EBERHARD KARLS UNIVERSITÄT TÜBINGEN

WIRTSCHAFTS- UND SOZIALWISSENSCHAFTLICHE FAKULTÄT

Chair of Statistics, Econometrics and Empirical Economics

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S414
Advanced Mathematical Methods

Exercises

WS 2018/19

Probability and Distribution Theory

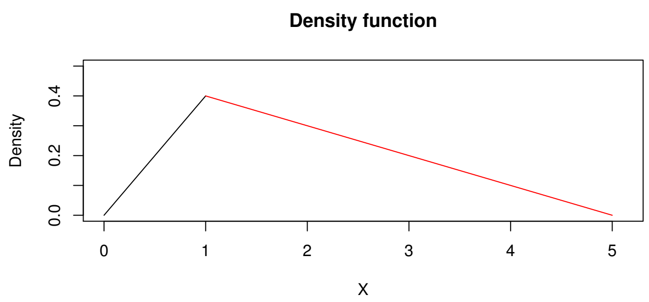
Exercise 1 **Probability and Distribution Theory**

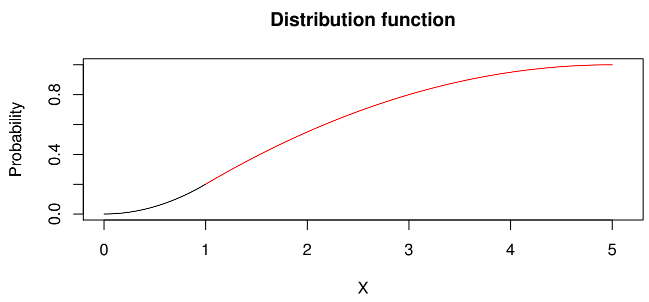
Given a continuous random variable X with:

$$f(x) = \begin{cases} 4ax & 0 \leq x < 1 \\ -ax + 0.5 & 1 \leq x \leq 5 \\ 0 & \text{else} \end{cases}$$

Determine the parameter a such that $f(x)$ is a density function of X. Calculate the corresponding distribution function and sketch it. Compute the expectation and the variance of X.

Solution:





The density function is given by:

$$f(x) = \begin{cases} 4ax & \text{for } x \in [0;1[\\ -ax + 0.5 & \text{for } x \in [1;5] \\ 0 & \text{else} \end{cases}$$
$$= f_a(x)\mathbb{1}(x \in [0;1[) + f_b(x)\mathbb{1}(x \in [1;5])$$

where $\mathbb{1}$ is the indicator function. The distribution function can be found by integrating:

$$\begin{aligned} F(x) &= \int_{-\infty}^{x} f(u)du \\ &= F_a(x)\mathbb{1}(x \in [0;1[) + (F_b(x) - F_b(1) + F_a(1))\,\mathbb{1}(x \in [1;5]) + \mathbb{1}(x > 5) \end{aligned}$$

where

$$\begin{aligned} F_a(x) &= 2ax^2 \\ F_b(x) &= -0.5x^2 + 0.5x \end{aligned}$$

and thus

$$F(x) = \begin{cases} 0 & \text{for } x < 0 \\ 2ax^2 & \text{for } x \in [0;1[\\ -\frac{1}{2}ax^2 + 0.5x \underbrace{+\frac{1}{2}a - 0.5}_{-F_b(1)} + \underbrace{2a}_{F_a(1)} & \text{for } x \in [1;5] \\ 1 & \text{for } x > 5 \end{cases}$$

In order for $f(x)$ to be a proper density function it has to be always positive, hence $a \geq 0$ such that $f_a(x) > 0$ and simultaneously $a \leq 0.1$ such that $f_b(x) > 0$. Additionally, the integral of $f(x)$ over all values of x has to be 1 in order for $f(x)$ to be a proper density function. Thus,

$$\begin{aligned} F(5) &\stackrel{!}{=} 1 \\ -10a + 2 &= 1 \\ \Rightarrow \quad a &= \frac{1}{10} \end{aligned}$$

