Laser Physics and Applied Optics

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Contents

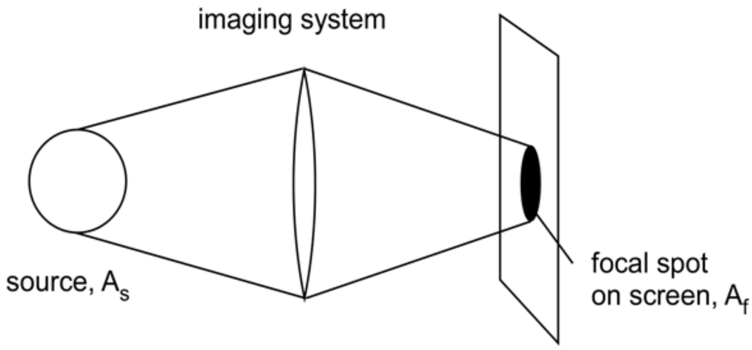
1. What is a laser?

1.1 Laser light

Laser light differs in two fundamental properties from the usual thermal light as for instance from the sun or a light bulb.

- Focusing

  Laser light can be focused much better than thermal light. We calculate the smallest focal spot that can be created with a thermal source. Light of a thermal source with a surface A_s is focused by a lens on focal spot with area A_f. An imaging system transmits a factor ε of the light that the source radiates. The material in the focal spot is absolutely black and heats up to Temperature T_0.

  The heat radiated by the focal spot is given by the Stefan-Boltzman law

  $$P_{em} = A_f \cdot \sigma \cdot T_0^4.$$

  The heat absorbed by the focal spot is given by

  $$P_{abs} = \varepsilon \cdot \underbrace{A_s \cdot \sigma \cdot T_s^4}_{\text{power radiated by the source}}$$

  In thermal equilibrium, the spot radiates as much heat as it absorbs,

  $$P_{em} = P_{abs.}$$

  Solving for A_f leads to

  $$A_f = \varepsilon \cdot A_s \cdot \left(\frac{T_s}{T_0}\right)^4.$$

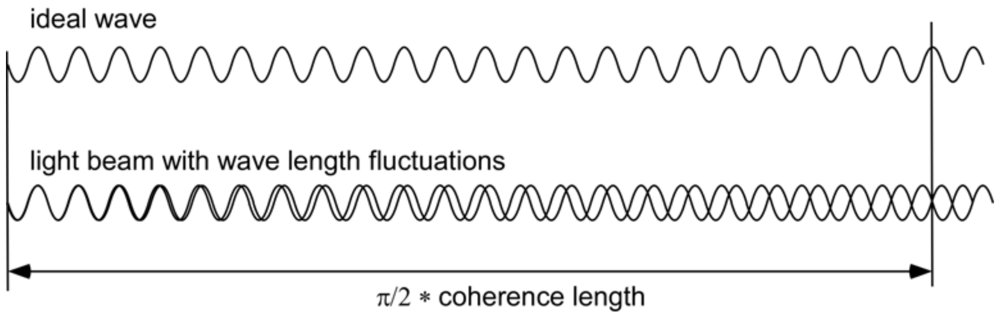
  Since heat always flows from hot to cold $T_0 \leqslant T_s$ and thus

  $$A_f \geqslant \varepsilon \cdot A_s$$

If $\varepsilon = 1$ (complete energy transfer) the focal spot can never be smaller than the source! Laser light is not thermal light and can be focussed much better. As we will see later, for laser light the fundamental limit is given by the wavelength of the light which limits the area of the focal spot to about $\simeq \lambda^2$(see chapter about Gaussian beams).

- Laser light is monochromatic

  This property can be quantified by the coherence length. It is determined by the statistical fluctuations Δk of the lasers's wave number k.



The coherence length L_c is defined as

$$L_c := \frac{1}{|\Delta k|}.$$

By using the relation between frequency ω, wavelength λ, and wave number k

$$\omega = 2\pi\nu = c \cdot k = c \cdot \frac{2\pi}{\lambda},$$

we get various expressions for the coherence length

$$L_c = \frac{1}{|\Delta k|} = \frac{c}{|\Delta\omega|} = \frac{1}{2\pi}\frac{c}{|\Delta\nu|} = \frac{1}{2\pi}\frac{\lambda^2}{|\Delta\lambda|}.$$

Here, we used the frequency fluctuations $\Delta\nu$, which are connected to wavelength fluctuations $\Delta\lambda$ by

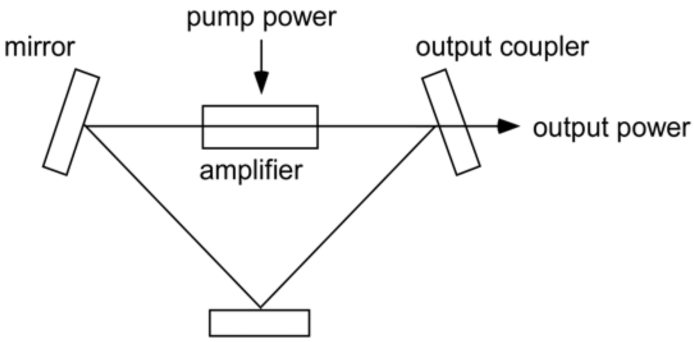
$$\Delta\nu = \frac{d\nu}{d\lambda}\Delta\lambda = -\frac{c}{\lambda^2}\Delta\lambda.$$

In other words: After propagating a distance $\pi/2 \cdot L_c$ the statistical deviation from the ideal wave is a quarter of a wavelength. Sometimes you find other definitions for the coherence length without the factor 2π. Lasers can have coherence lengths between a few mm and some 100 km. The relative frequency fluctuations range from a few % to 10^{-14}. Lasers can be extremely good oscillators, comparable only to the atomic transitions which are used to define the second.

1.2 How does a laser work?

- optical feedback

  Lasers are optical amplifiers, which starts oscillating when the output of the amplifier is fed back to the input. The resulting closed optical path forms an optical resonator.



- Amplification

  We assume that there are optically transparent materials which can amplify light. This means that the transmitted power P_t behind the medium is given by the incident power P_{in} multiplied by a factor larger than one,

$$P_t = (1 + g)P_{in}. \tag{1}$$

  The gain parameter g quantifies the amplification. Every amplifier has a maximum output power. This means that the gain must drop for increasing input power. The amplifier "saturates". Later in this chapter we will discuss a model that describes this saturation and see that the gain parameter behaves like

$$g = \frac{g_0}{1 + P_{in}/P_{sat}}, \tag{2}$$

$$\begin{aligned} g_0 &: = \text{small signal gain} \\ P_{sat} &: = \text{saturation power} \end{aligned}$$

  These two parameters specify the material used for optical amplification.

- Feedback

  The light power that is fed back from the output to the input of the amplifier is proportional to the transmitted power reduced by the losses occurring during the round trip.

$$P_{in} = (1 - T) \cdot (1 - L) \cdot P_t$$

The round trip losses are determined by the transmission T of the output coupler and unwanted extra losses L due to absorption in the optical elements or scattering due to dust and inhomogeneities of the mirror coating. We describe the losses by introducing the round trip reflectivity

$$R_m := (1-T)\cdot(1-L)\,.$$

The feedback relation then simply reads,

$$P_{in} = R_m \cdot P_t. \tag{3}$$

- Threshold

  The three relations 1, 2, and 3 can be used to eliminate P_{in},

$$P_t = \left(1 + \frac{g_0}{1+\frac{R_m P_t}{P_{sat}}}\right) R_m P_t$$

Solve for P_t

$$\begin{aligned}
1 &= \left(1 + \frac{g_0}{1+\frac{R_m P_t}{P_{sat}}}\right) R_m \\
1 + \frac{R_m P_t}{P_{sat}} &= \left(1 + \frac{R_m P_t}{P_{sat}}\right) R_m + R_m g_0 \\
\left(1 + \frac{R_m P_t}{P_{sat}}\right)(1-R_m) &= R_m g_0 \\
P_t &= \frac{P_{sat}}{R_m}\left(\frac{R_m}{1-R_m} g_0 - 1\right) \\
&= \frac{P_{sat}}{1-R_m}\left(g_0 - \frac{1-R_m}{R_m}\right)
\end{aligned}$$

A positive solution requires that

$$g_0 > \frac{1-R_m}{R_m}$$

This is the threshold condition. It quantifies the minimum gain parameter required for the laser to oscillate.

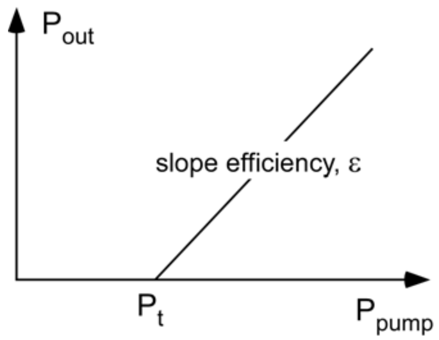
Usually the round trip reflectivity is close to one, $R_m \simeq 1$ and one can approximate

$$g_0 > \frac{1-R_m}{R_m} \sim 1 - R_m.$$

The threshold is reached if the small signal gain just cancels the round trip losses.

- Laser output power

  The small signal gain g_0 depends on the pump power P_{pump} that drives the amplifier. We assume a linear relation,
  $$g_0 = K \cdot P_{pump}$$
  with a constant K that depends on the details of the amplifier. The laser output power behind the output coupler is
  $$\begin{aligned} P_{out} &= T \cdot P_t \\ &= T \cdot P_{sat} \frac{1}{1-R_m} \left(K \cdot P_{pump} - \frac{1-R_m}{R_m} \right) \\ &= P_{sat} \frac{T}{1-R_m} K \left(P_{pump} - \frac{1-R_m}{K R_m} \right) \end{aligned}$$
  or
  $$\begin{aligned} P_{out} &= \varepsilon (P_{pump} - P_{th}) \\ \varepsilon &:= K \cdot P_{sat} \cdot \frac{T}{1-R_m} \quad \text{("slope efficiency")} \\ P_{th} &:= \frac{1}{K} \frac{1-R_m}{R_m} \quad \text{("threshold power")} \end{aligned}$$



1.3 Resonators

- resonance condition

  The frequency spectrum of the laser is determined by the frequencies the resonator supports. A light wave can exist in the laser resonator only if it interferes constructively with itself after one round trip. The phase accumulated during one round trip is given by the wave number k of the light wave and the length l of the round trip,
  $$\Delta\varphi = k \cdot l.$$

Constructive interference requires (n being a natural number)

$$\Delta\varphi = n \cdot 2\pi$$

such that

$$\begin{aligned} k_n \cdot l &= n \cdot 2\pi \\ k_n &= n \cdot \frac{2\pi}{l} \end{aligned}$$

For the angular frequency $\omega = c \cdot k$ one obtains

$$\omega_n = c \cdot k_n = n\frac{2\pi}{l} \cdot c.$$

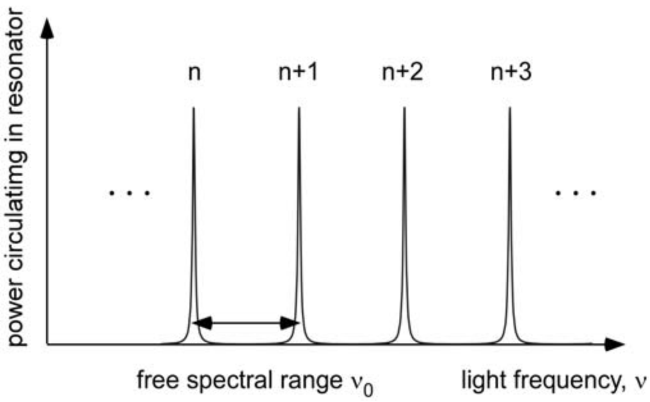
More common is the frequency $\nu = \omega/(2\pi)$

$$\nu_n = n \cdot \frac{c}{l} = n \cdot \nu_0$$

The frequency

$$\nu_0 := \frac{c}{l}.$$

is called "free spectral range" of the resonator. A resonator supports a series of frequencies separated by the free spectral range ν_0.



Depending on the band width of the gain material the laser may operate on several resonances simultaneously. The construction of a "single mode" laser operating on a single resonance may require the development of extra, sometimes quite sophisticated methods.

- Tuning the laser frequency

Fine tuning of the laser frequency is possible by changing the round trip length. Lets assume the round trip length l changes by a tiny amount Δl we obtain for the frequency change

$$\begin{aligned}
\Delta\nu_n &= \frac{d\nu_n}{dl}\Delta l \\
&= -n \cdot \frac{c}{l^2}\Delta l \\
&= -\nu_0 \cdot \frac{n}{l}\Delta l \\
&= -\nu_0 \cdot \frac{\nu_n}{c}\Delta l \\
\frac{\Delta\nu_n}{\nu_0} &= -\frac{\Delta l}{\lambda_n},
\end{aligned}$$

Here

$$\lambda_n = \frac{c}{\nu_n}$$

is the wave length of the n^{th} resonance. This means that changing the round trip length by one wavelength tunes the resonance frequency by one free spectral range. Typical numbers are $l = 10\,\mathrm{cm}$, $\nu_0 = 3\,\mathrm{GHz}$, $\lambda = 1\,\mu\mathrm{m}$. The change in the round trip length is usually realized by mounting one of the mirrors on a piezoelectric transducer. This "piezo" is a little tube that expands by a few μm if a voltage of about 100 V is applied. Piezo tuning only works if the round trip length of the passive laser resonator is sufficiently stable and does not fluctuate in the μm-range due to acoustic or seismic noise.

- line width and finesse

The resonances of an empty passive resonator have a certain frequency width. This "line width" is determined by the round trip losses. The ratio of the free spectral range ν_0 and the full width at half maximum δ_{FWHM} is called "finesse" F of the resonator.

$$F := \frac{\nu_0}{\delta_{FWHM}}.$$

As we will see in the chapter 5, the finesse is connected to the $1/e$-life time T_{res} of the light power inside the resonator

$$F = 2\pi\nu_0 T_{res}.$$

There is also a simple approximate relation between the finesse and the round trip losses

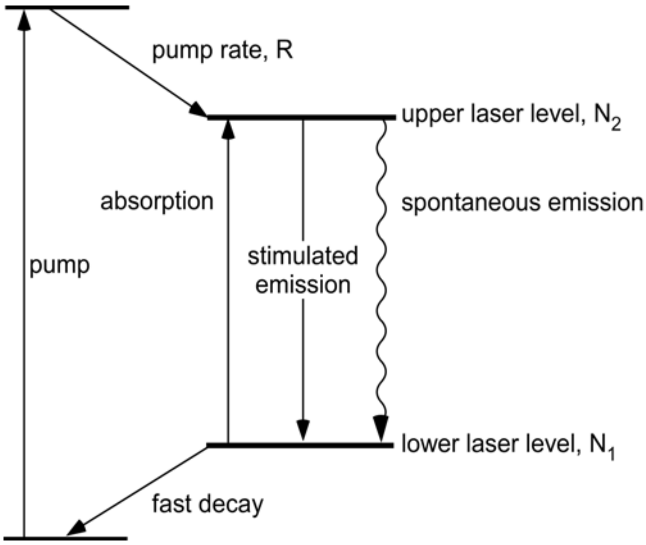
$$F \simeq \frac{\pi}{1 - R_m}.$$

Further details in chapter 5.

1.4 Rate model description of optical amplifiers

- Four level laser scheme

  Optical amplifiers may be constructed by using gases, liquids, or solids that have transitions in the optical range. The active particles are usually atoms or molecules. The typical laser scheme consists of the upper and the lower laser level. The number of atoms in the two levels are denoted N_2 and N_1. A suitable pumping mechanism increases the number of atoms in the upper level at a rate R. This pumping mechanism can be physically very different for each laser type and must be discussed separately. Atoms in the lower laser level should decay very quickly into the ground state, such that the number of atoms in the lower laser level remains small. This ensures that inversion is maintained, i.e. that there are more atoms in the upper level than in the lower level.



  In addition there are three optical processes.

- Absorption

  A photon is absorbed. The energy of the photon is removed from the laser beam and an atom is transferred from the lower to the upper laser level. The absorption rate is proportional to the occupation number N_1, the photon flux density j and the absorption cross section σ_a.

$$\dot{N}_1 = -\sigma_a \cdot j \cdot N_1$$

  The power in the laser beam changes by

$$\Delta P = \hbar\omega\dot{N}_1,$$

  Here, ω is the angular frequency of the laser light.

- Stimulated emission

  A photon is created in the laser beam and an atom is transferred from the upper to the lower laser level. This transition is triggered by a photon that already exists in the laser beam circulating in the resonator. If there is no photon in the resonator, there is no stimulated emission. Stimulated emission enhances the energy of the stimulating laser beam by the energy of a single photon. All other properties (frequency, polarization, beam shape, phase etc.) remain unchanged. In short one can say, that stimulated emission creates a "clone" of the triggering photon. The emission rate is proportional to the emission cross section σ_e.

  $$\dot{N}_2^{(st)} = -\sigma_e \cdot j \cdot N_2.$$

  The power in the laser beam changes by

  $$\Delta P = -\hbar\omega \dot{N}_2^{(st)}$$

- Spontaneous emission

  This decay process needs no trigger. It happens statistically at a given rate which is proportional to the number of atoms in the upper level. The rate coefficient can be written as the inverse of the "natural life time" τ,

  $$\dot{N}_2^{(sp)} = -\frac{1}{\tau} N_2$$

  The emitted photon propagates in an arbitrary direction. Only a very small part of the emitted photons happens to be captured by the resonator. The change in the laser power can thus be neglected.

- Amplification

  The total change of the laser power is

  $$P_t - P_{in} = \Delta P = -\hbar\omega \dot{N}_2^{(st)} + \hbar\omega \dot{N}_1 = \hbar\omega(\sigma_e \cdot j \cdot N_2 - \sigma_a \cdot j \cdot N_1).$$

  For a simple two level system, absorption and stimulated emission are equally strong and we can set

  $$\sigma_e = \sigma_a =: \sigma.$$

  With that we obtain

  $$\begin{aligned} \Delta P &= \hbar\omega \cdot \sigma \cdot j \cdot (N_2 - N_1) \\ &= \hbar\omega \cdot j \cdot A \cdot \frac{\sigma}{A}(N_2 - N_1) \end{aligned}$$

  Here we have introduced the cross-section area A of the laser beam in the resonator. Since $j \cdot A$ is the number of photon that passes the area A in one second the expression

  $$\hbar\omega \cdot j \cdot A$$

can be interpreted as the light energy that enters the laser medium per second which is just the input power P_{in},

$$\Delta P = P_{in} \cdot \frac{\sigma}{A}(N_2 - N_1).$$

With this we obtain

$$P_t = P_{in} + \Delta P = P_{in}\left(1 + \frac{\sigma}{A} \cdot (N_2 - N_1)\right) = (1+g)\,P_{in}$$

By comparing this with the above definition of the gain (equ. 1),

$$P_t = (1+g)\,P_{in},$$

we find that

$$g = \frac{\sigma}{A} \cdot (N_2 - N_1). \tag{4}$$

The gain is positive for inverted population with

$$N_2 > N_1.$$

- Inversion

  In order to calculate the inversion $N_2 - N_1$ we write down the total rate at which N_2 changes

  $$\dot{N}_2 = -\sigma \cdot j \cdot N_2 - \frac{1}{\tau} \cdot N_2 + \sigma \cdot j \cdot N_1 + R.$$

  In equilibrium

  $$\dot{N}_2 = 0$$

  resulting in

  $$-N_2(\sigma j + \frac{1}{\tau}) + \sigma j N_1 + R = 0.$$

  Since the lower laser level decays rapidly we approximately can neglect its occupation number,

  $$N_1 \simeq 0.$$

  With that one obtains

  $$N_2 = \frac{R}{\sigma j + \frac{1}{\tau}}. \tag{5}$$

- small signal gain

  Inserting equ. 5 into equ. 4 yields

  $$\begin{aligned} g &= \frac{\sigma}{A}(N_2 - N_1) \simeq \frac{\sigma}{A} N_2 = \frac{\sigma}{A}\frac{R}{\sigma j + \frac{1}{\tau}} \\ &= \frac{\sigma\tau}{A}\frac{R}{\sigma\tau j + 1}. \end{aligned}$$

We compare that to the ansatz (equ. 2)

$$g = \frac{g_0}{1 + P_{in}/P_{sat}},$$

and obtain expressions for the small signal gain and the saturation power.

$$\begin{aligned} \frac{P_{in}}{P_{sat}} &= \sigma\tau j \\ g_0 &= \frac{\sigma\tau}{A} R. \end{aligned}$$

By using the definition of the photon flux density,

$$j = \frac{P_{in}}{\hbar\omega} \cdot \frac{1}{A},$$

one finally gets

$$\begin{aligned} \frac{P_{in}}{P_{sat}} &= \sigma\tau \frac{P_{in}}{\hbar\omega} \cdot \frac{1}{A} \\ P_{sat} &= \hbar\omega \cdot A \cdot \frac{1}{\sigma\tau} \end{aligned}$$

and

$$g_0 = \frac{\hbar\omega}{P_{sat}} \cdot R.$$

- Pump power

  In order to model the pump rate R we look again at the four level scheme. Very often the system has a state with an energy $\hbar\omega_p$ above that of the upper laser level. This upper pump state rapidly decays into the upper laser level at the pump rate R. We can thus relate the pump rate to the pump power P_{pump} by assuming that each pump event consumes an energy $\hbar\omega_p$.

  $$R = \eta \cdot \frac{P_{pump}}{\hbar\omega_p}$$

  The pre factor η describes the pump efficiency. It is 1 only if every consumed energy $\hbar\omega_p$ really leads to an excitation of the upper laser level. For real lasers this is not the case. Usually, the excitation of an atom into the upper pump level requires far more pump energy than $\hbar\omega_p$. In fact, optimizing the pumping mechanism and maximizing η is one of the main tasks of laser developers.

  With this expression for R we obtain a linear relation between the pump power and the small signal gain

  $$g_0 = \frac{1}{P_{sat}} \cdot \frac{\omega}{\omega_p} \cdot \eta \cdot P_{pump}.$$

We now can explicitly write down the proportionality constant

$$K = \frac{\omega}{\omega_p} \cdot \frac{\eta}{P_{sat}}.$$

The slope efficiency and the laser threshold now finally reads

$$\begin{aligned} P_{out} &= \varepsilon(P_{pump} - P_{th}) \\ \varepsilon &= \eta \cdot \frac{\omega}{\omega_p} \cdot \frac{T}{1-R_m} \\ P_{th} &:= \frac{1}{\eta} \cdot \frac{\omega_p}{\omega} \cdot \frac{1-R_m}{R_m} \cdot P_{sat} \end{aligned} \tag{6}$$

The ratio $\frac{\omega_p}{\omega}$ is also called "quantum defect".

- Interpretation

  Equ.6 contains the resonator properties (T, R_m) , the efficiency of the pumping mechanism (η), the property of the laser medium (P_{sat}) and the energy structure of the laser scheme (ω_p/ω, quantum defect).

  If the laser has no unwanted losses, $L = 0$, the round trip reflectivity is solely given by the output coupler, $R_m = 1 - T$. If, in addition also the pump efficiency is maximum, $\eta = 1$, the slope efficiency acquires a maximum value

  $$\varepsilon_{\max} = \frac{\omega}{\omega_p}.$$

  This means that above threshold every additional pump quantum is transformed into a photon of the laser output beam.

  Lets finally have a look at the parameter P_{sat} that quantifies the laser material. We divide it by the cross-section area of the laser beam and obtain a saturation intensity that only depends on the microscopic parameters of the lasers system

  $$I_{sat} := \frac{P_{sat}}{A} = \hbar\omega \cdot \frac{1}{\sigma\tau}.$$

  The product $\sigma \cdot \tau$ turns out to be the crucial entity for characterizing the laser medium. A low threshold requires a small saturation intensity which in turn is obtained for a large $\sigma\tau$ product. Consequently, a good laser medium should have a long life time of the upper laser level and a large emission cross-section.

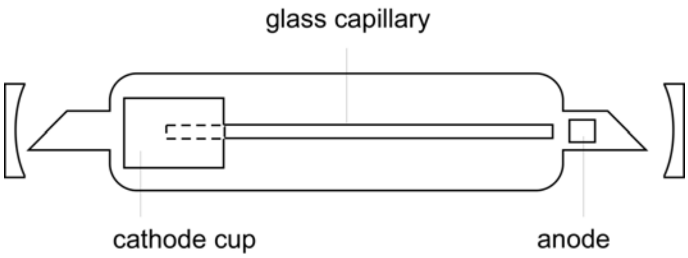
2. The laser zoo

Since the invention of the laser in the early 1960s, the development of ever better lasers is still in full swing and there is now a huge "zoo" of different types of lasers. It contains small semiconductor chips of only a few tenths of a millimeter in size but there are also large specialized laser facilities costing billions, such as the XFEL free electron laser in Hamburg, which emits in the X-ray range. Here we take a short tour through the laser zoo and look at some prominent examples. The the most important type of laser, the semiconductor laser, is excluded from the tour. A separate chapter is dedicated to it.

2.1 Gas laser

- The Helium-Neon Laser

  A mixture of helium and neon is confined in a sealed glass tube with a helium pressure of 10 mbar and a neon pressure of 1 mbar. A high voltage of about 2 kV is applied between two electrodes which generates a gas discharge.

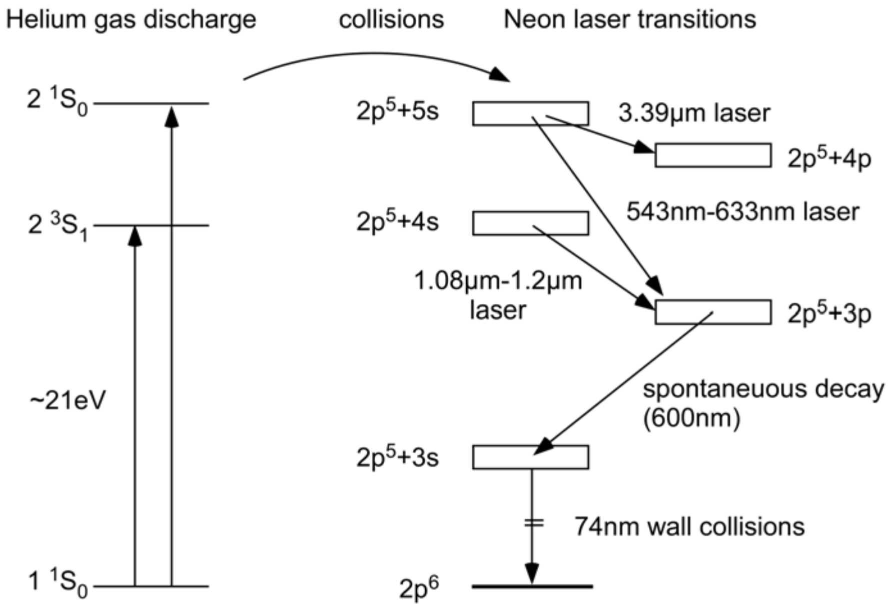
  

  The fast electrons in the discharge collide with the helium atoms and excite one of the two helium electrons into states with principle number n = 2. The excited triplet states (2^3S_1) and singlet states (2^1S_0) both cannot decay optically into the ground state (1^1S_0) because matrix elements between states with same orbital angular momentum vanish. It is not possible to generate a population inversion this way since electron collisions can also deexcite an excited helium atom such that the population of the ground state and the excites states is equal at maximum. Therefore, a helium discharge alone is not suitable as a laser medium.

  Here, neon comes into play. Lets take a brief look at the electron structure of neon. In the ground state all three orbitals of the 2p state are doubly occupied ($2p^6$). An excited state is formed if one of the six electrons is excited into the 3s, 4s or 5s state. Since the 5s-state has about the same energy as the excited singlet state (2^1S_0) of helium, the two atomic species can efficiently exchange energy in a collision ("collision of second kind"). During such a collision, the helium decays into the ground state and one of the neon electrons gets excited into the 5s state. The neon gas is now inverted because there is no neon in the lower laser level but some neon atoms in the upper one. The majority of neon is in the ground state but that does not affect the inversion of the two laser levels.

  The excited neon state ($2p^5$ $+5s$) decays via the laser transitions into the lower laser state ($2p^5$ $+3p$), which is quickly decays into the state $2p^5+3s$ by spontaneous emission.

This state cannot further decay optically. If there would be no other decay mechanism, the laser would stop after all neon atoms would have gathered in this state. Fortunately, the $2p^5 + 3s$-state can decay into the ground state by collisions with the glass wall. Therefore, the gas discharge burns in a thin glass capillary in which the neon atoms frequently collide with the wall.



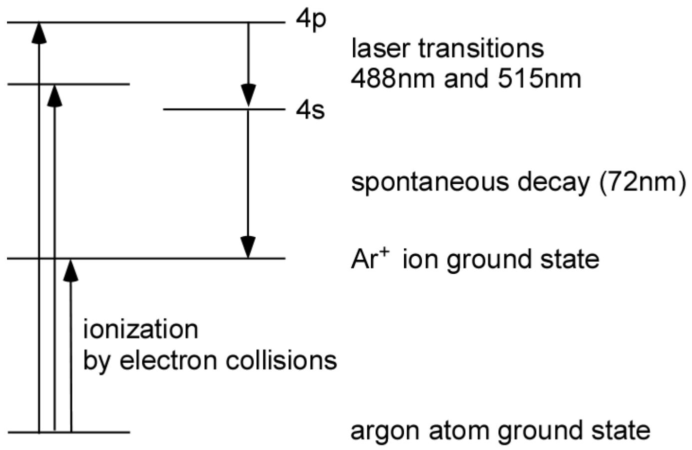
Neon provides several laser transitions. The strongest transition is in the infrared at 3.39 μm. There are also a number of lines in the visible range. Best known is the red line at 633 nm.

The maximum output power of about 50 mW is limited by the number of wall collisions. To increase the collisions one could try to make the laser tube longer. However, the length of the capillary is limited to a few 10 cm by unavoidable divergence of the laser beam (see chapter 5).

The selection of the wavelengths is accomplished by the mirror coating which selectively reflects light only at specific wave lengths. One can also use wave length selective elements inside the resonator as for instance a prism.

- Ion laser

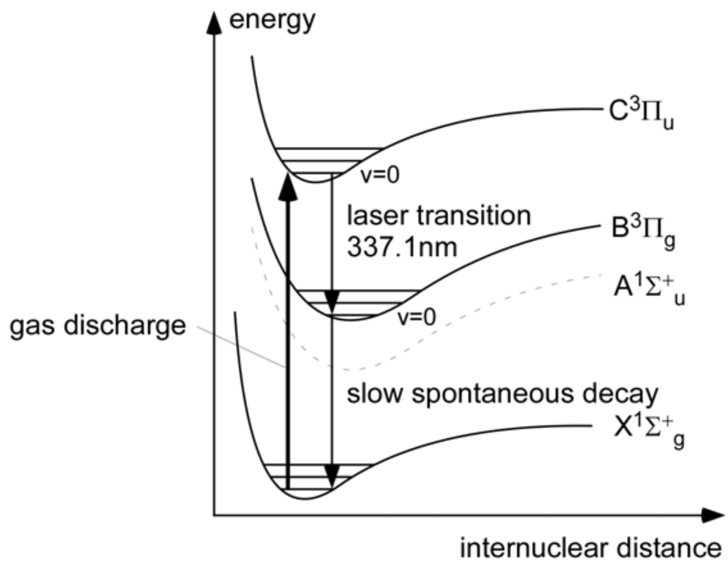
  The laser medium of the argon laser are ions rather than neutral atoms as in the He-Ne laser. Excited ions in the 4p-state are generated in the gas discharge via electron collisions with the argon atom. During this ionization process an electron is removed form the atom and flies away. Therefore, the inverse process (electron + excited ion $\rightarrow$ atom) is not possible and the excited ion remains inverted. High output powers is obtained if the plasma is compressed with a magnetic field. This ensures a good overlap with a focused laser beam. A small cross-section area guaranties a small threshold.



In addition to strong laser lines at 488 nm and 514 nm, there are a number of weaker laser lines ranging from the near infrared to the ultra violet: 1092.3 nm (infrared), 528.7 nm (green), 501.7 nm (green), 496.5 nm (turquoise), 476.5 nm (blue), 472.7 nm (blue), 465.8 nm (blue), 457.9 nm (blue), 454.5 (blue), 363.8 nm (UV), 351.1 nm (UV). Besides the argon ion laser the krypton ion laser is a second popular ion laser with a series of transitions at 406.7 nm, 413.1 nm, 415.4 nm, 468.0 nm, 476.2 nm, 482.5 nm, 520.8 nm, 530.9 nm, 568.2 nm, 647.1 nm, and 676.4 nm. The ion lasers are not tunable and mainly serve as pump lasers for other laser types. The laser output power can be as much as 20 W for the very strong lines. Typical are rather some 100 mW. The pumping mechanism is very inefficient, because the gas discharge and the electromagnet for plasma compression must be cooled with up to 30 kW of cooling power.

- The nitrogen laser

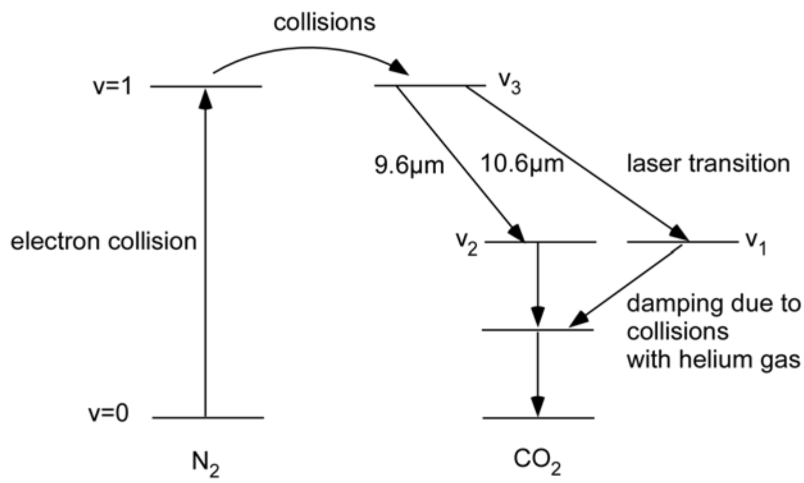
  Here, the laser media consists of electronically excited molecules. Nitrogen is a homonuclear dimer with two electrons. In the ground state ($X\,^1\Sigma_g^+$) both electrons occupy the σ-type molecular orbital. According to Pauli's principle the electron spins are antiparallel (singlet-state). The gas discharge excites one of the electrons into a state with orbital angular momentum and flipped spin. The resulting triplet state ($C^3\Pi_u$) cannot optically decay into the ground state because an optical transition cannot flip the spin (unless the fine structure is strong enough). However, the upper and the lower laser level ($C^3\Pi_u$, $B^3\Pi_g$) have the same spin and different inversion symmetries (ungerade and gerade). Therefore, the superposition of the two states generates a dipole moment that oscillates rectangular to the inter-nuclear axis. Via this dipole the molecule efficiently couples to the laser light. Due to its very small matrix element the lower laser level decays only very slowly and lives for about 1 ms. If the lower laser level has filled up the laser stops. Thus only pulsed operation is possible.



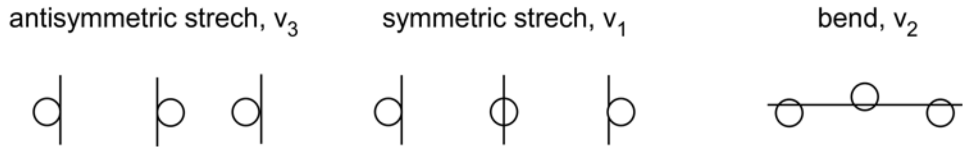
The laser radiates in the ultra violet at 337.1 nm and emits light pulses with a peak power of 1 MW and a duration of $1 - 10$ ns. The energy per pulse is $1 - 10$ mJ. A typical pulse repetition rate is 100 Hz. The gain of the laser medium is such high that one does not even need a resonator. Therefore, nitrogen is sometimes called "super emitter". There are detailed instructions in the internet how to built a nitrogen laser that operates with the nitrogen molecules in air. Nitrogen lasers are mainly used to pump other laser.

- The CO_2 laser.

  This laser does not need electronic excitations. It rather uses the relative motion of the three atoms in the molecule.



This motion is described by three normal modes of the CO_2-molecule with index v_1, v_2, v_3. Similar to the He-Ne laser, the CO_2-laser is pumped with a second gas, namely nitrogen. Also in nitrogen the atoms may oscillate against each other in a single normal mode. At room temperature, only the vibrational ground state is occupied. Again, a gas discharge excites the first vibrational state. The nitrogen oscillation energy is then transferred to the CO_2 molecule via collisions. The antisymmetric stretching oscillation v_3 caries an oscillating dipole moment and radiates laser light. While loosing energy, the oscillation changes its character and decays into either the bending oscillation (v_2) or a symmetrical stretching oscillation (v_1). Finally, damping to the ground state is achieved by collisions with a buffer gas as for instance helium.



The upper laser level has a long life time (milliseconds) because of the small energy separation between the two laser levels which suppresses radiative decay. Furthermore, the oscillation dipole moment of the upper laser level is much larger than in atoms. Both effects lead to a large product $\sigma\tau$ which helps to build an efficient laser. Consequently, very high laser output power can be generated (20 kW). In the laser the CO_2-gas is consumed by chemical reactions and must be constantly refilled. The CO_2 laser is mainly used for material processing such as laser cutting, drilling, and welding.

2.2 Solid state laser

- In a solid state laser the laser medium is a transparent crystal (host material) that is doped with optically active ions. The host crystal should be chemically, mechanically and optically stable and robust. They should have good thermal conductivity and their production should be as simple as possible. The laser ions should fit into the host lattice and must provide an efficient four level laser scheme. The pumping is usually done by light, either from other lasers, especially diode lasers, or from flash lamps.

- Common materials

host material	chemical formula	optically active ions
Garnet (YAG)	$Y_3Al_5O_{12}$	$Nd, Er, Cr.Yb$
Vanadate (YVO)	YVO_4	Nd, Er, Cr
Flouride (YLF)	$LiYF_4$	Nd, Yb
Sapphire	Al_2O_3	Ti, Cr
Glass	SiO_2	Nd

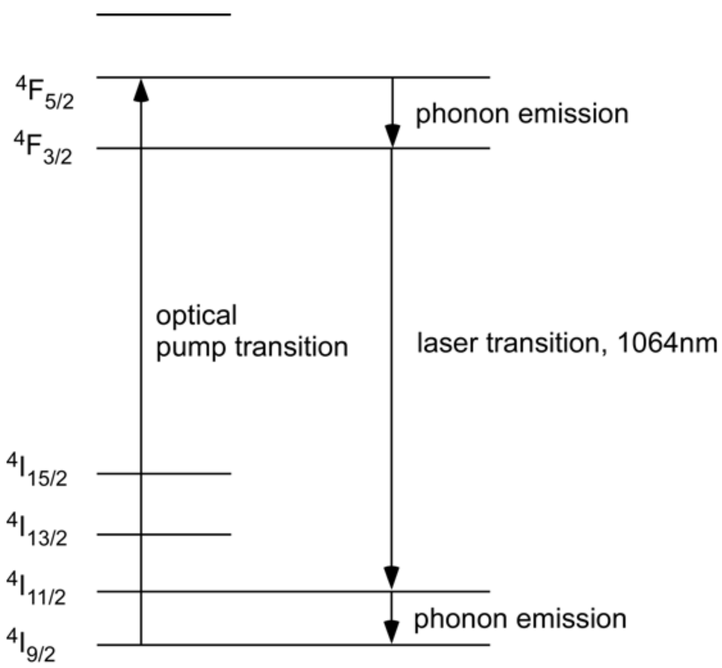
Suitable laser ions are rare earth metals (Nd, Yb, Er) or the transition metals chromium (Cr) and titanium (Ti). The laser properties are very different for both classes. We look at typical examples for each class.

- The Neodymium-doped Yttrium Aluminum Garnet Laser (Nd: YAG)

  In rare earth lasers, the laser transitions take place within the 4f-shell of the triple ionized rare earth atoms. For an isolated ion these transitions are dipole-forbidden. If the ion is part of a crystal the transitions become allowed due to the influence of the electric field of the neighboring atoms ("ligands"), the so-called "crystal field". However, in rare earth lasers the partly occupied 5f-shell acts as a Faraday cage and shields the 4f-electrons from the crystal field such that the transitions are only slightly allowed and the excited states have long lifetimes of up to ms. A wide range of 4-level systems exist. The most important example is the Nd: YAG laser which we discuss here in more detail.

  The neodymium ion consists of a xenon noble gas core with 3 additional electrons in the 4f-shell

$$Nd : [Xe]\, 4f^3$$



  The Coulomb energy is smallest if both the total spin and the total angular momentum have maximum values. In the ground state the electron spins couple to the maximum spin, $S = 3/2$ with a spin multiplicity of $2S + 1 = 4$. Since in this state the three electrons all have the same spin, Pauli's principle requires that the orbital angular momenta of the three electrons must differ. All electrons are in the 4f-state, which means that only the z-components of their orbital angular momentum can differ. The maximum total orbital angular momentum therefore amounts to $L = 1 + 2 + 3 = 6$. Consequently, the possible values for total angular momenta are $|S - L| \leq j \leq |S + L|$, resulting in $j = 9/2...15/2$. The ground state multiplet thus has the notation ${}^4I_{9/2}$ to ${}^4I_{15/2}$

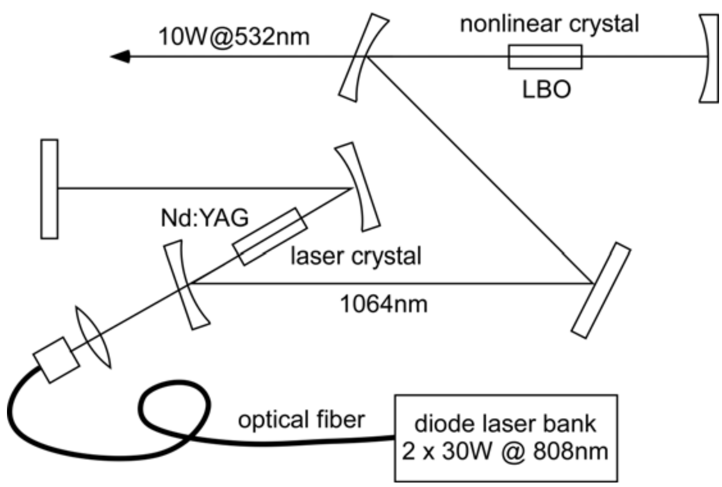
$$\begin{array}{rl} & {}^4I_{9/2} \;...\; {}^4I_{15/2} \\ \text{spin multiplicity}\;: & 2s+1=4 \\ \text{orbital angular momentum:} & L = 6 \leftrightarrow I \\ \text{total angular momentum:} & j = 9/2 \;...\; 15/2 \end{array}$$

The states of the I- and F-multiplet are split due to the crystal field.

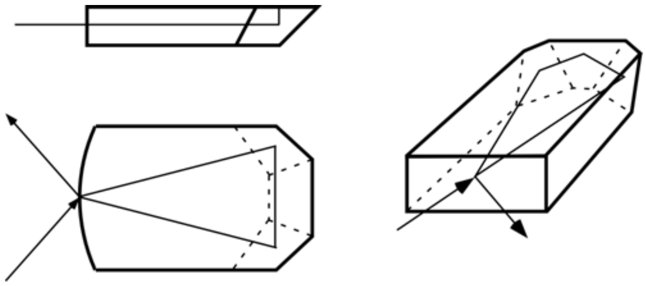
The Nd:YAG laser is a compact, high-power continuous wave laser and is mainly used for material processing. The laser is very efficiently pumped with a semiconductor laser resonant with the transition at 808 nm resulting in a high "wall plug efficiency (large ratio of the optical output power to the electric input power).

The long lifetime of the upper laser level can be used to generate intense laser pulses: The resonator is blocked with an optical shutter until the maximum inversion is reached. After unblocking the laser emits a strong but short light pulse. This method is called "Q-switching". The Q stands for the "quality factor" of the resonator.

A widely used commercial product is the frequency-doubled YAG laser which radiates in the green near 532 nm. A nonlinear crystal (lithium triborate, LBO) is placed inside the laser resonator and doubles the frequency of the circulating light. Despite the complexity of the setup the technical problems can be solved sufficiently well to make this laser a robust product suitable also for industrial production lines. In research this laser is used to pump titanium-sapphire lasers which we will discuss below.



The most compact Nd:YAG laser is obtained if the laser crystal itself is cut in the right way to form the resonator. At the mirror-coated front facet the pump light enters the crystal and the laser light is coupled out. At the other surfaces the laser light is reflected by total internal reflection.



