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THE COMPLETENESS OF INTUITIONISTIC LOGIC WITH RESPECT TO A VALIDITY CONCEPT BASED ON AN INVERSION PRINCIPLE

In his well-known programmatic remarks on the characteristics of the inference rules of his 'Calculus of Natural Deduction' Gentzen states that

> The introductions represent, as it were, the 'definitions' of the symbols concerned, and the eliminations are no more, in the final analysis, than the consequences of these definitions. This fact may be expressed as follows: In eliminating a symbol, we may use the formula with whose terminal symbol we are dealing only 'in the sense afforded it by the introduction of that symbol'. . . . By making these ideas more precise it should be possible to display the E-inferences as unique functions of their corresponding I-inferences, on the basis of certain requirements (Gentzen 1935, ed. Szabo 1969, pp. 80–81).

Several attempts have been undertaken to elaborate this program, especially by Prawitz (1965) in his formulation of an inversion principle for natural deduction rules (thus generalizing an idea of Lorenzen (1955), which was formulated only for calculi without an operation of assumption elimination ("logistic calculi" in Gentzen's terminology)) and by Prawitz (1971, 1973) in his definition of validity for inference rules and derivations. For a philosophical discussion of an intuitionistic meaning theory for the logical constants which is closely connected with this program, see Dummett (1975, 1977), Prawitz (1977).

Prawitz (1973) conjectured that the inference rules of minimal logic (in a natural deduction formulation) were complete with respect to his validity concept based on introduction rules (p. 246). This seems to be obvious from the standpoint of intuitive reasoning, as Prawitz (1979, p. 37) mentions, but he was not able to prove it.

In the following we shall propose a modified concept of validity based on introduction rules with respect to which the completeness of intuitionistic logic can be established. Our definition does not employ the *constructive* interpretation of derivations from assumptions (and therefore of the implication sign) according to which such derivations are justified by constructions transforming proofs of the assumptions into a proof of the

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end-formula. (This was favoured by Dummett and by Prawitz (1971 and later)). Rather we retain the *operative* interpretation which underlies Gentzen's natural deduction rules. Valid inference rules are considered not to be rules transforming valid derivations of the premisses into a valid derivation of the conclusion but rules *inverting* introduction rules in some sense. So we go back to the inversion principle of Prawitz (1965) in order to obtain a criterion for the validity of derivations.[1]

In part I we define the general concept of an inference rule suitable for natural deduction systems and the derivability of inference rules. Part II then gives the definitions of canonical derivations and valid rules. In part III we prove that the usual introduction and elimination rules of intuitionistic first-order logic are valid and that all valid rules are derivable in intuitionistic logic. Part IV discusses some features of our concept of validity.

I. THE CONCEPT OF AN INFERENCE RULE FOR NATURAL DEDUCTION SYSTEMS

In order to define a concept of validity for arbitrary inference rules we have first to state a general schema for inference rules which includes the usual introduction and elimination rules. Such a schema has to take into account the fact that by application of an inference rule assumptions may be discharged. Furthermore, the conditions on the eigenvariables of quantifier rules have to be taken into account. Therefore we propose inference rules to have the form

$$(1)\qquad \frac{\begin{matrix}\Gamma_1 & & \Gamma_n \\ \vdots\,\underline{x}_1 & & \vdots\,\underline{x}_n \\ A_1 & \ldots\ldots & A_n\end{matrix}}{A}\qquad (n \geqslant 0)$$

where the Γ's are (possibly empty) systems of formulas of the object language, the A's are formulas of the object language, and the $\underline{x}$'s are (possibly empty) systems of distinct individual variables. A *system* is understood as a list of signs which are separated by commas (cf. Lorenzen 1955, §12). So a system of signs is itself a sign. For a system of formulas Γ we denote by {Γ} the *set* containing exactly the formulas belonging to the system Γ. A system Δ is called a *subsystem* of Γ if $\{\Delta\} \subseteq \{\Gamma\}$. (1) is to be read as:

If for all i $(1 \leqslant i \leqslant n)$ derivations of A_i are given, possibly depending on Γ_i and other systems of formulas Γ_i' where none of the variables of $\underline{x}_i$ occurs free in any formula of Γ_i', then we may immediately infer A, whereby the resulting derivations depends on $\cup_{i=1}^{n} \{\Gamma_i'\}$.

