}\ \in\mathrm{E}_g\ (m)}{C}\ \in\mathrm{I}_g\ (n) \qquad \overset{\in_G-\mathrm{Red}}{\rhd}$$

$$\overset{\in_G-\mathrm{Red}}{\rhd} \qquad \dfrac{\begin{matrix}\mathscr{D}\\ A(t/x)\end{matrix} \qquad \begin{matrix}\mathscr{D}\\ [A(t/x)]\,\overset{n}{[t \in \{x : A\}]}\\ \mathscr{D}'\\ C\end{matrix}}{C}\ \in\mathrm{I}_g\ (n)$$

By removing all occurrences of $\neg A \supset A$ and of $A \supset \neg A$ from von Plato's $\mathbf{E}'_g$ and replacing all occurrences of A with occurrences of ρ, all applications of $\supset\mathrm{E}_g$ with major premises $\neg A \supset A$ or $A \supset \neg A$ in $\mathbf{E}'_g$ are turned into applications of the following instances of $\in\mathrm{I}_g$ and $\in\mathrm{E}_g$ respectively:

$$\dfrac{\neg\rho \qquad \begin{matrix}[\rho]\\ C\end{matrix}}{C}\ \in\mathrm{I}_g \qquad\qquad \dfrac{\rho \qquad \begin{matrix}[\neg\rho]\\ C\end{matrix}}{C}\ \in\mathrm{E}_g$$

Thus $\mathbf{E}'_g$ becomes the following derivation of $\bot$ in the system obtained by extending $\mathtt{NM}_g$ with $\in\mathrm{I}_g$ and $\in\mathrm{E}_g$ (we call it $\mathtt{NM}_G$):[11]

$$\dfrac{\dfrac{\dfrac{\overset{3}{\rho} \qquad \dfrac{\overset{2}{\neg\rho} \qquad \overset{3}{\rho} \qquad \overset{1}{\bot}}{\bot}\ \supset\mathrm{E}_g\ (1)}{\bot}\ \in\mathrm{E}_g\ (2)}{\neg\rho}\ \supset\mathrm{I}\ (3) \qquad \dfrac{\overset{6}{\rho} \qquad \dfrac{\overset{5}{\neg\rho} \qquad \overset{6}{\rho} \qquad \overset{4}{\bot}}{\bot}\ \supset\mathrm{E}_g\ (4)}{\bot}\ \in\mathrm{E}_g\ (5)}{\bot}\ \in\mathrm{I}_g\ (6) \qquad (\mathbf{R}'_G)$$

The derivation $\mathbf{R}'_G$ in fact contains an $\in_G$-redundant formula occurrence (encircled). To eliminate this redundancy we can apply the following instance of $\in_G$-Red:

[11] To obtain a derivation in a system in which all rules are in general form, one should have to add an extra discharged premise in correspondence to the application of $\supset$I so to turn it into an application of $\supset\mathrm{I}_g$:

$$\dfrac{\begin{matrix}\overset{1}{[A]}\\ \mathscr{D}\\ \bot\end{matrix}}{\neg A}\ \supset\mathrm{I}\ (1) \qquad \dashrightarrow \qquad \dfrac{\begin{matrix}\overset{1}{[A]}\\ \mathscr{D}\\ \bot\end{matrix} \qquad \overset{2}{\neg A}}{\neg A}\ \supset\mathrm{I}_g\ (1, 2)$$

$$
\dfrac{\begin{array}{c}\mathscr{D}\\ \neg\rho\end{array} \qquad \dfrac{\begin{array}{c}\overset{n}{\rho}\end{array} \qquad \begin{array}{c}\overset{m}{[\neg\rho]}\overset{n}{[\rho]}\\ \mathscr{D}'\\ C\end{array}}{C}\ {\in}\mathrm{E}_g\ (m)}{C}\ {\in}\mathrm{I}_g\ (n)
\quad \rhd \quad
\dfrac{\begin{array}{c}\mathscr{D}\\ \neg\rho\end{array} \qquad \begin{array}{c}\mathscr{D}\\ {[\neg\rho]}\overset{n}{[\rho]}\\ \mathscr{D}'\\ C\end{array}}{C}\ {\in}\mathrm{I}_g\ (n)
$$

As the reader can check, one thereby introduces a new $\supset_g$-redundant formula occurrence. By getting rid of this redundancy using $\supset_g$-`Red` one gets back the derivation $\mathbf{R}'_G$ from which one started.