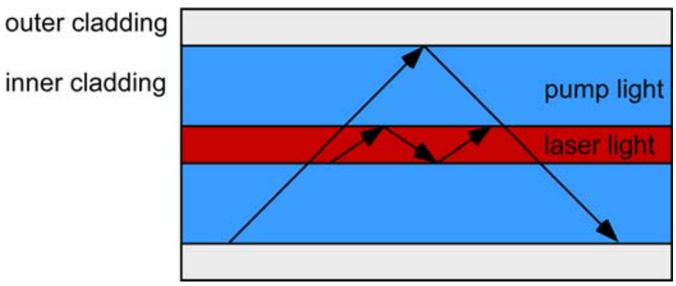
NPRO: Non Planar Ring Oscillator

Due to the large free spectral range of the tiny resonator the laser operates on a single resonance ("single mode"). In a ring resonator the light can propagate the optical path in two counter propagating round trips. If nothing distinguishes the two directions of propagation, the laser hops between them statistically and is not very useful. Therefore, the light beam does not circulate in one plane but hits the upper surface at one point. This has the effect that the orientation of the light polarization is slightly tilted after one round trip. By applying a magnetic field the Faraday effect (see chapter about optical components) in the crystal can be exploited to compensate this tilt. This only works for one of the two possible round trip directions. For the round trip in the opposite direction the Faraday effect and the geometric tilt add up and the polarization is even further tilted. The constructive interference of consecutive round trips is messed up and the laser cannot work. Laser operation is therefore only possible for one direction and the laser operates "single mode".

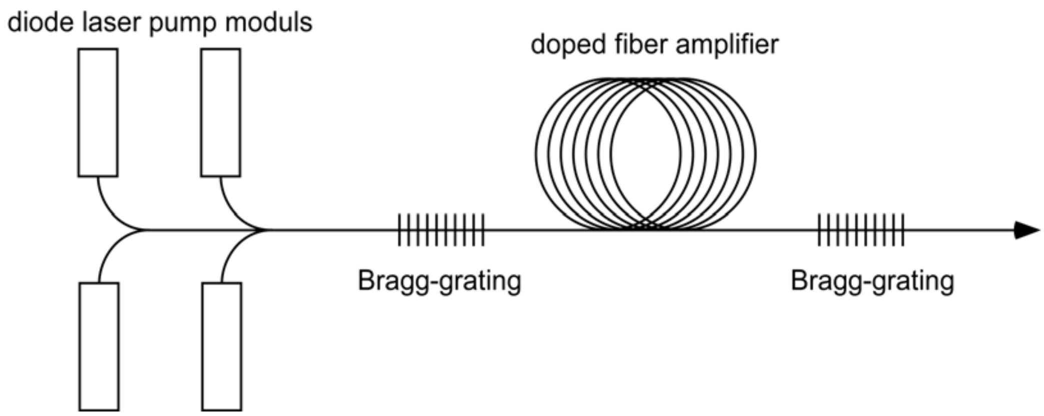
By slightly bending the laser crystal with a piezo element the length of the round trip can by varied by a fraction of a wavelength. This tunes the laser frequency over a significant fraction of the free spectral range (several 100 MHz). Slow tuning over a wider range is achieved by varying the temperature. This changes the index of refractive of the crystal and with it the resonance frequency of the resonator.

- Fiber Laser

  Optical fiber laser have an ion-doped core, which forms the laser amplifier. A second glass layer, the "inner cladding", surrounds the core. Its refractive index is smaller than that of the core. Thus the laser light is guided in the core by total internal reflection .



One can also use the cladding to guide the pump light as well. To this end an outer cladding with even smaller index of refraction contains the pump light in the fiber. Both, the laser beam and the pump beam propagate over long distances held together at a small cross section. This makes the laser very efficient. The resonator is formed by two Bragg gratings, which are inscribed into the fiber with an intense laser.

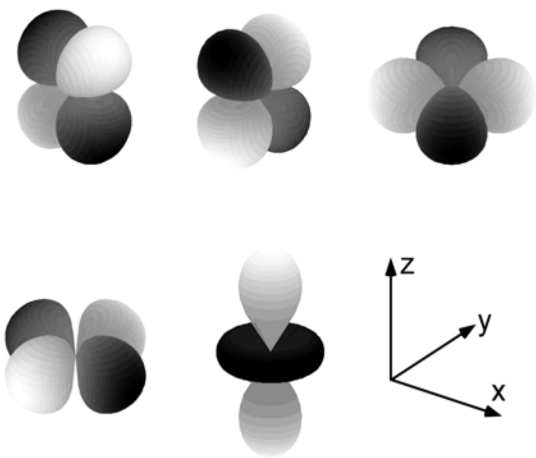


Erbium fiber lasers are mainly used for optical communication and data transmission. Erbium is a good amplifier at a wavelength near 1550 nm, where optical fibers have the lowest losses. Erbium is pumped with laser diodes at 970 nm.

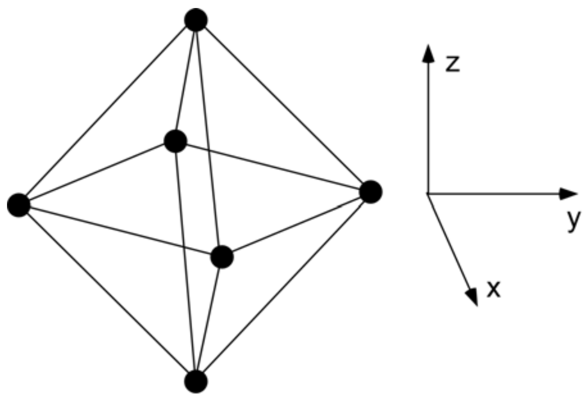
Ytterbium fiber laser emit between 1030-1120 nm and are pumped by laser diodes near 940 nm. Ytterbium doping can be higher as in the erbium laser and the Ytterbium level scheme has a smaller quantum defect. Continuous wave operation with high power of several 10 kW can been achieved. Fiber laser may replace $C0_2$-lasers for material processing.

- The titanium-sapphire laser (TiSa)

  The laser material consists of a sapphire crystal doped with triple ionized titanium atoms. The electrons of the titanium ion form an argon shell with a single additional electron in one of the five 3d-orbitals. Lets look at the shape of these 3d-orbitals.



Three of them form clovers (upper line in the image), whose leaves are oriented along the three space diagonals. These states are the T-states (T for triplet). The fourth cloverleaf is oriented along the x- and y-axis. The fifth orbital forms an hourglass (with a belt) oriented along the z-axis. The last two states are called E-states (E for excited). The ion is surrounded by 6 aluminum atoms, which form a symmetric octahedron.

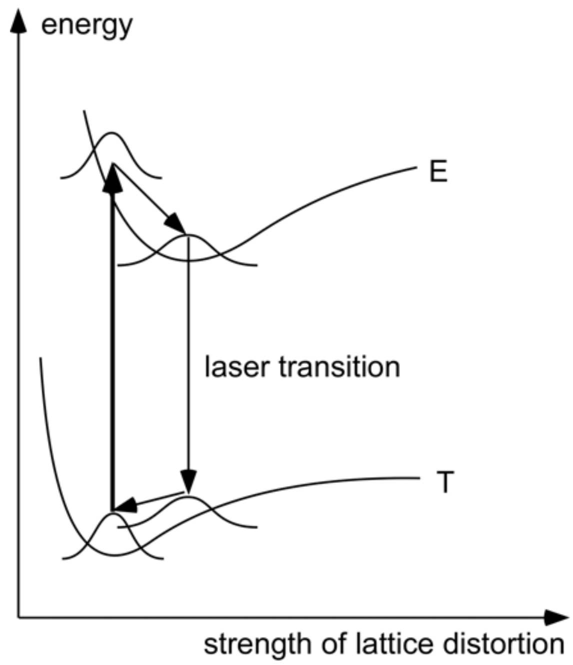


Different to rare earth laser titanium has no orbital shell that shields the 3d-electron from the crystal field of the aluminum ions. It strongly feels the crystal field and tries to find the energetically most favorable orientation within the octahedron. From classic electrodynamics you would expect the maximum probability for the electron to appear at the positions of the aluminum ions. However in a quantum description you have to take the kinetic energy into account: the stronger the confinement the higher the localization energy. In total the lowest energy is obtained if the maximum probability for the electron lies at the position between the ions.

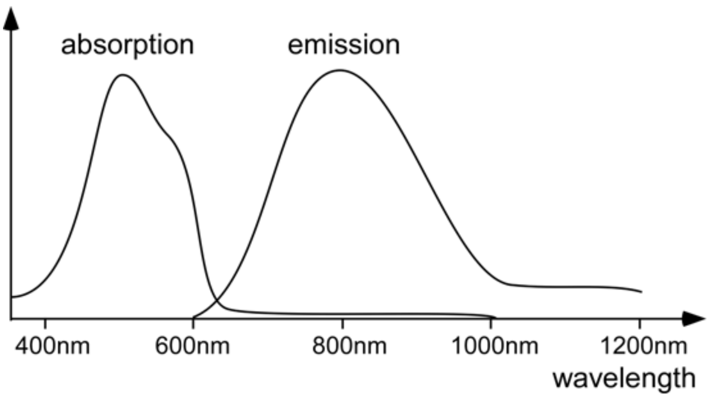
This nicely works for the three T-states: Shift the orbitals into the octahedron and then turn them all by the same angle around the z-axis until each of the leaves are placed between two aluminum atoms. For the so determined relative orientation of the titanium orbital system within the octahedron, the electron position probability of the two E-states is energetically less favorable. Their energy is shifted upwards by about $19000\,\mathrm{cm}^{-1}$.

The second important ingredient is the reaction of the aluminum ions. In the E-states the aluminum ions try to reduce the energy at least a little bit by moving along the z-axis. This slight lattice distortion reduces the energy of badly fitting E-states.

The four level laser scheme now works as follows: In the ground state the titanium electron occupies one of the three T-states. Right after excitation into one of the two ill-fitting E-states, the lattice remains still undistorted. This situation establishes the upper pump state of the four-level system. The lattice then gives way and very rapidly swings into the new state of equilibrium with distorted lattice. This is the upper laser level. During this relaxation, the released energy is converted into lattice vibrations (phonons) and the crystal heats up. Immediately after the laser transition from the E- into the T-state the titanium electron orbital has changed its shape into a well fitting cloverleaf, but the lattice is still distorted. This is the lower laser level. In the last step the lattice quickly relaxes into the ground state and emits a phonon which dissipates as heat. A laser that is based on the fast decay of lattice vibrations is called "vibronic laser".



As compared to rare earth laser the optical transition matrix elements are stronger because there is no crystal field shielding. The natural lifetime of the upper laser level thus amounts to only 3 μs. In contrast, the relaxations of the lattice distortion occur on the ps-scale and are very fast. Because of the very fast vibronic decay both the excitation spectrum and the absorption spectrum are extraordinarily broad. This allows to tune the Laser over a wide wave length range from 700 nm to 1100 nm which makes the TiSa an almost optimal laser crystal. The only disadvantage is that it cannot be pumped very efficiently with laser diodes, because there are no powerful single mode diodes with wavelengths around 500 nm. Usually the TiSa-laser is pumped by a frequency doubled Nd:YAG laser.



3. Laser Diodes

During the last decades the semiconductor laser has become the most important laser type. The development was originally driven by optical data storage on CD or DVD, which became a mass market with the spread of PCs and home videos. The development of ever better laser diodes was very successful and today cheap, compact, efficient and reliable diodes are available. This has led to the development of a second industry that uses such diode chips for more specialized devices. Today, it is almost impossible to come up with a complete list of diode laser applications. They are used in many different fields such as telecommunication, as laser pointers, in laser printers, for confocal microscopy, as a tunable light source in spectroscopy, as pump sources for other lasers, etc.. There are high power diode blocks with several kW output power but there are also cheap and compact laser diodes with integrated diffraction grating that determines the exact wave length of the laser. The development continues unabated and it is to be expected that other types of lasers will become obsolete step by step or will survive only for very special applications. Although the laser diode is now a cheap mass component, it consists of a sophisticated quantum technology. In order to understand the laser diode, we therefore need to go back a little. We begin with a short qualitative reminder of the theory for electrons in a solid state materials.

3.1 Energy bands

- 1D periodic potential

  A crystalline solid state material consists of a periodic lattice of ion cores surrounded by a gas of valence electrons. Valence electrons are the electrons that are not tightly bound to the ion cores but may hop from core to core. This hopping dynamics determines the optical and electric properties of the material. For a single electron the repulsive forces due to the other valence electrons average out such that a single electron mainly feels the periodic electrostatic potential of the ion lattice. Therefore, we first consider a single electron in a one-dimensional periodic attractive potential. To be specific, we use a one-dimensional model potential

  $$U(x) = -U_0 \cos Kx.$$

- Free electron

  We first look at a free electron ($U_0 = 0$). The solution of the Schrödinger equation for a free electron are complex plane waves

  $$\psi(x) = |k\rangle = \psi_0 e^{ikx},$$

  with wave number k as only parameter and a normalization constant ψ_0. We consider a 1D situation with the x-axis as only direction of propagation. The energy is purely kinetic

  $$E(k) = \frac{\hbar^2 k^2}{2m}.$$

  The wave number $k \in \Re$ is a continuous index which labels the solutions. It is the quantum number of the problem. The function $E(k)$ is the spectrum of the problem.

- Brillouin-Zone

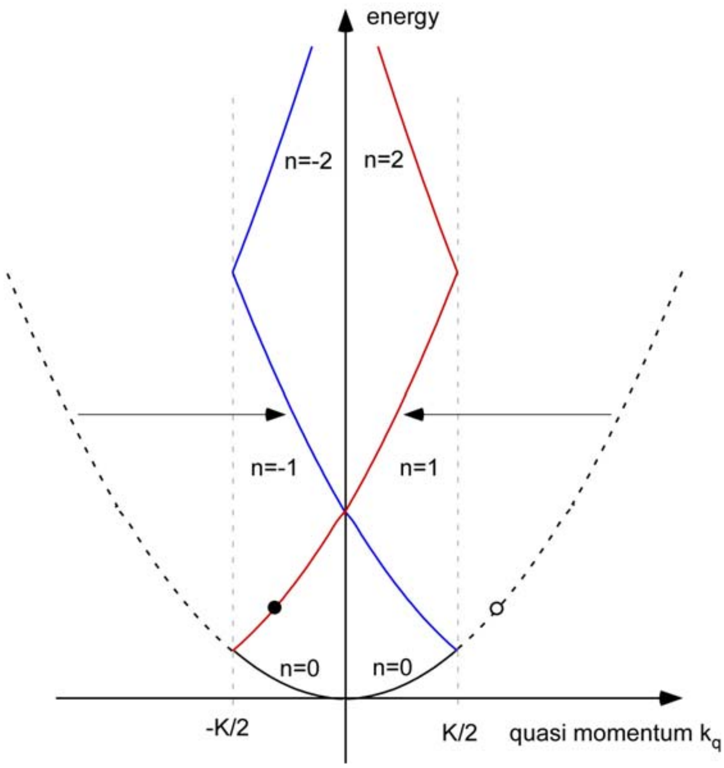
  Next, we change book keeping and introduce the "quasi momentum" k_q. It is limited to the interval

  $$-K/2 < k_q < K/2.$$

  This interval is called the "first Brillouin zone". In principle, the "reciprocal lattice vector" K can be freely chosen. For a free electron the choice is not important. In a crystal $K = 2\pi/a$ with a being the lattice constant, i.e. the spatial distance between two neighboring ion cores. We now write the wave vector k by means of k_q and K.

  $$k = k_q + nK,$$

  The integer n indicates by how many lattice vectors K the actual wave number k differs from the quasi momentum k_q. We plot the energy of a free electron in terms of the new entities k_q and K,



For example, in the new scheme the state marked with the white circle on the original energy-momentum parabola (dashed line) has a negative quasi momentum (dark circle).

- Edges of the Brillouin zone

  At the right edge of the Brillouin zone two states of same energy occur. This is the state $|n = 0, k_q = K/2\rangle$ which corresponds to the state $|k = K/2\rangle$ (plane wave propagating to the right) and the state $|n = -1, k_q = K/2\rangle$, which corresponds to the the state $|k = K/2 - K\rangle = |k = -K/2\rangle$ (plane wave propagating to the left). The two degenerate states can be combined to superpositions with same energy. In particular one can construct two orthogonal standing waves,

  $$\psi_\pm = \frac{1}{\sqrt{2}} \left(|K/2\rangle \pm |-K/2\rangle\right) = \frac{\psi_0}{\sqrt{2}} \left(e^{iK/2\cdot x} \pm e^{-iK/2\cdot x}\right).$$

  The analog situation holds for the states at all points of degeneracy. They are the states with $k_q = 0, \pm K/2$ (except $k_q = 0$, $n = 0$).

- Weak potential

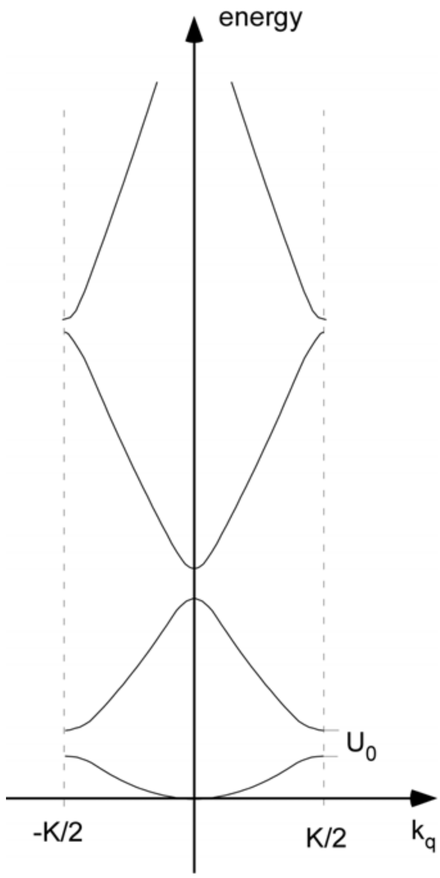
  So far we have only described the energy of a free electron in a new scheme. Next, we slowly turn on the potential of the ion cores. As a first consequence, the two degenerate standing wave states at the edge of the Brillouin zone split energetically. This splitting can be explained by looking at the spatial density distribution of the electron for the two possible standing waves.

$$\begin{aligned}\rho_+ \quad &: \quad = \left|\psi_+\right|^2 = 2\psi_0^2 \cos^2(K/2 \cdot x) = \psi_0^2 \left(1 + \cos(K \cdot x)\right) \\ \rho_- \quad &: \quad = \left|\psi_-\right|^2 = 2\psi_0^2 \sin^2(K/2 \cdot x) = \psi_0^2 \left(1 - \cos(K \cdot x)\right).\end{aligned}$$

We compare theses distributions with the potential generated by the ions,

$$U(x) = -U_0 \cos Kx.$$

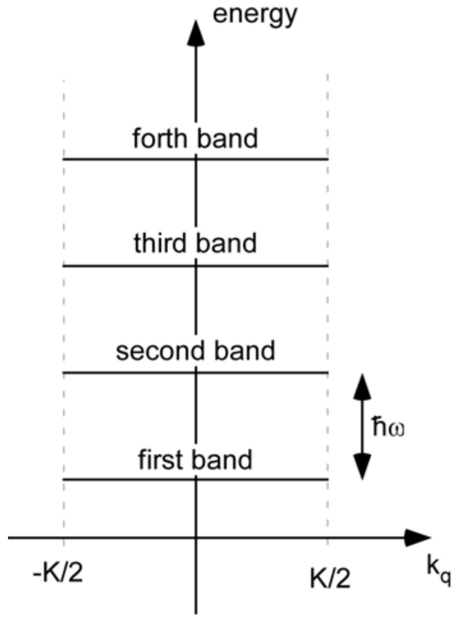
ρ_+ has a maximum at the position where $U(x)$ has a minimum which is at the position of the ions cores. This is energetic more favorable than the distribution ρ_-, which has its maxima between the ion cores. If one calculates the energy difference of the two states for our model potential one finds $\Delta E = U_0$. The spectrum now looks like this



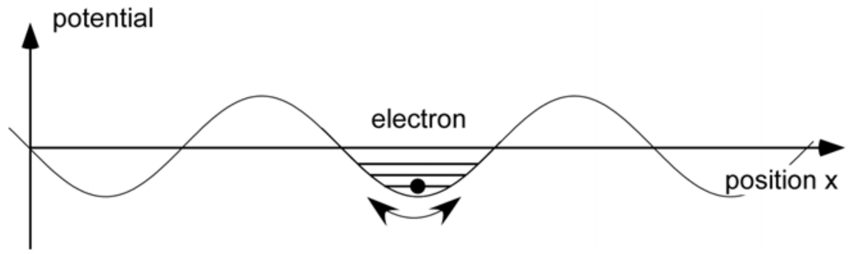
The black solid lines are called "bands". The energy splitting at $k_q = 0, \pm K/2$ create gaps that separate the various bands. In a lattice there are no states with energies that fall into the gaps. This is a very important finding. As we will see, the existence of band gaps determines whether the solid is a conductor, an isolator or a semiconductor.

- Tight binding potential

  By further increasing U_0 the gaps widen and the bands become more flat. For $U_0 >> E_{kin}$ one obtains bands that are almost completely flat.



  The energy distance between the bands become equal and all quasi momenta of the same band are degenerate. This is plausible if we look at a deep lattice where the electron is tightly confined in one of the lattice sites and tunneling between neighboring lattice sites is completely suppressed. Close to its minimum we can approximate the lattice potential by an harmonic oscillator potential.



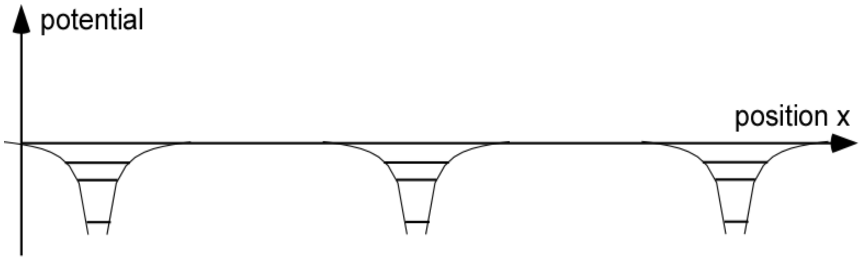
  In an harmonic oscillator the states are separated by the same energy, $E_n = n\hbar\omega$ with $n \in N$ trapping frequency ω. This is consistent with equally separated flat bands.

- Tunneling and number of states

  If the potential well is not too deep the electron can tunnel between the wells even if it sits in the lowest harmonic oscillator state. If we include tunneling, the state of the electron becomes a superposition of the harmonic oscillator ground states of all wells. Theses states can be superimposed with different relative phases. The relative phases are determined by the quasimomentum k_q of the particular superposition state. For details please consult standard text books. For a qualitative discussion is is sufficient to note that there are as many superposition states as we have oscillator ground states. Therefore, the number N of states in which the electrons can sit equals the number of wells i.e. the number of ion cores. Since the electron can has two spin orientations the total number of states for the electron is $2N$.

- Real lattice potentials

  In a real crystal the form of the potential is no longer a cosine function. The electron is trapped in the combined $1/r$-Coulomb potentials of the ion cores. In the tight binding limit, the harmonic oscillator solutions have to be replaced by hydrogen type wave functions. The spectrum is no longer equidistant but rather resembles the spectrum of the hydrogen atom.

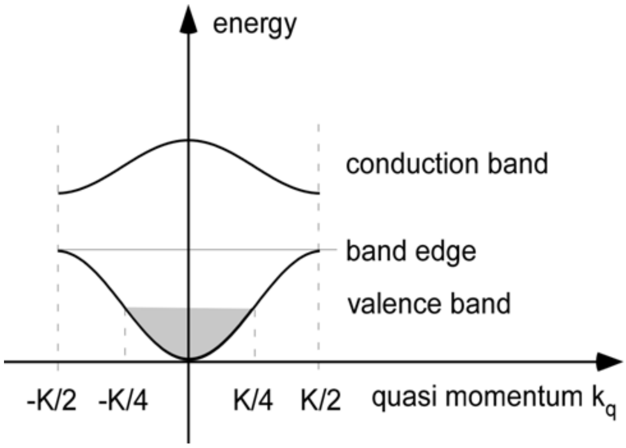
  

  As with the harmonic oscillator, the lowest band is formed from the lowest hydrogen like state. Each additional band corresponds to an excited hydrogen type state.

- Lithium

  Let us take lithium crystal as example. Lithium has 3 electrons. The ion core thus consists of two of these electrons tightly bound to the nucleus (3 protons and 4 neutrons). The third electron is relatively loosely bound and can be described with hydrogen type orbitals. If lithium is in the ground state the third electron occupies the 2s-orbital.

  In the crystal the 2s-electrons can tunnel between the ion cores and form the lowest band of the lithium crystal, the 2s band. Since each electron can have two spin orientations, a crystal with N ion core provides $2N$ states for all the 2s-electrons. The energetically lowest states are filled first and according to the Pauli principle each state can only be occupied once. The 2s-band is therefore half filled, which means that all states with $|k_q| < K/4$ are occupied with an electron.



States with positive quasi momentum are equally occupied as states with negative quasi momentum and there are as many electrons moving to the right as moving to the left. Consequently there is no net current. In order to induce an electron current in the positive direction, for instance by applying a voltage, electrons must be lifted into states with higher energy and positive quasi momentum. In Lithium, this is easily possible because there are still N unoccupied states in the valence band. The electron gas thus responds to smallest voltages. Lithium is a conductor.

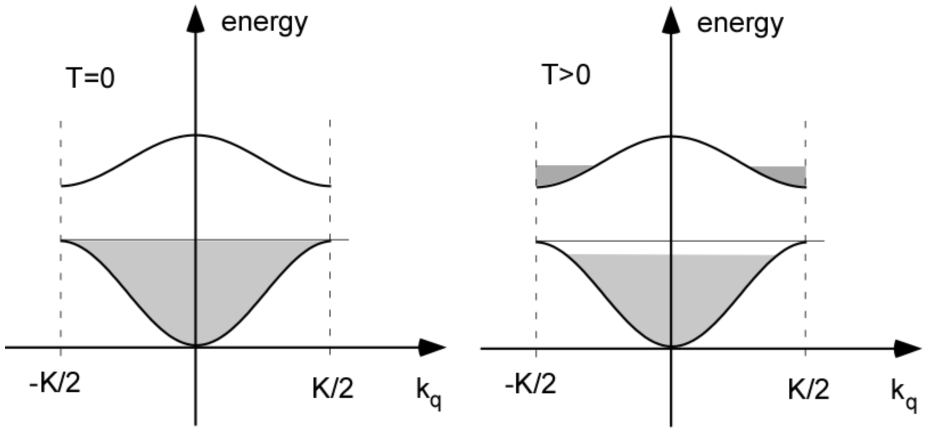
3.2 Conductors, semiconductors, and insulators

- Conductor and insulator

  Depending on the degeneracy of the states, the lowest band can be partially filled or completely filled. If it is completely filled, it is no longer easy to transfer energy to the electrons. The first unoccupied state lies in the conduction band and is separated by the band gap. Normally the band gap is in the eV range and cannot be overcome at voltages, which are normally applied from outside. The material does not react to such voltages and behaves as an insulator.

- Semiconductor

  For some materials with a completely filled valence band, the band gap is particularly small. At room temperature, some of the electrons can then be thermally excited into the conduction band and the valence band is no longer completely filled. The unfilled states are called "holes".



Now, the electrons can respond to small external voltages, because there are unoccupied states in the conduction band and in the valence band. Such materials are called semiconductors.

- n-doped semiconductors

  One can design interesting artificial materials by replacing a small percentage of the atoms in the lattice by atoms that carry one additional electron than the normal lattice atom. Each such donor atom brings an additional electron into the band structure and the corresponding ion core carries an extra positive charge. In a semiconductor this electron can only enter the conduction band, because the valence band is full. In the conduction band the electrons are free to move. In this way, a donor-doped or n-doped semiconductor becomes a conductor even without thermal excitation. "n" stands for "negative" because negative charges are added to the band structure.

- p-doped semiconductors

  Similarly one replacing a small percentage of the atoms by atoms that carry one electron less (acceptor) than the normal lattice atom. Such an acceptor atom captures an electron from the filled valence band and integrates it into the ion core. This generates holes in the valence band, which can also move freely. Such a p-semiconductor also becomes a conductor.

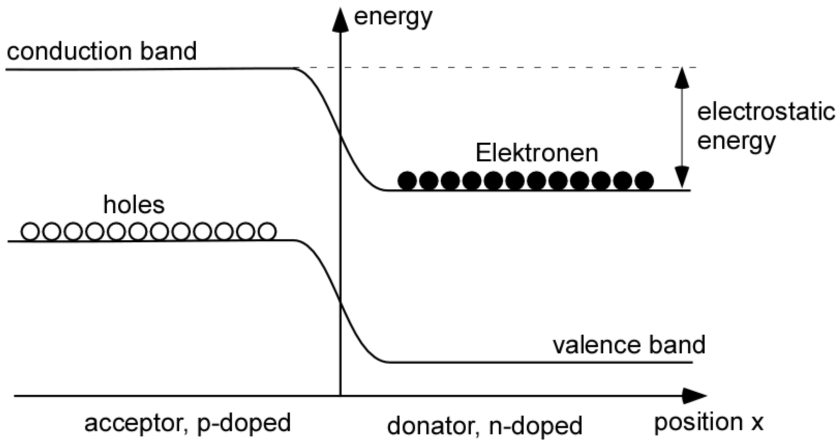
- Donor and acceptor states

  Strictly speaking, the additional electrons of a n–doped semiconductor do not end up in the conduction band but in states just below its band edge. Since the donor ion core carries an extra charge, the electrons still feel the additional attraction to the donor core. This creates new electron states slightly below the conduction band. However, this energy difference can easily be overcome thermally, so that the additional electrons can be found with high probability in the conduction band. Similarly, the hole states of p-doped semiconductors are slightly shifted above the edge of the valence band.

3.3 P-N-junction

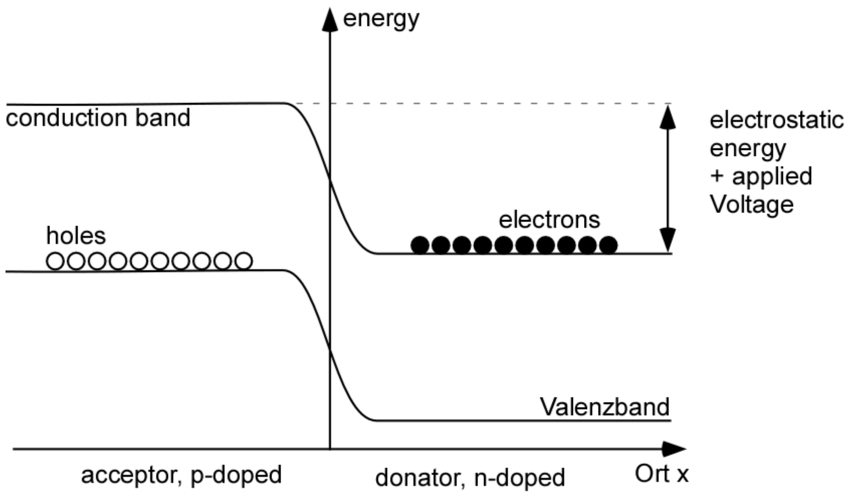
- Recombination current

  If an n-doped semiconductor is brought into contact with a p-doped semiconductor, the electrons form the conduction band of the n-semiconductor drift across the interface to the p-semiconductor and fill the holes in the valence band. In the process the n-semiconductor charges up positively and the p-semiconductor negatively. To bring an electron from n- to p-doped semiconductor thus costs increasingly more electrostatic energy. As a result the energy levels of the p-semiconductor are shifted upwards by just this electrostatic energy. Close to the junction the energy of the bands vary with position as qualitatively shown on the drawing.



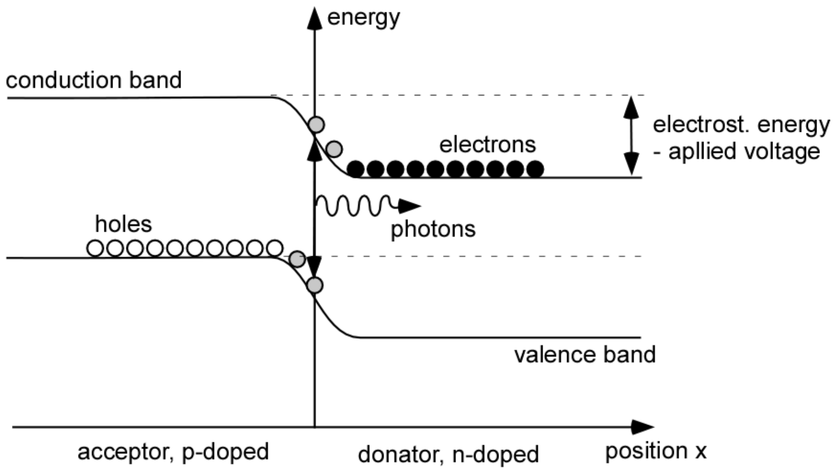
- Diode in blocking direction

  If a negative voltage is applied to the p-doped material from outside, the potential step increases at the junction where the resistance is largest. It is even more difficult for the electrons to get from n to p.



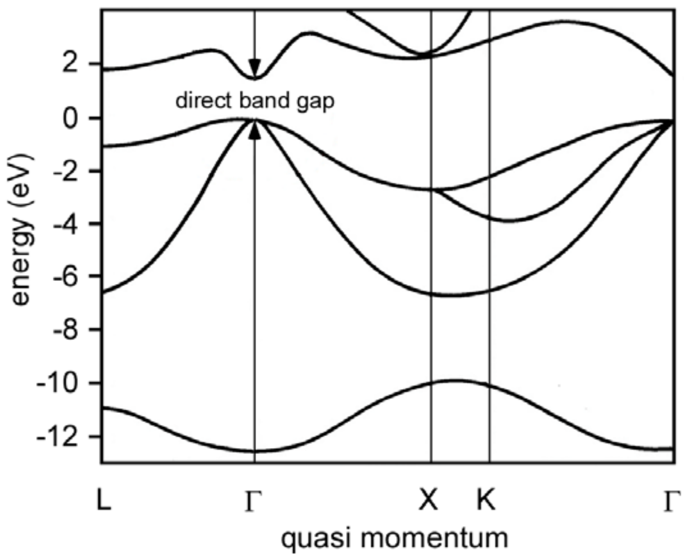
- Diode in passage direction

  If the p-material is set to a positive voltage, the potential step from p to n decreases and the electrons and holes can approach the interface more easily. At the junction they recombine and release energy that corresponds to the band gap. This can be optical energy, i.e. light. Laser diodes are operated in the direction of passage.



- Direct transition

  Since the photon momentum is very small as compared to the quasi momentum, an optical transition preserves the quasi momentum. Furthermore optical transitions take electrons in states close to the minimum of the conduction band and use them to fill holes close to the maximum of the valence band. An optical transition thus takes place only in semiconductors with direct band gaps, where maximum of the valence band and minimum of the conduction band have the same quasi-momentum. A semiconductor with a direct band gap is gallium arsenide (GaAs). Mixtures with aluminum and indium (aluminum gallium arsenide, AlGaAs or indium gallium arsenide, InGaAs) widen the band gap. These materials are used for red and near infrared laser diodes. Here the band structure of GaAs:

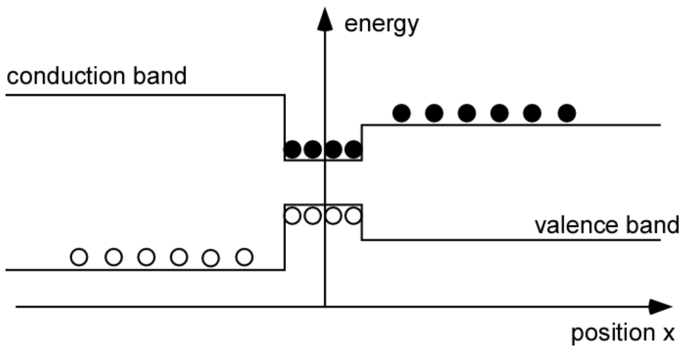


The arrows mark the band gap where the transition takes place.

3.4 Hetero structures

- Tailored potentials

  By combining materials with different band gaps, more sophisticated potentials for electrons can be tailored. If you place layers of different materials next to each other, a potential step is created at each junction. In this way, electrons can be strongly confined in the x-direction. This results in a two-dimensional electron gas spread out in the y-z-plane.

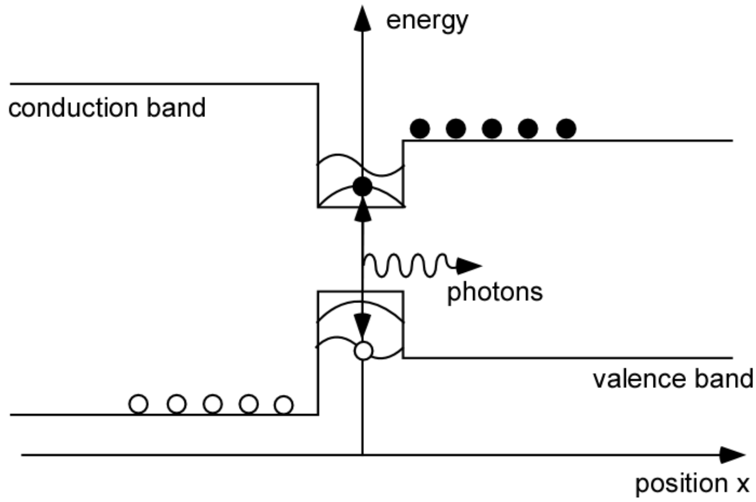


This increases the density of the electrons and the holes in the region where they can recombine. The smaller band gap in the middle layer also increases the refractive index of the light and it is possible to guide light in the middle layer like in a waveguide.

This leads to a very good spatial overlap between the light and the inverted region of the semiconductor. Now we can identify a four level laser system with the conduction band at $x > 0$ as pump level and the valence band for $x < 0$ as ground state. The two states in the middle layer form the upper and lower laser level.

- Quantum Wells

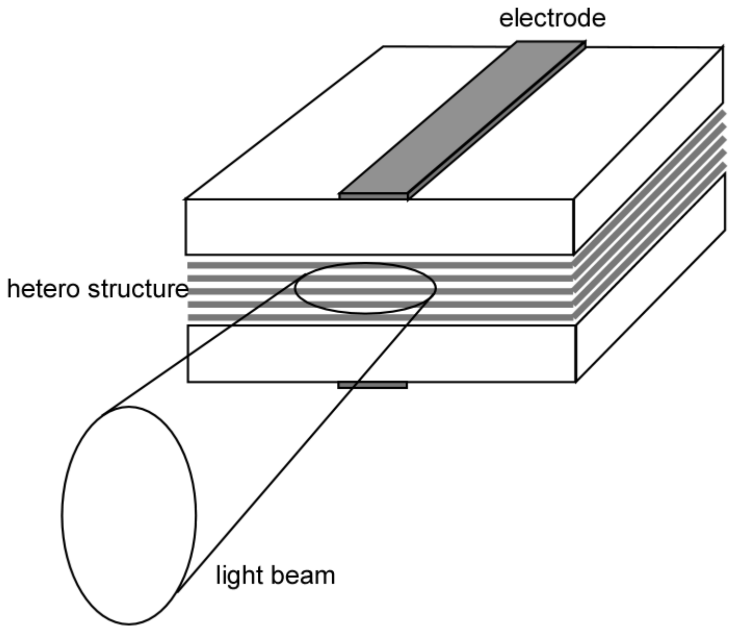
  If the middle layer is very thin on the order of 10 nm, the electrons are so strongly confined in space that the uncertainty relation comes into play and the electrons have to be described as quantum mechanical particles in a box potential. So-called quantum wells with discrete energy states are formed for the electrons and the holes.



  The transition frequency is determined by the position of the energy levels and thus by the layer thickness.

- Multiple Quantum Wells

  Since the radius of a light beam cannot be smaller than its wavelength, the cross-section of a light beam covers quantum wells. Obviously, it is advantageous to stack several quantum wells in such a way that they all interact with the same light beam. Modern laser diodes consist of such multiple quantum well structures.



Electrodes at the two surface planes provide and extract the electrons. The electrodes also guide the light in the layer plane: Between the electrodes inversion is strongest, which locally changes the index of refraction (see chapter 6). This index profile acts as optical wave guide very similar to an optical fiber.

- Laser diodes

  To construct a laser with such hetrostructured chips the back facet of the chip is coated with a mirror coating that reflects almost 100% of the light. The front facet acts as second mirror and closes the standing wave resonator. Due to the large index of refraction of GaAs ($n = 3.2$) about 30% of the light would be reflected at the front facet. However, the gain of the chip is very high and the optimum reflectivity of the output coupler is more in the range of a few %. The output facet is therefore coated with a dielectric coating that reduces the reflectivity and brings it close to the optimum value.

- Single mode laser diodes

  In single mode diodes the optical waveguide in the diode chip has a small cross-section with a diameter of about one optical wavelength. The emitted laser beam has a Gaussian shaped intensity profiles and oscillates at a single frequency. It is diffraction-limited in both vertical and horizontal direction, i.e. the angle at which the laser beam propagates is inversely proportional to the dimension of the output surface. Typically, the wave guide is wider than high, so that the beam diverges about three times faster in the direction perpendicular to the layer plane of the hetero structures than in the direction parallel to it. Typical output power amounts to several 10 mW.

- Applications

  Laser diodes are available at different wavelengths. Typical applications are

  980 nm pump light for erbium laser

  808 nm pump light for Nd:YAG laser

  780 nm DVD player,CD player

  630 nm DVD data storage

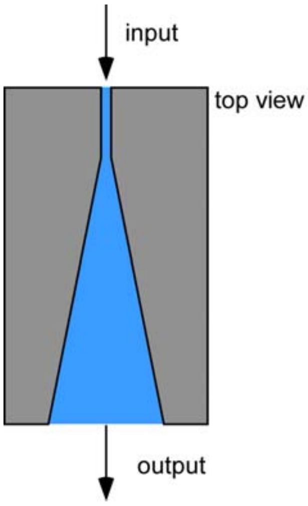
  395 nm bluray discs

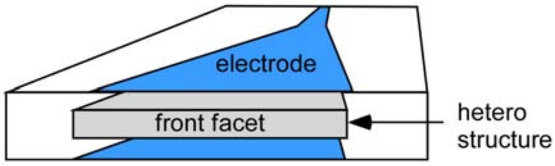
- High Power diode laser

  At power levels above 100 mW the intensity at the front facet of a single mode laser diode becomes too high and there is unavoidable optical damage when the light exits the material. This limits the maximum power of standard single mode quantum well laser diodes. However, higher total power is possible if the front facet is extended in the horizontal direction. The intensity is kept below the damage threshold but the total power increases. Such broad emission area lasers may deliver up to several watts of output power. Stacks of such lasers can provide up to several kW of optical power. However, the shape of the light beam radiated by a broad emitter diode is not very well defined and it is difficult to focus.

- Tapered amplifiers

  Tappered amplifiers avoid this problem. They are multiple quantum well diode laser whose electrodes widens as they approach the front facet. The back and front facet are coated for maximum transmission. A weak Gaussian beam that enters the chip at the back facet and is amplified on it way to the front facet. While it is amplified, it adiabatically increases its horizontal width parallel to the widening of the electrodes. A diffraction-limited high power beam leaves the diode through the front facet. Because of the large cross section of the front facet the intensity stays below the critical intensity for optical damage. Well shapes beams with powers up to 10 W are possible.

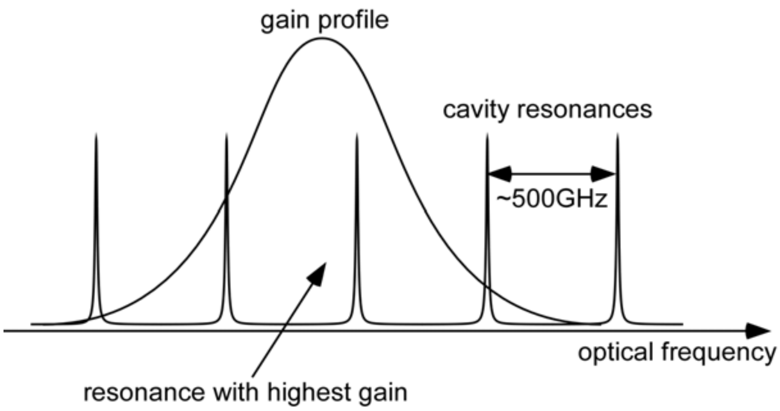




3.5 ECDL (external cavity diode laser)

- Tuning the laser frequency by temperature

  The frequency at which a single mode laser diode oscillates is determined by the gain profile and by the resonances of the chip.