The system of variables $\underline{x}_i$ in

$$\begin{array}{l} \Gamma_i \\ \vdots \underline{x}_i \\ A_i \end{array}$$

indicates a kind of generalization of that premiss: if A_i is derived from Γ_i and Γ_i' where no formula of Γ_i' contains any variable of $\underline{x}_i$ free, then we have for all systems of terms $\underline{t}_i$ of the same length as $\underline{x}_i$ derivations of $A_i[\underline{x}_i/\underline{t}_i]$ from $\Gamma_i[\underline{x}_i/\underline{t}_i]$ and Γ_i' at our disposal, where $[\underline{x}_i/\underline{t}_i]$ is the operation of simultaneously substituting the terms of $\underline{t}_i$ for the variables of $\underline{x}_i$ within the formulas of Γ_i (provided, as always in the following, that the terms of $\underline{t}_i$ are free for the corresponding variables of $\underline{x}_i$). (We shall speak simply of $\underline{x}$, $\underline{t}$, Γ when variables of $\underline{x}$, terms of $\underline{t}$ and formulas of Γ are meant.) This interpretation of the variable conditions presupposes that it is guaranteed that

(2) if $\Gamma \vdash A$, then $\Gamma[\underline{x}/\underline{t}] \vdash A[\underline{x}/\underline{t}]$ for arbitrary systems of formulas Γ, formulas A, systems of variables $\underline{x}$ and systems of terms $\underline{t}$ of the same length as $\underline{x}$ (provided $\underline{t}$ is free for $\underline{x}$ in Γ, A.

This can be achieved, for example, by the requirement that the calculus considered contains with each inference rule of the form (1) also

$$(3) \qquad \frac{\begin{array}{c} \Gamma_1[\underline{x}_1/\underline{y}_1]\,[\underline{x}/\underline{t}] \\ \vdots \underline{y}_1 \\ A_1[\underline{x}_1/\underline{y}_1]\,[\underline{x}/\underline{t}] \end{array} \quad \cdots\cdots \quad \begin{array}{c} \Gamma_n[\underline{x}_n/\underline{y}_n]\,[\underline{x}/\underline{t}] \\ \vdots \underline{y}_n \\ A_n[\underline{x}_n/\underline{y}_n]\,[\underline{x}/\underline{t}] \end{array}}{A[\underline{x}/\underline{t}]}$$

where the $\underline{y}_i$ are systems of distinct variables of the same length as $\underline{x}_i$ not occurring in Γ_i or A_i, such that $\underline{x}$ and $\underline{t}$ contain no variable of $\underline{y}_i$ $(1 \leqslant i \leqslant n)$. Another way is to consider (1) to be a rule for which each application has the form (3), i.e., a rule which is applied under substitution of terms for variables. In the following we assume that with each inference rule of the form (1) we have rules of the form (3) at our disposal. It should be noted that the requirement (2) is not artificial and reflects the sense of the use of free individual variables.

Obviously the schema (1) includes all instances of the introduction and elimination rules of natural deduction systems. For example, each instance of the $\rightarrow$ introduction rule has the form

$$\frac{\begin{matrix} A \\ \vdots \\ B \end{matrix}}{A \rightarrow B},$$

each instance of $\forall$ introduction the form

$$\frac{\begin{matrix} \vdots y \\ A[x/y] \end{matrix}}{\forall x A}$$

where y is not free in $\forall x A$, and each instance of $\exists$ elimination the form

$$\frac{\exists x A \quad \begin{matrix} A[x/y] \\ \vdots y \\ B \end{matrix}}{B}$$

where y is not free in $\exists x A$ or B.