Thus, the expectation can be calculated by:

$$\begin{aligned} \mathbb{E}[x] &= \int_{-\infty}^{\infty} xf(x)dx \\ &= \int_0^1 4ax^2dx + \int_1^5 (-ax^2 + 0.5x)dx \\ &= \left[\frac{4}{3}\cdot\frac{1}{10}x^3\right]_0^1 + \left[-\frac{1}{3}\cdot\frac{1}{10}x^3 + \frac{1}{4}x^2\right]_1^5 \\ &= \left[\frac{4}{3}\cdot\frac{1}{10}\right] + \left[-\frac{1}{3}\cdot\frac{1}{10}\cdot 125 + \frac{1}{4}\cdot 25\right] - \left[-\frac{1}{3}\cdot\frac{1}{10} + \frac{1}{4}\right] \\ &= \frac{4}{30} + \left[-\frac{125}{30} + \frac{25}{4}\right] - \left[-\frac{1}{30} + \frac{1}{4}\right] \\ &= 2 \end{aligned}$$

For the variance, we need the second moment:

$$\begin{aligned}
\mathbb{E}[x^2] &= \int_{-\infty}^{\infty} x^2 f(x) dx \\
&= \int_0^1 4ax^3 dx + \int_1^5 (-ax^3 + 0.5x^2) dx \\
&= \left[\frac{1}{10}x^4\right]_0^1 + \left[-\frac{1}{4}\cdot\frac{1}{10}x^4 + \frac{1}{6}x^3\right]_1^5 \\
&= \frac{1}{10} - \frac{625}{40} + \frac{125}{6} + \frac{1}{40} - \frac{1}{6} \\
&= \frac{12}{120} - \frac{1875}{120} + \frac{2500}{120} + \frac{3}{120} - \frac{20}{120} \\
&= \frac{620}{120} = 5.1666\bar{6}
\end{aligned}$$

Thus, the variance is given by:

$$\text{var}(x) = \mathbb{E}[x^2] - \mathbb{E}[x]^2 = \frac{31}{6} - 4 = \frac{7}{6} = 1.1666\bar{6}$$

EXERCISE 2 Probability and Distribution Theory

The Federal Statistical Office assumes all values in the interval $2 \leq x \leq 3$ to be possible realizations of the random variable X : "Growth rate of the GDP". Moreover, the following function is assumed:

$$f(x) = \begin{cases} c \cdot (x-2) & 2 \leq x \leq 3 \\ 0 & \text{else} \end{cases}$$

a) Determine c such that the function $f(x)$ is a density function of the random variable X.

b) Compute the distribution function of the random variable X.

c) Compute $P(X < 2.1)$ and $P(2.1 < X < 2.8)$.

d) Compute $P(-4 \leq X \leq 3 | X \leq 2.1)$ and show that the events $\{-4 \leq X \leq 3\}$ and $\{X \leq 2.1\}$ are statistically independent.

e) Compute the expectation, median and the variance of X.

Solution:

a) In order for $f(x)$ to be a density function it must hold:

- $f(x) > 0$
- $\int_{\mathbb{R}} f(x)dx = 1$

$\Rightarrow$

$$\int_{\mathbb{R}} f(x)dx = \int_2^3 f(x)dx = \left[\frac{1}{2}cx^2 - 2cx\right]_2^3 \overset{!}{=} 1$$
$$\frac{1}{2}c9 - 2c3 - \frac{1}{2}c4 + 2c2 = 1$$
$$\frac{1}{2}c = 1$$
$$c = 2$$

b) The distribution function can be found by integrating $f(x) = 2x - 4$, which yields $F(x) = \int_{-\infty}^{x} f(u)du = x^2 - 4x + h$ – where h is the undefined integration constant. To make $F(x)$ a proper distribution function, we require $F(2) = 0$ and $F(3) = 1$ as the whole probability mass lies in the interval $x \in [2; 3]$. This gives us $h = 4$. Hence, the distribution function is given by:

$$F(x) = \int_{-\infty}^{x} f(u)du$$
$$= \begin{cases} 0 & \text{for } x < 2 \\ x^2 - 4x + h & \text{for } 2 \leq x \leq 3 \\ 1 & \text{for } x > 3 \end{cases}$$

c) $P(X < 2.1) = F(2.1) = 2.1^2 - 4 * 2.1 + 4 = \underline{\underline{0.01}}$
$P(2.1 < X < 2.8) = F(2.8) - F(2.1) = 2.8^2 - 4 * 2.8 + 4 - 0.01 = \underline{\underline{0.63}}$