Since the free spectral range of the diode chip is not very much smaller than the width of the gain profile of the semiconductor material there is usually only one resonance close to the maximum of the gain profile. This resonance has the highest gain and the smallest threshold. It thus starts lasing first. The frequency of this resonance can be tuned by varying the temperature. This is because the length of the diode chip changes due to thermal expansion and also the refraction index n of the semiconductor changes with temperature. The resonance condition

$$n(T)\,k \cdot d(T) = \pi$$

(phase accumulation of a multiple of 2π after one round trip) is fulfilled for the wavevector

$$k(T) = \frac{\pi}{n(T) \cdot d(T)}$$

and the laser frequency $\nu = c \cdot k/2\pi$ now also depends on temperature.

Once the frequency is tuned across the gain profile the next resonance approaches the maximum of the gain profile and eventually takes over. The laser undergoes a "mode hop". This limits the tuning range to about one free spectral range of the diode chip which is about 500 GHz.

- Laser tuning by injection current

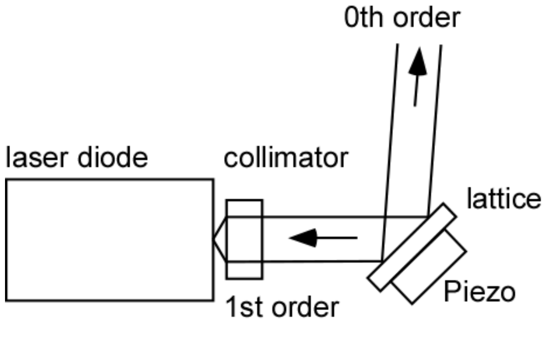
  The frequency can also be tuned over a small range by adjusting the electrical pump current flowing through the diode. The current changes the inversion and thus the refractive index of the semiconductor (see chapter 6), which in turn changes the resonant frequency of the resonator. Since the inversion quickly follows the current the laser can be tuned within nano seconds which is orders of magnitude faster than temperature tuning. Modulation frequencies of up to one GHz are possible.

- Laser line width

  Due to the electromagnetic vacuum, the frequency of diode lasers fluctuates by about 30-100 MHz. For many applications in atomic spectroscopy such a spectral line width is too large. "Laser diodes" can be refined to become "diode lasers" by adding an external resonator.

- ECDL Laser

  There are different versions of such external cavity diode laser (ECDL). One popular version uses an optical grating which is arranged so that the first diffraction order is diffracted back into the diode.

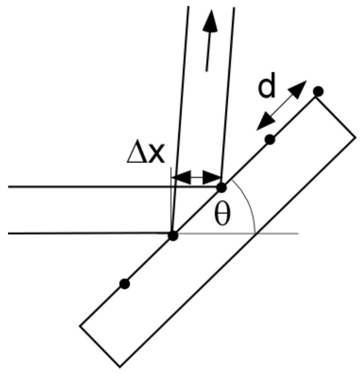


This so called Littrow-configuration requires an angle of incidence θ of

$$d \cos\theta = \frac{\lambda}{2},$$

with lattice constant d and wavelength λ. For this angle the waves emitted by the lattice lines constructively interfere in the reverse direction.

$$2\Delta x = 2d\cos\theta = \lambda.$$



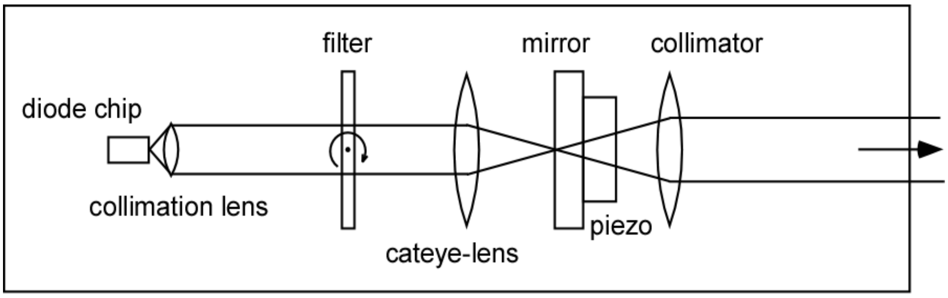
The laser resonator is now formed by the back facet of the diode chip by and the grating. The front facet is antireflection coated. The resonator is closed only within a narrow wave length range given by the tilt angle of the grating. This wave length range is sufficiently small to select a single resonances of the external resonator and suppress all other longitudinal modes.

- Piezo element

  The frequency of this resonance can be fine tuned by moving the grating toward the diode by means of a piezoceramic actuator. This changes the length of the resonator. If the current of the diode is also changed synchronously, the laser frequency may be controlled electronically over a range of 10 GHz within a few milliseconds. The wavelength of the laser is preset by mechanically tilting the grating with a millimeter screw. The output beam is diffracted upwards as zeroth diffraction order. With this setup the line width can be reduced to about 100 kHz. Such lattice-stabilized diode lasers are the most commonly used lasers in quantum optics.

- Cateye laser

- The grating can be replaced by a lens and a partially transparent mirror. If the mirror is placed at a distance from the lens that equals its focal length one obtains a "cateye". This is an optical element that reflects any incoming light beam back to itself independent of the incident angle. Tedious fine alignment becomes unnecessary. The laser is tuned with an optical filter between mirror and laser, which transmits light only in a narrow frequency range. The frequency of the transmission maximum changes when the filter is slightly tilted. This presets the laser wave length. Fine tuning is achieved with a piezo actuator that changes the resonator length. A second lens behind the mirror collimates the divergent output beam.



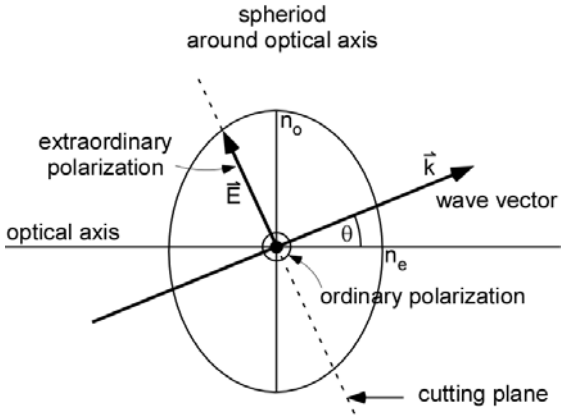
4. Optical Elements

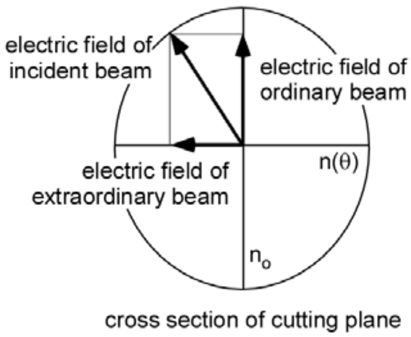
4.1 Polarization Optics

Birefringent crystals such as quartz can be used to manipulate and control the polarization of a laser beam.

- Indicatrix

  In birefringent crystals, the refraction index depends on the orientation of the electric field vector $\vec{E}$ of the light field relative to the axes of the crystal lattice. We can thus plot the refraction index in a polar diagram: we plot a line from the origin in the direction of $\vec{E}$. The length of this line is given by the refraction index. The endpoints of the lines for all possible orientations of $\vec{E}$ form an spheroidal surface centered at the origin. This surface is called "Indicatrix". The orientation of the Indicatrix in space is given by the orientation of the crystal axes in space.



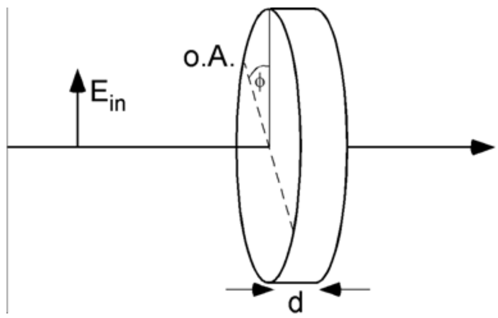


- uniaxial crystals

  We limit our considerations to the case that the Indicatrix is rotational symmetric around one axis. This axis is called "optical axis". Crystals with rotational symmetric Indicatrix are called "uniaxial". Light fields with a polarization ($\vec{E}$ orientation) perpendicular to the optical axis all have the same index of refraction. Its value n_o is called "ordinary index of refraction". Light fields polarized parallel to the optical axis have an index of refraction of n_e which is called the "extraordinary index of refraction". In the following we assume that $n_o > n_e$.Since the orientation of $\vec{E}$ is always perpendicular to the wave vector $\vec{k}$ the possible values for the refraction index depends on the direction in which the light beam propagates. The relevant parameter is the angle θ between $\vec{k}$ and the optical axis. There is always a polarization with refraction index n_o (out of plane in the above sketch). The orthogonal polarization (in plane) has a refraction index that varies with θ between n_o and n_e.

- Retardation plates

  We consider the case with $\theta = 90°$. ($\vec{k}$ perpendicular to the optical axis) and cut the crystal such that it forms a thin plate with the optical axis parallel to the surface.



The plate can be rotated relative to the polarization of the input field by the angle ϕ. If $\phi = 0$ the light experiences a refraction index n_e, if $\phi = \pi/2$ the refraction index is n_o.

For an arbitrary angle ϕ the linear polarization of the input field can be decomposed into two field components parallel to the optical axis and perpendicular to it. The two components experience different refraction indices with the values n_e and n_o, respectively.

After having passed the plate the two field components have accumulated different phase shifts,

$$\begin{aligned}\alpha_o &= n_o \cdot \frac{\omega}{c} \cdot d \\ \alpha_e &= n_e \cdot \frac{\omega}{c} \cdot d\end{aligned}$$

The output beam is the superposition of both components with relative phase shift of

$$\alpha_o - \alpha_e.$$

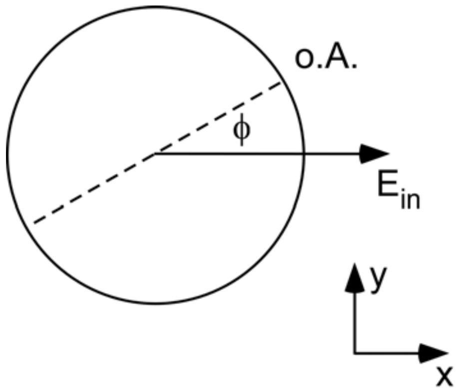
If $\alpha_o - \alpha_e = \pi/2$ the output light is circularly polarized. If $\alpha_o - \alpha_e = \pi$ the output beam is still linearly polarized but its orientation has changed depending on ϕ. To see this in detail, we need a formalism that describes the polarization.

- Jones matrices

  We write the light field in the basis of linearly polarized waves, which propagate along the z-axis.

$$\vec{E} = \begin{pmatrix} E_x e^{i(kz-\omega t)} \\ E_y e^{i(kz-\omega t)} \end{pmatrix}.$$

  The field vector of the input beam is polarized horizontally (x-axis) and the optical axis of the retardation plate is rotated left (downstream view) by an angle ϕ relative to the x-axis.



To calculate the field vector behind the plate, the field strength vector is first transformed from the laboratory basis to the basis of the retardation plate, phase-shifted in this basis and then transformed back to the original basis.

$$\vec{E}_{out} = D(\Phi) \begin{pmatrix} e^{i\alpha_e} & 0 \\ 0 & e^{i\alpha_o} \end{pmatrix} D(-\Phi) \cdot \vec{E}_{in}$$

The transformation is described by the matrix D ,

$$D(\Phi) = \begin{pmatrix} \cos\Phi & \sin\Phi \\ -\sin\Phi & \cos\Phi \end{pmatrix}.$$

We obtain

$$\begin{aligned} \vec{E}_{out} &= \begin{pmatrix} \cos\Phi & \sin\Phi \\ -\sin\Phi & \cos\Phi \end{pmatrix} \begin{pmatrix} e^{i\alpha_e} & 0 \\ 0 & e^{i\alpha_o} \end{pmatrix} \begin{pmatrix} \cos\Phi & -\sin\Phi \\ \sin\Phi & \cos\Phi \end{pmatrix} \vec{E}_{in} \\ &= M\vec{E}_{in}. \end{aligned}$$

with

$$M = \begin{pmatrix} e^{i\alpha_e}\cos^2\Phi + e^{i\alpha_o}\sin^2\Phi & (e^{i\alpha_o} - e^{i\alpha_e})\sin\Phi\cos\Phi \\ (e^{i\alpha_o} - e^{i\alpha_e})\sin\Phi\cos\Phi & e^{i\alpha_e}\sin^2\Phi + e^{i\alpha_o}\cos^2\Phi \end{pmatrix}$$

We now look at special cases.

- Half-wave-plate

  A half-wave-plate shifts the phase of the ordinary polarization relative to the extraordinary polarization by π. The thickness of the plate is chosen such that

$$\alpha_o = \alpha_e + \pi$$

or

$$e^{i\alpha_o} = -e^{i\alpha_e}.$$

With this we obtain

$$\begin{aligned}
M_{\lambda/2} &= e^{i\alpha_o}\begin{pmatrix} -\cos^2\Phi + \sin^2\Phi & 2\sin\Phi\cos\Phi \\ 2\sin\Phi\cos\Phi & -\sin^2\Phi + \cos^2\Phi \end{pmatrix} \\
&= e^{i\alpha_o}\begin{pmatrix} -\cos 2\Phi & \sin 2\Phi \\ \sin 2\Phi & \cos 2\Phi \end{pmatrix} \\
&= -e^{i\alpha_o}\begin{pmatrix} \cos 2\Phi & \sin 2\Phi \\ -\sin 2\Phi & \cos 2\Phi \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\end{aligned}$$

The common pre-factor $-e^{i\alpha_o}$ does not play a role for the polarization and can be ignored. The two matrix operations contain the action of a half wave plate. It first reflects the polarization at the x-axis and a subsequently rotates the polarization by 2Φ. Since the input beam is polarized along the x-axis (this is how the x-axis was defined) the reflection has no influence on the polarization. What remains is the rotation by twice the rotation angle of the plate.

$$\vec{E}_{out} = \begin{pmatrix} \cos 2\Phi & \sin 2\Phi \\ -\sin 2\Phi & \cos 2\Phi \end{pmatrix} \cdot \vec{E}_{in}$$

This operation is equivalent to a reflection at the optical axis.

- Quarter wave plate

  The relative phase change of a quarter wave plate is $\pi/2$,

  $$\alpha_o = \alpha_e + \frac{\pi}{2}$$

  and

  $$e^{i\alpha_e} = -ie^{i\alpha_o}.$$

  For the matrix we now obtain

  $$M_{\lambda/4} = e^{i\alpha_o}\begin{pmatrix} -i\cos^2\Phi + \sin^2\Phi & (1+i)\sin\Phi\cos\Phi \\ (1+i)\sin\Phi\cos\Phi & -i\sin^2\Phi + \cos^2\Phi \end{pmatrix}.$$

  We look at the special case of $\Phi = \frac{\pi}{4}$. The products of the trigonometric function all become $\frac{1}{2}$

  $$\begin{aligned}
  \begin{pmatrix} -i\frac{1}{2} + \frac{1}{2} & (1+i)\frac{1}{2} \\ (1+i)\frac{1}{2} & -i\frac{1}{2} + \frac{1}{2} \end{pmatrix} &= \frac{1}{2}\begin{pmatrix} 1-i & 1+i \\ 1+i & 1-i \end{pmatrix} \\
  &= e^{-i(\pi/4)}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}.
  \end{aligned}$$

  This means that the an input vector parallel to the x-axis gets circularly polarized

  $$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}.$$

The y-component of the output vector contains a phase factor $i = e^{i\pi/2}$ i.e. the phase of the y-component is shifted by 90° relative to the x-axis. By rotating the plate in the opposite direction, $\Phi = -45°$, one obtains circular polarization with the opposite sense of rotation,

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix}.$$

- Zero order and multiple order retardation plates

  A mechanically stable plate must have a minimum thickness of several 100 μm. Each polarization component therefore collects a multiple of 2π in addition to the desired phase (multiple order). This phase offset is a problem because the n_0 and n_e vary with temperature in a different ways. The phase difference becomes very temperature dependent and a $\lambda/4$-plate may drift into a $\lambda/2$ plate and vice versa.

  This problem is solved by zero order wave plates which consist of two plates glued together with optical axes tilted by 90° relative to each other. For plates of the same thickness the phase offset and the relative phase shift would both cancel exactly. If, however, the thickness of the plates is slightly different, the desired phase shift can be achieved while most of the phase offset still cancels.

- Stockes parameter and Poincare-sphere

  The polarization of a light field is typically described by means of the so called Poincare-sphere. In general arbitrarily polarized light can be written as

$$\vec{E} = a_1\vec{e}_{\|}\cos(kx - \omega t + \delta_1) + a_2\vec{e}_{\perp}\cos(kx - \omega t + \delta_2).$$

  The values of a_1, a_2, δ_1, and δ_2 determine whether the polarization is circular or linear. It is useful to combine these parameter and define the Stokes parameters

$$\begin{aligned} s_0 &:= a_1^2 + a_2^2 \\ s_1 &:= a_1^2 - a_2^2 \\ s_2 &:= 2a_1a_2\cos(\delta) \\ s_3 &:= 2a_1a_2\sin(\delta) \\ \delta &:= \delta_2 - \delta_1. \end{aligned}$$

  The intensity of the light field is proportional to

$$I \sim \vec{E}^2 = a_1^2 + a_2^2 = s_0$$

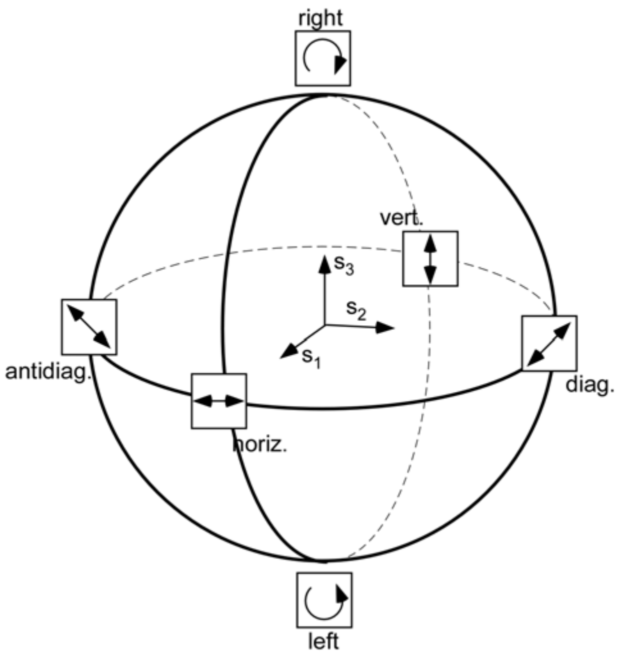
  Therefore the parameter s_0 does not change with the state of polarization. Furthermore

$$s_0^2 = s_1^2 + s_2^2 + s_3^2.$$

This means that the vector

$$\vec{s} = \begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix}$$

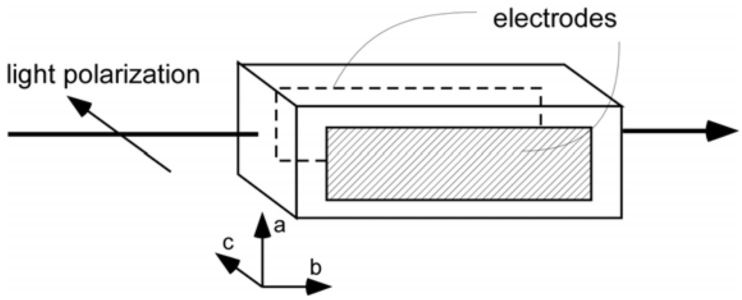
points to the surface of a sphere with fixed radius s_0. If we normalize $\vec{s}$ to its length s_0, the radius of this so called Poincare-sphere is 1. It is used to visualize the polarization state of the wave. If $\vec{s}$ points to the north pole or the south pole the light is circularly polarized with right rotation or left rotation, respectively. The equator corresponds to linearly polarized light. As the tip of $\vec{s}$ travels along the equator the polarization changes its direction from vertical to antidiagonal to horizontal to diagonal.



4.2 Optical Modulators

- Electrooptic modulator (EOM)

  There are crystals which change their refractive index if placed in an external electric field.



The primitive axis that span the crystal lattice are denoted a, b, c. The external electric field is generated by two electrodes and points along the c-axis. The refractive index is changed only for light that is linearly polarized also along the c-axis i.e. parallel to the external electric field. The phase accumulated by passing the crystal thus depends on the strength of the external field. If this field is varied in time, the phase of the light changes accordingly. Such a device is called electrooptic phase modulator. Light that is linearly polarized along the x- or y-axis remains unaffected.

- Half wave voltage

  The index change Δn is proportional to the applied voltage and the distance d between the electrodes,

  $$\Delta n = rn^3 \frac{1}{2} \underbrace{\frac{U}{d}}_{\text{field}} .$$

  The material constant r is called 'electrooptic coefficient'. Typical values are $r = 11\,\text{pm} / \text{V}$ and $n = 1.51$ (for a KDP crystal). The 'half wave voltage' $U_{\lambda/2}$ is defines as the voltage required to shift the phase by π.

  $$\Delta\varphi = \Delta n \cdot k \cdot L = \left(rn^3 \frac{1}{2} \frac{U_{\lambda/2}}{d} \right) \cdot k \cdot L = \pi \ .$$

  Here, k is the wave vector of the light and L the length of the crystal. Solving for the half wave voltage yields

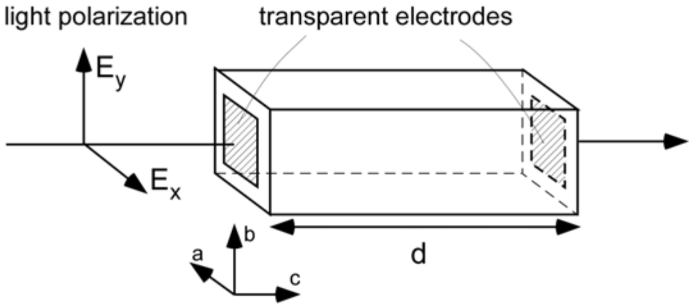
  $$U_{\lambda/2} = \lambda \cdot \frac{d}{L} \cdot \frac{1}{rn^3}$$

  For a crystal with a ratio $L/d = 10$ and the wavelength of the helium neon laser ($\lambda = 633nm$) the half wave voltage is $U_{\lambda/2} \simeq 1700V$. Such high voltages are typical.

- Transparent electrodes

  The electric field and the polarization of the light do not have to be parallel. For example, there are crystals where the refractive index also changes when the direction of the applied electric field is perpendicular to the polarization of the light beam and parallel to the c-axis of the crystal. For horizontal and vertical polarizations of the light field, the pre-factor now has a opposite sign.

$$\begin{aligned}\Delta n_x &= -\frac{1}{2} r n^3 \frac{U}{d} \\ \Delta n_y &= -\Delta n_x.\end{aligned}$$



The half wave voltage $U_{\lambda/2}$ is now independent of the crystal length

$$\Delta\varphi_y = \underbrace{\underbrace{r n^3 \frac{1}{2} \frac{U_{\lambda/2}}{d}}_{\text{index change}} \cdot \quad k \cdot d}_{\text{accumulated phase}} = \pi$$

- Polarization modulator

  For light linearly polarized along the diagonal between the x- and y-axis, the x- and y-component are phase shifted with opposite sign. If the relative phase shift is $\pi/2$ the polarization is rotated by 90°. The modulator is now a switchable half wave plate. The required voltage is given by the condition

  $$2 \cdot r n^3 \frac{1}{2} \frac{U}{d} \cdot \frac{2\pi}{\lambda} \cdot d = \frac{\pi}{2},$$

  resulting in

  $$U_{\lambda/2} = \frac{\lambda}{4 r n^3}.$$

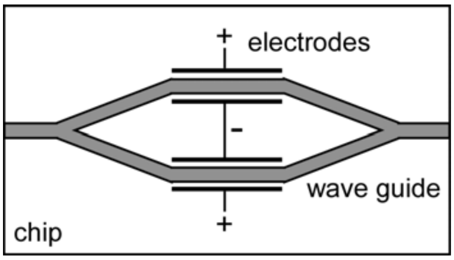
  For KDP and a red helium neon laser one obtains $U_{\lambda/2} = 2100V$, which is quite high. Usually four of such modulators are used in a row such that each modulator only needs $U_{\lambda/2}/4 = 525\,\mathrm{V}$. This is more realistic. Nevertheless, electric drivers which can switch such a voltage in short time are expensive and nontrivial.

- Amplitude modulator

  Between two crossed polarizers an EOM acts like a fast amplitude switch. Rise times in the range of nano seconds are possible (see exercises).

- Amplitude modulator with optical waveguides

  Much lower half-wave voltages are obtained by integrated waveguides. In a nonlinear crystal, a waveguide is created e.g. by in-diffusion of titanium, which changes the refractive index locally. With such wave guides one can construct an interferometer that is tuned by an electric field with reversed polarity in the two arms. It shifts the phase of light in both branches in opposite directions.

  The phase shift switches the interferometer from constructive to destructive interference at the output splitter. Since the distance between the electrodes is very small, the half-wave voltage is reduced to a few volts. Fast intensity switching (within less than 1ns) is possible without expensive electronics. Such switches are used to generate light pulses for optical data communication.

- Phase modulator

  Electrooptic modulators are often used as phase modulators to generate optical sidebands. Such a "sideband" is a frequency component of the laser light in addition to the main frequency the so called "carrier". As in fm-radio technology a sideband can carry information and transport it either through air or via glass fibers.

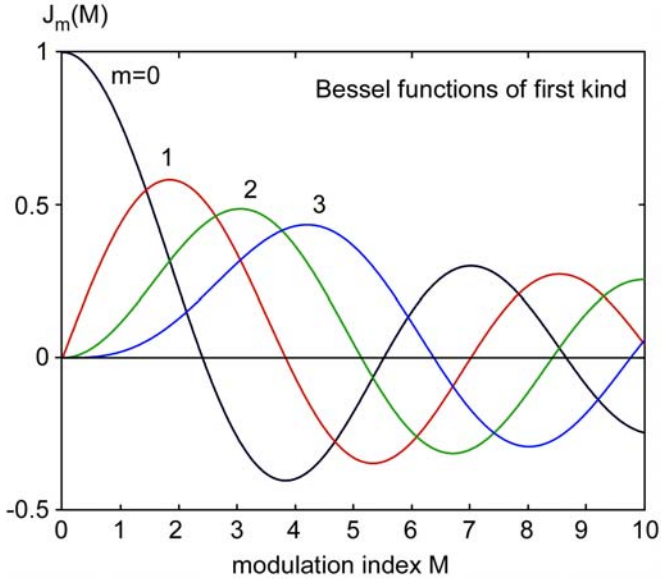
  We take a closer look at a phase modulated light field. The phase is modulated with a cosine-oscillation at a frequency Ω and a maximum phase shift of M

  $$E(t) = E_0 \cdot e^{i(\omega t + M\cos\Omega t)}.$$

  M is called "modulation index". The Fourier expansion can be written as (Jacobi-Anger-expansion):

  $$e^{i(\omega t + M\cos\Omega t)} = J_0\left(M\right)e^{i\omega t} + 2\sum_{n=1}^{+\infty} i^n \cdot J_n(M) \cdot \cos\left(n\Omega t\right) \cdot e^{i\omega t}$$

  with $J_n\left(M\right)$ being the Bessel functions of first kind.



The electric field is the real part,

$$\begin{aligned} E(t) &= E_0 \cdot \mathrm{Re}\left(J_0(M)\, e^{i\omega t} + 2\sum_{n=1}^{+\infty} i^n \cdot J_n(M) \cdot \cos(n\Omega t) \cdot e^{i\omega t}\right) \\ &= E_0 \cdot \left(J_0(M) \cdot \cos\omega t + 2\sum_{n=1}^{+\infty} J_n(M)\,\mathrm{Re}\left(e^{i\omega t + in\frac{\pi}{2}} \cos(n\Omega t)\right)\right) \end{aligned}$$

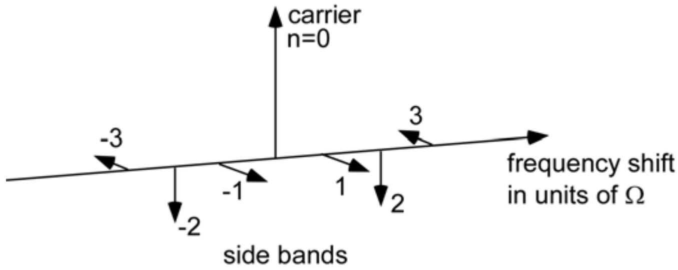
Using

$$\begin{aligned} \mathrm{Re}\left(e^{i\omega t + in\frac{\pi}{2}} \cos(n\Omega t)\right) &= \mathrm{Re}\left(e^{i\omega t + in\frac{\pi}{2}} \frac{1}{2}\left(e^{in\Omega t} + e^{-in\Omega t}\right)\right) \\ &= \frac{1}{2}\mathrm{Re}\left(e^{i(\omega+n\Omega)t + n\frac{\pi}{2}} + e^{i(\omega-n\Omega)t+n\frac{\pi}{2}}\right) \\ &= \frac{1}{2}\cos\left((\omega+n\Omega)\,t + n\frac{\pi}{2}\right) + \frac{1}{2}\cos\left((\omega-n\Omega)\,t + n\frac{\pi}{2}\right) \end{aligned}$$

we obtain

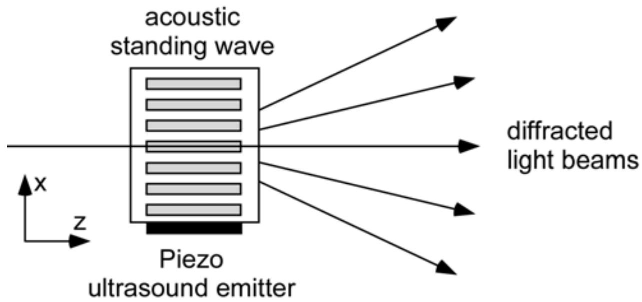
$$E(t) = E_0 \cdot \left(J_0(M) \cdot \cos\omega t + \sum_{n=1}^{\infty} J_n(M)\left(\cos\left((\omega+n\Omega)\,t + n\frac{\pi}{2}\right) + \cos\left((\omega-n\Omega)\,t + n\frac{\pi}{2}\right)\right)\right).$$

The first term is the carrier with amplitude $J_0(M)$. The sum contains pairs of side-bands with equal amplitude $J_n(M)$ and shifted in frequency by $\pm n\Omega$ relative to the carrier. The sidebands are phase shifted by $n \cdot 90°$ relative to the carrier. Odd sideband are thus shifted by $\pm 90°$. Even side bands are shifted by $0°$ or $180°$.



- Acousto optic deflector (AOD)

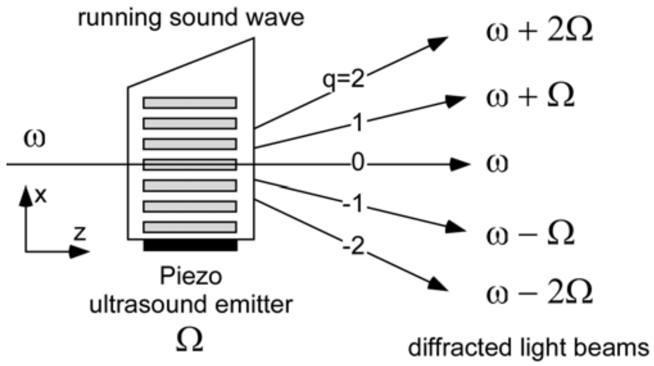
  Another way to control light is by optical diffraction from acoustic waves.



  In a transparent solid (glass, crystal), a piezoelectric material generates an ultrasonic wave at a fixed frequencies in the range of 10 MHz to 500 MHz. The associated density modulation leads to a modulation of the refractive index. This modulation acts like a diffraction grating.

  If the solid is cut such that the acoustic wave is reflected back at the end of the solid, a standing acoustic wave forms and the light is diffracted at a stationary lattice. The different diffraction orders have the same frequency as the input beam and one obtains a fast switch that deflects the light into various directions. Due to the limited velocity of the acoustic wave inside the solid the switch reacts with a delay of some micro seconds. The rise time depends on the size of the focus in the crystal and can be as fast as a few 10 ns.

- Acousto optic modulator (AOM)



You can also create running waves in the crystal, if the solid is cut at an angle. The acoustic wave is reflected towards the side and diffuses inside the solid without forming a standing wave. The initial piezo generated running density wave however generates diffraction orders with shifted frequencies. This can be seen by transformation into the rest frame of the sound wave, which moves perpendicular to the light beam along the x-direction. For plane light waves incident along of the z-axis the transformation does not change the optical wave and one still obtains diffraction from a standing lattice without frequency shift. In this frame the wave vectors of different diffraction orders are

$$\vec{k}_q = \begin{pmatrix} q \cdot K \\ 0 \\ k \end{pmatrix}$$

with the wave number K of the acoustic wave and the diffraction order q. The Modulation frequency is connected to K via the sound velocity v

$$\Omega = v \cdot K.$$

The electric field of the qth order is a plane wave with propagation direction along $\vec{k}_q$

$$E \simeq \exp\left(i\vec{k}_q\vec{r} - i\omega t\right).$$

Back-transformation into the lab system

$$r'_x = r_x - vt$$

yields

$$\begin{aligned}
\exp\left(i\vec{k}_q\vec{r} - i\omega t\right) &\rightarrow \exp\left(i\vec{k}_q\vec{r}' - i\omega t\right) \\
&= \exp\left(i\vec{k}_q\vec{r} - q \cdot K \cdot vt - i\omega t\right) \\
&= \exp\left(i\vec{k}_q\vec{r} - i\left(\Delta\omega + \omega\right)t\right).
\end{aligned}$$

Obviously the frequency is shifted by

$$\Delta\omega := q \cdot K \cdot v \ .$$

And with

$$v = \frac{\Omega}{K}$$

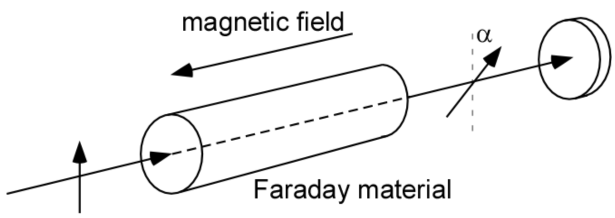
we finally obtain

$$\Delta\omega = q \cdot \Omega.$$

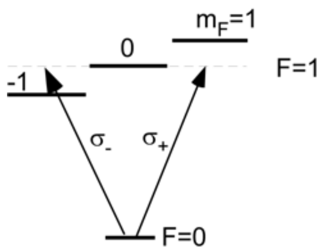
The AOM shifts the frequencies with a very high precision limited only by the electronic signal that generates the sound wave. Typical AOM's shift the frequency by $\Omega = 20 - 200\,\mathrm{MHz}$. An AOM can also control the light intensity by varying the amplitude of the sound wave.

- Faraday Isolator

  Often one wants to prevent light from getting back into the laser and disturbing it. Especially diode lasers are particularly sensitive to tiny feedback from elements placed somewhere down the laser beam. "Optical diodes" use the Faraday effect to filter out such reflections.

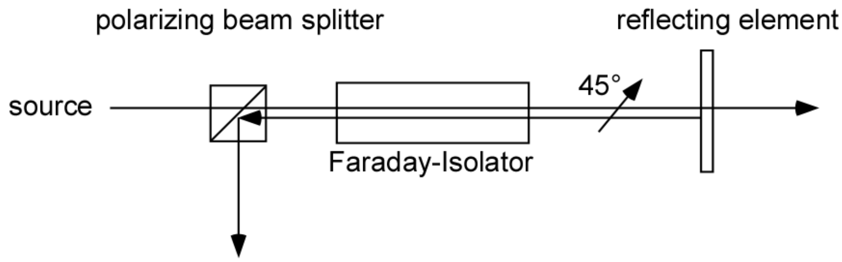


  A linear polarization is rotated by an angle $\alpha = V \cdot B \cdot l$ when passing a crystal, with Verdet constant V, crystal length l, and magnetic field B. The magnetic field generates a difference in the refractive indices for right and left circular light. For explanation we consider the analogy of an atomic gas in a cell with the following term scheme,



The atoms are in the state $F = 0$. The light is linear polarized perpendicular to the quantization axis and thus consists of two left and a right circularly polarized part which drive σ_-- and σ_+-transitions. Due to the Zeeman effects the two parts are blue- and red-detuned relative to the resonance frequency at $B = 0$. According to the Lorentz-model the refractive index is increased for red detuned light and decreased for blue detuned light. Behind the crystal the two circular components are thus phase shifted relative to each other and together they generate a linearly polarized light with its orientation turned by the angle α. If the light moves in the direction of the magnetic field, the polarization turns right. If the B-field changes direction, the light turns left. Similar physics also happens in crystals. Such devices are called "Faraday rotators".

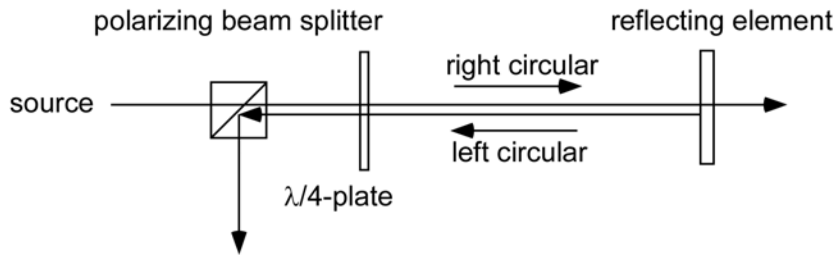
In the setup shown below the magnetic field is adjusted such that $\alpha = 45°$. Reflected light passes the Faraday rotator twice in opposite directions relative to the magnetic field orientation and is therefore turned by $2\alpha = 90°$. The light is then reflected at the polarizing beam splitter and cannot go back into the source.



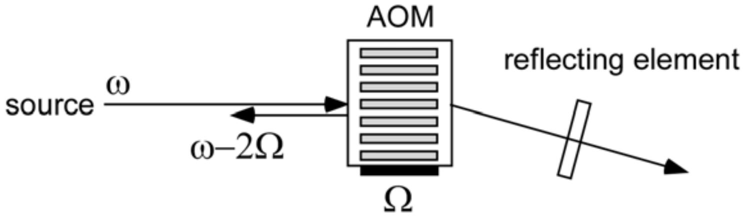
This setup is called "optical diode" or "Faraday isolator".

- Optical isolator based on polarization optics

  If circular polarized light is acceptable at the output one can construct an optical isolator also with polarizing elements alone.



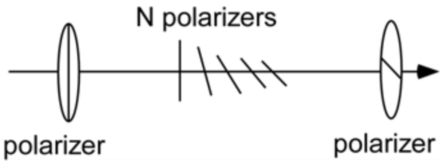
- Isolator based on an AOM



The unwanted reflection is shifted by twice the AOM modulation frequency. Sometimes such shifted light does not perturb the source any more. For instance, if the source is a laser with a high finesse resonator the shifted light is detuned from the resonator and cannot enter it any more. However, the diffraction efficiency of an AOM is typically not much better than 60% and a significant amount of light gets lost by diffraction into other orders.

- Rotating polarization by using filters

  Two crossed polarizer do not transmit light. If a third diagonally oriented polarizer is placed between them 25% of the input power is transmitted. If one uses N gradually tilted polarizers the transmission may be much higher.



In the limit of $N \to \infty$ the transmission is 100%. To see that, we discuss a simple model. In the vertical-horizontal basis a single polarizer is described by the matrix

$$M = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

This matrix obviously eliminates the y-component of the light field. Similar to our reasoning in connection with retardation plates, a rotated polarizer is described by

$$\begin{aligned} M &= D(-\theta) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} D(\theta) \\ &= \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \\ &= \begin{pmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{pmatrix} \end{aligned}$$

The Matrix rotates a vertically polarized input beam by the angle θ,

$$\vec{E}' = M \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \cos^2\theta \\ \cos\theta\sin\theta \end{pmatrix} = \cos\theta \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix},$$

and the transmitted intensity is reduced by

$$\begin{aligned} |E|^2 &= |E_x|^2 + |E_y|^2 = \cos^4\theta + \cos^2\theta\sin^2\theta \\ &= \cos^2\theta(\cos^2\theta + \sin^2\theta) \\ &= \cos^2\theta \sim 1 - \theta^2 + ... \end{aligned}$$

(Here, we assume the field vector to be of length 1). For small angles θ the intensity reduction is proportional to θ^2. If we use N polarizers to generate a total polarization rotation of $\theta_g = 90°$ each polarizer is tilted relative to its neighbor by the tilt angle θ such that the sum of all θ equals $\pi/2$,

$$\begin{aligned} \theta_g \stackrel{!}{=} \frac{\pi}{2} &= \sum_{i=1}^{N} \theta = N\theta \\ \theta &= \frac{1}{N}\frac{\pi}{2} \end{aligned}$$

The transmitted intensity can be calculated as

$$E|_g^2 = \left(1 - \theta^2\right)^N$$

In the limit $N \to \infty$ we obtain

$$\begin{aligned} |E|_g^2 &= \lim_{N\to\infty} \left(1 - \theta^2\right)^N \\ &= \lim_{N\to\infty} \left(1 - \left(\frac{\pi}{2}\frac{1}{N}\right)^2\right)^N \\ &= \lim_{N\to\infty} \exp\left(N \cdot \ln\left(1 - \left(\frac{\pi}{2}\frac{1}{N}\right)^2\right)\right) \end{aligned}$$

We expand the logarithm in the limit of large N i.e. small $1/N$,

$$\ln(1 - x) \simeq -x + ...$$

and obtain

$$\lim_{N\to\infty} \left(N \cdot \ln\left(1 - \left(\frac{\pi}{2}\frac{1}{N}\right)^2\right)\right) = -N\left(\frac{\pi}{2}\frac{1}{N}\right)^2 = -\left(\frac{\pi}{2}\right)^2 \frac{1}{N} = 0.$$

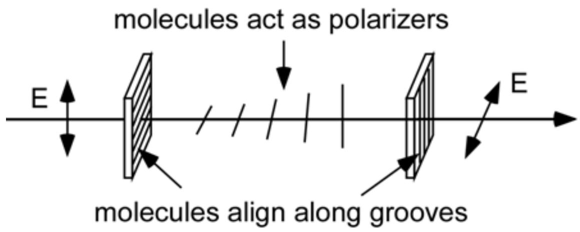
The transmitted intensity now becomes

$$\lim_{N\to\infty} |E|_g^2 = \exp(0) = 1.$$

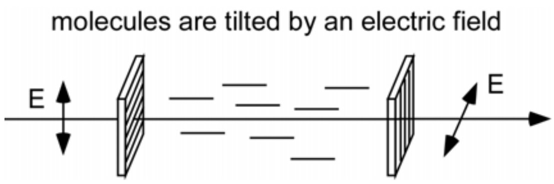
In the limit of many filters each tilted by a small angle relative to its neighbors the total transmission approaches 1!

- Liquid-Crystal-Display (LCD-display)

  Long chain molecules tend to line up parallel to each other and to grooves, which have been added to the surfaces of two crossed polarizers. If the two set of grooves are oriented orthogonal, the molecules form a spiral. This spiral acts like a series of polarizers, which rotates the light polarization. The light can pass the crossed polarizers and the pixel is bright.



  An additional electric field that points along the optical axis forces the molecules to turn and line up parallel to the field. They stop to act as polarizers and the pixel is dark now.



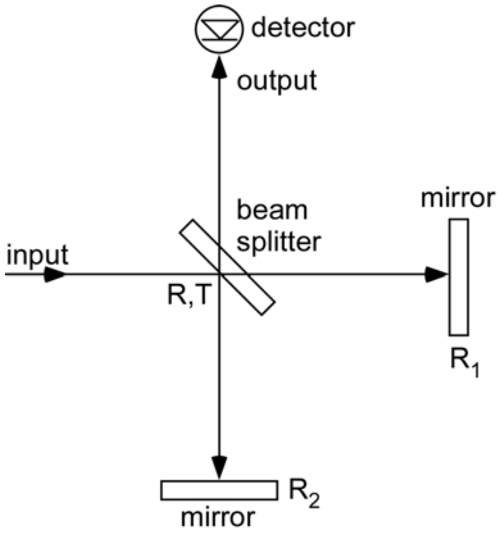
  With many of such modulators arranged in a plane one obtains a device that can display spatial patterns, numbers, pictures etc. Such kind of displays are found in all kind of electronic devices including computer screens and beamers.

- SLM

  Without polarizers the molecules acts as a medium with variable refractive index. Arranged in a plane one can control the wave front curvature of an incident light beam and generate various diffraction patterns. Such devices are called spacial light modulators (SLM).

4.3 Interferometers

- Michelson interferometer



In a Michelson interferometer the beam splitter at the center divides the input light beam into two beams and sends them to the two mirrors. When they return to the beam splitter the two beams recombine and interfere. The power reflectivities of the mirrors are R_1 and R_2 and the power transmittivity and power reflectivity of the beam splitter are T and R, respectively. We assume no losses such that $R = 1 - T$.

