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Chair of Statistics, Econometrics and Empirical Economics

PD Dr. Thomas Dimpfl


**S414**
**Advanced Mathematical Methods**
Exercises

WS 2018/19

# Difference Equations

## Exercise 1 **Difference Equations**

Find the solution for the following difference equations with the given values of $x_0$:

a) $x_{t+1} = 2x_t + 4, \quad x_0 = 1$

b) $3x_{t+1} = x_t + 2, \quad x_0 = 2$

c) $2x_{t+1} + 3x_t + 2 = 0, \quad x_0 = -1$

d) $x_{t+1} - x_t + 3 = 0. \quad x_0 = 3$

## Exercise 2 **Difference Equations**

Consider the difference equation $x_{t+1} = ax_t + b$ and explain how its solution behaves in each of the following cases, with $x^* = \frac{b}{1-a}$ (for $a \neq 1$):

a) $0 < a < 1, \quad x_0 < x^*$

b) $-1 < a < 0, \quad x_0 < x^*$

c) $a > 1, \quad x_0 > x^*$

d) $a < -1, \quad x_0 > x^*$

e) $a \neq 1, \quad x_0 = x^*$

f) $a = -1, \quad x_0 \neq x^*$

g) $a = 1, \quad b > 0$

h) $a = 1, \quad b < 0$

i) $a = 1, \quad b = 0$

## Exercise 3 **Difference Equations**

Consider the difference equation $x_t = \sqrt{x_{t-1} - 1}$ with $x_0 = 5$. Compute $x_1, x_2$ and $x_3$. What about $x_4$? (This problem illustrates that a solution may not exist if the domain of the function $f$ in (1) is restricted in any way.)

## Exercise 4 **Difference Equations**

Suppose that at time $t = 0$, you borrow \$100.000 at a fixed interest rate of 7% per year. You are supposed to repay the loan in 30 equal annual repayments so that after $n = 30$ years, the mortgage is paid off. How much is each repayment?

**Solution Exercise 1:**

a) $x_t = 5 \cdot 2^t - 4$

b) $x_t = \frac{1}{3}^t + 1$

c) $x_t = -\frac{3}{5} \cdot -\frac{3}{2}^t - \frac{2}{5}$

d) $x_t = -3t + 3$

**Solution Exercise 2:**

a) Monotone convergence to $x^*$ from below.

b) Damped oscilliations around $x^*$.

c) Monotonically increasing towards $\infty$

d) Explosive oscilliations around $x^*$

e) $x_t = x^*$ for all $t$

f) Oscilliations around $x^*$ with constant amplitude.

g) Monotonically (linearly) increasing towards $\infty$

h) Monotonically (linearly) decreasing towards $-\infty$

i) $x_t = x_0$ for all $t$

**Solution Exercise 3:**
$x_1 = 2$,
$x_2 = 1$,
$x_3 == 0$
$x_4 = \sqrt{-1}$

## Exercise 5 **Difference Equations**

Prove that $x_t = A + Bt$ is the general solution of $x_{t+2} - 2x_{t+1} + x_t = 0$.

**Solution Exercise 4:**
The yearly repayment is $a = \frac{0.07 \cdot 100000}{1-(1.07)^{-30}} \approx 8058.64$. In the first year the interest payment is $0.07B = 7000$, and so the principal repayment is $\approx 8058.64 - 7000 = 1058.64$. In the last year, the interest payment is $0.07b_{29} \approx 8058.64\left[1-(1.07)^{-1}\right] \approx 527.20$ and so the principal repayment is $\approx 8058.64 - 527.20 = 7531.44$.