d)

$$\begin{aligned} P\left(-4 \leq X \leq 3 | X \leq\right) &= \frac{P\left(\{-4 \leq X \leq 3\} \cap \{x \leq 2.1\}\right)}{P\left(X \leq 2\right)} \\ &= \frac{P\left(\{-4 \leq X \leq 2.1\}\right)}{P\left(X \leq 2\right)} \\ &= \frac{F(2.1) - F(-4)}{F(2.1)} \\ &= 1 - \frac{F(-4)}{F(2.1)} = \underline{\underline{1}} \end{aligned}$$

The events $A = \{-4 \leq X \leq 3\}$ and $B = \{x \leq 2.1\}$ are independent if

$$P(A \cap B) = P(A) \cdot P(B|A) = P(A) \cdot P(B)$$

Or simpler if $P(B|A) = P(B)$.
We have: $P(A \cap B) = P(A) \cdot P(B|A) = P(B) \cdot P(A|B)$
with $P(A) = P(-4 \leq X \leq 3) = 1$,
and $P(A|B) = P(-4 \leq X \leq 3 | x \leq 2.1) = 1$.
Hence: $P(A \cap B) = P(B|A) = P(B)$ q.e.d.

e)
- Median $\bar{x}[0.5]$:

  $F(\bar{x}[0.5]) = 0.5 \quad \Rightarrow \quad x^2 - 4x + 4 = 0.5$
  $\Rightarrow \quad x_1 = 2 + \frac{1}{\sqrt{2}} \vee x_2 = 2 - \frac{1}{\sqrt{2}}$
  Since only $x_1 \in [2; 3]$:
  $\underline{\underline{\bar{x}[0.5] = 2 + \frac{1}{\sqrt{2}} = 2.7071.}}$

- Expectation:
  $\mathbb{E}[x] = \int_{\mathbb{R}} x f(x) dx = \int_{\mathbb{R}} x(2x - 4) dx = [\frac{2}{3}x^3 - 2x^2]_2^3 = \underline{\underline{2.66\bar{6}}}$

- Variance:
  $\mathbb{E}[x^2] = \int_{\mathbb{R}} x^2 f(x) dx = \int_{\mathbb{R}} x^2(2x - 4) dx = [\frac{2}{4}x^4 - \frac{4}{3}x^3]_2^3 = \frac{43}{6} = 7.16\bar{6}$
  $\mathrm{var}[x] = \mathbb{E}[x^2] - \mathbb{E}[x]^2 = \frac{43}{6} - \left(\frac{8}{3}\right)^2 = \underline{\underline{\frac{1}{18} = 0.05\bar{5}}}$

Exercise 3 **Probability and Distribution Theory**

Show the Markov - inequality:

$$P(X \geq c) \leq \frac{E[X]}{c}$$

for every positive value of c with X being strictly non-negative.

Solution:

Recall the definition of $P(X \geq c)$:

$$P(X \geq c) = 1 - \int_{-\infty}^{c} f(x)dx = \int_{c}^{\infty} f(x)$$

One can also split the expectation integral into two parts:

$$\mathbb{E}[x] = \int_{-\infty}^{c} xf(x)dx + \int_{c}^{\infty} xf(x)dx$$

Hence, one can state that:

$$\mathbb{E}[x] > \int_{c}^{\infty} xf(x)dx$$

As c is the lower bound of the integral and thus all $x \in]c; \infty] > c$, one can write:

$$\begin{aligned} \mathbb{E}[x] &> c\int_{c}^{\infty} f(x)dx \\ &> cP(X \geq c) \\ \frac{\mathbb{E}[x]}{c} &= P(X \geq c) \text{ q.e.d.} \end{aligned}$$

Exercise 4 **Probability and Distribution Theory**

		X		
		1	2	3
Y	1	0.25	0.15	0.10
	2	0.10	0.15	0.25

a) Compute the expectation and the variance of X and Y.

b) Determine the conditional distributions of $X|Y = y$ and $Y|X = x$.

c) Determine the covariance and the correlation coefficient of X and Y.

d) Determine the variance of $X + Y$.