At the detector the beam that hits the mirror R_1 has an amplitude

$$E_{in}\sqrt{T}\sqrt{R_1}\sqrt{R}$$

The beam that hits mirror R_2 has an amplitude

$$-E_{in}\sqrt{R}\sqrt{R_2}\sqrt{T}e^{i\varphi}$$

The phase φ is the relative phase between the two beams at the point of interference at the beam splitter. Such a phase difference may be due to a different distance of the mirrors from the splitter. The negative sign of the second amplitude is caused by the phase jump during reflection at an optically thicker medium. The sum of both amplitudes at the detector is

$$E_{out} = E_{in}\sqrt{T}\sqrt{R_1}\sqrt{R} - E_{in}\sqrt{R}\sqrt{R_2}\sqrt{T}e^{i\varphi} = E_{in}\sqrt{TR}\left(\sqrt{R_1} - \sqrt{R_2}e^{i\varphi}\right)$$

From the field we calculate the intensity,

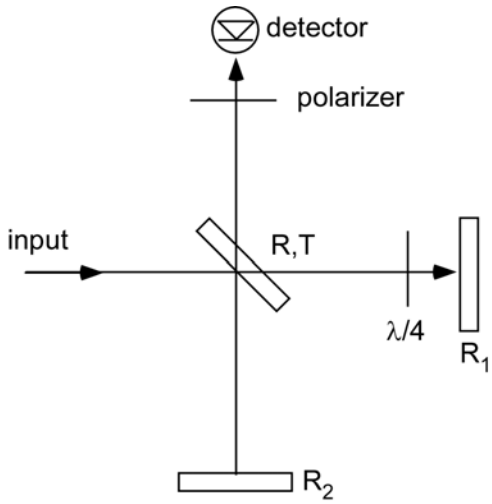
$$\begin{aligned}\frac{I_{out}}{I_{in}} &= TR\left(\sqrt{R_1} - \sqrt{R_2}e^{i\varphi}\right)\left(\sqrt{R_1} - \sqrt{R_2}e^{-i\varphi}\right)\\ &= TR\left(R_1 + R_2 - 2\sqrt{R_1R_2}\cos\varphi\right).\end{aligned}$$

One can define the "contrast" as the difference between the minimum and maximum signal normalized to its sum,

$$
\begin{aligned}
K \quad : &= \frac{I_{out}\left(\varphi_{\max}\right) - I_{out}\left(\varphi_{\min}\right)}{I_{out}\left(\varphi_{\min}\right) + I_{out}\left(\varphi_{\max}\right)} \\
&= \frac{\left(R_1 + R_2 + 2\sqrt{R_1 R_2}\right) - \left(R_1 + R_2 - 2\sqrt{R_1 R_2}\right)}{\left(R_1 + R_2 + 2\sqrt{R_1 R_2}\right) + \left(R_1 + R_2 - 2\sqrt{R_1 R_2}\right)} \\
&= \frac{2\sqrt{R_1 R_2}}{R_1 + R_2}.
\end{aligned}
$$

The contrast has a maximum value of $K = 1$ for $R_1 = R_2$. The contrast does not depend on the beam splitter i.e. the value of R. The maximum signal however is obtained only for $T = R = 1/2$.

- Quantum eraser



With an additional $\lambda/4$-plate in front of one of the two mirrors the contrast disappears, because at the detector the light fields from the two arms are orthogonally polarized and do not interfere. However, with a polarizer at 45° in front of the detector, one will recover an interference signal, because the two perpendicular polarizations are projected by the polarizer to a common diagonal direction.

The quantum-mechanical aspect becomes interesting when the source is made so weak that there is only one photon in the interferometer at a time. Whether interference occurs or not is apparently only decided by the type of detection. If you destroy the "which way" information with the polarizer, interference takes place again.

- Ether hypothesis and experiment of Michelson and Morley

In the experiment by Michelson and Morley, it was initially assumed that light propagates in a hypothetical medium, the so called ether.

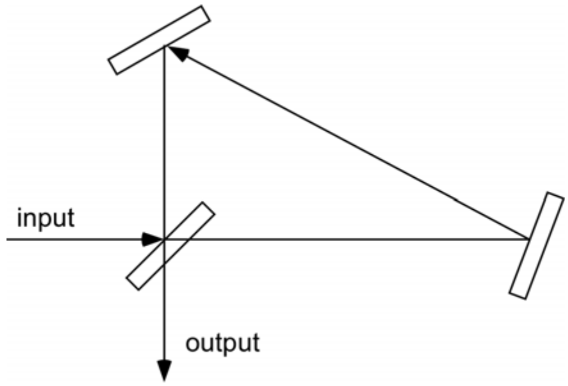
We assume that there is an ether and we look at a Michelson interferometer that moves at constant speed parallel to one of its arms. After one round trip of the light in this arm, the beam splitter and the mirror has moved such that the total round trip time is longer than for an interferometer at rest. It is longer because during the first part of the round trip the mirror moves away from the light and increases the time it takes to reach the mirror. On the way back this delay is partly compensated because now the beam splitter moves towards the light. However, since the return time is shorter, the beam splitter does not move the same distance as the mirror moves in the first part of the journey.

In the perpendicularly oriented arm, the round trip time is also a bit longer because the beam splitter moves sideways and therefore the light travels a longer triangular path. However, this effect is a bit smaller, and there remains a the net difference between the two arms.

If this net time difference is expressed in an effective path length differences, one expected about half a wavelength in the particular geometry of Michelson's experiment. He assumed that the velocity at which the interferometer moves inside the ether is given by the velocity with which the earth rotates around the sun (30km/s). By alternateingly orienting each arm parallel to the motion of the earth, one would expect a shift of the interference pattern from minimum to maximum. Based on the motion of the solar system around the center of the galaxy (220km/s) or even the cosmic background radiation (368 km/s), the signal should be much larger and a rotation should lead to the passage of several maxima.

In fact, Michelson and Morley did not observe any dependence on the orientation of the interferometer. The resolution of the original experiment of 1887 set an upper limit for the interferometer velocity relative to the ether of about 5km/s. As an explanation one simply assumed that close to its surface the earth dragged the ether with it, which reduces the relative velocity. With the help of Einstein's special relativity theory of 1905, the experiment could be explained without the ether hypothesis.

- Sagnac interferometer



The right and the left circulating path always interfere destructively at the exit! This is due to the 180° total phase jump connected to the reflections in the clockwise rotating path. However, there is a signal at the output when the interferometer is rotated as a whole. Then the orbital path in the direction of rotation is longer, since the beam splitter as the starting point of the path continues to move during the orbit. Accordingly, the distance in the opposite direction of rotation is effectively shorter. This results in an oscillating signal at the output, whose frequency is a measure for the rotational acceleration. The phase shift between the two directions of rotation is proportional to the angular velocity of the rotation Ω and inversely proportional to the frequency of the light frequency ω

$$\varphi = 4\frac{\Omega}{\omega}A.$$

(here without derivation). However, the area A enclosed by the interferometer must be quite large for the signal to be measurable.

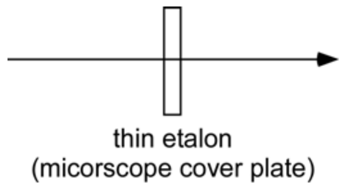
If, on the other hand, a ring laser is rotated, which emits in both directions of rotation simultaneously, the shortened or extended paths lead to different frequencies for the two modes. The beat of the two output beams oscillates with a frequency that is proportional to the angular frequency of the rotation. Since frequencies can be measured very accurately, one obtains a very useful and mechanics-free device for measuring rotation. Modern jet planes are equipped with such "laser gyroscopes" as navigation instruments.

- Frequency filters in laser resonators

  Laser typically emit on several resonator modes simultaneously. Single mode operation is possible with frequency filters inside the laser resonators that suppress unwanted modes. Here some examples.

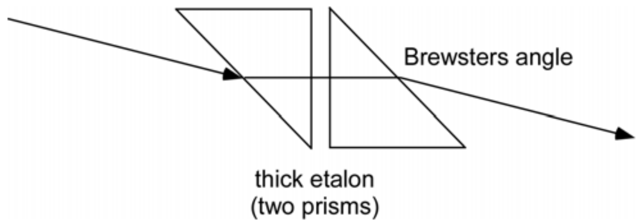
  Thin etalon:

  Thin glass plates as used in microscopy can be used as interference filter with very large free spectral range on the order of several hundred Gigahertz. Due to the small reflectivity of the glass surface of only 4% the directly transmitted light interferes with a small fraction that survives a full round trip inside the glass plate. The contrast of the filter is not very high but in many cases sufficient to push the unwanted modes below laser threshold. The filter can be tuned by tilting. The associated transverse shift of the optical beam can be neglected.



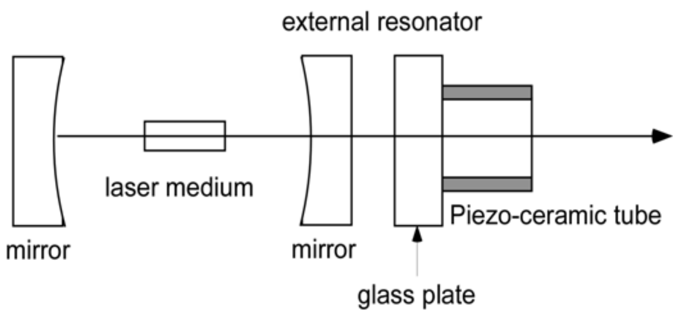
Thick etalon:

Filter with smaller free spectral range can be constructed from two glass prisms whose distance is tuned with a piezoelectric actuator.



Frequency selective output coupler:

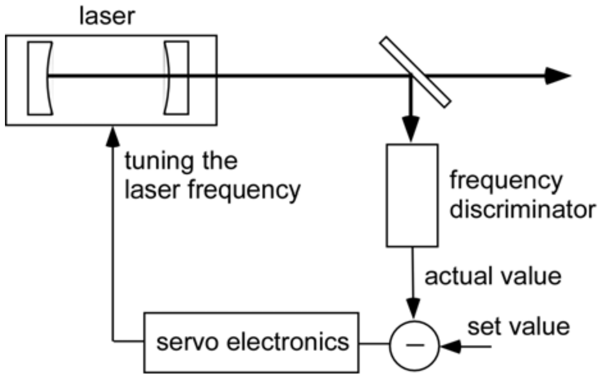
The transmission of the output coupling mirror can be changed with an additional glass plate positioned outside the resonator. Together with the output coupler the glass plate forms a low finesse resonator. The laser oscillates at frequencies where the losses are smallest i.e. the anti resonance of the output resonator. This resonator can be tuned by a Piezo element.



4.4 Stabilizing the frequency of a laser

- Feedback loop

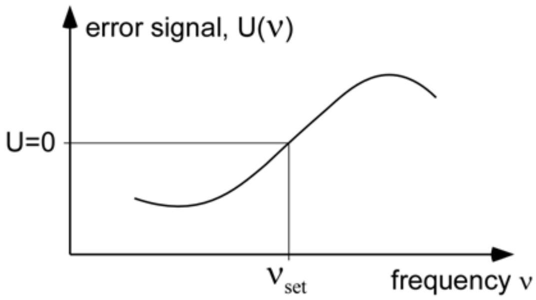
  The emission frequency of a laser is determined by the resonance frequency of its resonator. If the length of a round trip path changes by one wave length the laser varies its frequency by one free spectral range (GHz-range). To resolve the natural line width of a typical atom (MHz-range), each mirror must therefore be mechanically stable within a very small fraction of a wave length. This can usually be accomplished only be means of an active stabilization: A frequency discriminator detects the deviation of the actual laser frequency from the wanted "set value". An electronic feed back loop (sometimes called "servo loop") tunes the laser until the difference between the actual and the set value is zero.



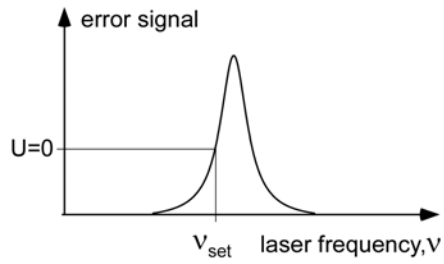
  In this section we discuss the details of such a setup.

- Error signal

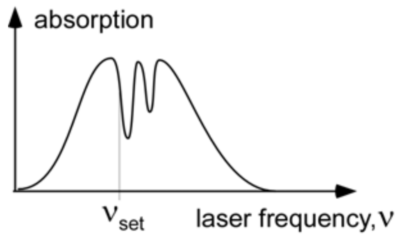
  The frequency discriminator generates an "error signal" which typically is a voltage $U(\nu)$ that is proportional to the difference of the actual laser frequency ν and the wanted set value ν_{set}. At ν_{set} the function $U(\nu)$ must have a nonzero slope which tells the servo which way to go with ν to reach ν_{set}.



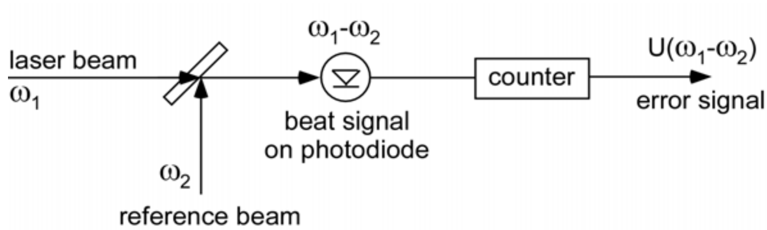
The maximum of some resonance (atomic, molecular, resonator whatever) would not work as discriminator since for positive and negative deviation the error signal would be the same. However the slope of a resonance would do.



An example could be the saturation spectrum of an alkali metal vapor



As reference one can also use another laser beam. The two beams are overlapped on a fast photo diode.



The photo current is proportional to the intensity of the total field,

$$\begin{aligned} I &= E^2 = (E_1 \cos \omega_1 t + E_2 \cos \omega_2 t)^2 \\ &= (E_1 \cos \omega_1 t)^2 + (E_2 \cos \omega_2 t)^2 + 2E_1 E_2 \cos \omega_1 t \cos \omega_2 t \\ &= (E_1 \cos \omega_1 t)^2 + (E_2 \cos \omega_2 t)^2 + E_1 E_2 \cos(\omega_1 t + \omega_2 t) + E_1 E_2 \cos(\omega_1 t - \omega_2 t)\,. \end{aligned}$$

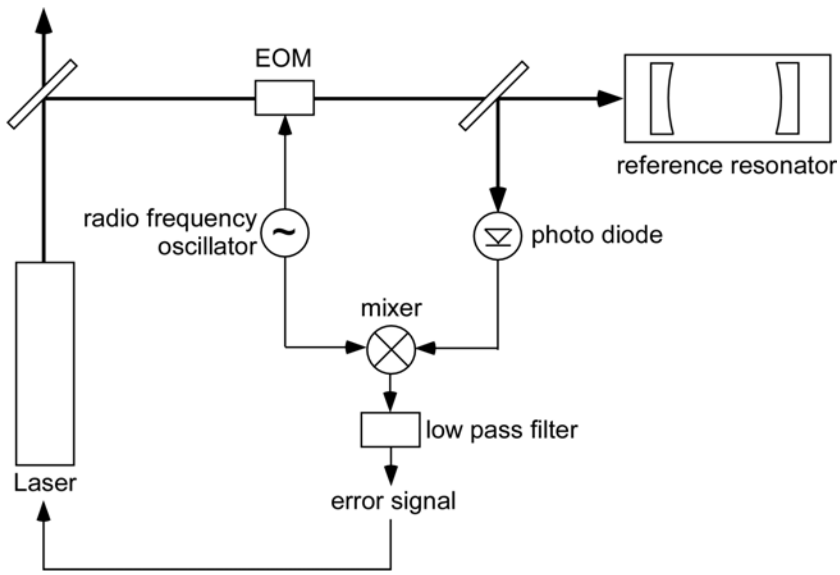
The first three terms oscillate at optical frequencies which even the fastest photodiode cannot resolve. They average out and only the last term generates a detectable time dependant signal. This so called optical beat signal oscillates with the frequency difference between the two beams. It can be counted with an electronic counter. The counter output is used as error signal to stabilizes the frequency difference to a fixed (nonzero) value.

- Amplitude and phase

  Near a resonance not only the amplitude of the oscillation changes but also its phase. This can be seen most easily in an optical resonator. If the laser is detuned relative to the resonator, two successive round trips of the light inside the resonator interfere with a small relative phase difference. The constructive interference is incomplete and the amplitude of the light field inside the resonator is reduced. More importantly, the incomplete interference also changes the phase φ between the total field inside the resonator and the incident laser light! This change has opposite signs at the two sides of the resonance since on one side the wave length is too long and at the opposite side it is too short for perfect constructive interference. At resonance the relative phase φ becomes zero.

  Some of the light that is reflected at the resonator comes from inside the resonator such that φ can be detected by looking at the phase shift of the reflected light relative to the incident light. While tuning the laser across the resonance this phase shift changes. This makes the phase useful as error signal. There are several methods to observe the phase. The best and most commonly used is the Pound-Drever-Hall method.

- Rf-spectroscopy according to Pound, Drever and Hall (1983)



  An electronic radio frequency generator (20-100 MHz) drives an electrooptic modulator (EOM) that modulates the phase of the laser light. The light reflected at the resonator is observed with a photodiode and its electronic signal is multiplied with the signal of the EOM-driver with an electronic mixer. A low pass filter suppresses the oscillating part of the mixer output. The result is a signal that is proportional to the phase of the field inside the resonator.

This needs some explanations. As shown in the above section about modulators, a phase modulator generates sidebands at frequencies $\omega + \Omega$ and $\omega - \Omega$. The modulation frequency Ω should be larger than the width of resonance curve of the reference resonator such that the side band frequencies are far detuned from resonance and do not enter the resonator. They are reflected at the input coupler and guided to the photo diode. In contrary, the carrier at frequency ω is close to the resonance and enters the resonator. The part of the carrier which finds its way back to the photodiode experiences the phase shift φ explained above. The total field at the photodiode is

$$E = E_0 e^{i\omega t + i\varphi} + E_1 e^{i(\omega+\Omega)t + i\pi/2} + E_2 e^{i(\omega-\Omega)t+i\pi/2}.$$

The diode current is proportional to the intensity

$$\begin{aligned} I &\propto EE^* \\ &= |E_0|^2 + |E_1|^2 + |E_2|^2 \\ &\quad + \left(E_0 e^{i\omega t+i\varphi} E_1 e^{-i(\omega+\Omega)t - i\pi/2}\right) + c.c \\ &\quad + \left(E_0 e^{i\omega t+i\varphi} E_2 e^{-i(\omega-\Omega)t - i\pi/2}\right) + c.c. \\ &\quad + \left(E_1 e^{i(\omega+\Omega)t} E_2 e^{-i(\omega-\Omega)t}\right) + c.c. \end{aligned}$$

Obviously, there are three mutual beat signals: between carrier and left side band, between carrier and right side band, and between both side bands. We look at the terms which oscillate with Ω. Since the sidebands are equally strong, $E_1 = E_2 =: E_s$, one can simplify the expression

$$\begin{aligned} I_\Omega &\propto E_0 E_s \left(e^{i\omega t+i\varphi} e^{-i(\omega+\Omega)t-i\pi/2} + c.c.\right) \\ &\quad + E_0 E_s \left(e^{i\omega t+i\varphi} e^{-i(\omega-\Omega)t-i\pi/2} + c.c.\right) \\ &= 2E_0E_s\left(\cos\left(-\Omega t + \varphi - \pi/2\right) + \cos\left(\Omega t + \varphi - \pi/2\right)\right) \\ &= -2E_0E_s\left(\sin\left(-\Omega t + \varphi\right) + \sin\left(\Omega t + \varphi\right)\right) \\ &= -4E_0E_s\cos(\Omega t)\cdot\sin\left(\varphi\right) \end{aligned}$$

The mixer multiplies the signal of the driver which is proportional to $\cos(\Omega t)$ and the signal of the diode. The band pass takes the temporal average and we obtain

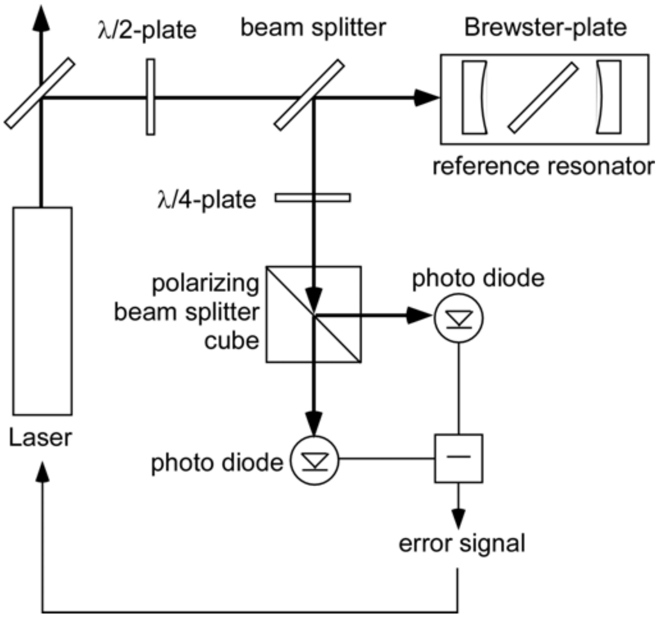
$$\begin{aligned} U &\propto \frac{1}{T}\int_0^T \cos(\Omega t)\cdot\left(-4E_0E_s\cos(\Omega t)\cdot\sin\left(\varphi\right)\right)dt \\ &= -4E_0E_s\sin\left(\varphi\right)\underbrace{\frac{1}{T}\int_0^T \cos^2(\Omega t)dt}_{1/2} \\ &= -2E_0E_s\cdot\sin\varphi \end{aligned}$$

The error signal is proportional to $\sin(\varphi)$ and has the required zero-crossing at the resonance where $\varphi = 0$ and is antisymmetric in φ.

The photo diode signal is only used in a small frequency range around Ω. All other frequency components play no role. This means that the error signal is insensitive to low frequency optical and electronic noise on the diode signal.

- Hänsch-Couillaud-method

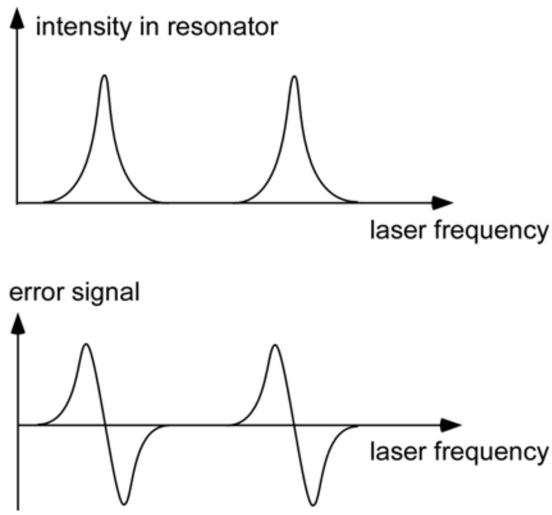
  A second method to observe the phase of an optical oscillator has been proposed by T. Hänsch.



The laser light is linearly polarized. The $\lambda/2$-plate tilts the polarization slightly relative to the Brewster plate in the resonator. The Brewster plate is a glass plate positioned inside the resonator at Brewsters angle. The Brewster plate transmits light with p-polarization (parallel to the plane of incidence) without reflection. In contrary, at Brewsters angle about 17% of s-polarized light is reflected at each glass surface such that 68% of the light leaves the resonator per round trip (4 glass-air interfaces per round trip). Because of these strong losses, the resonator only supports p-polarized light and shifts its phase according to the detuning. The s-polarized component of the incident light is reflected at the input coupler without phase shift.

The remaining optical elements detect the phase difference between the s- and p-polarized light reflected at the resonator. A qualitative understanding is possible if we first assume no phase shift i.e. the laser is in resonance with the resonator. The s- and p-polarized light are oscillating in phase and the light that enters the quarter wave plate is linearly polarized. Behind the quarter wave plate the light is circularly polarized. The polarizing beam splitter cube decomposes the circular light into two equally strong linear components each hitting one of the diodes. After electronically taking the difference of the two diode signals the error signal is zero.

If now the cavity is detuned the s- and p-components are slightly out of phase and the light contains a circular component. This component is transformed into a linear component by the quarter wave plate. The splitter cube sends it to one of the diodes and the electronic difference signal becomes nonzero. Again one obtains an error signal which is zero at resonance and antisymmetric in phase. A full calculation of the error signal is possible by means of the stokes matrices discussed above.



Since polarization optics is sensitive to changes of the ambient temperature, the zero crossing of the error signal may drift and its slope may vary. Therefore, the Pound Drever Hall method is preferred whenever possible.

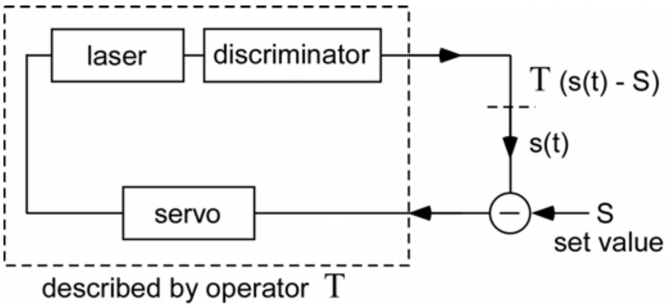
- General servo problem

  Once the error signal is generated it has to be fed back to the laser. The main problem in such feedback loops is its stability. If the signal is fed back with the wrong phase, perturbations are not suppressed but enhanced. Even without external perturbation, loops with positive feed back tend to oscillate starting from tiny noise which is always present. The construction of stable servos thus requires a closer look at the general feed back problem which we do here.

  We start with the signal $s(t)$ provided by the discriminator and call it the "actual value". Its difference from a set value S is electronically generated. This difference enters the "feedback loop", which including an analog or digital servo electronics, the element that controls the laser frequency, and the discriminator that provides the error signal $s(t)$. The action of the feedback loop is mathematically described by the transfer operator $\hat{T}$. This operator can include the derivative of $s(t)$, its integral or any other operation. Here we restrict ourselves to linear transfer operators,

$$\hat{T}\left(\alpha f(t) + \beta g(t)\right) = \alpha \hat{T}\left(f(t)\right) + \beta \hat{T}\left(g(t)\right).$$

After completing the loop the function $s(t)$ is transformed into the function $T(s(t) - S)$.



- Proportional servo control

  In a proportional servo system, the operator $\hat{T}$ is simply given by multiplication with a factor $-G$,

$$\hat{T}(s(t) - S) = -G \cdot (s(t) - S).$$

  The constant G is called servo gain.

  In equilibrium $s(t)$ is reproduced after one full round trip

$$\begin{aligned} s(t) &= \hat{T}(s(t) - S) \\ &= -G \cdot (s(t) - S). \end{aligned}$$

  We solve for $s(t)$ and obtain,

$$s(t) = \frac{G}{1+G} \cdot S.$$

  The actual value equals the set value only if the gain is infinitely large because then

$$\lim_{G \to \infty} \frac{G}{1+G} = 1.$$

  A proportional servo alone is obviously no good solution.

- Integral servo control

  We now try an integrator which transforms a function $f(t)$ into its integral,

$$\hat{T} \cdot f(t) = -G \int_0^t f(t')\, dt'.$$

For our feedback loop we obtain

$$s(t) = \hat{T}(s(t) - S) = -G\int_0^t (s(t') - S)dt'.$$

We look at the derivative of this equation and obtain the differential equation

$$\dot{s} = -G(s - S).$$

In equilibrium $\dot{s} = 0$ and therefor $s = S$ also for finite gain G. The integral servo control is the most simple control that balances the actual value with the set value.

- Perturbations

  Somewhere in the system we have unavoidable perturbation. Maybe one of the laser mirrors shake due to acoustic or seismic noise. We thus add a perturbation term $R(t)$ to the loop,

$$s(t) = -G\int_0^t (s(t') - S)dt' + R.$$

Now we obtain the equation

$$\dot{s} = -G(s(t) - S) + \dot{R}.$$

In equilibrium $\dot{s} = 0$ and consequently

$$s(t) = \frac{\dot{R}}{G} + S.$$

If we assume an oscillating perturbation with frequency ω and amplitude $R(\omega)$

$$R(t) = R(\omega) \cdot e^{i\omega t}$$

we get

$$\dot{R} = i\omega \cdot R(\omega) \cdot e^{i\omega t}.$$

and

$$s(t) = i\frac{\omega}{G}R(\omega)e^{i\omega t} + S.$$

The influence of the perturbation is linear in ω. Low frequency perturbations are less important than high frequency perturbations or, in other words, low frequency noise is linearly suppressed. This is good since most perturbations take place at low frequencies (1/f-noise).

- complex transfer function

  We now describe the system in Fourier-space and write $s(t)$ as a superposition of oscillations $e^{i\omega t}$,

  $$s(t) = \int s(\omega)\, e^{i\omega t} d\omega.$$

  We further require the transfer operator to be diagonal in Fourier space

  $$\hat{T}\left(e^{i\omega t}\right) = T(\omega) \cdot e^{i\omega t},$$

  such that its action is equivalent to the multiplication with the eigenvalues $T(\omega)$,

  $$\begin{aligned}\hat{T}(s(t)) &= \hat{T}\left(\int s(\omega)\, e^{i\omega t} d\omega\right) \\ &= \int s(\omega)\, \hat{T}\left(e^{i\omega t}\right) d\omega \\ &= \int s(\omega)\, T(\omega)\, e^{i\omega t} d\omega.\end{aligned}$$

  The complex eigenspectrum $T(\omega)$

  $$T(\omega) = |T(\omega)|\, e^{i\varphi(\omega)}$$

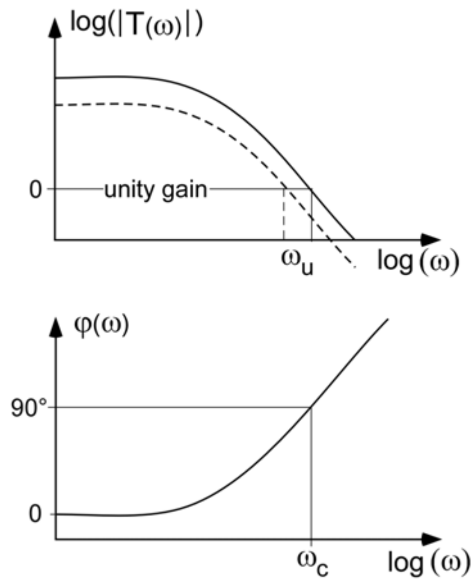
  is called "transfer function". It completely describes the servo loop.

- Stability and maximum gain

  With increasing frequency the feedback loop gets less efficient. If for instance a piezo element is used to control the laser resonator, it can not react any more at high frequencies. Also the servo electronic may be too slow or there is a delay due to the limited reaction time of the system that generate the error signal. Therefor, it is a good assumption that the modulus of the transfer function drops as high frequencies and that the signal accumulates a phase lag $\varphi(\omega)$ after one round trip. It is described by the phase factor of the transfer function

  $$T(\omega) = |T(\omega)|\, e^{i\varphi(\omega)}.$$

  It is helpful to plot the modulus and the phase of the transfer function as function of frequency. Usually $|T(\omega)|$ is plotted in a double logarithmic plot. A typical system may look like this:



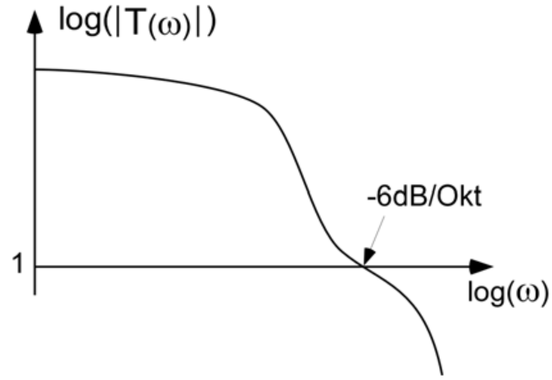
Above a critical frequency ω_c the phase lag φ exceeds $90°$,

$$\varphi(\omega_c) = 90°.$$

At this point the negative feedback becomes positive and the system may start to oscillate if the round trip gain is larger than one, $|T|_{\omega_c} > 1$. The frequency where the modulus of $T(\omega)$ becomes 1 is called "unity gain"-frequency ω_u. The system is stable as long as the unity gain frequency is smaller than the critical frequency, $\omega_u < \omega_c$. The unity gain frequency can be adjusted by the gain of the servo electronics. If the gain is reduced also the unity gain frequency drops (see dashed line in figure above).

- Stability and transfer function

  In control theory the general transfer function is analyzed in order to derive practical stability criteria. We don't go into detail here but only mention some of the results. First one finds that the modulus $|T(\omega)|$ and the phase $\varphi(\omega)$ of the transfer function are not independent. In fact, a certain shape of $|T(\omega)|$ determines the phase lag at various frequencies. As a result it is possible to derive the stability behavior by looking only at $|T(\omega)|$.



In this plot the slope at a given point describes the power at which $|T(\omega)|$ changes with ω. The slope is given in $dB\ /Oct$. A decibel (dB) is a unit that describes the relative change of a "power"-type quantity. It is a tenth of a B (Bel) and xB means that a power quantity has changed by a factor of 10^x,

$$\frac{|T(\omega)|^2}{|T_0(\omega)|^2} = 10^x$$

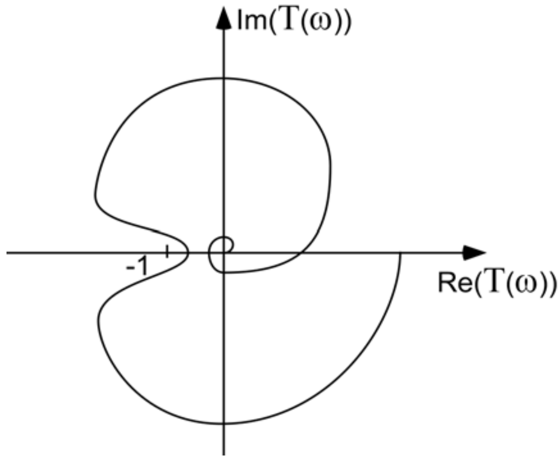
Here, "power" is a general expression for the modulus square of a complex quantity. In our case "power" is the modulus square of the transfer function. Therefore, xB means that $|T(\omega)|$ changes by a factor of $10^{x/2}$

since

$$\frac{|T(\omega)|}{|T_0(\omega)|} = \sqrt{\frac{|T(\omega)|^2}{|T_0(\omega)|^2}} = 10^{x/2}.$$

The unit "octave" (Oct) means that a quantity has changed by a factor of 2. A slope of $-6dB\ /Oct$ thus means that if ω doubles, $|T(\omega)|$ changes by a factor of $10^{-0.6/2} = 0.5012 \simeq 0.5$. The modulus of the transfer function and the frequency are inverse proportional. Control theory now finds that a system is stable if the modulus of the transfer function crosses the unity gain with a slope of $-6dB\ /Oct$. The rest of the transfer function far away from unity gain is not important for stability and can have any shape.

- Another way to visualize the complex $T(\omega)$ is to look at its values in the complex plane for various frequencies. One obtains a curve in the complex plain parameterized by the frequency.

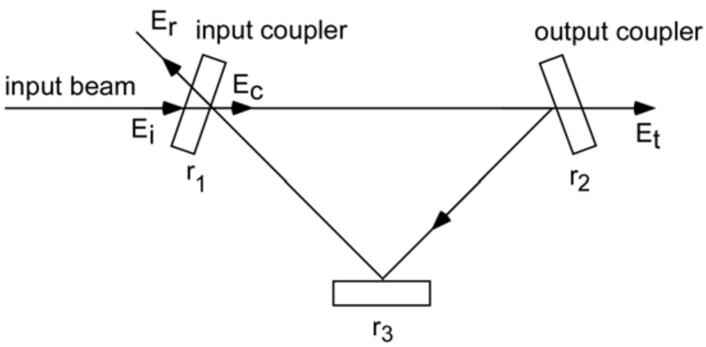


For small frequencies the phase lag φ is zero and T is a point on the real axis. With increasing frequency the negative phase lag generates an imaginary part and the curve starts to rotate clockwise. For very high frequencies the modulus gets smaller since the control system cannot react any more. The curve spirals in and eventually hits the origin. In this picture the stability criteria can be expressed by the condition that the curve does not include the point -1.

5. Optical Resonator and Gaussian Beams

5.1 Intensity and field in optical Resonators (Equilibrium solutions)

We consider a ring resonator with light circulating between a set of mirrors. Such resonators may be combined with optical gain elements which would make it a laser. In this chapter, however, we only consider the empty resonator and ask about the power and the shape of the circulating light beam.



We look at the following scenario. Light enters the resonator through a partially transmitting mirror ("input coupler") and interferes with the electric field already circulating in the resonator. In the resonant case this interference is constructive and thus also the circulating light power increases. On the other hand, there are losses either due to unwanted scattering at the mirrors or due to transmission through another partially transmittive mirror of the resonator ("output coupler"). Since the amount of lost power increases with circulating power, an equilibrium is reached when the losses balance the input. The circulating light power can be up to 10^5 times larger than the power of the input beam.

- Field inside the resonator

  We look at the phase and the amplitude of the circulating electric field at a position right behind the input coupler. If the resonator is in equilibrium the amplitude and the phase have to stay the same after one round trip in the resonator. If E_n denotes the complex electric field amplitude after n round trips this equilibrium condition can be written as

  $$E_{n+1} = E_n.$$

  During one cycle the field accumulates a certain phase, which is the product of the wave number k and the length of a round trip l,

  $$\varphi = kl.$$

  Furthermore, the field is attenuated by a factor of

  $$r_m := r_1\sqrt{1-L},$$

  with the field reflection coefficient r_1 of the input coupler and the factor L by which the light power is reduced during one cycle. The loss factor L includes all the losses (output coupler, mirrors scattering, etc.) except the losses due to transmission at the input coupler, which are already accounted for by r_1. Additionally, field enters the resonator at the input coupler according to the field transmission coefficient of the input coupler t_1,

  $$t_1 E_i = \sqrt{1-r_1^2}E_i,$$

  where E_i is the field amplitude of the input beam. It total one obtains

  $$E_{n+1} = r_m e^{i\varphi}E_n + t_1 E_i = E_n := E_c.$$

  Solved for E_c one gets

  $$\frac{E_c}{E_i} = \frac{t_1}{1 - r_m e^{i\varphi}}.$$

  Here the index "c" stands for " cavity " which is used as another name for resonator. Near the resonance, when $\varphi \ll 1$ (modulo 2π), one can approximate the exponential function to obtain

  $$\frac{E_c}{E_i} = \frac{t_1}{1 - r_m - r_m i\varphi}.$$

- Temporal phase and spatial phase

  It is easy to get confused about the sign of the phase that is accumulated after one round trip. Since the light seems "older" after one cycle, one could think that the phase φ should be negative. However, this thought confuses spatial and temporal phases. If one compares the temporal oscillation E_1 of a wave at a given spatial point with the temporal oscillation E_2 of the same wave after an extra travel distance d (e.g. one round trip), then E_2 is shifted by a positive phase compared to E_1. This becomes evident if you look at a "snapshot" of the situation at a fixed time and follow the spatial phase of the wave from E_1 to E_2.

  A wave traveling to the right (positive wave vector) is given by $e^{i(kz-\omega t)}$. With $e^{i(kd-\omega t)} = e^{i\omega(kd/\omega - t)} = e^{-i\omega(t+t_0)}$ we can express the spatial phase connected to the distance d by the phase after the time $t_0 = -kd/\omega$, which is negative. Therefore, a positive spatial phase corresponds to the phase of the field at a time in the past.

- Circulating power

  The power averaged over one oscillation period is given by the absolute square of the complex field amplitude,

$$\frac{P_c}{P_i} = \frac{|E_c|^2}{|E_i|^2} = \frac{t_1^2}{(1-r_m e^{i\varphi})(1-r_m e^{-i\varphi})} = \frac{t_1^2}{1-2r_m\cos\varphi + r_m^2}$$

  Close to resonance, $\varphi \ll 1$, one can write the cosine as $\cos(x) \simeq 1 - x^2/2$ and

$$\frac{P_c}{P_i} \simeq \frac{t_1^2/r_m}{(1-r_m)^2/r_m + \varphi^2}.$$

  This has the form of a Lorentzian function with respect to the phase φ.

- Impedance matching

  On resonance one gets for $\varphi = 0$

$$\frac{P_c}{P_i} = \frac{t_1^2}{(1-r_m)^2} = \frac{1-r_1^2}{\left(1-r_1\sqrt{1-L}\right)^2}.$$

  We introduce the power related quantities

$$\begin{aligned} R_1 &:= r_1^2 \\ t_1^2 &=: T_1 = 1 - R_1 \\ R_m &:= r_m^2 = R_1(1-L) \end{aligned}$$

  and obtain

$$\frac{P_c}{P_i} = \frac{1-R_1}{\left(1-\sqrt{R_m}\right)^2}.$$

We calculate the power reflectivity R_1 of the input coupler, for which the circulating power has a maximum

$$\begin{aligned}
\frac{d}{dR_1}\frac{1-R_1}{\left(1-\sqrt{R_1(1-L)}\right)^2} &= \frac{\sqrt{-R_1(L-1)}+L-1}{\left(-1+\sqrt{-R_1(L-1)}\right)^3\sqrt{-R_1(L-1)}} = 0\\
\sqrt{-R_1(L-1)} &= 1-L\\
R_1(1-L) &= (1-L)^2\\
R_1 &= 1-L.
\end{aligned}$$

This is the "impedance-matched" case. The "enhancement" is defined as

$$A := \frac{P_{\max}}{P_i}.$$

In the impedance matched case one obtains

$$A = \frac{P_{\max}}{P_i} = \frac{1}{1-R_1} = \frac{1}{L}.$$

With 1% losses and perfect impedance matching, the circulating power is 100 times larger than the power of the input beam.

- Reflected power

  The reflected field consists of a part that is directly reflected at the input coupler and the part that leaves the resonator,

$$\begin{aligned}
E_r &= -r_1E_i + E_c\sqrt{1-L}e^{ikl}t_1\\
&= -r_1E_i + E_i\frac{t_1}{1-r_1\sqrt{1-L}e^{i\varphi}}\sqrt{1-L}t_1e^{i\varphi}
\end{aligned}$$

The negative sign of the first term stems from the fact that a phase jump of π occurs if the field is reflected from a medium with higher refraction index (see Fresnel's formulas for reflection at an optical interface). This sign has its fundamental origin in the principle of time reversal symmetry and shows up for any beam splitter independent of its specific construction. Optical resonators are typically made of dielectric mirrors which in itself consist of a sophisticated stack of thin layers. The phase shift in reflection can have any value and even changes with the wavelength. However, if one adds up the two phase shifts from the reflection of both sides of the mirror one always obtains π. Here we assume that the phase shift for the reflection from one side is π and from the other is 0.

Introducing the round trip reflectivity for the field amplitude

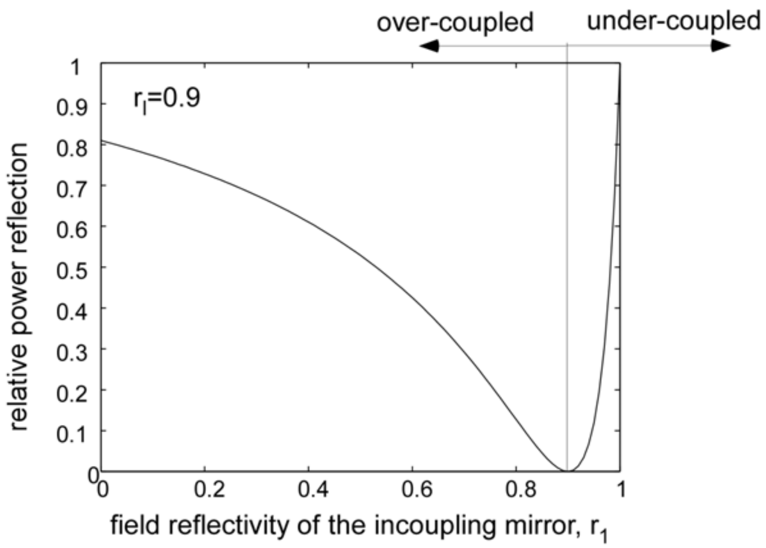
$$r_l := \sqrt{1-L}$$

and assuming resonance ($\varphi = 0$) one obtains the simple expression

$$\begin{aligned}\frac{E_r}{E_i} &= -r_1 + \frac{t_1^2 r_l}{1 - r_1 r_l} \\ &= -r_1 + \frac{(1 - r_1^2)\, r_l}{1 - r_1 r_l} \\ &= \frac{-r_1 (1 - r_1 r_l) + (1 - r_1^2)\, r_l}{1 - r_1 r_l} \\ &= \frac{r_l - r_1}{1 - r_1 r_l}.\end{aligned}$$

And for the power we get

$$\frac{P_r}{P_i} = \frac{|E_r|^2}{|E_i|^2} = \left(\frac{r_l - r_1}{1 - r_1 r_l}\right)^2.$$



For perfect impedance matching, i.e. $r_1 = r_l$, the reflection disappears and all light is coupled into the resonator. In the overcoupled regime one can lower the reflection by introducing losses.