Since the Γ's and A's in (1) were chosen as (systems of) formulas of the object language they are not permitted to contain *syntactical* variables. Thus (1) includes only the *instances* of inference rules in the usual sense, not these rules themselves. But the distinction between rules with syntactical variables for formulas and instances of such rules (obtained by substitution of the syntactical variables) is not necessary in our context. We are interested only in the derivability and validity of inference rules: A definition of these properties for inference rules with syntactical variables would refer to the set of their instances which are inference rules in the sense of (1) or to inference rules in which the syntactical variables are replaced by schematic letters (belonging to the object language) which are also inference rules in the sense of (1). Therefore when we speak of an introduction or an elimination rule of a natural deduction system we mean the set of their instances, when we speak of an application of such a rule in a derivation we mean an application of an instance of that rule which is suitable in that situation.

We do not generally require inference rules to be closed under substitutivity of formulas. Contrary to the substitutivity of individual variables, expressed by (2), this is not necessary for the (motivation of the) definition of derivability and validity of inference rules. The usual logical systems of course obey the principle of substitutivity of formulas since their inference rules are instances of certain rules which contain syntactical variables for formulas.

As a definition of derivability of an inference rule R of the form (1) in a certain calculus $\mathscr{K}$ we could propose: "R is derivable in $\mathscr{K}$ iff for all systems of assumptions Γ: If for all i $(1 \leqslant i \leqslant n) A_i$ is derivable in $\mathscr{K}$ from Γ_i and Γ_i' where Γ_i' is a subsystem of Γ not containing a variable of $\underline{x}_i$ free, then A is derivable in $\mathscr{K}$ from Γ." However, the derivability of a rule R can then (in general) be established only by metalogical considerations, not by a derivation in $\mathscr{K}$ itself, contrary to what we would expect from a concept of *derivability*. A more adequate definition of derivability has to treat the existence of a derivation of A_i from Γ_i as a kind of assumption in a derivation of A, such that the derivability of R can be considered to be a derivation of A from assumptions of this kind (in analogy with the derivability of a rule

$$\frac{\begin{matrix}\vdots & \vdots \\ A_1 & \ldots A_n\end{matrix}}{A} \quad \text{in } \mathscr{K} \text{ as } A_1, \ldots, A_n \vdash_{\mathscr{K}} A).$$

Such an extension of the concept of assumption can be achieved in the following way. We define objects $\Gamma \underset{\underline{x}}{\Rightarrow} A$ for a system of formulas Γ, a formula A and a system of individual variables $\underline{x}$, which we call *assumption rules*. If Γ or $\underline{x}$ is empty, an assumption rule may have the form $\Gamma \Rightarrow A$, $\underset{\underline{x}}{\Rightarrow} A$, $\Rightarrow A$. We identify an assumption rule $\Rightarrow A$ with the formula A taken as an assumption. In an assumption rule $\Gamma \underset{\underline{x}}{\Rightarrow} A$ the variables $\underline{x}$ are considered to be bound. $A_1, \ldots, A_n \underset{\underline{x}}{\Rightarrow} A$ may enter a derivation as an additional assumption according to the schema

$$(4) \qquad \frac{A_1, \ldots, A_n \underset{\underline{x}}{\Rightarrow} A \quad \begin{matrix}\vdots \\ A_1[\underline{x}/\underline{t}]\end{matrix} \; \ldots\ldots \; \begin{matrix}\vdots \\ A_n[\underline{x}/\underline{t}]\end{matrix}}{A[\underline{x}/\underline{t}]}$$

where $\underline{t}$ is of the same length as $\underline{x}$ and free for $\underline{x}$ in $A_1, \ldots, A_n, A$. Here $A[\underline{x}/\underline{t}]$ depends on $A_1, \ldots, A_n \underset{\underline{x}}{\Rightarrow} A$ along with the undischarged assumptions of the derivations of the $A_i[\underline{x}/\underline{t}]$. This schema justifies the name 'assumption rule': We use the rule $A_1, \ldots, A_n \underset{\underline{x}}{\Rightarrow} A$ as an assumption when we apply it to certain derived premisses $A_i[\underline{x}/\underline{t}]$ in order to obtain $A[\underline{x}/\underline{t}]$. Assumption rules are treated as a subclass of assumptions. The application $\Rightarrow A/A$ of $\Rightarrow A$ is identified with the introduction of A as an assumption.