Solution:

a) $\mathbb{E}[x] = 0.35 \cdot 1 + 0.3 \cdot 2 + 0.35 \cdot 3 = 2$
$\mathbb{E}[x] = 0.5 \cdot 1 + 0.5 \cdot 2 = 1.5$

b) NB: $f(x|y) = \frac{f(x,y)}{f(y)}$

		$x = 1$	$x = 2$	$x = 3$
Conditional	$y = 1$	0.5	0.3	0.2
distribution $f(x\|y)$	$y = 2$	0.2	0.3	0.5

		$x = 1$	$x = 2$	$x = 3$
Conditional	$y = 1$	5/7	1/2	2/7
distribution $f(y\|x)$	$y = 2$	2/7	1/2	5/7

c) $\operatorname{cov}[x, y] = \mathbb{E}[x \cdot y] - \mathbb{E}[x]\,\mathbb{E}[y]$

$\mathbb{E}[x \cdot y] = 0.25 + 0.15 \cdot 2 + 0.1 \cdot 3 + 0.1 \cdot 2 + 0.15 \cdot 4 + 0.25 \cdot 6 = 3.15$

$\Rightarrow \quad \operatorname{cov}[x, y] = 3.15 - 2.15 = \underline{\underline{0.15}}$

d) $\operatorname{var}[x + y] = \operatorname{var}[x] + \operatorname{var}[y] + 2\operatorname{cov}[x, y]$

$\mathbb{E}[x^2] = 0.35 + 0.3 \cdot 2^2 + 0.35 \cdot 3^2 = 4.7$

$\mathbb{E}[y^2] = 0.5 + 0.5 \cdot 2^2 = 2.5$

$\operatorname{var}[x] = 4.7 - 2^2 = 0.7$

$\operatorname{var}[y] = 2.5 - 2.25 = 0.25$

$\Rightarrow \quad \operatorname{var}[x + y] = 0.7 + 0.25 + 2 \cdot 0.15 = 1.25$

Exercise 5 Probability and Distribution Theory

The joint probability function of X and Y is given by:

$$f(x,y) = \begin{cases} e^{-2\lambda} \cdot \frac{\lambda^{x+y}}{x!y!} & x,y \in \{0,1,\ldots\} \\ 0 & \text{else} \end{cases}$$

a) Determine the marginal distributions of X and Y.

b) Determine the conditional distributions of $X|Y=y$ and $Y|X=x$ and compare them to the marginal distributions.

c) Determine the covariance of X and Y.

Solution:

a) $$f(x) = \sum_y f(x,y) = e^{-2\lambda}\frac{\lambda^x}{x!}\underbrace{\sum_y \frac{\lambda^y}{y!}}_{e^\lambda} = \underline{\underline{e^{-\lambda}\frac{\lambda^x}{x!}}}$$

$$f(y) = \sum_x f(x,y) = e^{-2\lambda}\frac{\lambda^y}{y!}\underbrace{\sum_x \frac{\lambda^x}{x!}}_{e^\lambda} = \underline{\underline{e^{-\lambda}\frac{\lambda^y}{y!}}}$$

b) $$f(x|y) = \frac{f(x,y)}{f(y)} = \frac{\frac{\lambda^{x+y}}{x!y!}e^{-2\lambda}}{e^{-\lambda}\frac{\lambda^y}{y!}} = e^{-\lambda}\frac{\lambda^x}{x!} = f(x)$$

$$f(y|x) = f(y)$$

$$\Rightarrow X \perp\!\!\!\perp Y$$

c) Since $X \perp\!\!\!\perp Y$ the covariance has to be 0. This can also be shown by:

$$\mathbb{E}[x] = \sum_{x=1}^{\infty} x e^{-\lambda}\frac{\lambda^x}{x!} = \lambda e^{-\lambda}\underbrace{\sum_{x=1}^{\infty}\frac{\lambda^{x-1}}{(x-1)!}}_{e^\lambda} = \lambda$$

$$\mathbb{E}[y] = \lambda$$

$$\mathbb{E}[x\cdot y] = \sum_{x=1}^{\infty}\sum_{y=1}^{\infty} x\cdot y\frac{\lambda^{x+y}}{x!y!}e^{-2\lambda} = \lambda^2 e^{-2\lambda}\underbrace{\sum_{x=1}^{\infty}\frac{\lambda^{x-1}}{(x-1)!}}_{e^\lambda}\underbrace{\sum_{y=1}^{\infty}\frac{\lambda^{y-1}}{(y-1)!}}_{e^\lambda}$$

$$\Rightarrow \operatorname{cov}[x,y] = \mathbb{E}[x\cdot y] - \mathbb{E}[x]\,\mathbb{E}[y] = \lambda^2 - \lambda^2 = 0$$