- Hypothetical case of a lossless resonator

  If the field only escapes from the resonator via the output coupler and if there are no other losses, i.e.

$$L = T_2$$

  one gets for the power behind the outcoupling mirror

$$P_{trans} := P_c \cdot T_2 = P_{in} \cdot \frac{1}{L} \cdot T_2 = P_{in} \cdot \frac{1}{T_2} \cdot T_2 = P_{in}.$$

In the case of optimal impedance matching and the absence of losses, all light passes through the resonator, regardless of the mirror reflectivities. This also applies for mirrors with 100% reflectivity! In this limit it takes an infinitely long time until an equilibrium is established. When equilibrium is reached (after an infinitely long time), the circulating power is infinite and even if only 0% of it leaves the resonator through the output coupler (100% reflectivity), the transmitted power is finite and identical to the input power.

5.2 Equation of motion

- Differential equation for the field

  In order to describe temporal processes outside equilibrium we derive a differential equation by looking at the field amplitude after one additional round trip time τ,

$$
\begin{aligned}
E(t+\tau) &= E(t) r_m e^{i\varphi} + t_1 E_i \\
E(t+\tau) - E(t) &= E(t)\left(r_m e^{i\varphi} - 1\right) + t_1 E_i \\
\frac{E(t+\tau) - E(t)}{\tau} &= E(t)\left(r_m e^{i\varphi} - 1\right)\frac{1}{\tau} + \frac{t_1}{\tau} E_i
\end{aligned}
$$

  We are interested in dynamic changes which take much longer than the very short round trip time. In this limit we can replace the difference quotient by the differential quotient,

$$
\lim_{\tau \to 0} \frac{E(t+\tau) - E(t)}{\tau} = \frac{dE}{dt} \simeq E(t)\left(r_m e^{i\varphi} - 1\right)\frac{1}{\tau} + \frac{t_1}{\tau} E_i .
$$

  Close to the resonance, for $\varphi \ll 1$ (modulo 2π), we can expand the exponential function and obtain

$$
\begin{aligned}
\frac{dE}{dt} &\simeq E\left(r_m\left(1 + i\varphi\right) - 1\right)\frac{1}{\tau} + \frac{t_1}{\tau} E_i \\
&= E\left(r_m i\varphi + r_m - 1\right)\frac{1}{\tau} + \frac{t_1}{\tau} E_i .
\end{aligned}
$$

  This equation has the form of a an exponential equation with complex parameter,

$$
\begin{aligned}
\frac{dE}{dt} &= \left(i\Delta - \kappa\right) E + \eta \\
\Delta &:= r_m \frac{\varphi}{\tau}, \text{ detuning} \\
\kappa &:= \frac{1 - r_m}{\tau}, \text{ field decay rate} \\
\eta &:= \frac{t_1}{\tau} E_i, \text{ pump rate of the field}
\end{aligned}
$$

- Low loss resonators

We introduce

$$t_m := \sqrt{1 - r_m^2} = \sqrt{1 + r_m}\sqrt{1 - r_m}$$

and solve for

$$1 - r_m = \frac{t_m^2}{1 + r_m}.$$

If losses inside the resonator are small and r_m is close to 1, we arrive at often used expressions for the detuning Δ and the field decay rate κ,

$$\begin{aligned}\Delta &= r_m \frac{\varphi}{\tau} \simeq \frac{\varphi}{\tau} \\ \kappa &= \frac{1 - r_m}{\tau} \simeq \frac{t_m^2}{2\tau}.\end{aligned}$$

- Solution

The solution of the differential equation is given by

$$E(t) = \frac{\eta}{\kappa - i\Delta} + Ce^{(i\Delta - \kappa)t},$$

where the constant C is determined by the initial conditions. For long times the second term of the equation damps out and one reproduces the solution of the above self-consistency approach.

$$\begin{aligned}E_g(t) &= \frac{\eta}{\kappa - i\Delta} \\ &= \eta \frac{i\Delta + \kappa}{(\kappa - i\Delta)(\kappa + i\Delta)} \\ &= \eta \frac{i\Delta}{\Delta^2 + \kappa^2} + \eta \frac{\kappa}{\Delta^2 + \kappa^2}\end{aligned}$$

Like a driven harmonic oscillator, the field oscillates with a dispersive and an absorptive amplitude. The phase relative to the pump field is

$$\Theta = \arctan \frac{\Delta}{\kappa}.$$

For $\Delta = \pm\kappa$ the phase shift is $\pm\pi/4$. For large "blue detuning" ($\Delta \gg \kappa$), the light in the resonator is shifted by a factor of π relative to light with large "red detuning" ($\Delta \ll \kappa$). This behavior is similar to the classical driven harmonic oscillator with the difference that the phase shift on resonance is not $\frac{\pi}{2}$ but 0. The change of the phase with detuning is the key property used in the method by Pound, Drever and Hall to generate an electronic error signal (see chapter 4).

5.3 Line width, Finesse, Decay rate

- Line width

  The line width is defined as the full width at half maximum (FWHM). To determine it, we use the equation for power,

$$\frac{P_c}{P_i} = \frac{t_1^2}{1 - 2r_m \cos\varphi + r_m^2}$$

in the small angle approximation

$$\frac{P_c}{P_i} \simeq \frac{t_1^2}{1 - 2r_m\left(1 - \frac{1}{2}\varphi^2\right) + r_m^2} = \frac{t_1^2}{\left(1 - r_m\right)^2 + r_m\varphi^2}.$$

Half of the maximum power is reached for the phase $\varphi_{1/2}$ with

$$r_m\varphi_{1/2}^2 = \left(1 - r_m\right)^2,$$

since for this phase the denominator has just doubled. We express the phase by means of the above definition of the detuning Δ

$$\Delta = r_m\frac{\varphi}{\tau} = r_m\varphi\frac{c}{l} = r_m\varphi\nu_0.$$

Solving for φ

$$\varphi = \frac{\Delta}{r_m\nu_0}$$

and changing from angular frequency Δ to frequency $\delta = \Delta/2\pi$

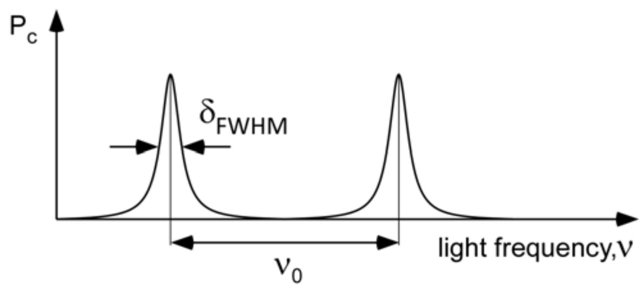
$$\varphi = \frac{2\pi\delta}{r_m\nu_0},$$

we obtain

$$\frac{\varphi_{1/2}}{2\pi} = \frac{\delta_{1/2}}{r_m\nu_0},$$

and

$$\delta_{FWHM} = 2\delta_{1/2} = \frac{1}{\pi}r_m\nu_0\varphi_{1/2} = \frac{1}{\pi}\sqrt{r_m}\left(1 - r_m\right)\nu_0.$$



- Finesse

  The finesse is the ratio of the free spectral range and full line width at half maximum:

$$F = \frac{\nu_0}{\delta_{FWHM}} = \frac{\pi}{\sqrt{r_m}\,(1-r_m)}$$

  With the definition of the Loss factor L from above

$$r_m = r_1\sqrt{1-L}$$

  and in the case of impedance matching,

$$r_1 = \sqrt{1-L},$$

  we obtain

$$r_m = 1-L.$$

  Inserting into the expression for F

$$F = \frac{\pi}{\sqrt{1-L}\,(1-(1-L))} = \frac{\pi}{\sqrt{1-L}L}$$

  For small losses, $L \ll 1$, one gets a very simple and useful estimation for the Finesse as π times enhancement A,

$$F \simeq \frac{\pi}{L} \simeq \pi A.$$

- Decay rate

  The relationship between line width and power decay rate γ_P is obtained from the field decay rate κ, which is half as large as the power decay rate,

$$\gamma_P := 2\kappa = 2\frac{1-r_m}{\tau}.$$

  The inverse of the decay rate is the resonator 1/e-life time T_{res}

$$T_{res} := \frac{1}{\gamma_P}.$$

  We insert this into the expression for the line width

$$\delta_{FWHM} = \frac{1}{\pi}\sqrt{r_m}\,(1-r_m)\,\nu_0 = \frac{1}{\pi}\sqrt{r_m}\frac{1-r_m}{\tau}.$$

  and obtain the useful relations

$$\delta_{FWHM} = \sqrt{r_m}\frac{\kappa}{\pi} = \sqrt{r_m}\frac{\gamma_P}{2\pi} = \sqrt{r_m}\frac{1}{2\pi}\frac{1}{T_{res}}.$$

  and

$$F = \frac{\nu_0}{\delta_{FWHM}} = \frac{1}{\sqrt{r_m}} 2\pi\nu_0 T_{res}.$$

For low losses one can drop the $\sqrt{r_m}$-terms,

$$\delta_{FWHM} \simeq \frac{\kappa}{\pi} = \frac{\gamma_P}{2\pi} = \frac{1}{2\pi}\frac{1}{T_{res}}$$
$$F = \frac{\nu_0}{\delta_{FWHM}} \simeq 2\pi\nu_0 T_{res}.$$

5.4 Geometrical optics

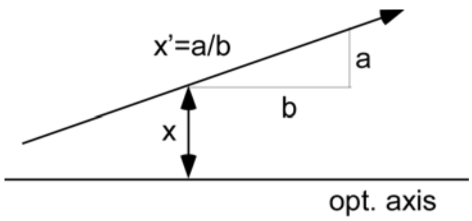
Laser beams are waves and behave fundamentally different than the rays of geometric optics. Nevertheless, the shape of laser beams that passes a combination of optical elements can be calculated if one knows how the rays of geometric optics behave. We thus briefly discuss the matrix-representation of optical elements in geometric optics.

- Ray vector

  A ray at a given point is described by the vector

  $$\vec{x} := \begin{pmatrix} x \\ x' \end{pmatrix}.$$

  The first component, x, is the distance of the ray from the optical axis at a given point on the optical axis. The second component is the slope of the ray at the given point.

- ABCD-Matrices

  The change of the ray vector along the optical axis is described by the so called ABCD-matrix or ray transfer matrix,

  $$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

  The ray vector at point 2 is a linear function of the ray vector at point 1,

  $$\vec{x}_2 = M\,\vec{x}_1.$$

In general this linear Ansatz is not exact but it is a good approximation for rays with small slopes

$$x' \ll 1.$$

This regime is called "paraxial approximation".

- Propagation

  Lets look at some examples. The propagation of the ray along a distance d is described by the matrix

$$M_d = \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix}.$$

  This is because

$$\begin{pmatrix} x_2 \\ x_2' \end{pmatrix} = \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_1' \end{pmatrix} = \begin{pmatrix} x_1 + d \cdot x_1' \\ x_1' \end{pmatrix}.$$

  The matrix leaves the slope unchanged and changes the distance to the optical axis according to the slope at point 1.

- Thin lenses

  For a thin lens of focal length f, the ABCD-matrix is (by convention focusing lenses have a positive focal length),

$$M_d = \begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix},$$

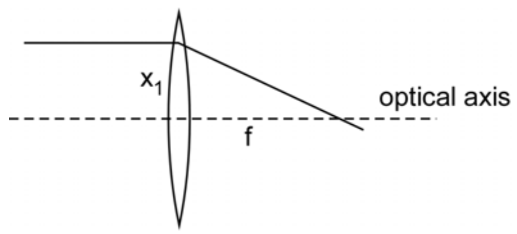
  The transformation yields

$$\begin{pmatrix} x_2 \\ x_2' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_1^{'} \end{pmatrix} = \begin{pmatrix} x_1 \\ x_1' - \frac{x_1}{f} \end{pmatrix}.$$

  The matrix leaves the distance to the optical axis unchanged but changes the slope. To see how the focal length f comes into play, let's look at the transformation of a parallel input beam,

$$\begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ 0 \end{pmatrix} = \begin{pmatrix} x_1 \\ -\frac{x_1}{f} \end{pmatrix}.$$

  The slope of the ray has now changed such that the new ray will intersect the optical axis at a distance f from the lens. This is what we expect from a focusing lens.



- Interface between two media

  In a similar way, the Fresnel equations can be used to calculate the matrix for the interface between of two media with different refraction index. The calculation is not difficult but lengthy. Here, we just look at the result. A curved interface with a radius of curvature R between a medium with a refractive index n_1 and a medium with a refractive index n_2 is the matrix is

$$M_R = \begin{pmatrix} 1 & 0 \\ \frac{n_1-n_2}{Rn_2} & \frac{n_1}{n_2} \end{pmatrix}.$$

  For a plane interface, $R = \infty$ the matrix reduces to

$$M_p = \begin{pmatrix} 1 & 0 \\ 0 & \frac{n_1}{n_2} \end{pmatrix}.$$

  This means that the slope changes by n_1/n_2.

$$x_2' = \frac{n_1}{n_2} x_1'.$$

  If we recall Snell's law

$$\frac{\sin\theta_r}{\sin\theta_i} = \frac{n_1}{n_2}$$

  and use the small angle approximation

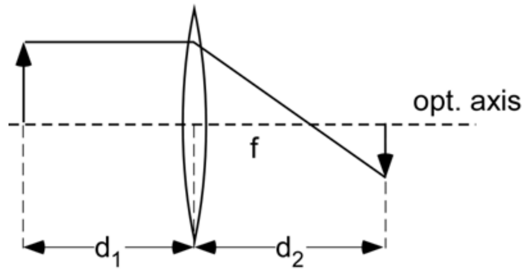
$$\sin\theta = \tan\theta = x'$$

  we find that the Matrix M_p represents Snell's' law in the paraxial approximation.

- Systems

  Systems of combined optical elements are described by the product of the matrices that represent the elements. The sequence - distance d_1, lens f, distance d_2- is described by the matrix

$$M = \begin{pmatrix} 1 & d_2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix} \begin{pmatrix} 1 & d_1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 - \frac{d_2}{f} & \left(1 - \frac{d_2}{f}\right) d_1 + d_2 \\ -\frac{1}{f} & 1 - \frac{d_1}{f} \end{pmatrix}.$$



- Glass plate

  Another example is a plane parallel glass plate of thickness d. It is formed by two plane interfaces and a distance in between.

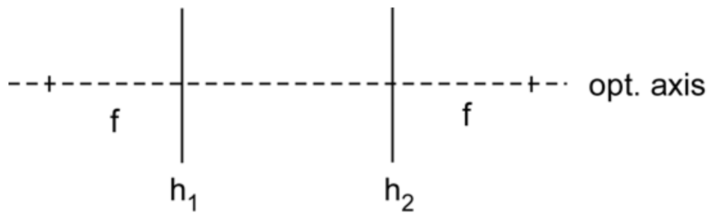
$$M_n = \begin{pmatrix} 1 & 0 \\ 0 & n \end{pmatrix} \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{n} \end{pmatrix} = \begin{pmatrix} 1 & \frac{d}{n} \\ 0 & 1 \end{pmatrix}.$$

  It transforms the ray the same way as the propagation along an effective distance $d' = d/n$. It is as if the Medium would compress the ray along the optical axis by a factor $1/n$.

- Focal length and main plains

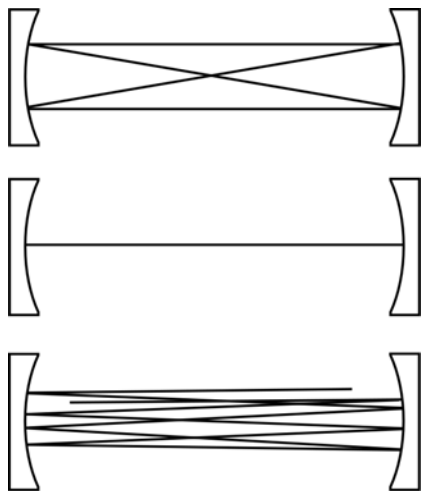
  For those who are familiar with the concept of main planes it may be interesting to note that there is a simple connection between the elements of the ABCD-matrix and the main planes of a given optical system. The focal length f and the main planes h_1 and h_2 are given by

$$f = -\frac{1}{c}$$
$$h_1 = \frac{D-1}{C}$$
$$h_2 = \frac{A-1}{C}$$

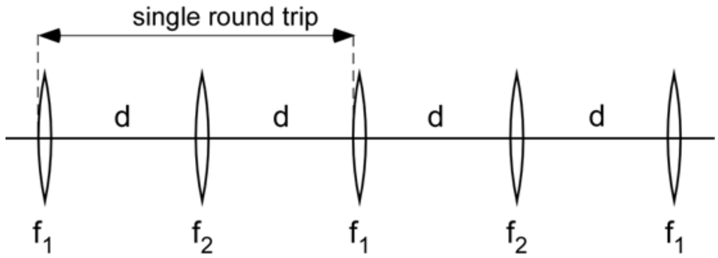


- Resonator and lens guide

  Which ray do you get if you place two mirrors opposite to each other?



A standing wave resonator made of two mirrors equivalent to a series of lenses that repeats periodically,



We use the convention that concave mirrors have a positive radius of curvature. Therefore, the related focal length is

$$f = \frac{r}{2}.$$

One cycle is represented by the matrix

$$\begin{aligned} M &= M_d \cdot M_{f_2} \cdot M_d \cdot M_{f_1} \\ &= \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{2}{r_2} & 1 \end{pmatrix} \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{2}{r_1} & 1 \end{pmatrix} \\ &= \begin{pmatrix} \left(1 - 2\frac{d}{r_2}\right)\left(1 - 2\frac{d}{r_1}\right) - 2\frac{d}{r_1} & \left(1 - 2\frac{d}{r_2}\right) d + d \\ -\frac{2}{r_2}\left(1 - 2\frac{d}{r_1}\right) - \frac{2}{r_1} & 1 - 2\frac{d}{r_2} \end{pmatrix}. \end{aligned}$$

A number of n cycles is thus described by

$$M^n = \left(M_d \cdot M_{f_2} \cdot M_d \cdot M_{f_1}\right)^n$$

- Sylvester theorem

  The Sylvester theorem allows to calculate of powers of matrices,

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^n = \frac{1}{\sin\theta}\begin{pmatrix} A\sin(n\cdot\theta) - \sin((n-1)\cdot\theta) & B\,(\sin n\cdot\theta) \\ C\sin(n\cdot\theta) & D\sin(n\cdot\theta) - \sin((n-1)\cdot\theta) \end{pmatrix}$$

  with

$$\cos\theta := \frac{1}{2}(A+D).$$

  The number of round trips n appears in the argument of the sin-function which causes periodic behavior with "frequency" θ.

- Stability

  Sylvester's theorem is applicable only if the $\cos\theta$ exists, i.e. $\frac{1}{2}(A+D)$ is inside the interval $[-1,1]$. Otherwise, the ray is unbound and leaves the resonator. Stable resonators thus require

$$-1 < \frac{1}{2}(A+D) < 1$$

  In the specific case of a standing wave resonator one obtains

$$\begin{aligned}
-1 &< \frac{1}{2}\left(\left(1-2\frac{d}{r_2}\right)\left(1-2\frac{d}{r_1}\right) - 2\frac{d}{r_1} + 1 - 2\frac{d}{r_2}\right) < 1 \\
-1 &< 1 - 2\frac{d}{r_1} - 2\frac{d}{r_2} + 2\frac{d^2}{r_2 r_1} < 1 \\
0 &< 2 - 2\frac{d}{r_1} - 2\frac{d}{r_2} + 2\frac{d^2}{r_2 r_1} < 2 \\
0 &< 1 - \frac{d}{r_1} - \frac{d}{r_2} + \frac{d^2}{r_2 r_1} < 1
\end{aligned}$$

  It is useful to introduce the parameters

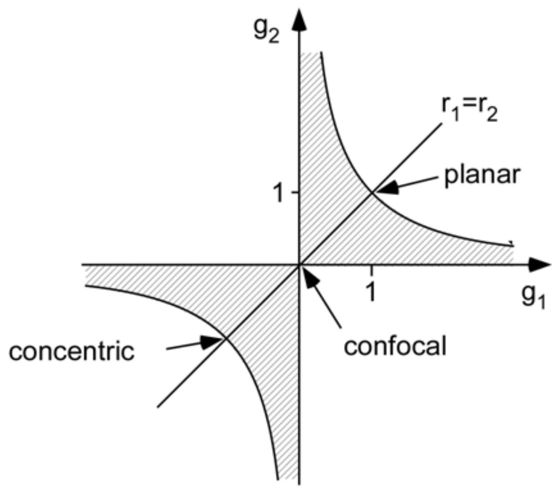
$$\begin{aligned}
g_1 &:= 1 - \frac{d}{r_1} \\
g_2 &:= 1 - \frac{d}{r_2}
\end{aligned}$$

  The stability condition may now be written as

$$0 < g_1 g_2 < 1.$$

- Stability diagram

  We visualize the right boundary $g_1 g_2 = 1$ by plotting $g_2 = \frac{1}{g_1}$ as function of g_1,



Since the product $g_1 g_2$ has to be positive, stable resonators are represented by points $(g_1 g_2)$ in the first and third quadrant (shaded area).

- Identical mirrors

  For two identical mirrors, $g_1 = g_2$, the points $(g_1 g_2)$ lie on the bisecting line. For the resonators at the stability limits one obtains

$$g_1 g_2 = \left(1 - \frac{d}{r}\right)\left(1 - \frac{d}{r}\right) = 1$$
$$1 - \frac{d}{r} = \pm 1$$

  with the two solutions

$$\frac{d}{r} = 0$$
$$\frac{d}{r} = 2$$

- Planar resonator

  The first solution

$$\frac{d}{r} = 0$$

  implies that the radii of curvature of the mirrors, r, become infinite. We obtain two plane mirrors facing each other. A ray with a nonzero slope would increase its distance to the optical axis with each round trip and finally leaves the resonator. An arbitrarily large but finite r would be sufficient to eventually reflect the ray back to the optical axis.

- Concentric resonator

$$\frac{d}{r} = 2$$

The radius of curvature is exactly half the distance between the mirrors. The mirror surfaces are part of a common sphere. Rays that go through the center of the sphere are stable, all others are not. If the mirror distance is reduced by an infinitesimal amount, these unstable rays would become stable. In contrast to the plane resonator, this type of instability is not immediately obvious and a nontrivial result of our analysis.

- Confocal resonator

The left part of the above stability condition requires that

$$0 < g_1 g_2.$$

We look at the stability edge where

$$g_1 g_2 = 0.$$

For two identical mirrors one obtains

$$1 - \frac{d}{r} = 0$$
$$d = r.$$

This means that the focal points of the mirrors coincide. Such confocal resonators are stable since a slight change of d would keep the resonator within the shaded area. Confocal resonators only become unstable if the radii of the two mirrors slightly differ. However, mirrors can be manufactured with high reproducibility, so that this instability is practically irrelevant.

5.5 Gaussian beam (Kogelnik, Li, Applied Optics 5, 1550, (1966))

Geometric optic helps to analyze optical systems but the "rays" used in this model are not a good description for laser beams. Laser beams are waves. Usually waves are discussed as superpositions of plane waves. Plane waves are mathematically very useful but they are not physical objects since they are infinite in space and time. One may start to construct realistic waves as superpositions of plane waves but this leads to lengthy and unpractical expressions. How can you then describe a laser beam? In this section we introduce the most simple but still realistic type of wave that is finite in space, at least in transverse direction. This "Gaussian beam" has the additional advantage that it is compatible with the boundary conditions of an optical resonator with spherical mirrors.

- Wave equation

We start with usual wave equation for the electric field

$$\nabla^2 \vec{E}(\vec{r},t) = \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\vec{E}(\vec{r},t)$$

which can be derived from Maxwell equations in vacuum.

- Variable separation

  We look for stationary solutions of the form

$$\vec{E}(\vec{r},t) = E(t) \cdot \vec{u}(\vec{r})$$

Furthermore, we restrict ourselves to solutions with homogenous polarization, i.e. the field vector $\vec{E}$ points to the same direction independent of $\vec{r}$,

$$\vec{u}\left(\vec{r}\right) = u\left(\vec{r}\right)$$

We insert this Ansatz into the wave equation and get

$$\begin{aligned}\nabla^2\left(E(t)\cdot u(\vec{r})\right) &= -\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\left(E(t)\cdot u(\vec{r})\right)\\ E(t)\nabla^2 u(\vec{r}) &= u(\vec{r})\frac{1}{c^2}\frac{\partial^2}{\partial t^2}E(t)\\ \frac{\nabla^2 u(\vec{r})}{u(\vec{r})} &= \frac{1}{c^2}\frac{1}{E(t)}\frac{\partial^2}{\partial t^2}E(t)\end{aligned}$$

Since the left part only depends on $\vec{r}$, a general solution for any r and t requires that the time dependent part is constant. As we will see in a moment this constant should be negative. We therefore write it as $-k^2$. One obtains the two equations

$$\begin{aligned}\frac{1}{c^2}\frac{1}{E(t)}\frac{\partial^2}{\partial t^2}E(t) &= -k^2\\ \nabla^2 u(\vec{r}) &= -u(\vec{r})k^2.\end{aligned}$$

The first equation

$$\frac{\partial^2}{\partial t^2}E(t) = -c^2k^2E(t)$$

has the solution

$$E\left(t\right) = Ee^{\pm i\omega t}$$

with

$$\omega^2 = k^2c^2$$

This is why we require the constant $-k^2$ to be negative. Positive k^2 would lead to exponential damping or explosion without oscillation. The parameter k turns out to be the wave vector, which fixes the wavelength of the light beam

$$\lambda = 2\pi/k.$$

- Helmholtz-equation

  The second equation is the so called Helmholtz-equation

$$-\nabla^2 u(\vec{r}) = k^2 u(\vec{r})$$

  This is the eigenequation of the operator $-\nabla$. Its eigenvectors determine the spatial shape of the electric field ("optical modes") and the eigenvalues k determine the frequency of the modes.

- Paraxial wave equation

  We look for solutions that describe a beam and make the Ansatz,

$$u(\vec{r}) = \Psi(x, y, z) \cdot e^{-ikz}.$$

  It consists of a plane wave propagating in z-direction and a function $\Psi(x, y, z)$, which describes the shape of the beam. We require that the rapid spatial oscillation along the z-direction should be completely described by the e^{-ikz}-term and that $\Psi(x, y, z)$ changes only little with z on the spatial scale of a wavelength $\lambda = 2\pi/k$. There are no restrictions in the x-y-plane such that oscillations and node structures in the transverse plane are allowed.

  By inserting the Ansatz into the wave equation we obtain the "paraxial wave equation",

$$\frac{\partial^2}{\partial x^2}\Psi + \frac{\partial}{\partial y^2}\Psi - 2ik\frac{\partial \Psi}{\partial z} = 0.$$

  Here, terms proportional to $\frac{\partial^2 \Psi}{\partial^2 z}$ are neglected according to our above assumption of a "slowly varying amplitude" in the z-direction..

- Fundamental mode

  The most important solution is the called "fundamental mode",

$$\Psi(r, z) = \Psi_0 \cdot e^{-i\left(P(z) + \frac{k}{2q(z)} \cdot r^2\right)}.$$

  This Ansatz is rotational symmetric around the optical axis with a radius r defines as

$$r^2 := x^2 + y^2.$$

  By inserting in the paraxial wave equation one finds that the function $q(z)$ and $P(z)$ fulfill the differential equations

$$\frac{dq(z)}{dz} = 1$$

  and

$$\frac{dP(z)}{dz} = -\frac{i}{q(z)}.$$

From

$$\frac{dq}{dz} = 1$$

follows

$$q = q_0 + z$$

where q_0 being a complex parameter. Its real part can be eliminated by shifting of the origin along the z-axis by the negative value of the real part $\mathrm{Re}\,(q_0)$. Then, q_0 becomes purely imaginary. As we shall see, the origin then lies at the focus of the beam where the beam radius is smallest. We introduce the real valued "Rayleigh-length" z_0, and write

$$q_0 = iz_0.$$

The Rayleigh-length will turn out to be the only free parameter of the fundamental mode (besides the wave vector k). It fully describes the shape of the beam.

- Beam waist, Rayleigh length and confocal parameter

  Substituting $q = iz_0 + z$ into the solution Ψ yields

$$\begin{aligned}
\Psi &= \Psi_0 \cdot e^{-iP(z)} \cdot \exp\left(-i\frac{k \cdot q^*}{2q \cdot q^*} r^2\right) \\
&= \Psi_0 \cdot e^{-iP(z)} \cdot \exp\left(-i\frac{k}{2} \cdot \frac{z - iz_0}{z^2 + z_0^2} r^2\right) \\
&= \Psi_0 \cdot \exp\left(-iP(z) - i\frac{k}{2}\frac{r^2 z}{z^2 + z_0^2}\right) \cdot \exp\left(-\frac{r^2}{w^2}\right)
\end{aligned}$$

with

$$w^2 := \frac{2}{k} \cdot \frac{z^2 + z_0^2}{z_0} = \frac{2z_0}{k} \cdot \left(1 + \left(\frac{z}{z_0}\right)^2\right).$$

The amplitude of the fundamental mode has a Gaussian profile $\exp\left(-\frac{r^2}{w^2}\right)$ with $1/e$-beam radius of

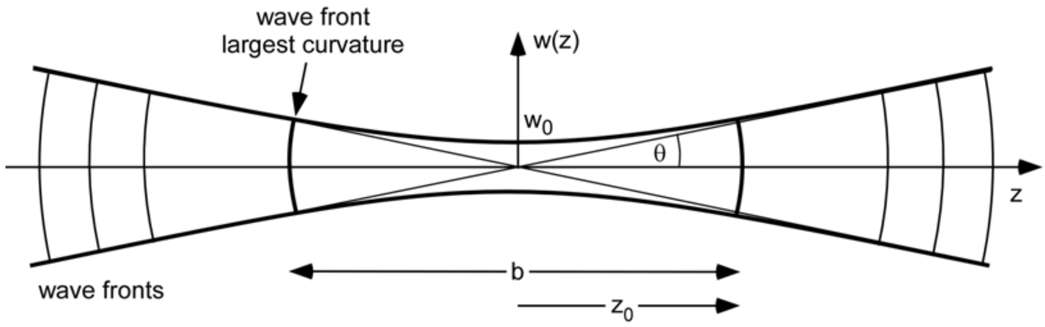
$$w\,(z) = w_0 \cdot \sqrt{1 + \left(\frac{z}{z_0}\right)^2}$$

The beam waist w_0 is the beam radius at position $z = 0$ and obeys

$$w_0^2 = \frac{2z_0}{k} = \frac{b}{k}.$$

Twice the Rayleigh length is sometimes called "confocal parameter",

$$b := 2z_0.$$



- far field angle

  For large $|z| \gg |z_0|$ the 1/e-radius may approximately be written as

$$w(z) = w_0 \cdot \sqrt{1 + \left(\frac{z}{z_0}\right)^2} \simeq \frac{w_0}{z_0}\, z$$

  The radius w increases linearly with z with a slope

$$\frac{w}{z} \simeq \frac{w_0}{z_0}$$

  We introduce the "far-field angle" θ

$$\tan\theta := \frac{w_0}{z_0} = \sqrt{\frac{2}{kz_0}} = \sqrt{\frac{\lambda}{\pi \cdot z_0}} = \frac{\lambda}{\pi \cdot w_0}$$

- Wavefronts

  The differential equation for $P(z)$

$$\frac{dP}{dz} = -\frac{i}{q} = -\frac{i}{z + iz_0}$$

  is solved by

$$iP(z) = \ln(1 - i\frac{z}{z_0}),$$

  because

$$\frac{dP}{dz} = \frac{1}{i}\frac{1}{1 - i\frac{z}{z_0}}\left(-i\frac{1}{z_0}\right) = -\frac{1}{z_0 - iz} = -\frac{i}{iz_0 + z}.$$

  Decomposition into real and imaginary part leads to:

$$\begin{aligned}\mathrm{Re}\,(iP(z)) &= \frac{1}{2}\left(\ln(1 - i\frac{z}{z_0}) + \ln(1 + i\frac{z}{z_0})\right)\\ &= \frac{1}{2}\ln\left((1 - i\frac{z}{z_0})(1 + i\frac{z}{z_0})\right)\\ &= \frac{1}{2}\ln\left(1 + (\frac{z}{z_0})^2\right) = \ln\sqrt{1 + (\frac{z}{z_0})^2}\end{aligned}$$

and

$$
\begin{aligned}
\mathrm{Im}(iP(z)) &= \frac{1}{2i}\left(\ln(1-i\frac{z}{z_0}) - \ln(1+i\frac{z}{z_0})\right) \\
&= \frac{1}{2i}\ln\left(\frac{1-i\frac{z}{z_0}}{1+i\frac{z}{z_0}}\right) \\
&= -i\ln\left(\sqrt{\frac{1-i\frac{z}{z_0}}{1+i\frac{z}{z_0}}}\right) \\
&= -\arctan\frac{z}{z_0}
\end{aligned}
$$

(Note: $\arctan x = i\ln\sqrt{\frac{1-ix}{1+ix}}$). Inserting the solution

$$
iP(z) = \ln\sqrt{1+(\frac{z}{z_0})^2} - i\arctan(\frac{z}{z_0}).
$$

into the exponent of the fundamental mode yields

$$
\begin{aligned}
-i\left(P(z) + \frac{k}{2}\frac{r^2 z}{z^2+z_0^2}\right) = &-\ln\sqrt{1+(\frac{z}{z_0})^2} \\
&+ i\left(\arctan(\frac{z}{z_0}) - \frac{k}{2}\frac{r^2 z}{z^2+z_0^2}\right).
\end{aligned}
$$

Finally the full solution for the electric field reads,

$$
u(\vec{r}) = \Psi(x,y,z)\cdot e^{-ikz}
$$

$$
\begin{aligned}
u(\vec{r}) &= \Psi(x,y,z)\cdot e^{-ikz} \\
&= \Psi_0 \cdot \underbrace{\frac{1}{\sqrt{1+(\frac{z}{z_0})^2}}}_{\text{energy conservation}} \cdot \underbrace{e^{-(\frac{r}{w(z)})^2}}_{\text{Gaussian beam profile}} \cdot \underbrace{e^{-i\left(kz-\arctan(\frac{z}{z_0})\right)}}_{\text{phase factor}} \cdot \underbrace{e^{-i\frac{k}{2}\cdot\frac{r^2}{R(z)}}}_{\text{curved wavefront}}
\end{aligned}
$$

- Radius of curvature

$$
R(z) := z\cdot\left(1+(\frac{z_0}{z})^2\right)
$$

The function $R(z)$ describes the radius of curvature of the wave fronts at the optical axis. This can be seen as follows. Solve the spherical shell equation

$$
z^2 + r^2 = R^2
$$

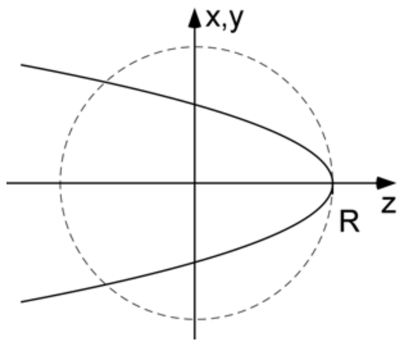
for z

$$z=\sqrt{R^2-r^2}=R\sqrt{1-\frac{r^2}{R^2}}$$

and make the approximation for small distances to the optical axis, $r^2 < R^2$.

$$z=R\sqrt{1-\frac{r^2}{R^2}}=R-\frac{1}{2}\frac{r^2}{R}+...$$

For $z\,(r)$ one gets a parabola with curvature $-1/R$ and vertex position at $z=R$.



We compare this to the phase fronts of the fundamental mode as given by the terms with imaginary exponents

$$e^{-ikz}\cdot e^{-i\frac{k}{2}\cdot\frac{r^2}{R(z)}}=e^{-i\left(\frac{k}{2}\cdot\frac{r^2}{R(z)}+kz\right)}.$$

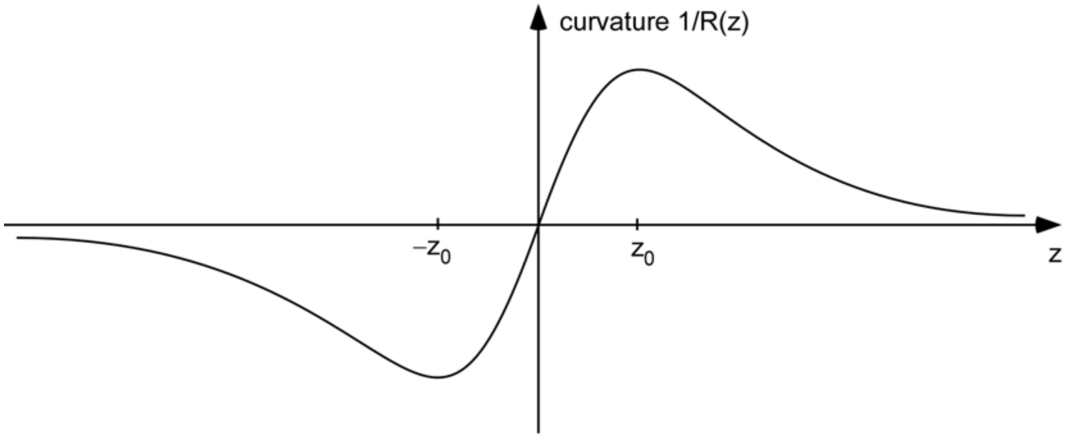
The phase fronts are defined by the points of constant phase,

$$\frac{k}{2}\cdot\frac{r^2}{R(z)}+kz=const$$

$$z(r)=\frac{const}{k}-\frac{1}{2}\cdot\frac{r^2}{R(z)}.$$

Again we get a parabola $z(r)$ with curvature $-1/R$. The function $R\,(z)$ can thus be interpret as the radius of a spherical shell on which the wave fronts lie.

If you plot $1/R\,(z)$ versus z, you get the curvature of the wave fronts along the beam.



The wave fronts are maximally curved at $\pm z_0$ where the radius of curvature has a minimum. For large distances values of z, the radius is

$$R \simeq |z| \,,$$

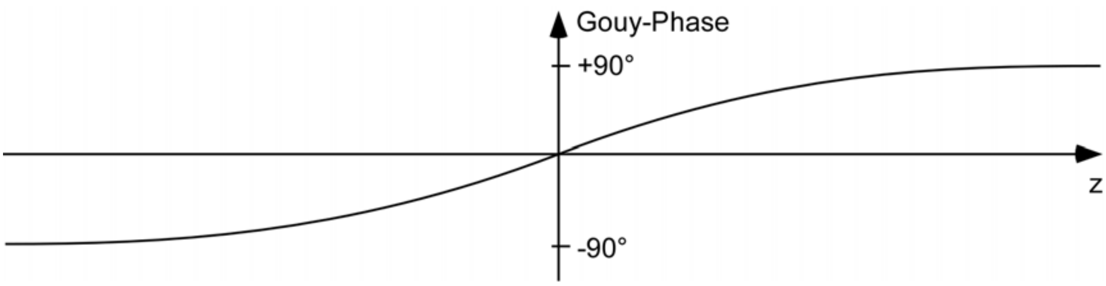
and the wave fronts approach spherical shells around the origin. As we know that spherical waves are a solution of Maxwell's equations, the deviation of the parabola from the spherical shell expression is caused by the paraxial approximation.

- Gouy-phase

  The phase spacing of the wave fronts lags a little behind that of a plane wave. The responsible phase term in the exponent of the complex terms

$$\arctan(\frac{z}{z_0})$$

  is called the Gouy-phase. When passing through the focus, the Gaussian beam loses the phase π compared to a plane wave.



- Absorption and Gouy phase (side note)

We look at an atom placed in the waist of a Gaussian beam. If the atom is optically excited to a state of higher energy, this energy is taken from the incident light beam which consequently becomes less intense. This intensity reduction is due to interference between the field of the beam and the field that is emitted by the atomic dipole. In the near field this interference pattern is complicated but in the far field the wavefronts lie on a sphere and may destructively interfere with the spherical wave emitted by the dipole. The dipole is excited by the incident field and oscillates with a phase delay of 90° relative to the exciting field (for resonant excitation). Destructive interference, however, requires a relative phase of 180°. The missing 90° come from the Gouy phase. At the beam waist where the atom sits the Gouy phase is 0° but in the far field where the interference takes place the Gouy phase amounts to 90°.

- Field strength and intensity

  We have derived a complex solution for the real paraxial wave equation. Thus the complex conjugate is also a solution. We combine both solutions to describe the physical realvalued electric field E

$$E = \Psi e^{i(kz \pm \omega t)} + \Psi^* e^{-i(kz \pm \omega t)}.$$

  We write the complex number Ψ as amplitude and phasefactor

$$\Psi = |\Psi| e^{i\varphi}$$

  and get

$$\begin{aligned} E &:= |\Psi| e^{i(kz \pm \omega t + \varphi)} + |\Psi| e^{-i(kz \pm \omega t + \varphi)} \\ &= 2 |\Psi| \cos(kz \pm \omega t + \varphi) \end{aligned}$$

  The amplitude $2|\Psi|$ of the real electric field of a propagating wave depends on the time-averaged intensity

$$\begin{aligned} I &= \frac{1}{2} c\varepsilon_0 n \left(2|\Psi|\right)^2 = 2cn\varepsilon_0 |\Psi|^2 \\ &= 2nc\varepsilon_0 \frac{\Psi_0^2}{1 + (\frac{z}{z_0})^2} e^{-2(\frac{r}{w(z)})^2} \end{aligned}$$

  At the waist the relationship between the prefactor Ψ_0 and the intensity is

$$\begin{aligned} I &= I_0 e^{-2(\frac{r}{w_0})^2} \\ I_0 &= 2c\varepsilon_0 n \Psi_0^2. \end{aligned}$$

- Power and intensity and Ψ_0

The power is the integral of the intensity across the beam profile. We calculate this integral at the waist where $z = 0$,

$$
\begin{aligned}
P &= \int I\left(r\right) r dr d\varphi \\
&= 2\pi \int I\left(r\right) r dr \\
&= 2\pi I_0 \int_0^\infty e^{-2\left(\frac{r}{w_0}\right)^2} r dr \\
&= \frac{1}{2}\pi w_0^2 I_0
\end{aligned}
$$

The ratio between power and beam cross-section can be defined as the average intensity,

$$\bar{I} := \frac{P}{\pi w_0^2}.$$

The peak intensity is twice the average intensity,

$$I_0 = 2\bar{I}.$$

The pre-factor Ψ_0 of the complex Gaussian beam may be derived from the observed beam power P according to

$$P = c\varepsilon_0 n\pi w_0^2 \Psi_0^2.$$

- Transverse modes

  In addition to the fundamental mode, there are other solutions. One set of solutions are the Gauß-Hermite-modes,

  $$\Psi = g\left(\frac{x}{w}\right) \cdot h\left(\frac{y}{w}\right) e^{-i\left(P(z) + \frac{k}{2q(z)}\left(x^2+y^2\right)\right)}.$$

  The functions $g\left(x/w\right)$ and $h\left(y/w\right)$ are connected to the solutions of the Hermite equations

  $$\frac{d^2}{dx^2} H_m - 2x\frac{d}{dx} H_m + 2mH_m = 0$$

  by

  $$g_m\left(\frac{x}{w}\right) \cdot h_n\left(\frac{y}{w}\right) = N_m \cdot H_m\left(\sqrt{2}\frac{x}{w}\right) \cdot N_n \cdot H_n\left(\sqrt{2}\frac{y}{w}\right).$$

  The normalization constant is

  $$N_j = \frac{1}{\sqrt{\sqrt{\frac{2}{\pi}} w \cdot 2^j j!}}.$$

The first Hermitian polynomials are:

$$
\begin{aligned}
H_0(x) &= 1 \\
H_1(x) &= 2x \\
H_2(x) &= 4x^2 - 2 \\
H_3(x) &= 8x^3 - 12x
\end{aligned}
$$

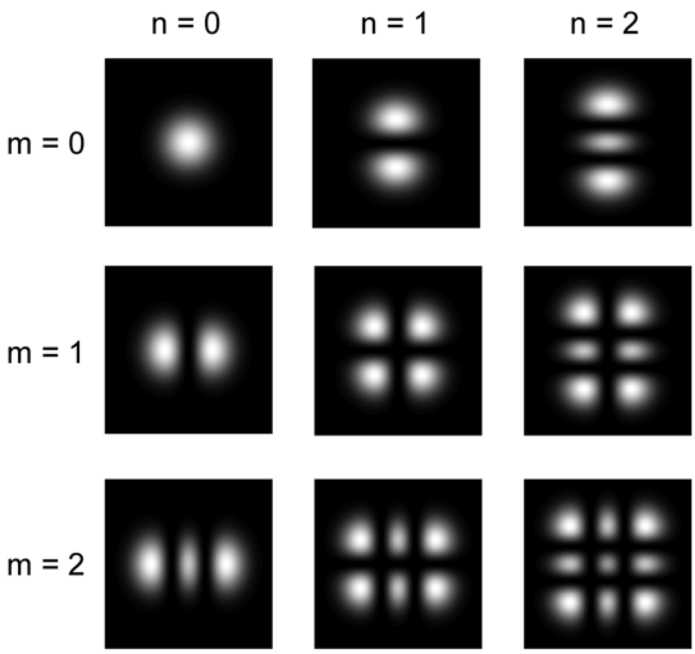
The Gouy phase for Hermite modes depends on m and n,

$$\phi(m, n, z) = (m + n + 1) \arctan(\frac{z}{z_0}).$$

The Hermite modes are also called "transverse electrical modes" , TEM_{nm}.