Thus, we can define a concept of derivability of formulas from formulas and assumption rules in a calculus $\mathscr{K}$. We denote it by $\Delta \vdash_{\mathscr{K}} A$ for a system of assumptions (formulas and assumptions rules) Δ and a formula A. (Assumption rules which are not identified with formulas may not be discharged by the application of inference rules; if a formula of a derivation depends on an assumption rule, then the end-formula of the derivation depends on it.[2])

An assumption rule $A_1, \ldots, A_n \underset{\underline{x}}{\Rightarrow} A$ can now represent the hypothesis that A can be derived from $A_1, \ldots, A_n$ and additional assumptions Δ not containing $\underline{x}$ free. Each application of the form (4) of an assumption $A_1, \ldots, A_n \underset{\underline{x}}{\Rightarrow} A$ in a derivation, where the $A_i[\underline{x}/\underline{t}]$ depend on Δ, is replaceable by a derivation of $A[\underline{x}/\underline{t}]$ from $A_1[\underline{x}/\underline{t}], \ldots, A_n[\underline{x}/\underline{t}]$ and Δ. Such a derivation is obtained from a derivation of A from $A_1, \ldots, A_n$ and Δ if Δ does not contain $\underline{x}$ free (by condition (2) which remains valid, when there are assumption rules as additional assumptions). Conversely we obtain a trivial derivation of A from $A_1, \ldots, A_n$ and $A_1, \ldots, A_n \underset{\underline{x}}{\Rightarrow} A$ (where $A_1, \ldots, A_n \underset{\underline{x}}{\Rightarrow} A$ does not contain $\underline{x}$ free).

Thus, the intuitive idea of the derivability of an inference rule of the form (1) by means of a derivation of A on the object-language level from the hypotheses that A_i can be derived from Γ_i can be explicated as follows: A is derivable from the assumption rules $\Gamma_1 \underset{\underline{x}_1}{\Rightarrow} A_1, \ldots, \Gamma_n \underset{\underline{x}_n}{\Rightarrow} A_n$. So we define:

DEFINITION. An inference rule of the form (1) is derivable in a calculus $\mathscr{K}$ iff $\Gamma_1 \underset{\underline{x}_1}{\Rightarrow} A_1, \ldots, \Gamma_n \underset{\underline{x}_n}{\Rightarrow} A_n \vdash_{\mathscr{K}} A$.

To demonstrate by an example how this definition works we consider a calculus $\mathscr{K}$ having

$$(\mathrm{R}_1) \quad \frac{\begin{matrix}\vdots \\ C[x/t]\end{matrix} \quad \begin{matrix}\vdots \\ A\end{matrix}}{B} \qquad \text{and} \qquad (R_2) \quad \frac{\begin{matrix}A \\ \vdots \\ B\end{matrix}}{D}$$

as inference rules where A, B, D are formulas without free variables, C a formula containing exactly one free variable x and t a constant term. We want to show that the inference rule

$$(R_3) \qquad \frac{\begin{array}{c} E \\ \vdots\, x \\ C \end{array} \quad \begin{array}{c} \vdots \\ E[x/t] \end{array}}{D}$$

is derivable in $\mathcal{K}$, where E is like C a formula containing x as the only free variable. According to the definition we have to show that

$$(5) \qquad E \underset{x}{\Rightarrow} C,\ E[x/t] \vdash_{\mathcal{K}} D.$$

This can be proved by means of the following derivation:

$$(6) \qquad R_2\,\dfrac{R_1\,\dfrac{X\,\dfrac{E \underset{x}{\Rightarrow} C \quad E[x/t]}{C[x/t]} \qquad \overset{1}{(A)}}{B}}{D}\,1$$

Here X denotes the application of the assumption rule $E \underset{x}{\Rightarrow} C$ according to the schema (4), and the numeral 1 indicates that A is discharged by a corresponding application of R_2.