EXERCISE 6 **Probability and Distribution Theory**

Suppose that x_u is the u percentile of the random variable X, that is, $F(x_u) = u$. Show that if $f(-x) = f(x)$, then $x_{1-u} = -x_u$

Solution:
$F(x_u) = u$ and $F(x_{1-u}) = 1 - u$.
If $f(x) = f(-x)$ then $\displaystyle\int_{-\infty}^{-x_u} f(z)dz = \int_{x_u}^{\infty} f(z)dz$.
From which follows that:

$$F(-x_u) = 1 - F(x_u) = 1 - u$$

Hence, $-x_u = x_{1-u}$ q.e.d.

Exercise 7 Probability and Distribution Theory

If $X \sim N(1000, 400)$ find:

a) $P(X < 1024)$

b) $P(X < 1024|X > 961)$

c) $P(31 < \sqrt{X} < 32)$

Solution:
NB:

$$\frac{z-\mu}{\sigma} \sim N(0,1)$$

a) $z = \dfrac{1024-1000}{20} = 1.2 \quad \Rightarrow P(X < 1024) = F_N(1.2) = 0.8849$

b) $P(X < 1024|X > 961) = \dfrac{P(961 < X < 1024)}{P(X > 961)}$

$$\begin{aligned} P(X > 961) &= 1 - F_N\left(\frac{961-1000}{20}\right) \\ &= 1 - F_N(-1.95) = 0.9744 \\ P(961 < X < 1024) &= F_N\left(\frac{1024-1000}{20}\right) - F_N\left(\frac{961-1000}{20}\right) \\ &= F_N(1.2) - F_N(-1.95) \\ &= 0.8849 - 0.0256 = 0.8593 \end{aligned}$$

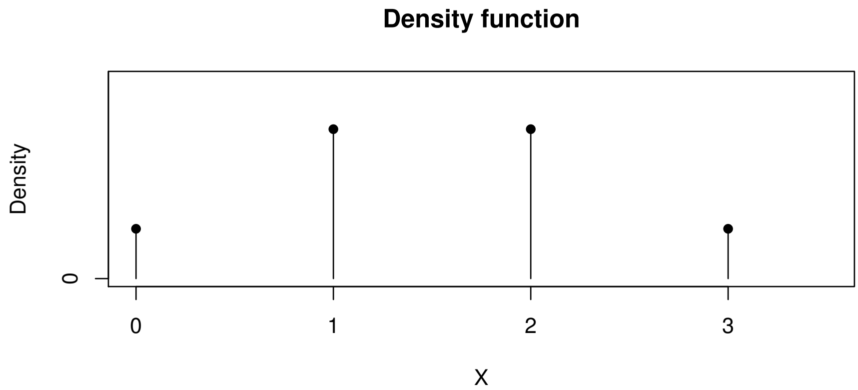
$$\begin{aligned} P(X < 1024|X > 961) &= \frac{P(961 < X < 1024)}{P(X > 961)} \\ &= \frac{0.8593}{0.9744} = 0.8819 \end{aligned}$$

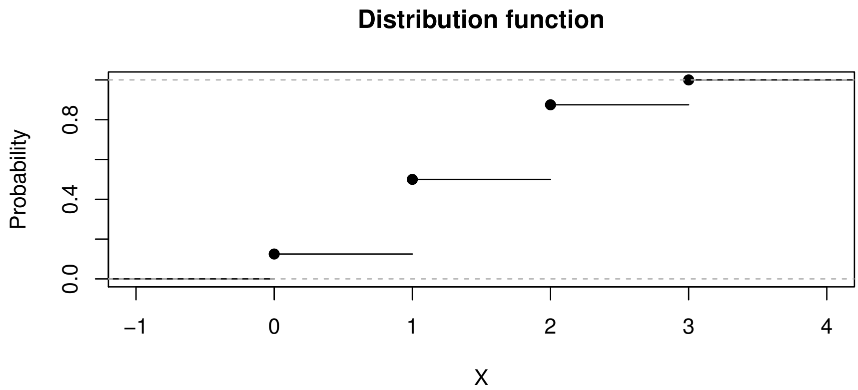
c) $P(31 < \sqrt{x} < 32) = P(31^2 < x < 32^2) = P(961 < X < 1024) = 0.8593$