- Intensity profiles

  The intensity profiles of Hermite modes exhibit nodal lines where the field exactly vanishes,



The indices m and n count the number of vertical and horizontal nodal lines, respectively. If one passes a nodal line, the sign of the electric field changes.

5.6 ABCD-law and mode matching

- ABCD-law

  If a Gaussian beam passes through an optical system, its q-parameter changes. The q-parameters of the Gaussian beams in front (q_1) of and behind (q_2) an optical system are linked by the ABCD law,

$$q_2 = \frac{Aq_1 + B}{Cq_1 + D}.$$

  A, B, C, D are the elements of the ray transfer matrix of geometric optics. A derivation of the ABCD-law can be found in the book of A. Siegman, "Laser", University Science Books, 1986, chapter 20.

  A complex q-parameter of the input beam

$$q = z + iz_0$$

  is transformed to the q-parameter of the output beam by

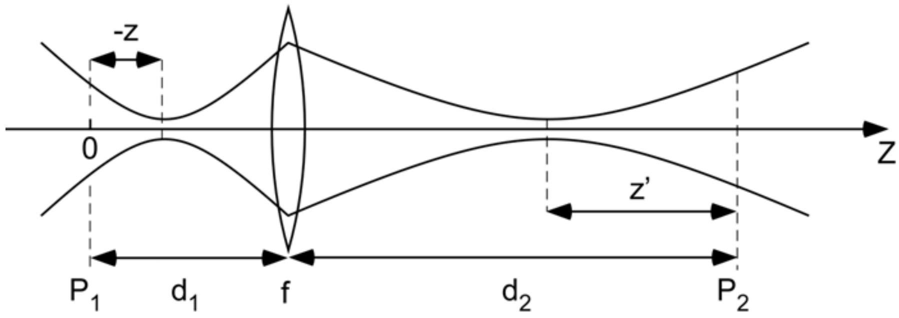
$$\begin{aligned} q_2 &= \frac{Az + iAz_0 + B}{Cz + iCz_0 + D} = \frac{(Az + iAz_0 + B)(Cz - iCz_0 + D)}{(Cz + iCz_0 + D)(Cz - iCz_0 + D)} \\ &= \frac{Az^2C + AzD + Az_0^2C + BCz + BD + i(Az_0D - BCz_0)}{(Cz + D)^2 + C^2z_0^2} \\ &= \frac{AC(z^2 + z_0^2) + (AD + BC)z + BD}{(Cz + D)^2 + C^2z_0^2} + i\frac{(AD - BC)z_0}{(Cz + D)^2 + C^2z_0^2} \\ &= z' + iz_0' \end{aligned}$$

  The distance to the waist of the output beam z' and its Rayleigh length z_0' thus are

$$\begin{aligned} z' &= \frac{AC(z^2 + z_0^2) + (AD + BC)z + BD}{(Cz + D)^2 + C^2z_0^2} \\ z_0' &= \frac{(AD - BC)z_0}{(Cz + D)^2 + C^2z_0^2} \end{aligned}$$

- Interpretation of z'

  To calculate the ABCD Matrix we start at a Position P_1 and analyze what happens to the beam until it arrives at position P_2. The Parameters z' and z_0' are the parameters of the new beam at P_2. The value z' thus tells us, how far the new beam has already propagated from its beam waist. This means, that the beam waist of the new beam lies at a distance $-z'$ from P_2. Here is an example (don't mix the parameters z and z_0 with the variable Z for the position along the optical axis):



The starting position P_1 is left of the focus of the input beam. From its waist the input beam must propagate a distance $|z|$ to the left to get to the origin. The input parameter z is thus negative. After transformation with a Matrix $M = M_{d_2} M_f M_{d_1}$ to the position P_2, the new beam has propagated from its focus a distance z' to the right to reach P_2. The focus of the new beam thus lies left from P_2 by a distance $|z'|$. If the calculated $z\prime$would be negative, the new focus would ly to the right of P_2.

- Resonator

  We describe the resonator by its ray transfer matrix for a single round trip

$$M = \begin{pmatrix} A & B \\ D & C \end{pmatrix}.$$

  In a stable resonator the Gaussian beam must be mapped onto itself after one round trip. Starting at a given reference point (for instance at the surface of one of the mirrors) the q parameter must reproduce after one round trip. Using the ABCD-law we obtain the q-parameter for the reference point as

$$q = \frac{Aq + B}{Cq + D}.$$

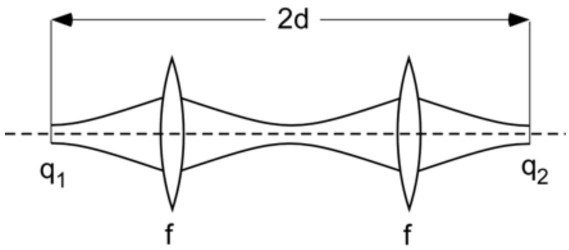
  Solving for the for the real and imaginary part of $q = z + iz_0$ yields

$$z = \frac{1}{2}\frac{A - D}{C}$$
$$z_0 = \sqrt{-z^2 - \frac{B}{C}}.$$

  The real part z of the beam parameter describes the position of the focus relative to the reference point. The imaginary part z_0 describes the Rayleigh length of the beam inside the resonator.

- Standing wave resonator with two curved mirrors

  A round trip in a standing wave resonator is equivalent to the lens guide with two lenses.



We write down the ray transfer matrix starting at the center between the mirrors as reference point,

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & \frac{d}{2} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1/f_1 & 1 \end{pmatrix} \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1/f_2 & 1 \end{pmatrix} \begin{pmatrix} 1 & \frac{d}{2} \\ 0 & 1 \end{pmatrix}.$$

From this expression we obtain the matrix elements, which then can be inserted into the general solution for z and z_0. For a resonator with two identical mirrors ($f_1 = f_2$) we get,

$$z_0 = \frac{1}{2}\sqrt{d(2r-d)}$$
$$z = 0.$$

Since $z = 0$, the waist is located at the reference point in the middle between the two mirrors.

- Stability

We replace f by the radius of the mirror

$$r = 2f.$$

and express other length quantities in units of r. For the mirror separation d we use the dimensionless parameter

$$g := 1 - \frac{d}{r}.$$

In the expression for z_0 we eliminate d by g

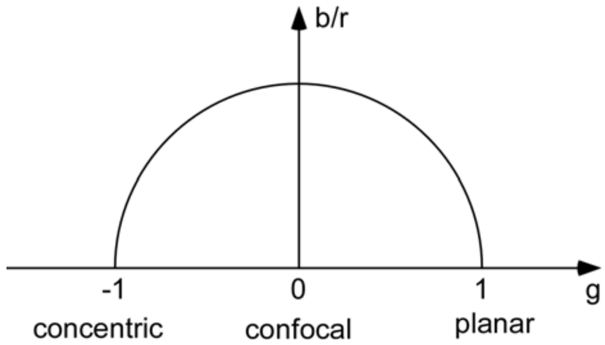
$$z_0 = \frac{1}{2}\sqrt{(1-g)\,r(2r-(1-g)\,r)}$$

and rearrange the terms

$$\left(\frac{2z_0}{r}\right)^2 = \left(\frac{2\cdot\frac{1}{2}\sqrt{(1-g)\,r(2r-(1-g)\,r)}}{r}\right)^2 = 1-g^2$$

$$g^2 + \left(\frac{2z_0}{r}\right)^2 = 1.$$

The separation parameter g and the confocal parameter $b = 2z_0$ normalized to r lie on a half circle with radius 1.



For $|g| > 1$ the resonator is unstable.

- Resonances and mode indices

  Maxima of the power in a resonator occur if the phase of the field reproduces after one round trip such that successive round trips interfere constructively. The phase the field accumulates while propagating from one mirror to the other is given by the exponent of the complex phase factor of the mode,

$$\varphi = k \cdot d - \underbrace{2(m+n+1)\arctan(\frac{d/2}{z_0})}_{\text{Gouy-phase}}.$$

  A full round trip changes the phase by twice that value and an intensity maximum is reached if this total phase is a multiple of 2π,

$$2\varphi = k \cdot 2d - 4(m+n+1)\arctan(\frac{d/2}{\frac{1}{2}\sqrt{d(2r-d)}}) = 2\pi q.$$

  Such a maximum is called "resonance" of the resonator. Each resonance is associated with a mode specified by the three integers q, m, n. Such a mode has $q+1$ longitudinal nodes (intersections with the optical axis) and $m+n$ nodal lines in the transverse plane.

- Spectrum of a standing wave resonator

  We solve the resonance condition for the wavenumber

$$k_{q,m,n} = \frac{\pi}{d}q + \frac{2}{d}(m+n+1)\arctan(\frac{d/2}{\frac{1}{2}\sqrt{d(2r-d)}})$$

and multiply with $c/2\pi$ to get the oscillation frequency $\nu(q,m,n)$,

$$\nu(q,m,n) = \frac{c}{2d}q + \frac{c}{2d}\frac{1}{\pi}(m+n+1)\cdot 2\arctan(\frac{1}{\sqrt{\frac{2r}{d}-1}}).$$

The arctan-function can be expressed as

$$2\arctan(\frac{1}{\sqrt{\frac{2}{x}-1}}) = \arccos(1-x)$$

and by introducing the free spectral range

$$\nu_0 := \frac{c}{2d}.$$

and the transverse oscillation frequency

$$\nu_t := \nu_0 \cdot \frac{1}{\pi}\arccos(1-\frac{d}{r})$$

we finally obtain the so called spectrum of the resonator

$$\nu\,(q,m,n) = q\nu_0 + (m+n+1)\nu_t.$$

Here, the mode indices q, m, n very much resemble the quantum numbers of an atomic state as for instance a hydrogen state with principle quantum number n, and angular momentum quantum numbers l and m. In both cases theses integers specify the solution of a wave equation and its oscillation frequency. In the quantum theory of light, the oscillation frequency ν of a mode is associated to an energy quantum $E = h\nu$, which is called "photon". If the light in the resonator is reduced to this smallest energy quantum, one can say that the resonator contains a single photon in a state (orbital) with quantum numbers q, m, n and energy $h\nu\,(q,m,n)$.

(Remark: In our description a value of $q = 0$ makes no sense and $q \in (1, 2..)$. In literature, q is sometimes defined including zero. In this case, $q\nu_0$ in the above expression for the spectrum is replaced by $(q+1)\,\nu_0$.)

- Planar resonator

  For a planar resonator with

$$\frac{d}{r} = 0$$

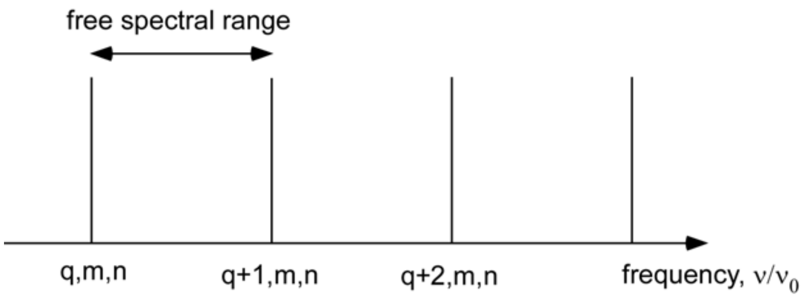
  the transverse oscillation frequency vanishes

$$\nu_t = \nu_0 \cdot \frac{1}{\pi}\arccos(1-0) = 0$$

and the spectrum is

$$\frac{\nu(q,m,n)}{\nu_0} = q.$$

The Gouy-phase can be neglected and the resonance frequency is independent of m and n. All transverse modes are degenerate.



- Concentric resonator

  For a concentric resonator with

  $$\frac{d}{r} = 2$$

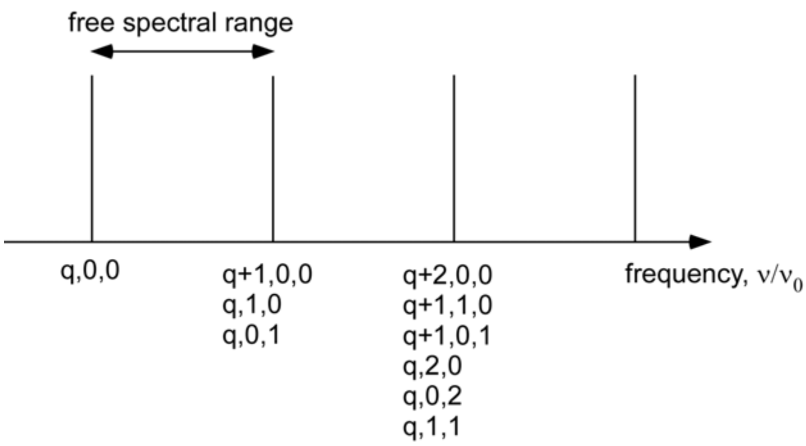
  the transverse oscillation frequency equals the free spectral range,

  $$\begin{aligned}\nu_t &= \nu_0 \cdot \frac{1}{\pi}\arccos(1-2)\\ &= \nu_0 \cdot \frac{1}{\pi}\arccos(-1) = \nu_0.\end{aligned}$$

  The spectrum simplifies to

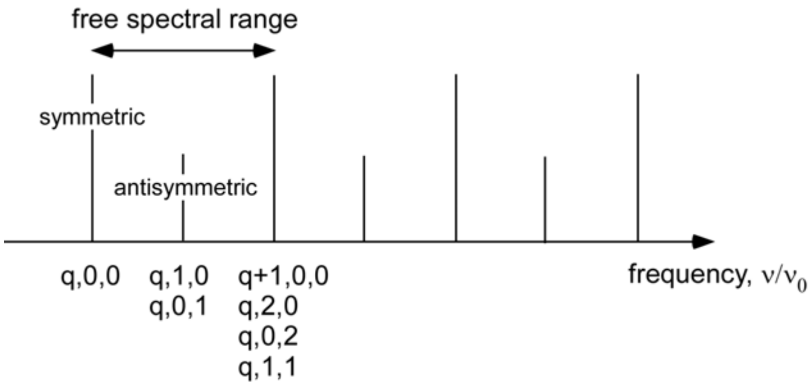
  $$\frac{\nu(q,m,n)}{\nu_0} = q + (m+n+1).$$

  Transverse modes with the same number of transverse nodal lines $(m+n)$ are degenerate.



- Confocal resonator

  For a confocal resonator with
  $$\frac{d}{r} = 1$$
  the transverse oscillation frequency is half the free spectral range,
  $$\begin{aligned}\nu_t &= \nu_0 \cdot \frac{1}{\pi} \arccos(1-1) \\ &= \nu_0 \cdot \frac{1}{\pi} \arccos(0) = \nu_0 \cdot \frac{1}{\pi}\frac{\pi}{2} \\ &= \frac{\nu_0}{2}.\end{aligned}$$
  For the spectrum we obtain
  $$\frac{\nu(q,m,n)}{\nu_0} = (q+1) + \frac{1}{2}(m+n+1).$$
  The spectrum consists of two equidistant parts shifted by half a free spectral range , $\nu_0/2$. One part contains all modes with an even number transverse modal lines including the fundamental mode with zero nodal line. The other part contains all modes with an odd number of transverse nodal lines. The wave functions of the first part are symmetric with respect to an inversion of the transverse plane $(x, y \to -x, -y)$. The wave functions of the second part are antisymmetric.



- Transverse modes and harmonic oscillator

  We can compare the transverse motion of light rays of geometric optics inside a resonator with the spectrum of the modes. It turns out, that the transverse motion of a ray forms an oscillation around the optical axis with a frequency that is identical to the transverse frequency ν_t from above. Qualitatively this makes perfect sense: For a planar resonator, this frequency is zero and a beam tilted towards the optical axis, runs away in the transversal plane and never comes back. A small mirror curvature is sufficient for the ray to return after a large number of round trips. In this case, the transverse oscillation frequency is small and the transverse modes are almost degenerate. In a confocal resonator, each ray returns to its starting point after two round trips. The transverse frequency is half the inverse round trip time and the transverse mode spectrum has a mode spacing of half the free spectral range. For a concentric resonator, every ray (which goes through the center) returns to its starting point after one round trip. The transverse mode spacing is as large as the free spectral range.

As we have seen, that the light distribution in the resonator is given by Hermite polynomials, which are also the solutions of the Schrödinger equation for the harmonic oscillator. The "wave-function" of the photons in the resonator is therefore the same as that of a massive particle in an harmonic potential. One can say that a resonator with curved mirrors forms a transverse harmonic trap for photons.

- More complex resonators

  The physics of resonators is surprisingly rich and there are many interesting topics still to be discussed. For instance, what happens in a ring resonator with two degenerate counter propagating modes? Is the spectrum different for the two polarization of the light? What if the mirrors do not lie in the same plane? What if the two modes are coupled by something (glass plate, grain of dust, atoms)? What if the mirrors are slightly distorted and form an ellipsoidal surface. What about elements in the resonator which change their optical properties with circulating power?... All theses systems can be analyzed with the methods introduced in this chapter. Much has been investigated in the last decades but there is still active research on various resonator systems with active and passive elements. In the appendix we discuss some more aspects of resonators for those who didn't get enough.

Appendix

(special topics, not part of the examination)

A1. Asymmetrical modes in a confocal resonator

One may think that for symmetry reasons in a standing wave resonator with two identical mirrors the waist of the modes must be at the center of the resonator half way between the two mirrors. However, in a confocal resonator this is not the case. Let's have a closer look.

In a resonator with two identical mirrors with radius r, the curvature of the Gaussian beam and the curvature of the mirror has to match,

$$r = R(d_1) = d_1\left(1 + \left(\frac{z_0}{d_1}\right)^2\right)$$
$$-r = R(d_2) = -d_2\left(1 + \left(\frac{z_0}{-d_2}\right)^2\right)$$

The lengths d_1 and d_2 denote the distance between the beam waist and the two mirrors. We will see that d_1 and d_2 are not necessarily identical in a confocal resonator. In fact the waist can be anywhere on the optical axis.

Solving for d_1 and d_2

$$d_1 = \frac{1}{2}r \pm \frac{1}{2}\sqrt{(r^2 - 4z_0^2)}$$
$$d_2 = \frac{1}{2}r \pm \frac{1}{2}\sqrt{(r^2 - 4z_0^2)}$$

provides four values for the mirror separation,

$$d_{++} = d_{1+} + d_{2+} = r + \sqrt{(r^2 - 4z_0^2)}$$
$$d_{--} = d_{1-} + d_{2-} = r - \sqrt{(r^2 - 4z_0^2)}$$
$$d_{+-} = d_{1+} + d_{2-} = r$$
$$d_{-+} = d_{1-} + d_{2+} = r$$

For the two cases with the separation d_{++} and d_{--}, the waist is a the center since $d_1 = d_2$. These cases correspond to the usual solutions of a standing wave resonator which obeys

$$z_0 = \frac{1}{2}\sqrt{2rd - d^2}.$$

(to check it, solve for d). The other two cases with $d_{+-} = d_{-+} = r$ represent confocal resonators with $d_1 \neq d_2$.

We calculate the Rayleigh length z_0 of the resonator by means of the two curvature condition from above which we combine to obtain

$$d_1\left(1 + \left(\frac{z_0}{d_1}\right)^2\right) = d_2\left(1 + \left(\frac{z_0}{-d_2}\right)^2\right)$$
$$d_1 + \frac{z_0^2}{d_1} = d_2 + \frac{z_0^2}{d_2}$$
$$z_0^2\left(\frac{1}{d_1} - \frac{1}{d_2}\right) = d_2 - d_1$$
$$z_0^2 = d_1 d_2$$

We introduce the total length

$$d := d_1 + d_2$$

and the waist displacement from the center

$$\Delta := \frac{1}{2}(d_1 - d_2)$$

to rewrite

$$\begin{aligned} d_1 &= \frac{d}{2} - \Delta \\ d_2 &= \frac{d}{2} + \Delta. \end{aligned}$$

With this, we obtain a simple connection between Rayleigh length and waist displacement

$$\begin{aligned} z_0^2 = d_1 d_2 &= \left(\frac{d}{2} - \Delta\right)\left(\frac{d}{2} + \Delta\right) \\ &= \left(\frac{d}{2}\right)^2 - z_0^2 \\ z_0^2 + \Delta^2 &= \left(\frac{d}{2}\right)^2 . \end{aligned}$$

The Rayleigh length z_0 and the waist displacement Δ lie on a circle with a radius of half the resonator length $d/2$.

We now look at the spectrum for the asymmetric case. With the Gouy phase

$$\varphi_G = -(m + n + 1)\arctan\left(\frac{z}{z_0}\right)$$

the phase collected along the path from the left to the right mirror is

$$\varphi = kd_2 - (m + n + 1)\arctan\left(\frac{d_2}{z_0}\right) + kd_1 - (m + n + 1)\arctan\left(\frac{d_1}{z_0}\right).$$

Resonances are obtained for $2\varphi = q \cdot 2\pi$, with q being an integer number,

$$q\pi = kd - (m + n + 1)\arctan\left(\frac{d_2}{z_0}\right) - (m + n + 1)\arctan\left(\frac{d_1}{z_0}\right).$$

With the free spectral range

$$\nu_0 := \frac{c}{2d}$$

and

$$\nu = \frac{c}{2\pi}k$$

we obtain

$$\frac{c}{2\nu_0} \cdot \frac{2\pi\nu}{c} = (m+n+1)\arctan\left(\frac{d_2}{z_0}\right) + (m+n+1)\arctan\left(\frac{d_1}{z_0}\right) + q\pi$$

$$\frac{\nu}{\nu_0} = \frac{1}{\pi}(m+n+1)\left(\arctan\left(\frac{d_1}{z_0}\right) + \arctan\left(\frac{d_2}{z_0}\right)\right) + q.$$

Using

$$\arctan(x) + \arctan(y) = \arctan\left(\frac{x+y}{1-xy}\right)$$

we get

$$\arctan\left(\frac{d_1}{z_0}\right) + \arctan\left(\frac{d_2}{z_0}\right) = \arctan\left(\frac{d/z_0}{1-d_1 d_2/z_0^2}\right).$$

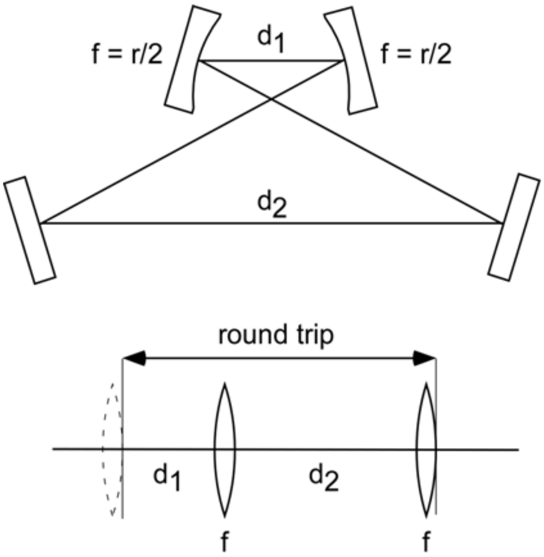
Inserting the relation $z_0^2 = d_1 d_2$ from above, the denominator on the right side becomes zero and $\arctan(\infty) = \pi/2$. The spectrum now reads

$$\frac{\nu}{\nu_0} = \frac{1}{2}(m+n+1) + q.$$

The asymmetric modes have the same spectrum as the symmetric modes. Consequently, the resonator simultaneously supports modes with different Rayleigh lengths and waist positions.

A2. Stability and Rayleigh lengths of bowtie resonators

Bowtie resonators are ring resonators with identical mirrors but different distances. They are often used in single-mode lasers and for frequency doubling with nonlinear crystals. There are two modes in the resonator. One mode propagates directly between the two curved mirrors. The second one reaches from one curved mirror via the two plane mirrors to the second curved mirror.



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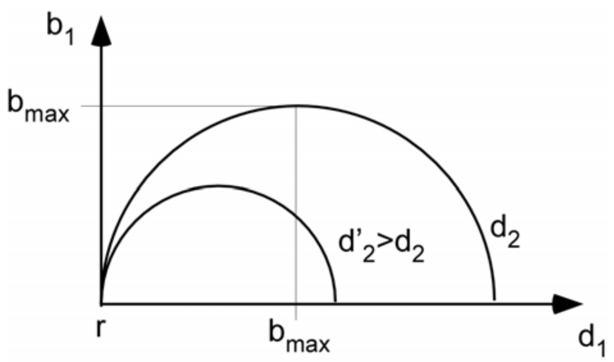
With the same parameters as introduced for standing wave resonators

$$g_1 = 1 - \frac{d_1}{r} \qquad g_2 = 1 - \frac{d_2}{r}$$

one can write the equations for the Rayleigh length of the two beams as (calculation as exercise)

$$\left(\frac{b_1}{r}\right)^2 = -g_1^2 + \frac{g_1}{g_2}$$
$$\left(\frac{b_2}{r}\right)^2 = -g_2^2 + \frac{g_2}{g_1}.$$

We plot the first equation and obtain a similar semi circle as for a standing wave resonator,



The maximum confocal parameter for the beam between the curved mirror is given by

$$b_{1\,\max} = \frac{1}{2}\frac{r^2}{d_2 - r}.$$

A very focused beam with a small beam waist is obtained for $d_2 \gg d_1$. Such resonators are used for the construction of titanium-sapphire lasers and whenever you need a small waist radius with high intensity.

A3. Astigmatism

The Gaussian beam profile

$$I = I_0 \exp(-2r^2/w^2)$$

can be written as the product of function of x and y,

$$I = I_0 \exp(-2x^2/w_x^2) \cdot \exp(-2y^2/w_y^2).$$

In general, the two associated beam radius functions $w_x(z)$ and $w_y(z)$ are not necessarily identical. If the waist of the two functions have different radii the beam is elliptic and if the position of the waists is different the beams are called astigmatic.

- Effective radius of curvature

  In a ring resonator, the beam may hit a curved mirror at an incident angle θ larger than zero (angle to the mirror normal). This leads to different effective curvatures for the x- and the y-part of the beam. For the direction that lies in the plane of incidence (y-direction) the radius of curvature is

  $$r_y = r \cdot \cos(\theta),$$

  and for the direction perpendicular to the plane of incidence (x-direction) it is

  $$r_x = \frac{r}{\cos(\theta)}.$$

  (See Jenkins and White, Fundamentals of Optics, McGraw-Hill, New York, 1957, page 95). The two radius functions $w_x(z)$ and $w_y(z)$ are calculated independently with ABCD-matrix formalism with the mirror radius r being replaced by the effective radii r_x and r_y.

- Crystal at Brewster's angle

  We consider a plane-parallel plate of thickness d made of a transparent optical material with refraction index n (glass, laser crystal etc.). The plate is tilted at Brewster's angle such that p-polarized light is not reflected. In this case the plate can be described by an ABCD-Matrix for an effective distance d_{eff} that differs for both directions:

  $$d_{eff}^{(x)} = d\frac{\sqrt{n^2+1}}{n^2}$$
  $$d_{eff}^{(y)} = d\frac{\sqrt{n^2+1}}{n^4}.$$

  The thickness d can also be expressed as the geometric path length ξ of the light in the plate. At Brewster angle ξ is longer than d by the factor

  $$\frac{\xi}{d} = \frac{\sqrt{n^2+1}}{n}$$

  such that

  $$d_{eff}^{(x)} = \frac{\xi}{n}$$
  $$d_{eff}^{(y)} = \frac{\xi}{n^3}.$$

A4. Mode spectra of ring resonators

In the ring resonator, the beam inevitably hits curved mirrors at an angle, whose effective radii differ for the x- and the y-variable. The beam loses its cylindrical symmetry and the transverse coordinates x and y have to be treated separately. This chapter is quite technical. Those who are not interested in the lengthy calculation can skip it and directly jump to the easy-to-interpret results.

- Ansatz

  We start with the Ansatz

$$\Psi(x, y, z) = g(x, z) f(y, z).$$

The paraxial wave equation

$$\frac{\partial^2}{\partial x^2}\Psi + \frac{\partial}{\partial y^2}\Psi - 2ik\frac{\partial \Psi}{\partial z} = 0$$

now reads

$$f\frac{\partial^2}{\partial x^2}g + g\frac{\partial}{\partial y^2}f - 2ik\left(f\frac{\partial}{\partial z}g + g\frac{\partial}{\partial z}f\right) = 0.$$

Dividing by $g \cdot f$ yields

$$\frac{1}{g}\frac{\partial^2}{\partial x^2}g + \frac{1}{f}\frac{\partial}{\partial y^2}f - 2ik\left(\frac{1}{g}\frac{\partial}{\partial z}g + \frac{1}{f}\frac{\partial}{\partial z}f\right) = 0$$

$$\left(\frac{1}{g}\frac{\partial^2}{\partial x^2}g - 2ik\frac{1}{g}\frac{\partial}{\partial z}g\right) + \left(\frac{1}{f}\frac{\partial}{\partial y^2}f - 2ik\frac{1}{f}\frac{\partial}{\partial z}f\right) = 0.$$

Since the first bracket only depends on x and the second only on y, both brackets must disappear independently. We first consider only the first bracket and solve the equation

$$\frac{1}{g}\frac{\partial^2}{\partial x^2}g - 2ik\frac{1}{g}\frac{\partial}{\partial z}g = 0$$

$$\frac{\partial^2}{\partial x^2}g - 2ik\frac{\partial}{\partial z}g = 0$$

- Equation for the x direction

  To solve the equation

$$\frac{\partial^2}{\partial x^2}g - 2ik\frac{\partial}{\partial z}g = 0$$

we try the Ansatz

$$g(x, z) = H\left(\sqrt{2}\frac{x}{w(z)}\right)e^{-i\left(P_x(z) + \frac{k}{2q_x(z)}\cdot x^2\right)}$$

with a real function $H\left(\sqrt{2}\frac{x}{w(z)}\right)$. The functions $q_x(z)$ we know already from the discussion of Gaussian beams. It obeys the differential equations

$$\frac{d}{dz}q_x = 1$$

with the solution

$$q_x = z + iz_{0x}.$$

The function $P_x(z)$ we discuss later.

The first derivative of $g(x,z)$ with respect to x is

$$\begin{aligned}
\frac{d}{dx}g(x,z) &= H\frac{d}{dx}e^{-i\left(P_x+\frac{k}{2q_x}\cdot x^2\right)} + e^{-i\left(P_x(z)+\frac{k}{2q_x(z)}\cdot x^2\right)}\frac{d}{dx}H \\
&= -i\frac{k}{q_x}x\cdot g(x,z) + g(x,z)\frac{1}{H}\frac{d}{dx}H \\
&= \left(-i\frac{k}{q_x}x + \frac{1}{H}\frac{d}{dx}H\right)g(x,z).
\end{aligned}$$

The second derivative is

$$\begin{aligned}
\frac{d^2}{dx^2}g(x,z) &= \frac{d}{dx}\left(-i\frac{k}{q_x}x + \frac{1}{H}\frac{d}{dx}H\right)g(x,z) + \left(-i\frac{k}{q_x}x + \frac{1}{H}\frac{d}{dx}H\right)\frac{d}{dx}g(x,z) \\
&= \left(\frac{d}{dx}\left(-i\frac{k}{q_x}x + \frac{1}{H}\frac{d}{dx}H\right) + \left(-i\frac{k}{q_x}x + \frac{1}{H}\frac{d}{dx}H\right)^2\right)g(x,z).
\end{aligned}$$

The derivative with respect to z is

$$\begin{aligned}
\frac{d}{dz}g(x,z) &= H\frac{d}{dz}e^{-i\left(P_x(z)+\frac{k}{2q_x(z)}\cdot x^2\right)} + e^{-i\left(P_x(z)+\frac{k}{2q_x(z)}\cdot x^2\right)}\frac{d}{dz}H \\
&= \left(-i\left(\frac{d}{dz}P_x - \frac{kx^2}{2q_x^2}\frac{d}{dz}q_x\right) + \frac{1}{H}\frac{d}{dz}H\right)g(x,z).
\end{aligned}$$

Inserting all this into the differential equation

$$\frac{\partial^2}{\partial x^2}g - 2ik\frac{\partial}{\partial z}g = 0$$

yields

$$\begin{aligned}
0 =& \left(\frac{d}{dx}\left(-i\frac{k}{q_x}x + \frac{1}{H}\frac{d}{dx}H\right) + \left(-i\frac{k}{q_x}x + \frac{1}{H}\frac{d}{dx}H\right)^2\right)g(x,z) \\
&- 2ik\left(-i\left(\frac{d}{dz}P_x - \frac{kx^2}{2q_x^2}\frac{d}{dz}q_x\right) + \frac{1}{H}\frac{d}{dz}H\right)g(x,z) \\
0 =& \frac{d}{dx}\left(-i\frac{k}{q_x}x + \frac{1}{H}\frac{d}{dx}H\right) + \left(-i\frac{k}{q_x}x + \frac{1}{H}\frac{d}{dx}H\right)^2 - 2k\left(\frac{d}{dz}P_x - \frac{kx^2}{2q_x^2}\frac{d}{dz}q_x\right) - 2ik\frac{1}{H}\frac{d}{dz}H.
\end{aligned}$$

The individual terms can be simplified. With

$$\frac{d}{dx}\left(-i\frac{k}{q_x}x+\frac{1}{H}\frac{d}{dx}H\right)=-i\frac{k}{q_x}-\frac{1}{H^2}\left(\frac{d}{dx}H\right)^2+\frac{1}{H}\frac{d^2}{dx^2}H$$

and

$$\left(-i\frac{k}{q_x}x+\frac{1}{H}\frac{d}{dx}H\right)^2=-\left(\frac{k}{q_x}x\right)^2-2i\frac{k}{q_x}x\frac{1}{H}\frac{d}{dx}H+\left(\frac{1}{H}\frac{d}{dx}H\right)^2$$

the first two terms can be written as

$$-i\frac{k}{q_x}-\left(\frac{k}{q_x}x\right)^2+\frac{1}{H}\frac{d^2}{dx^2}H-2i\frac{k}{q_x}x\frac{1}{H}\frac{d}{dx}H.$$

The differential equation now reads

$$-i\frac{k}{q_x}-\left(\frac{k}{q_x}x\right)^2+\frac{1}{H}\frac{d^2}{dx^2}H-2i\frac{k}{q_x}x\frac{1}{H}\frac{d}{dx}H-2k\frac{d}{dz}P_x+\frac{k^2x^2}{q_x^2}\frac{d}{dz}q_x-2ik\frac{1}{H}\frac{d}{dz}H=0$$

$$\left(\left(1-\frac{d}{dz}q_x\right)\frac{k^2}{q_x^2}x^2+2k\frac{d}{dz}P_x+i\frac{k}{q_x}\right)H-\frac{d^2}{dx^2}H+2i\frac{k}{q_x}x\frac{d}{dx}H+2ik\frac{d}{dz}H=0$$

Since $dq_x/dz=1$ the first term vanished. For the function $P_x(z)$ we require that

$$\left(2\frac{d}{dz}P_x+i\frac{1}{q_x}\right)k=-j\frac{2}{w^2(z)}$$

with j being an integer number. We discuss the consequences of this equation later.

- Hermit polynomials

  The differential equation now reads

$$-j\frac{2}{w^2(z)}H-\frac{d^2}{dx^2}H+2i\frac{k}{q_x}x\frac{d}{dx}H+2ik\frac{d}{dz}H=0.$$

  We rewrite the last term

$$2k\frac{d}{dz}H=2k\frac{dH}{d\left(\sqrt{2}\frac{x}{w(z)}\right)}\frac{d\left(\sqrt{2}\frac{x}{w(z)}\right)}{dz}=-2k\frac{dH}{d\left(\frac{x}{w(z)}\right)}\frac{x}{w^2(z)}\frac{d}{dz}w(z)$$

  where

$$\frac{d}{dz}w(z)=\frac{d}{dz}\left(w_0\sqrt{\left(1+\left(\frac{z}{z_{0x}}\right)^2\right)}\right)=w_0\frac{\frac{1}{2}}{\sqrt{\left(1+\left(\frac{z}{z_{0x}}\right)^2\right)}}\frac{d}{dz}\left(\left(\frac{z}{z_{0x}}\right)^2\right)=\frac{w_0^2}{w(z)}\frac{z}{z_{0x}^2}$$

and

$$\frac{dH}{d\left(\frac{x}{w(z)}\right)} = w\left(z\right)\frac{dH}{dx}.$$

With this we obtain

$$2k\frac{d}{dz}H = -2kw\left(z\right)\frac{dH}{dx}\frac{x}{w^2\left(z\right)}\frac{w_0^2}{w\left(z\right)}\frac{z}{z_{0x}^2} = -2\frac{zkw_0^2}{w^2\left(z\right)}\frac{1}{z_{0x}^2}x\frac{dH}{dx} = -\frac{4}{w^2\left(z\right)}\frac{z}{z_{0x}}x\frac{dH}{dx}$$

Inserting into the differential equation yields

$$-j\frac{2}{w^2\left(z\right)}H - \frac{d^2}{dx^2}H + 2i\frac{k}{q_x}x\frac{d}{dx}H - i\frac{4}{w^2\left(z\right)}\frac{z}{z_{0x}}x\frac{dH}{dx} = 0.$$

$$-j\frac{2}{w^2\left(z\right)}H - \frac{d^2}{dx^2}H + 2\left(i\frac{k}{q_x} - i\frac{2}{w^2\left(z\right)}\frac{z}{z_{0x}}\right)x\frac{dH}{dx} = 0.$$

The bracket can be simplified

$$\begin{aligned}\left(i\frac{k}{q_x} - i\frac{2}{w^2\left(z\right)}\frac{z}{z_{0x}}\right) &= i\left(\frac{kq_x^*}{q_xq_x^*} - \frac{2}{w^2\left(z\right)}\frac{z}{z_{0x}}\right)\\ &= i\left(\frac{k\left(z - iz_{0x}\right)}{z_{0x}^2\left(\frac{z^2}{z_{0x}^2}+1\right)} - \frac{2}{w^2\left(z\right)}\frac{z}{z_{0x}}\right)\\ &= i\left(2\frac{\left(z - iz_{0x}\right)}{z_{0x}w^2\left(z\right)} - \frac{2}{w^2\left(z\right)}\frac{z}{z_{0x}}\right)\\ &= \frac{2}{w^2\left(z\right)}.\end{aligned}$$

such that

$$-j\frac{2}{w^2\left(z\right)}H - \frac{d^2}{dx^2}H + 2\frac{2}{w^2\left(z\right)}x\frac{dH}{dx} = 0.$$

$$jH + \frac{w^2\left(z\right)}{2}\frac{d^2}{dx^2}H - 2x\frac{dH}{dx} = 0.$$

We replace the variable x by

$$x' = \sqrt{2}\frac{x}{w\left(z\right)},$$

which changes the second term into

$$\frac{dH^2}{dx^2} = \frac{dH^2}{dx'^2}\frac{2}{w^2\left(z\right)}.$$

With

$$m := \frac{j}{2}$$

we finally obtain

$$2mH(x') + \frac{d^2}{dx'^2}H(x') - 2x\frac{d}{dx'}H(x') = 0.$$

This is the Hermite equation. The solutions are the Hermite polynomials.

- Gouy-phase

  We now look at the equation for $P(z)$

$$\begin{aligned}
\left(2\frac{d}{dz}P_x + i\frac{1}{q_x}\right)k &= -j\frac{2}{w^2(z)} = -2m\frac{2}{w^2(z)} \\
\frac{d}{dz}P_x &= -i\frac{1}{2q_x} - 2m\frac{1}{kw^2(z)} \\
&= -i\frac{1}{2(z+iz_{0x})} - 2m\frac{1}{kw_{0x}^2\left(1+\frac{z^2}{z_{0x}^2}\right)} \\
&= -i\frac{1}{2}\frac{z-iz_{0x}}{z^2+z_{0x}^2} - 2m\frac{1}{2z_{0x}\left(1+\frac{z^2}{z_{0x}^2}\right)} \\
&= -i\frac{1}{2}\frac{z-iz_{0x}}{z^2+z_{0x}^2} - \frac{2m}{2}\frac{z_{0x}}{z^2+z_{0x}^2} \\
&= -i\frac{1}{2}\frac{z}{z^2+z_{0x}^2} - \frac{1}{2}\left(\frac{z_{0x}(1+2m)}{z^2+z_{0x}^2}\right)
\end{aligned}$$

With

$$\begin{aligned}
P_x &:= R + iQ \\
\frac{dP_x}{dz} &= \frac{dR}{dz} + i\frac{dQ}{dz} = -i\frac{1}{2}\frac{z}{z^2+z_{0x}^2} - \frac{1}{2}\left(\frac{z_{0x}(1+m)}{z^2+z_{0x}^2}\right)
\end{aligned}$$

we obtain differential equations for the real and imaginary part of P

$$\begin{aligned}
\frac{dR}{dz} &= -\frac{1}{2}\left(\frac{z_{0x}(1+2m)}{z^2+z_{0x}^2}\right) \\
\frac{dQ}{dz} &= -\frac{1}{2}\frac{z}{z^2+z_{0x}^2} = -\frac{1}{2}\frac{1}{z_{0x}}\frac{\frac{z}{z_{0x}}}{\frac{z^2}{z_{0x}^2}+1},
\end{aligned}$$

The solutions are

$$\begin{aligned}
R &= -\frac{1}{2}(1+m)\arctan\frac{z}{z_{0x}} \\
Q &= -\frac{1}{2}\ln\left(\sqrt{1+\left(\frac{z}{z_{0x}}\right)^2}\right).
\end{aligned}$$

Thus the phase factor of the initial Ansatz for $g\,(x,z)$becomes

$$-i\left(P(z)+\frac{k}{2}\frac{x^2 z}{z^2+z_{0x}^2}\right)=-\frac{1}{2}\ln\sqrt{1+(\frac{z}{z_{0x}})^2}$$
$$+\,i\left(\frac{1}{2}\,(1+2m)\arctan(\frac{z}{z_{0x}})-\frac{k}{2}\frac{x^2 z}{z^2+z_{0x}^2}\right)$$

- Elliptical Gaussian beam

  Up to this point, the calculation for the x variable is formally identical to that for the cylindrical Gaussian beam: for given effective mirror radii and distances the ABCD matrix can be calculated and self consistency yields the Rayleigh length and waist position for the x-direction. The parameters for y-direction are calculated analogous. The total field is thus

$$u(x,y,z)=\Psi(x,y,z)\cdot e^{-ikz}=g(x,z)f(y,z)\cdot e^{-ikz}$$
$$g(x,z)=H_m\left(\sqrt{2}\frac{x}{w_x\,(z)}\right)\left(e^{-\left(\frac{x}{w_x}\right)^2}e^{i\left(\frac{1}{2}(1+2m)\arctan(\frac{z}{z_{0x}})\right)}e^{-i\frac{k}{2}\cdot\frac{x^2}{R_x(z)}}e^{-\frac{1}{2}\ln\sqrt{1+(\frac{z}{z_{0x}})^2}}\right)$$
$$f(y,z)=H_n\left(\sqrt{2}\frac{y}{w_y\,(z)}\right)\left(e^{-\left(\frac{y}{w_y}\right)^2}e^{i\left(\frac{1}{2}(1+2n)\arctan(\frac{z}{z_{0y}})\right)}e^{-i\frac{k}{2}\cdot\frac{y^2}{R_y(z)}}e^{-\frac{1}{2}\ln\sqrt{1+(\frac{z}{z_{0y}})^2}}\right)$$

or

$$u(x,y,z)=N_{n,m}\,(z)\,H_m\left(\sqrt{2}\frac{x}{w_x}\right)H_n\left(\sqrt{2}\frac{x}{w_y}\right)e^{-\left(\frac{x}{w_x}\right)^2-\left(\frac{y}{w_y}\right)^2}e^{-i\frac{k}{2}\left(\frac{x^2}{R_x(z)}+\frac{y^2}{R_y(z)}\right)}e^{i\theta(z)}e^{-ikz}$$

with the Gouy-phase

$$\theta(z)=\frac{1}{2}\left((1+2m)\arctan(\frac{z}{z_{0x}})+(1+2n)\arctan(\frac{z}{z_{0y}})\right)$$

and the normalization factor

$$N\,(z):=\frac{1}{\sqrt{\sqrt{\frac{\pi}{2}}2^n n!w_{0x}}}\frac{1}{\sqrt{\sqrt{\frac{\pi}{2}}2^m m!w_{0y}}}\left(\sqrt[4]{\left(1+(\frac{z}{z_{0x}})^2\right)\left(1+(\frac{z}{z_{0y}})^2\right)}\right)^{-1}$$
$$=\sqrt{\frac{2}{\pi}}\frac{1}{\sqrt{2^n n!}}\frac{1}{\sqrt{2^m m!}}\frac{1}{\sqrt{w_x\,(z)\,w_y\,(z)}}.$$

For $z_{0x}=z_{0y}$ the filed of a standing wave resonator is reproduced. The Gouy phase is the arithmetic average of the two Gouy phases in x- and y-direction.