The relation between the relevant instances of $\in_G$-`Red` and of $\mathtt{Ekman}_g$ is exactly the same as that between the relevant instances of $\in$-`Red` and of `Ekman`. Thus, as Ekman's reduction can be seen as encoding Russell's paradox in `NM`, the general Ekman reduction we propose can be seen as encoding the version of Russell's paradox with general rules in $\mathtt{NM}_g$.

6 Conclusions and outlook

The addition of the conversion `Ekman` to the standard set of conversions for `NM` results in counterexamples to normalization. These can be viewed as simulations in the propositional setting of the counterexamples to normalization in the extension of `NM` with Prawitz's rules for naive set theory. The "safe" version of Russell's paradox proposed by Tennant (1978, 1982) faces the same problem as soon as one considers—besides reductions to get rid of introduction-elimination redundancies—the further reduction $\mathtt{Ekman}^=$.

Replacing standard elimination rules with their general versions does not help. As we have shown, it is possible to define general versions of the Ekmanesque reductions, that can be seen as simulating the reduction for general introduction and elimination rules for naive set theory. Using these reductions, Ekman's paradox and Tennant's safe version of Russell's paradox fail to normalize even when formulated using general elimination rules.

As already mentioned, we take these phenomena to call for a thorough investigation of criteria of acceptability for reduction procedures. In another place (see Schroeder-Heister and Tranchini 2017) we proposed as a natural criterion that reductions should preserve the identity of the proof (i.e. the process of reasoning) represented by the derivations to which they apply. The identity criteria for proofs to be adopted are those considered in categorial and computational approaches to proof-theory, as they apply to natural deduction using the Curry-Howard correspondence.

On such an understanding of reductions, neither `Ekman` nor its variants are acceptable, but only reductions to get rid of introduction-elimination patterns. Thus, Ekman's derivations do not qualify as paradoxical, nor does Tennant's safe version of Russell's paradox, independently of whether standard or general rules are adopted.

As remarked, the phenomenon observed by Crabbé is however unaffected by our proposed constraint on reductions, thus showing that further work is required for a thorough analysis of paradoxes along the lines of the Prawitz–Tennant analysis.

Further investigation is also needed to clarify the exact relatioship between the Prawitz–Tennant analysis of paradoxes based on normalization failure and the solution to paradoxes consisting in restricting the use of the cut rule in sequent calculus, a solution which goes back at least to Hallnäs (1991) and that has been recently brought up again by several authors, notably Ripley (2013).

Given the close correspondence between normalization in natural deduction and cut elimination in sequent calculus, the solution to paradoxes arising from the restriction to normalizable derivations can certainly be seen as anticipating current non-transitive sequent-calculus-based solutions. The adoption of general elimination rules called for by Tennant brings the two approaches even closer, given that general elimination rules more directly correspond to sequent calculus left rules than standard elimination rules.

The results presented in this paper, however, suggest that the relationship between two two approaches is not as obvious as one may assume. Von Plato's and Tennant's derivations correspond to cut-free derivations, and thereby it is *prima facie* unclear to which sort of transformation on sequent calculus derivations, the reductions we proposed correspond.

Moreover, whereas in natural deduction we have two kinds of derivations (normalizable and non-normalizable), in sequent calculus, by ruling out the cut rule from the outset (as Ripley, but also Tennant in his most recent work, recommend to do) no such distinction is available, and hence the original Prawitz–Tennant criterion for paradoxicality based on looping reduction sequences cannot immediately be reformulated in a cut-free setting.

Arguably, by allowing cut as a primitive rule, a distinction analogous to the one available in natural deduction can be formulated in sequent calculus as well (that is, between derivations for which the cut-elimination procedure does or does not enter a loop) and the reductions for general elimination rules can find a counterpart in the sequent calculus setting as well. A thorough investigation of these issues must be left for another occasion.

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