EXERCISE 8 **Probability and Distribution Theory**

A fair coin is tossed three times and the random variable X equals the total number of heads. Find and sketch $F_X(x)$ and $f_X(x)$.

Solution:





$$f_X(x) = 0.5^3 \binom{3}{x} = 0.5^3 \frac{3!}{x!(3-x)!}$$

$$F_X(x) = 0.5^3 \sum_{k=1}^{x} \binom{3}{k} = 0.5^3 \sum_{k=1}^{x} \frac{3!}{k!(3-k)!}$$

Exercise 9 **Probability and Distribution Theory**

The random variable X is $N(5,2)$ and $Y = 2X + 4$. Find μ_Y, σ_Y and $f_Y(y)$.

Solution:

$\mu_y = \mathbb{E}[y] = 2\,\mathbb{E}[X] + 4 = 14$

$\sigma_y = \sqrt{\text{var}[y]} = \sqrt{4\,\text{var}[x]} = 2\sqrt{2}$

$f(y) = \frac{1}{4\sqrt{\pi}} e^{-\frac{(y-14)^2}{16}}$

EXERCISE 10 **Probability and Distribution Theory**

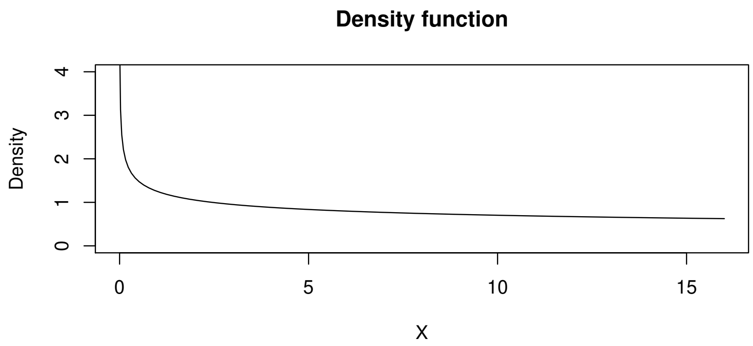
Find $F_Y(y)$ and $f_Y(y)$ if $Y = -4X + 3$ and $f_X(x) = 2e^{-2x}$ with $x \geq 0$.

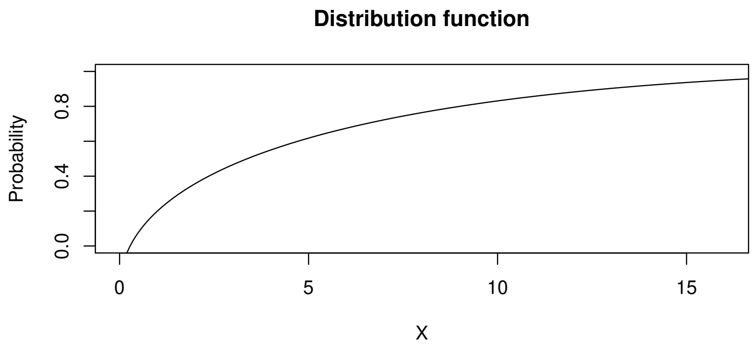
Solution:

EXERCISE 11 **Probability and Distribution Theory**

The random variable X is uniform in the interval $[-2c, 2c]$. Find and sketch $f_Y(y)$ and $F_Y(y)$ if $Y = g(X)$ and $g(X) = x^2$.

Solution:





As X is uniformly distributed in $[-2c, 2c]$, the density function is:

$$f_X(x) = \begin{cases} \frac{1}{4c} & \text{for } x \in [-2c, 2c] \\ 0 & \text{else} \end{cases}$$

EXERCISE 12 **Probability and Distribution Theory**

If X is $N(0,4)$ and $Y = 3x^2$, find μ_Y and σ_Y.