- Mode spectrum

We calculate the mode spectrum of a bowtie ring resonator with two identical curved mirror. It supports a weakly focused Gaussian beam along the long path (d_2) and more focused beam along the short path (d_1). Each of the two Gaussian beams is represented by a Rayleigh length for the x variable and one for the y variable, $z_{1x}, z_{1y}, z_{2x}, z_{2y}$. We calculate the phase accumulated in a single round trip of length $L = d_1 + d_2$,

$$\varphi = -k \cdot L + \theta(z).$$

The Gouy phase is

$$\begin{aligned}
\theta(z) &= 2\frac{1}{2}\left((1+2m)\arctan(\frac{d_1/2}{z_{1x}}) + (1+2n)\arctan(\frac{d_1/2}{z_{1y}})\right) \\
&+ 2\frac{1}{2}\left((1+2m)\arctan(\frac{d_2/2}{z_{2x}}) + (1+2n)\arctan(\frac{d_2/2}{z_{2y}})\right) \\
&= (1+2m)\left(\arctan\left(\frac{d_1/2}{z_{1x}}\right) + \arctan\left(\frac{d_2/2}{z_{2x}}\right)\right) \\
&+ (1+2n)\left(\arctan\left(\frac{d_1/2}{z_{1y}}\right) + \arctan\left(\frac{d_2/2}{z_{2y}}\right)\right)
\end{aligned}$$

On resonance the round trip phase is a multiple of 2π,

$$k \cdot L - \theta(z) = 2\pi q$$

Solving for k and multiplying by $c/2\pi$ provides the resonance frequencies

$$\begin{aligned}
\nu(q,m,n) &= q\nu_0 + \left(\frac{1}{2} + m\right)\nu_{tx} + \left(\frac{1}{2} + n\right)\nu_{ty} \\
\nu_{tx} &:= \nu_0 \cdot \frac{\arctan\left(\frac{d_1/2}{z_{1x}}\right) + \arctan\left(\frac{d_2/2}{z_{2x}}\right)}{4\pi} \\
\nu_{ty} &:= \nu_0 \cdot \frac{\arctan\left(\frac{d_1/2}{z_{1y}}\right) + \arctan\left(\frac{d_2/2}{z_{2y}}\right)}{4\pi}
\end{aligned}$$

in units of the free spectral range

$$\nu_0 := \frac{c}{L}.$$

- Geometric phase with an odd number of mirrors

  This spectrum is only valid for resonators with an even mirror number because we have neglected a geometric phase. This phase appears for transverse modes with nodal lines normal to the resonator plane (x-plane).With every reflection on one of the a mirrors (no matter if plane or curved) the mode profile is inverted with respect to the optical axis. For modes with even index m it does not matter because they have symmetrical wave functions

$$H_{m=even}(x) \to H_{m=even}(-x) = H_{m=even}(x).$$

Modes with an odd index m are antisymmetric and therefor change sign.

$$H_{m=odd}(x) \rightarrow H_{m=odd}(-x) = -H_{m=odd}(x).$$

This sign change represents an additional phase of π. The transverse spectrum for the modes in the x-direction are thus shifted by half a free spectral range.

An analogous argument can be applied to the polarization of the light. Each mirror changes the sign of a light field that is linearly polarized in the plane of the resonator (x-direction). Consequently, for resonators with an odd number of mirrors, the resonance frequencies of all modes with x-polarisation, including the fundamental mode, are shifted by half a free spectral range compared to the resonances for light polarized in the y-direction. Ring resonators with an odd number of mirrors are thus birefringent.

6. Semiclassical Laser Theory

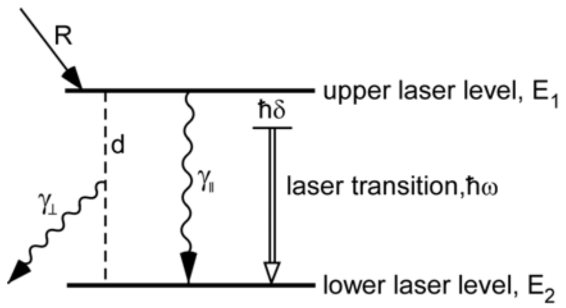
In this chapter, the laser is treated in the so-called semi-classical theory, in which atoms are described by the Schrödinger equation and light by Maxwell's equation. Atomic quantities such as electric polarization and population inversion are expressed by quantum mechanical expectation values. The light is described as electromagnetic wave. Photons do not occur in this picture. The result of the semiclassical laser model is a set of differential equations which, unlike the rate equations, also take into account the phase of the light field and the oscillation of the electric polarization inside the laser medium. From these differential equations, the rate equations, for example, can be derived and justified. In addition, one obtains an expression for the emission frequency of the laser, which may be shifted with respect to the resonant frequency of the optical resonator. This aspect is particularly important for the description of diode lasers. Furthermore, "injection locking" can be described, a phenomenon that occurs when light from a "master laser" is injected into a "slave laser". Under certain circumstances, the oscillation of the "slave" then follows that of the "master" in a phase-locked manner. Related effects are the "mode locking" technique for generating short pulses and the external cavity diode laser (ECDL), a device that uses commercially available laser diodes to realizes a tunable monochromatic light source.

6.1 Theoretical model

- Light

  The light is described as an electromagnetic wave according to Maxwells equations. The electric field $\vec{E}(\vec{r}, t)$ is the important part of the wave because the magnetic field hardly interacts with the atoms of the laser medium.

- Laser medium in two level approximation



The laser medium is approximated as a collection of two-level systems. A single system can carry an electric dipole that decays at a rate $\gamma_\perp$. The population if the excited state decays at a rate $\gamma_\|$. Moreover, the population of the upper laser level can be filled up at a "pump rate" R.

- Interaction

  The atoms form a distribution of Hertzian dipoles which can be driven by the optical field. The dipoles themselves radiate an additional optical field. Thus, the interaction of the light with the laser material takes place via the electric polarization wave $\vec{P}(\vec{r},t)$. It is formed by the density of the dipoles at the location $\vec{r}$ at time t. For its calculation we use the two level model. Field $\vec{E}\,(\vec{r},t)$ and polarization $\vec{P}(\vec{r},t)$ are written as complex quantities. The physically interpretable quantities are as usual the real parts of the complex values. More details of the two level description of atom light interaction is found in many text books. It is also discussed in the lecture notes "Atome, Moleküle und Licht".

6.2 Light modes

- Maxwell' equation

  The light wave is a solution of Maxwells equations. The electric field $\vec{E}(\vec{r},t)$ obeys the wave equation in polarizable matter:

$$\left(\nabla^2 - \frac{n^2}{c^2}\frac{\partial^2}{\partial t^2}\right)\vec{E}(\vec{r},t) = -\frac{1}{\varepsilon_0 c^2}\frac{\partial^2}{\partial t^2}\vec{P}(\vec{r},t).$$

- Separation of variables

  We look for solutions of the form

$$\begin{aligned}\vec{E}(\vec{r},t) &= \tilde{E}(t)\cdot\vec{u}(\vec{r})\\ \vec{P}(\vec{r},t) &= \tilde{P}(t)\cdot\vec{u}(\vec{r}).\end{aligned}$$

  The spatial part $\vec{u}(\vec{r})$ of field and electric polarization is *assumed* to be equal.

We further restrict ourselves to light that is everywhere polarized along the x-direction,

$$\vec{u}\,(\vec{r}) = \vec{e}_x \cdot u\,(\vec{r})\,.$$

With this ansatz one gets

$$\begin{aligned}
\tilde{E}(t) \cdot \nabla^2 \vec{u}(\vec{r}) - \vec{u}(\vec{r}) \cdot \frac{n^2}{c^2}\frac{\partial^2}{\partial t^2}\tilde{E}(t) &= -\frac{1}{\varepsilon_0 c^2}\vec{u}(\vec{r}) \cdot \frac{\partial^2}{\partial t^2}\tilde{P}(t) \\
\nabla^2 u(\vec{r})\vec{e}_x - u(\vec{r})\vec{e}_x \cdot \frac{1}{\tilde{E}(t)}\frac{n^2}{c^2}\frac{\partial^2}{\partial t^2}\tilde{E}(t) &= -\frac{1}{\tilde{E}(t)}\frac{1}{\varepsilon_0 c^2}u(\vec{r})\vec{e}_x \cdot \frac{\partial^2}{\partial t^2}\tilde{P}(t) \\
\frac{\nabla^2 u(\vec{r})}{u(\vec{r})} - \frac{1}{\tilde{E}(t)}\frac{n^2}{c^2}\frac{\partial^2}{\partial t^2}\tilde{E}(t) &= -\frac{1}{\tilde{E}(t)}\frac{1}{\varepsilon_0 c^2}\frac{\partial^2}{\partial t^2}\tilde{P}(t) \\
\frac{\nabla^2 \vec{u}(\vec{r})}{u(\vec{r})} &= \frac{1}{\tilde{E}(t)}\left(\frac{n^2}{c^2}\frac{\partial^2}{\partial t^2}\tilde{E}(t) - \frac{1}{\varepsilon_0 c^2}\frac{\partial^2}{\partial t^2}\tilde{P}(t)\right).
\end{aligned}$$

The left part depends only on the position in space and the right part only on time. The equation holds for all times and all points in space only if both parts are constant and equal. We call this constant K and obtain the two equations

$$\nabla^2 u(\vec{r}) = K u\,(\vec{r})$$

and

$$\frac{1}{\tilde{E}(t)}\left(\frac{n^2}{c^2}\frac{\partial^2}{\partial t^2}\tilde{E}(t) - \frac{1}{\varepsilon_0 c^2}\frac{\partial^2}{\partial t^2}\tilde{P}(t)\right) = K.$$

- Helmholtz equation

  The equation for the spatial part is the so called Helmholtz equation

  $$\nabla^2 u(\vec{r}) = K \cdot u(\vec{r}).$$

  It is an eigenequation with unknown eigenfunktions $u\,(\vec{r})$ and eigenvalues K. It has to be solved for the given boundary conditions set by the mirrors of the optical resonator. The eigenfunctions $u(\vec{r})$ are called "optical modes" of the problem.

- plane wave solutions

  The modes are typically Gaussian beams as discussed in chapter 5. Here we keep things formally simple and assume linearly polarized plane waves and flat mirrors. This means that we drop any field dependence on the coordinates rectangular to the optical axis and write

  $$u(\vec{r}) = u_0 e^{ikz}$$

  The mode propagates along the z-axis and its optical polarization is oriented along the x-axis. By inserting into the Helmholtz equation we get

  $$\begin{aligned}
  \nabla^2 e^{ikz} &= K \cdot e^{ikz} \\
  -k^2 e^{ikz} &= K e^{ikz} \\
  K &= -k^2
  \end{aligned}$$

The possible values for k are obtained from the condition of constructive interference after one round trip in the resonator. If this round trip has the length L, the accumulated phase should be a multiple of 2π,

$$Lk = q \cdot 2\pi.$$

The resulting wave number

$$k = 2\pi q \frac{1}{L}.$$

is connected to the resonance frequency of the "cold" resonator,

$$k = \frac{n}{c}\omega_k.$$

In a "cold" resonator the laser medium is present but not pumped. The refractive index n is that of the unpumped medium. In our model it completely fills the space between the mirrors. The integer number q counts the longitudinal modes. It is equal to the number of half waves that fit between the mirrors and thus on the order of 10^5. In the following we replace the constant K by

$$K = -k^2 = -\frac{n^2}{c^2}\omega_k^2$$

6.3 Differential equations for E and P

- Amplitude equations

  We look at the equation that describes the electric field,

$$\frac{n^2}{c^2}\frac{\partial^2}{\partial t^2}\tilde{E}(t) + \frac{n^2}{c^2}\omega_k^2 \cdot \tilde{E}(t) = \frac{1}{\varepsilon_0 c^2}\frac{\partial^2}{\partial t^2}\tilde{P}(t).$$

  We assume that field and polarization both oscillate at the same frequency which we call ω,

$$\begin{aligned} \tilde{E}(t) &= E(t)e^{-i\omega t} \\ \tilde{P}(t) &= P(t)e^{-i\omega t}. \end{aligned}$$

  We let the amplitudes $E(t)$ and $P(t)$ still change in time. Inserting this ansatz yields

$$\begin{aligned} &\frac{n^2}{c^2}\left(-\omega^2 E(t)e^{-i\omega t} - 2i\omega e^{-i\omega t}\frac{\partial}{\partial t}E(t) + e^{-i\omega t}\frac{\partial^2}{\partial t^2}E(t)\right) + \frac{n^2}{c^2}\omega_k^2 \cdot E(t)e^{-i\omega t} \\ = &\frac{1}{\varepsilon_0 c^2}\left(-\omega^2 P(t)e^{-i\omega t} - 2i\omega e^{-i\omega t}\frac{\partial}{\partial t}P(t) + e^{-i\omega t}\frac{\partial^2}{\partial t^2}P(t)\right) \end{aligned}$$

  which can be rearranged to

$$\left(\omega_k^2 - \omega^2\right)E(t) - 2i\omega\frac{\partial}{\partial t}E(t) + \frac{\partial^2}{\partial t^2}E(t) = \frac{1}{\varepsilon_0 n^2}\left(-\omega^2 P(t) - 2i\omega\frac{\partial}{\partial t}P(t) + \frac{\partial^2}{\partial t^2}P(t)\right).$$

- slowly varying amplitudes

  We assume that the amplitudes vary in time only slowly and that the fast optical oscillations are already described by ω. In this case the second derivatives can be neglected. We even go a step further and also neglected the first derivative for P,

  $$\frac{\partial^2}{\partial t^2}E(t) \simeq \frac{\partial^2}{\partial t^2}P(t) \simeq \frac{\partial}{\partial t}P(t) \simeq 0.$$

  This is the most drastic approximation which still yields a nontrivial result. We obtain

  $$\left(\omega_k^2 - \omega^2\right) E(t) - 2i\omega\frac{\partial}{\partial t}E(t) = -\frac{1}{\varepsilon_0 n^2}\omega^2 P(t).$$

  Since the laser frequency and the resonance frequency of the optical resonator are both optical frequencies and rather similar, we can simplify the prefactor in the first term,

  $$\omega_k^2 - \omega^2 = \left(\omega_k - \omega\right)\left(\omega_k + \omega\right) \simeq 2\omega\left(\omega_k - \omega\right)$$

  to obtain

  $$\frac{\partial}{\partial t}E(t) = i\left(\omega - \omega_k\right)E(t) + \frac{i}{2\varepsilon_0 n^2}\omega P(t).$$

- resonator losses

  The electric field in the resonator decays in time because light power is lost at the output coupler and because of other power losses such as scattering at the optical surfaces and absorption in optical elements. We add a heuristic loss term and obtain

  $$\frac{\partial}{\partial t}E(t) = i\left(\omega - \omega_k\right)E(t) + \frac{i}{2\varepsilon_0 n^2}\omega P(t) - \frac{\gamma_c}{2}E\left(t\right).$$

  The decay rate γ_c (c for "cavity") contains unavoidable but unwanted losses γ_{Loss} and losses γ_{out} connected with the transmission T of the output coupler

  $$\gamma_c = \gamma_{Loss} + \gamma_{out}.$$

  The rate γ_{out} is given by the power losses T per round trip time $\frac{c}{L}$

  $$\gamma_{out} = T\frac{c}{L}.$$

  For a cold resonator ($P(t) = 0$) we recover the equation for a resonator which we know from chapter 5

  $$\frac{\partial}{\partial t}E(t) = i\left(\omega - \omega_k + i\frac{\gamma_c}{2}\right)E(t).$$

  It has the solution

  $$E(t) = E_0 e^{i(\omega-\omega_k)t - \frac{\gamma_c}{2}t},$$

such that

$$\tilde{E}(t) = E_0 e^{i(\omega-\omega_k)t-\frac{\gamma_c}{2}t} e^{-i\omega t} = E_0 e^{-i\omega_k t-\frac{\gamma_c}{2}t}.$$

The intensity thus decays exponentially with γ_c:

$$I(t) \sim \left|\tilde{E}(t)\right|^2 = |E_0|^2 e^{-\gamma_c t}.$$

- Electric polarization

  The polarization is defined as the dipole density

$$P(t) = \frac{N_a}{V} \cdot \rho_a(t)$$

  Here, N_a is the number of two level systems in the volume V of the laser medium. The dipole ρ_a of a single two level system is a complex number

$$\rho_a(t) = d \cdot (u(t) + iv(t))^* .$$

  with real part $u(t)$ and complex part $v(t)$ and a prefactor d. (Do not mix ρ_a with the density matrix ρ in the next section). The prefactor is the modulus of the largest possible dipole moment the two level system may have. The unitless real functions $u(t)$ and $v(t)$ describe the dipole at a given time. They are called "quadrature" components of the dipole moment. We will calculate their values from the quantum model of the two level system.

6.4 Quantum description of a two level system

- Hamilton operator

  We model the laser medium as a collection of two level systems formed by the upper and the lower laser level. The Hamilton operator of a two level system that interacts with a light field is

$$H = \begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix} - \vec{E}_0 \cos(\omega t) \begin{pmatrix} 0 & \left\langle \vec{d} \right\rangle \\ \left\langle \vec{d} \right\rangle^* & 0 \end{pmatrix}.$$

  The first term is the energy of the non-interacting two level system with energy E_1 and E_2 of the upper and the lower state, respectively.

- Dipole interaction

  The second term describes the interaction with the light field. It is modelled as pure dipolar interaction: the electric field excites a dipole oscillation in the two level system which behaves like a Hertzian oscillator i.e. a little antenna. The interaction energy between a dipole $\vec{d}$ and an electric field $\vec{E}$ is taken from classical electrodynamics to be

$$-\vec{d} \cdot \vec{E}.$$

Since the two level system requires a quantum description we replace the dipole by its operator. This is the second matrix in the Hamilton. In most cases energy eigenstates do not carry a dipole moment. A dipole moment is developed only if the system is in a superpositon for the two energy eigenstates. This means that the diagonal elements of the dipole matrix are zero since the expecatation values for the two eigenstates must vanish. Furthermore the matrix must be hermitean such that there is only one independent matrix element left, which is the complex valued vector $\left\langle \vec{d} \right\rangle$. This is the so called "dipole matrix element"

In the case that the dipole moment is generated by an electron that moves around an atomic or molecular core, the matrix element can be calculated from the wave functions $\varphi_{1,2}(\vec{r})$ of the two electronic states that form the two level system

$$\left\langle \vec{d} \right\rangle = \langle \varphi_1 (\vec{r}) | \, e\vec{r} \, | \varphi_2 (\vec{r}) \rangle \, .$$

Here, we don't care about this calculation and simply take the vector $\left\langle \vec{d} \right\rangle$ as an open parameter of the theory.

- Limited dipole amplitude

  In the two level description, the maximum value of the dipole is limited by the dipole matrix element. However, in the real world there is no such limitation because one can simply increase the electric field that displaces the electron of an atom, molecule or hetero structure. Close to ionization the electron can be separated from the core by any value and the dipole can become arbitrarily large. This shows that the two level description is an approximation which is valid only for sufficiently weak electric fields.

- State

  The state $\vec{\psi}$ of the two level system is a two-dimensional vector. It obeys the Schrödinger-equation

$$i\hbar \frac{d\psi}{dt} = H\psi.$$

- Density matrix

  The two-level laser systems can be perturbed by the environment as for instance by collisions with other atoms or molecules in gas lasers or by coupling to lattice vibrations in solid state lasers. Usually such perturbation opens an additional decay channel into the ground state of the four level laser system. In particular, the population of the lower laser level is rapidly depleted and the dipole oscillation rapidly damps out. In the rate model of the laser, these are the "fast decays" which maintain the inversion. The best way to describe the interaction of a two-level system with the environment is to go to the density matrix description. For this we write the general state of the two level system in its energy eigenbasis as

$$\psi(t) = c_1(t) \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-itE_1/\hbar} + c_2(t) \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-itE_2/\hbar}.$$

The density matrix is defined as

$$\rho = \begin{pmatrix} |c_1|^2 & c_1 c_2^* \\ c_1^* c_2 & |c_2|^2 \end{pmatrix}.$$

Obviously, the diagonal elements are just the populations of the upper and lower laser level.

- Temporal evolution

  The equation of motion for the density matrix may be obtained by calculating the derivative

$$\dot{\rho} = \begin{pmatrix} \dot{c}_1 c_1^* + c_1 \dot{c}_1^* & \dot{c}_1 c_2^* + c_1 \dot{c}_2^* \\ \dot{c}_1^* c_2 + c_1^* \dot{c}_2 & \dot{c}_2 c_2^* + c_2 \dot{c}_2^* \end{pmatrix}.$$

  The expressions for the $\dot{c}_i$ are calculated by inserting the Ansatz $\psi(t)$ into the Schrödinger-equation for the two level system. This is discussed in detail in the lecture notes "Atome, Moleküle, Licht" or can be found in standard textbooks under the keywords "two level system" and "Bloch equations". Inserting the expressions for $\dot{c}_i$ into the matrix $\dot{\rho}$ yields the equation of motions for ρ,

$$\begin{aligned} \dot{\rho}_{1,1} &= -\dot{\rho}_{2,2} = i\frac{\Omega}{2}\left(e^{-i\delta t}\rho_{1,2} + e^{i\delta t}\rho_{2,1}\right) \\ \dot{\rho}_{1,2} &= \dot{\rho}_{2,1}^* = i\frac{\Omega}{2}e^{i\delta t}\left(\rho_{1,1} - \rho_{2,2}\right). \end{aligned}$$

  Here, the so called "Rabi-frequency" is defined as the scalar product

$$\Omega := -\frac{1}{\hbar}\left\langle \vec{d} \right\rangle \cdot \vec{E}_0.$$

  It determines the coupling strength between the two level system and the laser light. We further assume that the light is linearly polarized in z-direction and denote the z-component of the electric field amplitude $\vec{E}_0$ as E. The scalar product now reduces to

$$\Omega = -\frac{q}{\hbar}\left\langle \varphi_1 \right| z \left| \varphi_2 \right\rangle \cdot E_z = \frac{1}{\hbar} d \cdot E.$$

  The value of the complex number

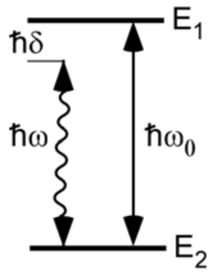
$$d := -q\left\langle \varphi_1 \right| z \left| \varphi_2 \right\rangle = e\left\langle \varphi_1 \right| z \left| \varphi_2 \right\rangle$$

is treated as a parameter of the model.

The detuning

$$\delta := \omega_0 - \omega$$

is the difference between the "Bohr frequency" ω_0 and the frequency ω of the laser light.



The Bohr frequency describes the energy difference between the upper and the lower laser level.

$$\omega_0 := \frac{1}{\hbar}\left(E_1 - E_2\right)$$

This derivation of the equations of motion of the density matrix contains the so called "rotating wave approximation". It neglects terms that oscillate at a frequency $\omega_0 + \omega$. This frequency is in the optical domain and averages out at the much longer time scale at which the c_i are changing. The rotating wave approximation is valid as long as the detuning δ is very much smaller than the laser frequency ω.

- Dipole moment

  The off-diagonal elements $c_1 c_2^*$ and $c_1^* c_2$ are called "coherences". One can show that they are connected to the oscillation of the dipole moment $\rho_a\left(t\right)$ of the two level system,

  $$\rho_a\left(t\right) = d \cdot \rho_{2,1}(t)$$

  (see also lecture notes "Atome, Moleküle, Licht"). As usual, the dipole oscillation is written as a complex function and only its real part is physically relevant.

- Rescaling the coherences

  The right side of the equations of motion explicitly depends on time. This can be simplified with the ansatz

  $$\rho_{1,2} = \tilde{\rho}_{1,2} e^{i\delta t}.$$

- Decay of population

The whole point of changing to the density matrix formalism is that the environment can be correctly taken into account by simply adding the two terms

$$\begin{array}{c} -\gamma_{\|}\rho_{1,1} \\ -\gamma_{\perp}\tilde{\rho}_{1,2} \end{array}$$

to the first and second equation, respectively. This is not obvious but result of a detailed "reservoir theory". As a reservoir we consider everything, which can statistically disturb the two-level system and has no "memory", i.e. is not influenced by the two-level system itself. Besides the noise of the electromagnetic vacuum, this can also be thermal noise, e.g. lattice vibrations or collisions, which influence the two-level system. Due to the coupling to the reservoir, the energy of the two-level system dissipates into the reservoir, i.e. the excited state decays into the lower state and the lower state decays into the ground state. In the process, also the dipole oscillation is damped.

The rate $\gamma_{\|}$ describes the population decay of the upper laser level into the lower laser level and the rate $\gamma_{\perp}$ describes the decay of the dipole moment. Since the dipole moment is proportional to the superposition of the two laser states, it cannot live longer than the lower laser level. A fast decay of the lower laser level thus results in a fast decay of the dipole moment and a large value of $\gamma_{\perp}$. In fact, in most laser schemes the lower laser level decays fast such that usually $\gamma_{\perp}$ is a large number.

- Equation of motion for the density matrix

  By using the scaled coherence and adding the decay terms one finally arrives at

$$\begin{aligned} \frac{d}{dt}\rho_{1,1} &= -\frac{d}{dt}\rho_{2,2} = i\frac{\Omega}{2}\tilde{\rho}_{1,2} - i\frac{\Omega}{2}\tilde{\rho}_{2,1} - \gamma_{\|}\rho_{1,1} \\ \frac{d}{dt}\tilde{\rho}_{1,2} &= \frac{d}{dt}\tilde{\rho}^{*}_{2,1} = -i\delta\tilde{\rho}_{1,2} + i\frac{\Omega}{2}\left(\rho_{1,1} - \rho_{2,2}\right) - \gamma_{\perp}\tilde{\rho}_{1,2}. \end{aligned}$$

- Optical Bloch equations

  A very compact form to write the equations of motion may be obtained by defining the "Bloch vector"

$$\vec{u} = \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} \tilde{\rho}_{1,2} + \tilde{\rho}_{2,1} \\ i\left(\tilde{\rho}_{2,1} - \tilde{\rho}_{1,2}\right) \\ \rho_{1,1} - \rho_{2,2} \end{pmatrix}.$$

  The first two components are twice the real and the imaginary part of $\tilde{\rho}_{1,2}$,

$$\begin{aligned} u &:= \tilde{\rho}_{1,2} + \tilde{\rho}_{2,1} = 2\,\mathrm{Re}\left(\tilde{\rho}_{1,2}\right) \\ v &:= i\left(\tilde{\rho}_{2,1} - \tilde{\rho}_{1,2}\right) = -i\left(\tilde{\rho}_{1,2} - \tilde{\rho}_{2,1}\right) = -i \cdot 2i\,\mathrm{Im}\left(\tilde{\rho}_{1,2}\right) = 2\,\mathrm{Im}\left(\tilde{\rho}_{1,2}\right). \end{aligned}$$

  The third component is the population inversion. In addition, we introduce the vector

$$\vec{\Omega} =: \begin{pmatrix} \Omega \\ 0 \\ \delta \end{pmatrix}$$

that contains the Rabi frequency Ω and the detuning δ. With this, the above equations of motion may be written as

$$\frac{d}{dt}\vec{u} = -\vec{\Omega} \times \vec{u}.$$

or

$$\begin{aligned}
\dot{u} &= \delta \cdot v - \gamma_\perp \cdot u \\
\dot{v} &= -\delta \cdot u - \gamma_\perp \cdot v + \Omega w \\
\dot{w} &= -\Omega v - \gamma_\parallel w.
\end{aligned}$$

These are the "optical Bloch equations".

6.5 Maxwell-Bloch Equations

- Inversion density

  We now are ready to combine the equation for the optical field with those for the laser medium. We first express the microscopic entities of the two level system as macroscopic entities of the laser medium. To this end we define the inversion density

  $$N(t) := \frac{N_a}{V} w(t),$$

  which may be interpreted as the mean number of inverted atoms per volume. N_a is the total number of two level systems in the Volume V of the laser medium.

- Dipole density

  Similarly we define the polarization of the laser medium as its dipole density,

  $$\begin{aligned}
  P(t) &= \frac{N_a}{V} \cdot d \cdot \rho_{2,1}(t) \\
  &= \frac{N_a}{V} \cdot d \cdot (u(t) + iv(t))^* \\
  &= \frac{N_a}{V} \cdot d \cdot (u(t) - iv(t)).
  \end{aligned}$$

- Maxwell-Bloch equations

With these definition we obtain for the derivative of the Polarization

$$\begin{aligned}
\dot{P}(t) &= \frac{N_a}{V} \cdot d \cdot (\dot{u}(t) - i\dot{v}(t)) \\
&= \frac{N_a}{V} \cdot d \cdot (\delta \cdot v - \gamma_\perp \cdot u - i(-\delta \cdot u - \gamma_\perp \cdot v + \Omega w)) \\
&= \frac{N_a}{V} \cdot d \cdot (\delta v - \gamma_\perp u + i\delta \cdot u + i\gamma_\perp v - i\Omega w) \\
&= \frac{N_a}{V} \cdot d \cdot ((i\delta - \gamma_\perp) u + (\delta + i\gamma_\perp) v - i\Omega w) \\
&= \frac{N_a}{V} \cdot d \cdot ((i\delta - \gamma_\perp) u - i(i\delta - \gamma_\perp) v - i\Omega w) \\
&= \frac{N_a}{V} \cdot d \cdot ((i\delta - \gamma_\perp)(u - iv) - i\Omega w) \\
&= (i\delta - \gamma_\perp) \frac{N_a d}{V}(u - iv) - i\frac{N_a d}{V}\Omega w \\
&= (i\delta - \gamma_\perp) P(t) - id\Omega N(t)
\end{aligned}$$

With the definition of Ω from above

$$\Omega = -\frac{q}{\hbar} \langle \varphi_1 | z | \varphi_2 \rangle \cdot E_z = \frac{1}{\hbar} d \cdot E.$$

one finally gets

$$\dot{P}(t) = (i\delta - \gamma_\perp) P(t) - i\frac{d^2}{\hbar} E(t) N(t).$$

Note that in our derivation the Rabi frequency Ω is a complex number since the electric field amplitude E is complex (the matrix element $-\frac{q}{\hbar} \langle \varphi_1 | z | \varphi_2 \rangle = \frac{e}{\hbar} \langle \varphi_1 | z | \varphi_2 > 0 \rangle$ is an observable and thus real. $q = -e$ is the charge of the electron). However, in the Bloch model the electric field amplitude is set to be real. The connection between our complex Ansatz for electric field amplitude $E(t)$ in the Maxwell equations and the real E of the Bloch equations can be understood by realizing that the only the phase differences between N, P, and E are important. The three entities oscillate with two time dependent relative phases. Physically, the overall phase of the total system is not important. We can use this free phase to set the phase of the electric field amplitude $E(t)$ alway zero. This makes $E(t)$ the real valued function that is used in the Bloch equations. Here we leave E complex but keep in mind that with the "free phase" we can always make the Rabi frequency a real positive number.

Similarly to the derivation of $\dot{P}$ we obtain for $\dot{N}$

$$\dot{N}(t) = -\frac{1}{\hbar} \operatorname{Im}(P(t)^* E(t)) - \gamma_\parallel (N(t) - N_0)$$

and for the electric field of the laser light

$$\dot{E}(t) = i\left(\omega - \omega_k + i\frac{\gamma_c}{2}\right) E(t) + \frac{i\omega}{2\varepsilon_0 n^2} P(t).$$

The three equations together are called Maxwell-Bloch equations. They describe a laser in the semiclassical model.

- Pump mechanism

  To describe the pump mechanism, we have included the real number N_0 into the Maxwell-Bloch equations. Its role is best understood if we assume that the resonator is suddenly blocked. The field E becomes zero and the equation for N reduces to

$$\dot{N}(t) = -\gamma_{\|} \left(N(t) - N_0\right).$$

  This is the equation for an exponential decay into a steady state. It is given by $\dot{N} = 0$ such that

$$N(t) = N_0.$$

  In this state pumping compensates the decay of the upper laser level. The value of N_0 thus quantifies the pump strength.

- Photon number

  It is useful to use dimensionless entities. To this end we normalize the electric field E to the field E_0 that a single photon would generate in the resonator,

$$\begin{aligned} a(t) &:= \frac{E(t)}{E_0} \\ E_0 &:= \sqrt{\frac{\hbar\omega}{\varepsilon\varepsilon_0 V_{Mode}}}. \end{aligned}$$

  Since the power is proportional the square of the electric field, $|a|^2$ can be interpreted as number of photons in the resonator. (In a quantized theory of the laser field a would be the field operator.)

- Dipole number

  The dipole density is normalized to the matrix element d and multiplied with the volume V of the laser medium,

$$\pi(t) := P(t)\frac{V}{d} = N_a \left(u\left(t\right) + iv\left(t\right)\right)^*.$$

  Obviously, $|\pi|$ expresses the polarization as the (hypothetical) number of two level systems with dipole d.

- Inversion number

  Similarly, one can express the inversion as the number of completely inverted two level systems in the medium,

$$n(t) := N(t) \cdot V = N_a w(t).$$

- Dimensionless entities

  The three equations now read,

$$
\begin{aligned}
\dot{a}(t) &= \left(-\frac{\gamma_c}{2} + i\left(\omega - \omega_k\right)\right) a(t) + i\frac{\tilde{g}}{2} \cdot \pi(t) \\
\dot{\pi}(t) &= -\gamma_\perp \left(1 + i\alpha\right) \pi(t) - i\tilde{g} \cdot a(t) n(t) \\
\dot{n}(t) &= -\gamma_{||} \left(n(t) - n_0\right) - \tilde{g} \cdot \mathrm{Im}\left(\pi(t)^* a(t)\right).
\end{aligned}
$$

  The detuning δ is written dimensionless in units of the dipole decay rate $\gamma_\perp$,

$$
\alpha := \frac{\delta}{\gamma_\perp} = \frac{\omega_0 - \omega}{\gamma_\perp}.
$$

  For the coupling strength between light and laser medium we use the abbreviation

$$
\tilde{g} := \frac{d}{\hbar}\sqrt{\frac{\hbar\omega}{\varepsilon\varepsilon_0 V_{Mode}}}.
$$

  It has the unit of a frequency and is sometimes called "single photon Rabi-frequency". It can be interpreted as the dipole energy in the electric field of a single photon. The relevant parameters of the system are now all written in units of frequency, s^{-1}. The pump rate is encoded in the steady state inversion

$$
n_0 := N_0/V
$$

  in a "cold" resonator. The frequency ω of the laser light is still unknown and needs to be determined later.

- Structure of the equations

  Without interaction i.e. for $\tilde{g} = 0$ the three equation are exponential differential equations of the form

$$
\frac{dx}{dt} = c\left(x - x_s\right)
$$

  with complex

$$
c = c_r + ic_i.
$$

  Its solution is

$$
x\left(t\right) = x_s + Ce^{ct} = x_s + C \cdot e^{c_r t} \cdot e^{ic_i t}
$$

  In all equations, c_r is negative and the solution decays to the steady state value x_s. It is zero for a and π and n_0 for n. In the equations for a and π the imaginary part c_i is nonzero which means that the entity oscillates during decay. Eventually, all three entities are zero. The interesting physics enters with the interaction terms. In the equation for a the polarization π simply enters as a time dependent source term and the equation remains linear. In the equations for π and n the source terms are products of the two remaining entities. This makes the equations nonlinear and a general analytic solution cannot be found any more.

6.6 Adiabatic elimination of polarization

- Fast decay of polarization

  One of the important features of the four level laser scheme is the fast decay of the lower laser level. As discussed above, this decays enters the model via the polarization decay rate $\gamma_\perp$. We now assume that its value is much larger than the decay rate $\gamma_\|$of the population and the decay rate γ_c of the resonator. This can be used to eliminate the equation for π.

  We consider the equation for the polarization,

$$\dot{\pi}(t) = -\gamma_\perp (1 + i\alpha)\, \pi(t) - i\tilde{g} \cdot a(t) \cdot n(t).$$

  If $\gamma_\perp$is large enough, its the first term that dominates the temporal change of π. If we neglect the last term for a moment we obtain an exponential decay of π with a decay time given by $1/\gamma_\perp$. As a good approximation we can assume that on this time scale the second term is constant in time. We this we obtain an exponential differential equation with an inhomogeneous term. Its general solution is the sum

$$\pi(t) = A \cdot \pi_h(t) + B \cdot \pi_p(t)$$

  of the homogeneous equation $\pi_h \sim \exp\left(-\gamma_\perp (1 + i\alpha)\, t\right)$ (with the second term set to zero) and a particular solution $\pi_p(t)$ with some constants A and B. One particular solution is the steady state solution

$$\begin{aligned} \dot{\pi}(t) &= 0 = -\gamma_\perp (1 + i\alpha)\, \pi_{st}(t) - i\tilde{g} \cdot a(t) \cdot n(t) \\ \pi_{st}(t) &= \frac{-i\tilde{g}}{\gamma_\perp (1 + i\alpha)} a(t) \cdot n(t). \end{aligned}$$

  On the time scale at which the particular solution changes, the homogeneous solution has decayed already. We can thus take the particular solution as a good approximation for $\pi(t)$ at time scales much longer than $1/\gamma_\perp$

$$\pi(t) \simeq \pi_{st}(t).$$

  This type of approximation is called "adiabatic approximation" or "adiabatic elimination".

- Approximate equation for the field

  Replacing $\pi(t)$ by $\pi_{st}(t)$ yields a simplified equation for the field

$$\begin{aligned} \dot{a}(t) &= i\left(\omega - \omega_k + i\frac{\gamma_c}{2}\right) a(t) + i\frac{\tilde{g}}{2} \cdot \frac{-i\tilde{g}}{\gamma_\perp (1 + i\alpha)} a(t) \cdot n(t) \\ &= \left(i(\omega - \omega_k) - \frac{\gamma_c}{2} + \frac{\tilde{g}^2 (1 - i\alpha)}{2\gamma_\perp (1 + \alpha^2)} \cdot n(t)\right) a(t). \end{aligned}$$

We collect the real and the imaginary parts,

$$\dot{a}(t) = \left(i\left(\omega - \omega_k - \frac{1}{2}\kappa \cdot n(t) \cdot \alpha\right) - \frac{1}{2}\left(\gamma_c - \kappa \cdot n(t)\right)\right) a(t),$$

and define a "scaled " coupling constant

$$\kappa := \frac{\tilde{g}^2}{\gamma_\perp \left(1 + \alpha^2\right)}.$$

- Approximate equation for the inversion

  Similarly we obtain for the inversion

$$\begin{aligned}
\dot{n}(t) &= -\gamma_\| \left(n(t) - n_0\right) - \tilde{g}^2 \cdot \mathrm{Im}\left(\frac{i}{\gamma_\perp (1 - i\alpha)}\right) n(t) \left|a(t)\right|^2 \\
&= -\gamma_\| \left(n(t) - n_0\right) - \kappa n(t) \left|a(t)\right|^2. \\
&= -\left(\gamma_\| + \kappa \left|a(t)\right|^2\right) n(t) + \gamma_\| n_0
\end{aligned}$$

- Laser equations

  In summary, we obtain a coupled set of equations for $a(t)$ and $n(t)$

$$\begin{aligned}
\dot{a}(t) &= \left(-\frac{1}{2}\left(\gamma_c - \kappa \cdot n(t)\right) + i\left(\omega - \omega_k - \frac{1}{2}\kappa \cdot n(t) \cdot \alpha\right)\right) a(t) \\
\dot{n}(t) &= -\left(\gamma_\| + \kappa \left|a(t)\right|^2\right) n(t) + \gamma_\| n_0.
\end{aligned}$$

  Because of the products $n(t)a(t)$ and $n(t)\left|a(t)\right|^2$ the equations are still nonlinear. The usual strategy is to look for steady state solutions and analyze small deviations from the steady state.

6.7 Steady state solutions

- Equilibrium of the field

  In equilibrium the field amplitude does not change any more,

$$\dot{a}(t) = 0 = \left(i\left(\omega - \omega_k - \frac{1}{2}\kappa \cdot n(t) \cdot \alpha\right) - \frac{1}{2}\left(\gamma_c - \kappa \cdot n(t)\right)\right) a(t).$$

  The first solution is $a(t) = 0$, i.e. the laser does not operate because it is not pumped strong enough. Above threshold for $a \neq 0$ the real and imaginary part of the expression in the large brackets must vanish,

$$\begin{aligned}
\gamma_c - \kappa \cdot n_{st} &= 0 \\
\omega - \omega_k - \frac{1}{2}\kappa \cdot n_{st} \cdot \alpha &= 0.
\end{aligned}$$

The first equation tells us that above threshold the resonator losses γ_c are compensated by the term $\kappa \cdot n_{st}$ which thus can be interpreted as the optical gain of the medium.

For the steady state inversion we obtain,

$$n_{st} = \frac{\gamma_c}{\kappa}.$$

- Laser frequency

  The second equation provides an expression for the yet unknown laser frequency ω,

  $$\omega = \omega_k + \frac{1}{2}\kappa \cdot n_{st} \cdot \alpha$$

  The second term shifts the laser frequency from the resonance frequency ω_k of the cold cavity mode. This so called "pulling effect" can be interpreted as a change of the refractive index in the inverted medium. We have discussed this effect already for laser diodes where it is used to guide the light within the hetero structure.

- Strength of the pulling effect

  We insert the first equation into the second and use the definition of α

  $$\alpha = \frac{\delta}{\gamma_\perp} = \frac{\omega_0 - \omega}{\gamma_\perp}$$

  to obtain

  $$\omega = \omega_k + \frac{1}{2}\gamma_c \cdot \frac{\omega_0 - \omega}{\gamma_\perp}.$$

  Solving for ω

  $$\omega = \frac{2\omega_k\gamma_\perp + \gamma_c\omega_0}{2\gamma_\perp + \gamma_c}.$$

  Except for laser diodes, $\gamma_c \ll \gamma_\perp$, and the pulling effect is usually quite small,

  $$\omega \simeq \omega_k + \frac{\gamma_c}{2\gamma_\perp}\omega_0 \simeq \omega_k.$$

- Frequency of laser diodes

  In laser diodes the approximation $\gamma_c \ll \gamma_\perp$ is not valid since the resonator is very short and open. The frequency deviation $\frac{1}{2}\kappa \cdot n_{st} \cdot \alpha$ can become quite large. Furthermore, the inversion n_{st} depends on the injection current which therefor can be used to tune the laser frequency. In fact, laser diodes can be tuned over a range of several 10 GHz by simply changing the current. With oscillating currents the laser frequency can be directly modulated with modulation frequencies of up to several 100 MHz. On the other hand, the high current sensitivity requires low noise current sources to avoid random frequency modulation.

- Threshold

  We now look at the steady state condition for the inversion and obtain an expression for the field amplitude,

$$0 = \dot{n} = -\kappa n \left|a\right|^2 + \gamma_{\|} \left(n_0 - n\right)$$

  The equation holds for

$$\begin{aligned} a &= 0 \\ n &= n_0. \end{aligned}$$

  This is the situation below threshold.

  A second solution is obtained by solving for a,

$$\left|a\right|^2 = \frac{\gamma_{\|}}{\kappa}\left(\frac{n_0}{n} - 1\right).$$

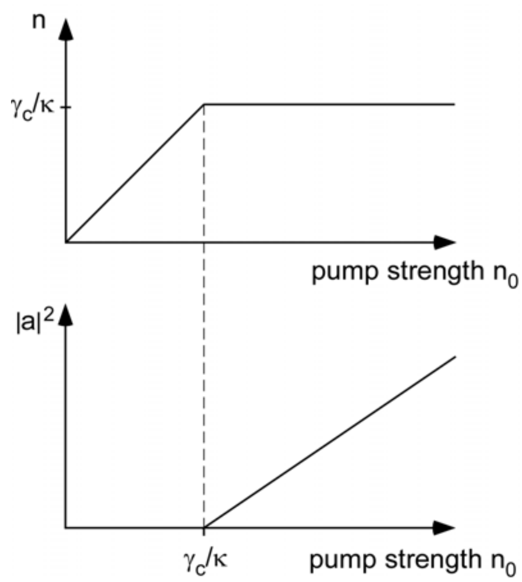
  Since the left side is always positive this solution only exists if the right side is also positive i.e. $n < n_0$. In other words, the pump strength, represented by n_0 must be large enough. If we take the steady state inversion $n_{st} = \gamma_c/\kappa$ from above we obtain

$$\left|a_{st}\right|^2 = \frac{\gamma_{\|}}{\kappa}\left(\frac{\kappa n_0}{\gamma_c} - 1\right)$$

  and the threshold condition reads

$$\frac{\kappa n_0}{\gamma_c} > 1$$

  We can summarize the situation by plotting the inversion and the field amplitude for various pump strength.



- Output power

  To obtain an expression for the output power we write the energy stored inside the resonator as the number of photons $|a_{st}|^2$ in the resonator mode times the energy of a photon $\hbar\omega$. This energy decays through the output coupler at a rate γ_{out} such that the output power is

$$P_{out} = \hbar\omega\gamma_{out}\,|a_{st}|^2$$

  Furthermore, we have to make the connection between n_0 and the pump rate R. For this we again take a look at the above equation for the inversion density with the laser light blocked,

$$\dot{N}(t) = -\gamma_{\|}\left(N(t) - N_0\right).$$

  We assume that in a four level scheme the lower laser level decays very fast and remains unpopulated such that the inversion is given by the population of the upper laser level alone and can never be negative. In this case the term $\gamma_{\|}N_0$ can be interpreted as the rate density with which the upper laser level is populated by the pump mechanism.

  If multiplied with the volume V of the medium we obtain an expression for the total pump rate rate R as introduced in the first chapter

$$R = VN_0\gamma_{\|} = n_0\gamma_{\|}.$$

  We use this expression to replace n_0 in the equation for the field amplitude

$$|a_{st}|^2 = \frac{\gamma_{\|}}{\kappa}\left(\frac{\kappa n_0}{\gamma_c} - 1\right)$$

  and obtain for the output power

$$\begin{aligned} P_{out} &= \hbar\omega\gamma_{out}\,|a_{st}|^2 = \hbar\omega\gamma_{out}\frac{\gamma_{\|}}{\kappa}\left(\frac{\kappa}{\gamma_c}\frac{R}{\gamma_{\|}} - 1\right) \\ &= \hbar\omega\frac{\gamma_{out}}{\gamma_c}\left(R - R_s\right). \end{aligned}$$

  Here we have defined the threshold rate

$$R_s := \frac{\gamma_c\gamma_{\|}}{\kappa}.$$

  Finally, we transform the pump rate into a power $R \cdot \hbar\omega_p$ by multiplying the pump rate with the energy $\hbar\omega_p$ absorbed in one pump event. Since not all the power that is applied to the medium is used for pumping we add the pump efficiency η to define the pump power

$$P_{pump} = \frac{R \cdot \hbar\omega_p}{\eta}$$

Similarly we express the threshold rate by the threshold power defined as

$$P_s := \frac{R_s \cdot \hbar\omega_p}{\eta}$$

With all this we can write the output power as

$$P_{out} = \eta \frac{\omega}{\omega_p} \frac{\gamma_{out}}{\gamma_c} \left(P_{pump} - P_s \right).$$

This is equivalent to the result obtained with the rate model in chapter 1.

6.8 Spiking and Relaxation-Oscillations

After having determined the steady state solutions we can ask about small perturbation around the equilibrium. If theses excursions are not to big we can linearize the nonlinear equations of motion around the equilibrium values. The resulting equations are very helpful to describe the dynamitic behavior of a laser.

- Equation of motion for the power

  If we are not interested in the phase of the light field but only in the light power we can simplify the laser equations even further.

  We ask for the derivative of the photon number

  $$n_p := |a|^2$$

  and calculate

  $$\dot{n}_p = \frac{d}{dt} |a|^2 = a \frac{d}{dt} a^* + a^* \frac{d}{dt} a.$$

  By means of

  $$\dot{a}(t) = \left(i \left(\omega - \omega_k - \frac{1}{2} \kappa \cdot n(t) \right) - \frac{1}{2} \left(\gamma_c - \kappa \cdot n(t) \right) \right) a(t),$$

  we get

  $$\begin{aligned} \frac{d}{dt} |a|^2 &= a \left(-i \left(\omega - \omega_k - \frac{1}{2} \kappa \cdot n \right) - \frac{1}{2} \left(\gamma_c - \kappa \cdot n \right) \right) a^* \\ &\quad + a^* \left(i \left(\omega - \omega_k - \frac{1}{2} \kappa \cdot n \right) - \frac{1}{2} \left(\gamma_c - \kappa \cdot n \right) \right) a \\ &= |a|^2 \left(-\gamma_c + \kappa \cdot n \right). \end{aligned}$$

  As result, we obtain a simple set of coupled nonlinear equations for the photon number n_p in the resonator and the inversion n,

  $$\begin{aligned} \dot{n}_p &= -\gamma_c n_p + \kappa n n_p \\ \dot{n} &= -\gamma_{\|} n - \kappa n n_p + R. \end{aligned}$$

- Equilibrium

  In equilibrium we obtain from the first equation

$$\bar{n} = \frac{\gamma_c}{\kappa}.$$

  and from the second equation

$$\bar{n}_p = \frac{R - \gamma_{\|}\bar{n}}{\kappa\bar{n}} = -\frac{\gamma_{\|}}{\kappa} + \frac{R}{\gamma_c}$$

- Linearization of the laser equations

  We now look at deviations from the steady state solutions

$$\begin{aligned} \delta n_p &:= n_p - \bar{n}_p \\ \delta n &:= n - \bar{n} \end{aligned}$$

  and calculate their derivatives

$$\begin{aligned} \delta\dot{n}_p &= \dot{n}_p \\ \delta\dot{n} &= \dot{n} \end{aligned}$$

  by means of the above simplified laser equations,

$$\begin{aligned} \delta\dot{n}_p &= -\gamma_c\left(\delta n_p + \bar{n}_p\right) + \kappa\left(\delta n + \bar{n}\right)\left(\delta n_p + \bar{n}_p\right) \\ \delta\dot{n} &= -\gamma_{\|}\left(\delta n + \bar{n}\right) - \kappa\left(\delta n + \bar{n}\right)\left(\delta n_p + \bar{n}_p\right) + R. \end{aligned}$$

  Now we make use of the assumption that the deviations are small and thus the product $\delta n \delta n_p$ is even smaller and can be neglected. We obtain

$$\begin{aligned} \delta\dot{n}_p &= -\gamma_c\delta n_p + \kappa\bar{n}_p\delta n + \kappa\bar{n}\delta n_p + \left(-\gamma_c\bar{n}_p + \kappa\bar{n}\bar{n}_p\right) \\ \delta\dot{n} &= -\gamma_{\|}\delta n - \kappa\delta n\bar{n}_p - \kappa\bar{n}\delta n_p + \left(-\gamma_{\|}\bar{n} - \kappa\bar{n}\bar{n}_p + R\right). \end{aligned}$$

  For the equilibrium values, the terms in brackets vanish because

$$-\gamma_c\bar{n}_p + \kappa\bar{n}\bar{n}_p = \left(-\gamma_c + \kappa\bar{n}\right)\bar{n}_p = \left(-\gamma_c + \kappa\frac{\gamma_c}{\kappa}\right)\bar{n}_p = 0$$

  and

$$\begin{aligned} -\gamma_{\|}\bar{n} - \kappa\bar{n}\bar{n}_p + R &= \left(-\gamma_{\|} - \kappa\bar{n}_p\right)\bar{n} + R \\ &= \left(-\gamma_{\|} - \kappa\left(-\frac{\gamma_{\|}}{\kappa} + \frac{R}{\gamma_c}\right)\right)\bar{n} + R \\ &= -\frac{\kappa R}{\gamma_c}\bar{n} + R = -\frac{\kappa R}{\gamma_c}\frac{\gamma_c}{\kappa} + R = 0 \end{aligned}$$

and we arrive at

$$\begin{aligned}\delta\dot{n}_p &= -\gamma_c\delta n_p + \kappa\bar{n}_p\delta n + \kappa\bar{n}\delta n_p \\ \delta\dot{n} &= -\gamma_{\|}\delta n - \kappa\bar{n}_p\delta n - \kappa\bar{n}\delta n_p.\end{aligned}$$

Inserting the equilibrium values for $\bar{n}$ and $\bar{n}_p$ from above we finally get for the first equation

$$\begin{aligned}\delta\dot{n}_p &= -\gamma_c\delta n_p + \kappa\left(-\frac{\gamma_{\|}}{\kappa} + \frac{R}{\gamma_c}\right)\delta n + \kappa\frac{\gamma_c}{\kappa}\delta n_p \\ &= -\gamma_{\|}\delta n + \kappa\frac{R}{\gamma_c}\delta n \\ &= \left(\frac{\kappa R}{\gamma_c} - \gamma_{\|}\right)\delta n\end{aligned}$$

and for the second equation

$$\begin{aligned}\delta\dot{n} &= -\gamma_{\|}\delta n - \kappa\left(-\frac{\gamma_{\|}}{\kappa} + \frac{R}{\gamma_c}\right)\delta n - \kappa\left(\frac{\gamma_c}{\kappa}\right)\delta n_p \\ &= -\gamma_{\|}\delta n + \kappa\frac{\gamma_{\|}}{\kappa}\delta n - \kappa\frac{R}{\gamma_c}\delta n - \gamma_c\delta n_p \\ &= -\frac{\kappa R}{\gamma_c}\delta n - \gamma_c\delta n_p.\end{aligned}$$

These equations are now linear and can be solved analytically.

- Laser as harmonic oscillator

  We can combine the two equations if we take the derivative of $\delta\dot{n}$

$$\delta\ddot{n} = -\frac{\kappa R}{\gamma_c}\delta\dot{n} - \gamma_c\delta\dot{n}_p$$

  and insert $\delta\dot{n}_p$ to obtain

$$\delta\ddot{n} = -\frac{\kappa R}{\gamma_c}\delta\dot{n} - \gamma_c\left(\frac{\kappa R}{\gamma_c} - \gamma_{\|}\right)\delta n.$$

  By introducing the scaled pump strength

$$\rho := \frac{\kappa R}{\gamma_{\|}\gamma_c} = \frac{R}{R_s}.$$

  we can write the equation even simpler as

$$\delta\ddot{n} + \gamma_{\|}\rho\delta\dot{n} + \gamma_c\gamma_{\|}\left(\rho - 1\right)\delta n = 0.$$

For $\rho > 1$, i.e. above threshold, the equation is that of a damped harmonic oscillator with a resonance frequency

$$\omega_f = \sqrt{\gamma_c \gamma_{\|} (\rho - 1)}$$

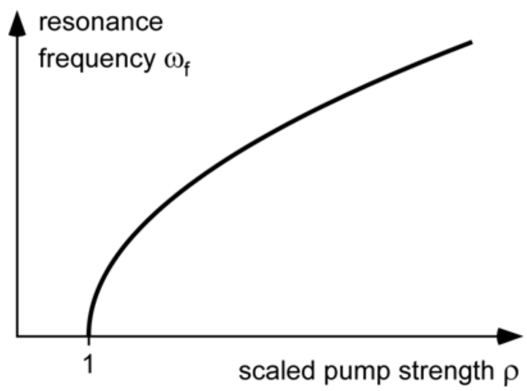
and a damping rate

$$\gamma_f = \frac{\gamma_{\|}}{2} \rho.$$

Obviously, deviations from the equilibrium behave like a pendulum. If there are external fluctuations they can be resonantly enhanced around the resonance frequency ω_f. This is important for laser diodes. Any noise on the injection current may be significantly enhanced.

- Mode softening

  At threshold, ω_f approaches zero and the damping vanishes. This "mode softening" is the typical behavior of a system near a phase transition.

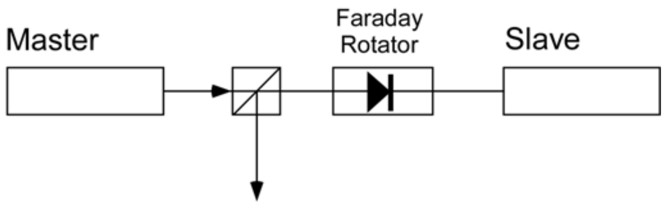


Near threshold the laser becomes noisy because the damping vanishes and external perturbations are typically strong at low frequencies (1/f noise).

6.9 Injection Locking

- Master slave system

  If the light of a weak "master laser" is radiated into the resonator of a strong "slave laser", the frequency of the slave follows that of the master in a phase-locked manner which means that the light in the slave resonator and the light of the master oscillate with a constant relative phase. That only works if the frequencies of the master laser and the non injected, "free-running" slave lasers is not too far apart. For most laser types, this "capture range" is very small and therefore "injection locking" is of limited practical use. Semiconductor lasers are again the exception. Master-slave systems are widespread here. The power of an external cavity stabilized diode laser is often limited to several 10 mW. To save the very good coherence length of the ECDL into the regime of higher power master-slave systems are used. Depending on the type of diode, a very small amount of power is often sufficient to inject a slave laser, even if its free-running wavelength is far shifted from the frequency of the master laser. The requirements for temperature and current stabilization are also less strict for the slave, so that a very stable and practicable setup is possible.



The light of the master is injected into the slave chip through a polarizing beam splitter and a Faraday rotator and comes back from there amplified. When passing through the rotator in the opposite direction, the polarization of the light is rotated so that it now leaves the beam splitter at the other port and does not return to the master. A slave output power of up to 100 mW with well-controlled frequency is possible with injected master laser powers of only 1 mW.

- Field equation

  We look at the field equation for the slave laser as derived above

$$\dot{a}(t) = \left(i\left(\omega - \omega_k - \frac{1}{2}\kappa \cdot n(t) \cdot \alpha\right) - \frac{1}{2}\left(\gamma_c - \kappa \cdot n(t)\right)\right) a(t) + D\left(t\right).$$

  To the right of the equation, we add a driving term $D\left(t\right)$. It is the field of the master that is coupled into the slave resonator. The master field

$$\tilde{E}_{ext}\left(t\right) = a_{ext}e^{i\omega_{ext}t+i\varphi}$$

  oscillates with an amplitude a_{ext}, a frequency ω_{ext}, and a phase φ relative to the field in the slave resonator. Above, when deriving the equation for the field in a laser resonator, we separated the field amplitude from the fast optical frequency ω. We have to do the same with the injected field. Furthermore, we introduce the rate $\gamma_{ext}/2$ at which the field of the master light is injected into the slave resonator (the power rate γ_{ext} is twice the field rate. See also the end of section 5.3). For the driving term we thus obtain

$$D\left(t\right) = \frac{1}{2}\gamma_{ext}a_{ext}e^{i(\omega_{ext}-\omega)t+i\varphi}$$

- Steady state

Is there a steady state solution? Equilibrium with $\dot{a}(t) = 0$ and $\dot{n}(t) = 0$ is only possible if the driving term is constant in time which means that

$$\omega_{ext} = \omega.$$

The slave oscillates with the same frequency as the master. With this condition we can write an equation for the real part and for the imaginary part,

$$\gamma_c - \kappa n + \gamma_{ext} \frac{a_{ext}}{a} \cos\varphi = 0,$$

$$\omega_{ext} - \omega_k - \frac{1}{2}\kappa n \cdot \alpha - \frac{1}{2}\gamma_{ext}\frac{a_{ext}}{a}\sin\varphi = 0.$$

If these conditions are fulfilled, the slave follows the master with a constant phase, same frequency, and constant amplitude.

- Capture range

  The first equation can be solved for κn

$$\kappa n = \gamma_c + \gamma_{ext}\frac{a_{ext}}{a}\cos\varphi,$$

  which we insert into the second equation,

$$\omega_{ext} - \omega_k - \frac{1}{2}\left(\gamma_c + \gamma_{ext}\frac{a_{ext}}{a}\cos\varphi\right)\alpha - \frac{1}{2}\gamma_{ext}\frac{a_{ext}}{a}\sin\varphi = 0.$$

$$\omega_{ext} - \omega_k - \frac{1}{2}\gamma_c\alpha = \frac{1}{2}\gamma_{ext}\frac{a_{ext}}{a}\left(\sin\varphi + \cos\varphi\alpha\right).$$

  We rearrange this equation to obtain

$$\omega_{ext} - \left(\omega_k + \frac{1}{2}\gamma_c\alpha\right) = \frac{1}{2}\gamma_{ext}\frac{a_{ext}}{a}\left(\alpha\cos\varphi + \sin\varphi\right).$$

  The bracket on the left side is the frequency the slave would have if there is no injection (see "mode pulling"in section 6.7). So we introduce the free running frequency of the slave laser without injection

$$\omega_f := \omega_k + \frac{1}{2}\gamma_c\alpha.$$

  By using the general identity

$$a\cos\varphi + b\sin\varphi = \sqrt{a^2+b^2}\cos(\varphi + \arctan(-b/a))$$

  we obtain for the bracket on the right side

$$\alpha\cos\varphi + \sin\varphi = \sqrt{1+\alpha^2}\cos(\varphi + \arctan(-1/\alpha)).$$

The equation now reads

$$\omega_{ext} - \omega_f = \frac{1}{2}\gamma_{ext}\frac{a_{ext}}{a}\sqrt{1+\alpha^2}\cos(\varphi - \arctan(1/\alpha)).$$

If the master frequency is detuned further and further away from the free running frequency the slave laser, the slave tries to follow the master as long as possible. The phase φ must change in order to provide the required value of the cosine. At some point, the cosine reaches the value 1 and cannot grow any further. For a larger detuning, the equation can no longer be fulfilled. The end of the capture range has been reached and the system falls out of balance. The maximum possible detuning with $\cos(\varphi + \arctan(1/\alpha)) = 1$ is the so called "capture range",

$$\Delta\omega = \frac{1}{2}\gamma_{ext}\frac{a_{ext}}{a}\sqrt{1+\alpha^2} = \frac{1}{2}\gamma_{ext}\sqrt{1+\alpha^2}\sqrt{\frac{I_{ext}}{I}}.$$

- Numbers for the capture range

  The capture range scales with the coupling rate γ_{ext}, which is typically the same rate as γ_{out} with which the light leaves the slave. This corresponds roughly to the line width of the cold resonator. For typical lasers, the cold line width is a few MHz. The two roots in the expression provide contributions on the order of equal or smaller than one ($\alpha \simeq 1$, and $I_{ext} < I$) so that the capture range remains in the regime of a few MHz. This corresponds roughly to the typical line width of the laser, so that the effect is difficult to observe, let alone technically usable.

  For diode lasers, however, the laser resonator is almost 10000 times shorter than the resonator of other lasers. The round trip time is shorter by the same factor. In addition, the gain is so high that the transmission of the out-coupling mirror is close to $80-90\%$. Therefore, the decay rate of a photon in the laser resonator is practically as large as one inverse round trip time i.e. one free spectral range of some 100 GHz. In addition, laser diodes have a large α of about 5. Even if the master is weak compared to the slave, $\sqrt{I_{ext}/I} < 1$, the capture range is still large and lies in the range of a nm.

- External Cavity Laser

  In an external cavity diode laser, part of the output of a laser diode is reflected back into the diode. This can be interpreted as a situation where the laser locks to its own light, which is basically indistinguishable from the light of a hypothetical master laser. Since in this case $\omega = \omega_{ext}$ always holds, the question arises at which frequency the laser oscillates.

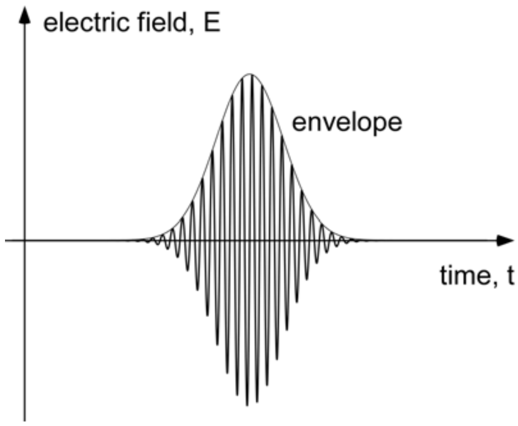
  The phase φ between the light in the laser resonator and the reflected injection light depends on the time it takes for the light to propagate from the diode to the reflecting element and back. As we have seen, this phase controls the frequency difference between the actual laser light and the free-running laser frequency. Since the free-running frequency is fixed by the laser diode (at a given temperature and injection current) one thus can control the diode output frequency with the help of the spacing L between the mirror and the diode. The return phase $\varphi = k \cdot 2L$ depends very sensitively on L: By moving the mirror by a quarter of a wavelength, the laser can be tuned over a frequency range that corresponds to twice the capture range. In a practical device, the mirror position is controlled with a piezoceramic actuator. Furthermore, it is important that the mechanical setup is very stable and that L does not drift with temperature.7. Pulsed Laser

Lasers can be constructed in a way that they emit trains of light pulses. In such lasers, a pulse circulates inside the resonator and part of it is transmitted through the output coupling mirror after each round trip. With quite moderate average output power in the range of Watts, the peak power in pulses can nevertheless be very large. The shorter the pulses the higher the peak power. In combination with the possibility to focus a laser beam to very small spots one can use pulsed lasers to explore physical phenomena that only take place at high light intensities. Furthermore, one can use short pulses to investigate very fast physical processes. If a sample is illuminated with a short pulse, it may for instance change its optical properties, which then can be probed with a second pulse. Obviously, the temporal resolution of such a "pump-probe" method is limited by the length of the applied light pulse. Since the invention of the laser, physicists have thus tried to develop methods for generating ever shorter light pulses. In this chapter we briefly discuss some aspects of pulsed lasers.

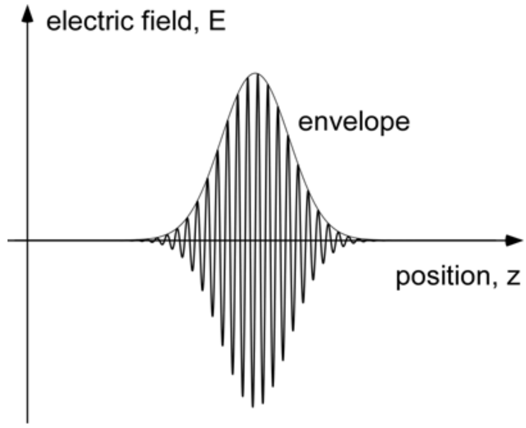
7.1 Mode Locking

- Pulses

  A light pulse can be viewed either as the temporal change of the electric field at a given position

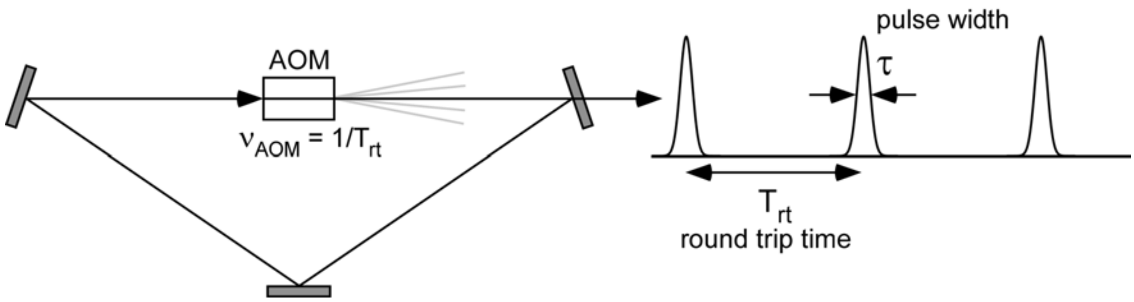


or as the spatial change of the electric field at a given time. Without dispersion (refractive index does not change with frequency) such a snapshot looks the same as the temporal change above.



- Pulse trains

  One can generate pulses by placing an acousto-optic modulator (AOM) inside the laser resonator. The AOM is periodically turned on and off with a period that corresponds to the round trip time T_{rt} of the resonator ($\nu_{AOM} = 1/T_{rt}$). When the AOM is turned on, it introduces losses by scattering light out of the resonator. A continuous laser beam experiences losses and is pushed below threshold. However, if the light circulates inside the resonator as a pulse that passes the AOM when it is turned off, no losses are experienced and the laser threshold may be crossed. The light leaves the resonator as a train of equally spaced pulses with a repetition rate $1/T_{rt}$.



- Spectrum of a puls train

  One may expect that a puls train is a periodic function that can be described as a Fourier-series of oscillations with discreet frequencies. However, this is not the case!

$$E(t) \neq \sum_{k=-\infty}^{k=\infty} A_k \exp\left(ik\left(\frac{2\pi}{T_{rt}}\right)t\right)$$

The envelope is in fact periodic with $1/T_{rt}$ but there is still the optical frequency ν, which is strictly synchronized with the envelope only if it is an integer multiple of $1/T_{tr}$.

$$\nu = q \cdot \frac{1}{T_{rt}}.$$

In an ideal resonator with no optical elements and ideal mirrors that reflect the light without phaseshift this would even be the case. The separation between resonances would be the free spectral range

$$\nu_{fsr} = \frac{1}{T_{rt}}$$

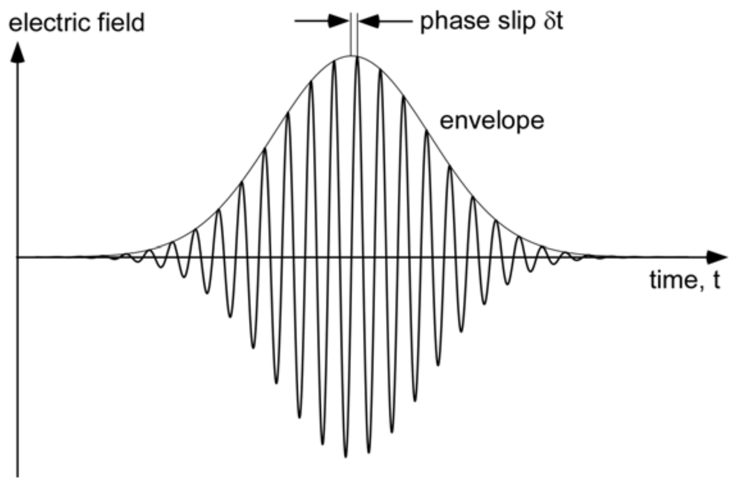
and the optical frequency would be an integer multiple

$$\nu = q \cdot \nu_{fsr} = q \cdot \frac{1}{T_{rt}}$$

but in the real world, there are optical elements with an refraction index that changes with optical frequency and the mirrors are dielectric mirrors with unknown phase shift.

$$\nu = q \cdot \nu_{fsr}(\nu) + \nu_0 \neq q \cdot \frac{1}{T_{rt}}$$

Consequently there is a phase slip between the position of the maximum field and the maximum of the envelope.



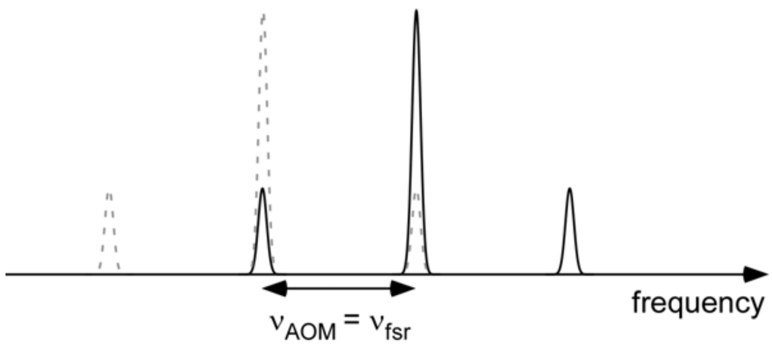
This shift varies from pulse to pulse. The field is thus not strictly periodic and the pulse must be described as a Fourier-transform

$$E(t) \sim \int_{-\infty}^{\infty} A(\omega) \exp(i\omega t)\, d\omega$$

- Mode locking

  A pulse train consists of many longitudinal modes oscillating at the resonance frequencies of the resonator, $\nu_n = q\nu_{fsr} + \nu_0$. All theses frequency components form a pulse only if the relative phase of the components is fixed such that constructive interference of all modes occurs at a distinct moment in time. If the phase relation is arbitrary one simple obtains a multi mode laser with a very noisy output. However, with an acousto optic modulator (AOM) it is possible to synchronize the modes and 'lock' their relative phases.

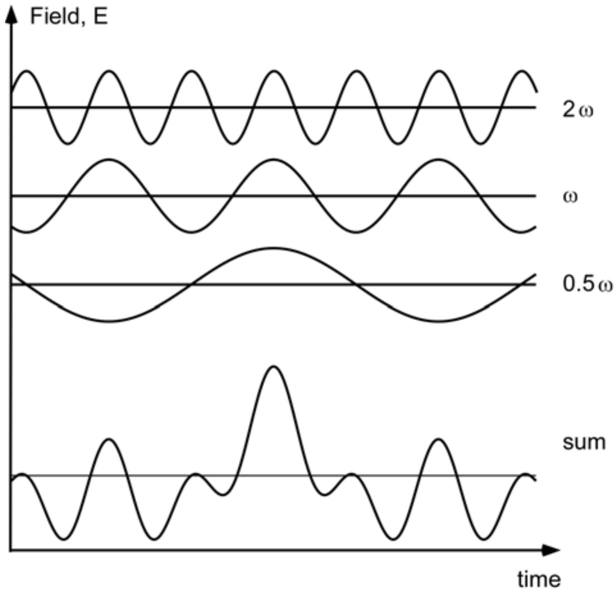
  In the frequency domain the AOM generates sidebands to each laser mode. Since $\nu_{AOM} = 1/T_{rt} \simeq \nu_{fsr}$, the frequency of the two sidebands coincide with the frequency of the neighboring longitudinal modes.

  These neighboring modes are thus injected by the sidebands similarly to a master-slave laser system. This optical injection lock guaranties a stable phase between the longitudinal modes and a circulating pulse is formed that passes the AOM at the right time. It is the nonlinear physics of the Maxwell-Bloch equations that causes the stable formation of pulses.

- Pulse width

  A puls can be considered as the superposition of many optical oscillations with different frequencies. All interfere constructively at a given moment. The electric field has a maximum at this moment. Before and after this constructive interference decays because of the different frequencies of the components.



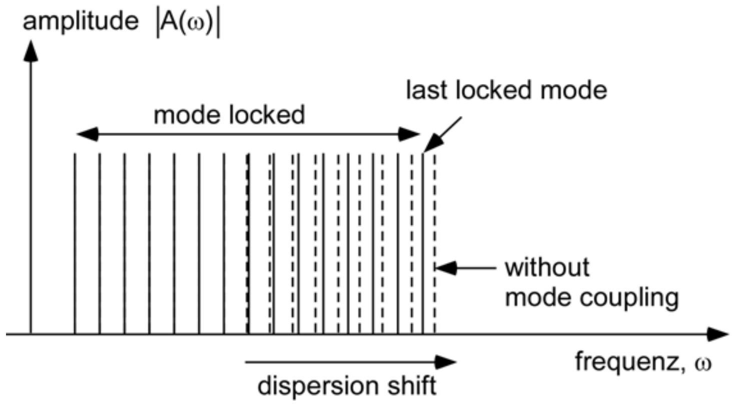
The fastest components quickly change their phase and start to destructively interfere with the rest. Thus the frequency difference between the fastest and the slowest component determines the temporal width of the puls: The temporal width τ of the puls in frequency space is inverse proportional to the spectral with $\Delta\omega$ of the pulse in frequency space.

$$\Delta\omega \simeq \frac{1}{\tau}$$

Therefore, the pulse width is given by the number of longitudinal modes that can be locked simultaneously. This number is limited by the width of the gain profile of the laser material. With its gain width of several hundred nanometers a Titanium Sapphire laser (TiSa) is thus a very good candidate for the generation of short pulses.

- Dispersion

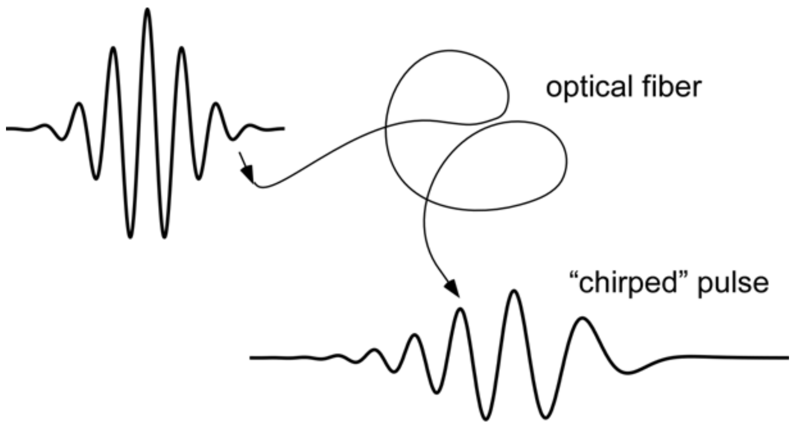
  Furthermore, the pulse width is limited by the dispersion of the refractive index: The value of the refractive index of any optical element inside the resonator increases with increasing optical frequency. As result the wave length also changes and with it the free spectral range of the resonator. Because of the limited capture range of the injection lock, there is always a last longitudinal mode that can be locked.



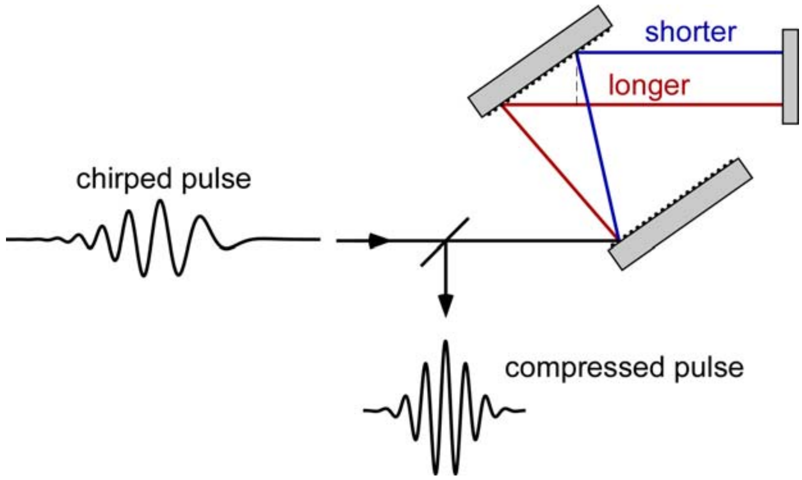
The dispersion effect can be compensated to a large extend with special dielectric mirrors.

- Chirped pulses

  The effect of a dispersive medium can also be discussed in the time domain. If a pulse enters a dispersive optical medium the red part of the pulse spectrum propagates faster than the blue part and leaves the medium earlier. The blue part is slower and comes later. As a result, behind the medium the frequency of the pulse increases with time. Such pulses are called chirped pulses.



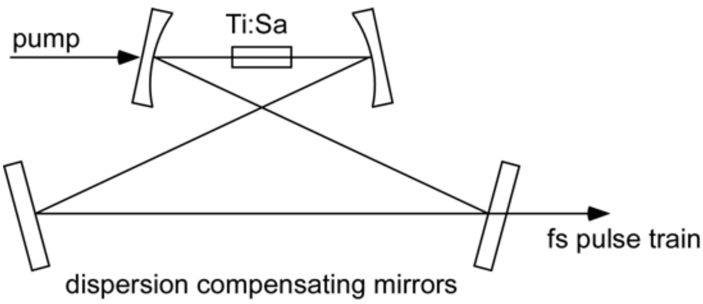
It is possible to compensate such "chirps" with two diffraction gratings which are set up such that the red part of the pulse spectrum travels a longer optical path than the blue part.



This technique was a breakthrough for the generation of very intense pulses. If a pulse is amplified to an extend that it starts to destroy its own laser medium a fundamental intensity limit seems to be reached. However, one can expand the pulse first by a dispersive medium or a pair of diffraction gratings, amplify it and than compress it again. For the invention of this method Donna Strickland and Gérard Mourou won the Nobel prize in 2018.

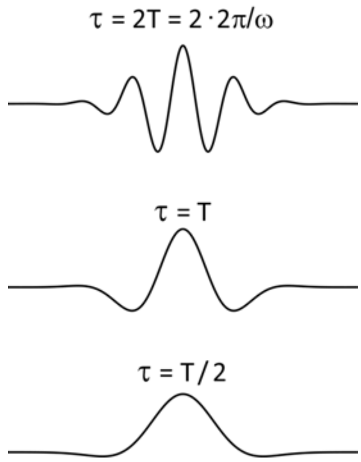
- Kerr lens mode locking

  The shortest pulses can be generated with a TiSa laser. Its broad gain bandwidth allows for some 10000 coupled longitudinal modes. If the resonator is designed properly, mode locking takes place even without AOM.



  This is because of the Kerr-effect: The refractive index of Sapphire increases with the light intensity. This effect is usually small but the peak intensity of a pulse can be very high such that the Kerr effect becomes important. Due to the Kerr-effect the transverse intensity profile of the pulse generates a transverse index profile that refocuses the Gaussian beam. A pulse inside the crystal propagates as if it would be guided in a glass fiber. This changes the ABCD-matrix of the resonator and the mirrors have to be moved slightly to keep the resonator within the stability range. For continuous operation without Kerr-effect the resonator is now unstable such that the laser can oscillate only in pulsed mode.

Since no further intracavity element is needed, the dispersion is small and the pulses can be very short. If the gain bandwidth covers a full octave (first mode has twice the frequency of the last mode) the spacial width of the envelope can be as small as one (average) wavelength and the temporal width is on the order of femtometer. The envelope may not even cover a full oscillation. The spectrum of such pulse trains covers the entire visible range and since the pulse repetition rate is in the range of 100 MHz one cannot distinguish single pulses with the eye. Beams of Kerr-modelocked lasers thus look like normal laser beams only that their color is white. Nevertheless, they can by focused to spots of the size of a wave length. If focused in air one observes electric discharge sparks caused by photoionization of the oxygen an nitrogen molecules of the air.



7.2 Frequency comb

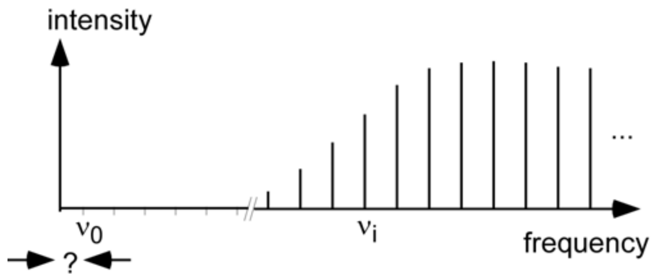
- Pulse train as frequency ruler

  For long time it was a big problem in metrology (the science of units and their realization) to connect the microwave regime where the second is defined (as hyperfine oscillation of cesium) to the optical domain where the oscillations are many orders of magnitude faster. Precision measurements of optical frequencies required sophisticated frequency chains. In such chains oscillators at various frequency ranges were connected by nonlinear frequency doublers that increased the frequency by a factor of two. To bridge five orders of magnitude, 16 of such steps were necessary with equipment that filled several large optical laboratories.

  A pulsed laser turned out to be the better alternative. Since the modes are phase-locked they all have a fixed frequency difference and the relation

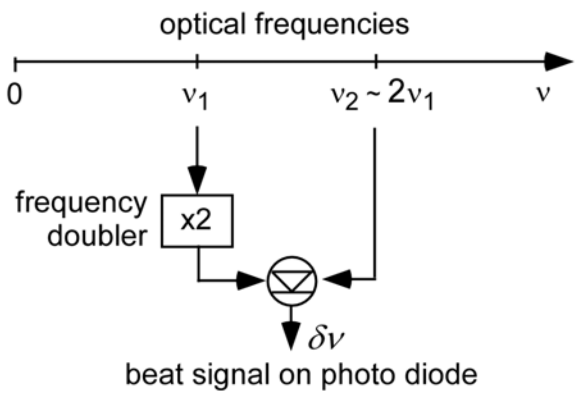
$$\nu_n = q\nu_{fsr} + \nu_0$$

strictly holds. To determine the optical frequency of mode n one needs to know the integer number q, the free spectral range ν_{fsr}, and the frequency offset ν_0. The free spectral range can be measured precisely. One just has to observe the repetition rate of the pulse train with a photo diode and compare it to the cesium clock. The integer q can be determined by measuring the optical frequency with a less precise method that only resolves steps of ν_{fsr}. This is easily possible with a grating spectrometer ('wavemeter'). The crucial problem is the offset frequency ν_0. It cannot be derived from the resonator geometry and depends on the phase shift the light experiences when reflected at the mirrors of the resonator. It must be determined experimentally.



- Self referencing

  Around the year 2000 a trick was developed to determine ν_0 quite elegantly. If the number of modes covers a full octave one can use the following setup



The frequency ν_1 of a low frequency mode is doubled in a nonlinear crystal and compared to the frequency ν_2 of a mode at the blue side of the spectrum. If $\nu_2 \simeq 2\nu_1$ the beat signal between the two modes lies in the radio frequency range. It can be recorded with photo diode and measured with a cesium clock. From the measured values of ν_{fsr} and ν_0 the optical frequencies $\nu_1$and ν_2 can be easily calculated. One simply inserts the expressions for ν_1 and ν_2

$$\begin{aligned}\nu_1 &= q_1\nu_{fsr} + \nu_0 \\ \nu_2 &= q_2\nu_{fsr} + \nu_0\end{aligned}$$

into the equation for the beat signal

$$\delta\nu = 2\nu_1 - \nu_2$$

to obtain

$$\delta\nu = 2\left(q_1\nu_{fsr} + \nu_0\right) - \left(q_2\nu_{fsr} + \nu_0\right).$$

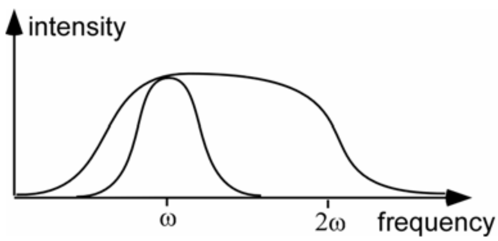
Solving for ν_0 yields

$$\nu_0 = \delta\nu + \left(q_2 - 2q_1\right)\nu_{fsr}.$$

The frequency offset ν_0 can thus be precisely determined from the measured frequencies $\delta\nu$ and ν_{fsr} which both lie in the radio frequency domain where the second is defined and sophisticated technology is easily available.

- self phase modulation

  Usually even a TiSa laser does not provide a spectrum that cover a full octave. However, a full octave can be generated by sending the light through an optical fiber. The Kerr effect in the glass of the fiber expands the spectrum significantly.



To see this, we look at a pulse that passes a piece of optical material of length L.

As the pulse propagates, it changes the refractive index in the material. We count the optical phase (think of it as the number of nodes) that enter the material within a time t

$$\varphi_{in} = \omega_{in} \cdot t$$

and compare them to the phase that leave the material.

$$\varphi_{out} = \omega_{out} \cdot t.$$

We also take into account the nodes inside the material

$$\varphi_m = k \cdot L = n\left(t\right) \frac{\omega_{in}}{c} \cdot L.$$

The phase at the output changes according to the phase change at the input minus the phase that is accumulated inside the material:

$$\frac{d}{dt}\varphi_{out} = \frac{d}{dt}\varphi_{in} - \frac{d}{dt}\varphi_m.$$

Inserting the equation for

$$\frac{d}{dt}\varphi_{in} = \omega_{in}$$

$$\frac{d}{dt}\varphi_{out} = \omega_{out}$$

and

$$\frac{d}{dt}\varphi_m = \frac{\omega_{in}}{c}\frac{d}{dt}n\left(t\right) \cdot L$$

yields

$$\begin{aligned}
\omega_{out} &= \omega_{in} - \frac{\omega_{in}}{c}\frac{d}{dt}n\left(t\right) \cdot L \\
&= \omega_{in}\left(1 - \frac{1}{c}\frac{d}{dt}n\left(t\right) \cdot L\right).
\end{aligned}$$

At the beginning of the pulse the intensity grows, $\dot{n} > 0$, and the frequency is shifted to smaller values. At the end of the puls, $\dot{n} < 0$, and the frequency is shifted to higher values. In total the spectrum is broadened. Although the Kerr effect is usually very small, the intensity in a single mode fiber is high enough and the broadening is very efficient. For the experimental demonstration of self referencing the Nobel price was awarded to Theodor Hänsch in Munich in 2005.