Solution:

$$\mu_Y = \mathbb{E}[Y] = \mathbb{E}[3x^2] = 3\int_{-\infty}^{\infty} x^2 f(x)dx = 3(\mathrm{var}(x) - \mathbb{E}(x)^2) = 12$$

$$\begin{aligned}
\mathbb{E}[Y^2] &= 9\,\mathbb{E}[X^4] \\
&= 9\int_{\mathbb{R}} x^4 \frac{1}{2\sqrt{2\pi}} e^{-\frac{x^2}{8}}\,dx \\
&= 9\frac{1}{2\sqrt{2\pi}}\int_{\mathbb{R}} x^4 e^{-\frac{x^2}{8}}\,dx \\
&= 9\frac{1}{2\sqrt{2\pi}}\int_{\mathbb{R}} \underbrace{\frac{-x^3}{2}}_{f(x)} \underbrace{-2xe^{-\frac{x^2}{8}}}_{g'(x)}\,dx
\end{aligned}$$

Integration by parts:

$$\int_{\mathbb{R}} f(x)g'(x)dx = [f(x)g(x)]_{-\infty}^{\infty} - \int_{\mathbb{R}} f'(x)g(x)dx$$

$f'(x) = \frac{-3}{2}x^2$
$g(x) = 8e^{-\frac{x^2}{8}}$

$$\begin{aligned}
\mathbb{E}[Y^2] &= 9\frac{1}{2\sqrt{2\pi}}\left(\underbrace{\left[\frac{-x^3}{2}8e^{-\frac{x^2}{8}}\right]_{-\infty}^{\infty}}_{=0} - \int_{\mathbb{R}} \frac{-3}{2}x^2 8e^{-\frac{x^2}{8}}\,dx\right) \\
&= 108\underbrace{\int_{\mathbb{R}} x^2 \frac{1}{2\sqrt{2\pi}} e^{-\frac{x^2}{8}}\,dx}_{\mathbb{E}[x^2]=4} \\
&= 432
\end{aligned}$$

Hence, the variance of Y is:

$$\mathrm{var}[Y] = \mathbb{E}[Y^2] - \mathbb{E}[Y]^2 = 432 - 144 = 288$$

EXERCISE 13 **Probability and Distribution Theory**

The random variables X and Y are $N(\mu_x, \sigma_x^2, \mu_y, \sigma_y^2, \rho_{xy}) = N(3, 4, 1, 4, 0.5)$. Find $f(y|x)$ and $f(x|y)$.

Solution:
The joint distribution is given by:

$$\mu = \big(\mu_x, \mu_y\big)' = \big(3, 1\big)' \qquad \mathbf{\Sigma} = \begin{pmatrix} \sigma_x^2 & \sigma_x\sigma_y\rho_{xy} \\ \sigma_x\sigma_y\rho_{xy} & \sigma_y^2 \end{pmatrix} = \begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix}$$

$$\begin{aligned}
f(y, x) &= (2\pi)^{-1} \mid \mathbf{\Sigma} \mid^{-\frac{1}{2}} \exp\left(-\frac{1}{2}(\mathbf{x}-\mu)'\mathbf{\Sigma^{-1}}(\mathbf{x}-\mu)\right) \\
&= \frac{1}{2\pi}\frac{1}{\sigma_x\sigma_y\sqrt{1-\rho_{xy}^2}} \exp\left(-\frac{1}{2\sigma_x^2\sigma_y^2(1-\rho_{xy}^2)}\begin{pmatrix} x-\mu_x & y-\mu_y \end{pmatrix}\begin{pmatrix} \sigma_y^2 & -\sigma_x\sigma_y\rho_{xy} \\ -\sigma_x\sigma_y\rho_{xy} & \sigma_x^2 \end{pmatrix}\begin{pmatrix} x-\mu_x \\ y-\mu_y \end{pmatrix}\right) \\
&= \frac{1}{2\pi}\frac{1}{\sigma_x\sigma_y\sqrt{1-\rho_{xy}^2}} \exp\left(-\frac{\big((x-\mu_x)^2\sigma_y^2 - 2(y-\mu_y)(x-\mu_x)\sigma_x\sigma_y\rho_{xy} + (y-\mu_y)^2\sigma_x^2\big)}{2\sigma_x^2\sigma_y^2(1-\rho_{xy}^2)}\right) \\
&= \frac{1}{2\pi}\frac{1}{\sigma_x\sigma_y\sqrt{1-\rho_{xy}^2}} \exp\left(-\frac{1}{2(1-\rho_{xy}^2)}\left(\frac{(x-\mu_x)^2}{\sigma_x^2} - \frac{2(y-\mu_y)(x-\mu_x)\rho_{xy}}{\sigma_x\sigma_y} + \frac{(y-\mu_y)^2}{\sigma_y^2}\right)\right)
\end{aligned}$$

The marginal distribution is given by:

$$f(y) = \frac{1}{\sqrt{2\pi}\sigma_y}\exp\left(-\frac{(y-\mu_y)^2}{2\sigma_y^2}\right)$$

With these information we can deduce the conditional distribution:

$$\begin{aligned}
f(x \mid y) &= \frac{f(x, y)}{f(y)} \\
&= \frac{\frac{1}{2\pi}\frac{1}{\sigma_x\sigma_y\sqrt{1-\rho_{xy}^2}} \exp\left(-\frac{\big((x-\mu_x)^2\sigma_y^2 - 2(y-\mu_y)(x-\mu_x)\sigma_x\sigma_y\rho_{xy} + (y-\mu_y)^2\sigma_x^2\big)}{2\sigma_x^2\sigma_y^2(1-\rho_{xy}^2)}\right)}{\frac{1}{\sqrt{2\pi}\sigma_y}\exp\left(-\frac{(y-\mu_y)^2}{2\sigma_y^2}\right)} \\
&= \frac{1}{\sigma_x\sqrt{2\pi(1-\rho_{xy}^2)}} \exp\left(-\frac{\big((x-\mu_x)^2\sigma_y^2 - 2(y-\mu_y)(x-\mu_x)\sigma_x\sigma_y\rho_{xy} + (y-\mu_y)^2\sigma_x^2\rho_{xy}^2\big)}{2\sigma_x^2\sigma_y^2(1-\rho_{xy}^2)}\right) \\
&= \frac{1}{\sigma_x\sqrt{2\pi(1-\rho_{xy}^2)}} \exp\left(-\frac{\left((x-\mu_x)^2 - 2(y-\mu_y)(x-\mu_x)\rho_{xy}\frac{\sigma_x}{\sigma_y} + (y-\mu_y)^2\left(\frac{\sigma_x^2}{\sigma_y^2}\rho_{xy}^2\right)\right)}{2\sigma_x^2(1-\rho_{xy}^2)}\right) \\
&= \frac{1}{\sigma_x\sqrt{2\pi(1-\rho_{xy}^2)}} \exp\left(-\frac{(x-\mu_x-(y-\mu_y)\rho_{xy}\frac{\sigma_x}{\sigma_y})^2}{2\sigma_x^2(1-\rho_{xy}^2)}\right) \\
&= \frac{1}{\sqrt{6\pi}}\exp\left(-\frac{(x-2.5-0.5y)^2}{6}\right)
\end{aligned}$$

$\Rightarrow X|Y \sim N\left(\mu_x + \rho_{xy}\frac{\sigma_x}{\sigma_y}(y-\mu_y); \sigma_x^2(1-\rho_{xy}^2)\right)$

$$
\begin{aligned}
f(y \mid x) &= \frac{f(x,y)}{f(x)} \\
&= \frac{1}{\sigma_y\sqrt{2\pi(1-\rho_{xy}^2)}} \exp\left(-\frac{(y-\mu_y-(x-\mu_x)\rho_{xy}\frac{\sigma_y}{\sigma_x})^2}{2\sigma_y^2(1-\rho_{xy}^2)}\right) \\
&= \frac{1}{\sqrt{6\pi}} \exp\left(-\frac{(y+0.5-0.5x)^2}{6}\right)
\end{aligned}
$$

$\Rightarrow Y|X \sim N\left(\mu_y + \rho_{xy}\frac{\sigma_y}{\sigma_x}(x-\mu_x); \sigma_y^2(1-\rho_{xy}^2)\right)$