If we had a derivation

$$\begin{array}{c} \Delta \quad E \\ \ddots\,\therefore \\ C \end{array} \quad \text{of } C$$

from E and from a system Δ of further assumptions not containing x free and a derivation

$$\begin{array}{c} \Delta' \\ + \\ + \\ + \\ E[x/t] \end{array} \quad \text{of } E[x/t]$$

from a system Δ' of assumptions, we would also have a derivation

$$\begin{array}{c} \Delta \quad E[x/t] \\ \ddots\,\therefore \\ C[x/t] \end{array} \quad \text{of } C[x/t]$$

from $E[x/t]$ and Δ (by condition (2) which is assumed to be valid for $\mathscr{K}$). Thus we could replace

$$\frac{E \underset{x}{\Rightarrow} C \quad E[x/t]}{C[x/t]} \quad \text{in (6) by} \quad \begin{array}{c} \quad\quad \Delta' \\ \quad\quad + \\ \quad\quad + \\ \quad\quad + \\ \Delta \quad E[x/t] \\ \ddots \; \therefore \\ C[x/t] \end{array}$$

and would obtain a derivation of D from Δ, Δ'. Therefore, (6) represents on the object level a procedure for transforming derivations of C from E and further assumptions Δ not containing x free and of $E[x/t]$ from Δ' into a derivation of D from Δ, Δ', what must be required from a definition of the derivability of (R_3). Conversely we would obtain (5) from such a procedure, if we assumed

$$\begin{array}{c} \Delta \quad E \\ \ddots \; \therefore \\ C \end{array} \quad \text{to be} \quad \frac{E \underset{x}{\Rightarrow} C \quad E}{C} \quad \text{and} \quad \begin{array}{c} \Delta' \\ + \\ + \\ E[x/t] \end{array} \quad \text{to be } E[x/t].$$

II. CANONICAL DERIVATIONS. THE DEFINITION OF VALIDITY

We assume a calculus for atomic formulas to be given. According to Prawitz (1973) we call it an atomic base $\mathscr{B}$. It is not necessary to give a closer specification of this system. It can be a Post system, but may also contain inference rules of the form (1). The only restriction we impose on the derivability relation $\vdash_{\mathscr{B}}$ of such a system is that it fulfil condition (2).

We want to extend the atomic base $\mathscr{B}$ by inference rules for the operators $\wedge$, $\vee$, $\rightarrow$, $\perp$, $\forall$, $\exists$ of intuitionistic first-order logic in such a way that these inference rules can be considered to determine the meaning of the operators. Following Gentzen's program we choose the introduction rules to be the central inference rules for that purpose:

$$\begin{array}{cc} \vdots & \vdots \\ A & B \end{array} \over A \wedge B \qquad \frac{\begin{array}{c}\vdots \\ A\end{array}}{A \vee B} \qquad \frac{\begin{array}{c}\vdots \\ B\end{array}}{A \vee B}$$

$$\frac{\begin{array}{c}A\\ \vdots\\ B\end{array}}{A \to B} \qquad [\bot \text{ has no introduction rule}]$$

$$\frac{\begin{array}{c}\vdots y\\ A[x/y]\end{array}}{\forall x A} \quad (y \text{ not free in } \forall x A) \qquad \frac{\begin{array}{c}\vdots\\ A[x/t]\end{array}}{\exists x A}$$

Let $\mathscr{C}$ be $\mathscr{B}$, extended by the introduction rules. $\vdash_{\mathscr{C}}$ obviously fulfils condition (2). According to a terminology used by Dummett and Prawitz in their philosophical discussions, we speak of a "canonical derivation" and of "canonical derivability" for a "derivation in $\mathscr{C}$" and for "derivability in $\mathscr{C}$", respectively. In doing so we consider the reference to introduction rules as the main feature of the notion of canonicality. With respect to other points our notion of canonical derivations is slightly different from that of Dummett and Prawitz.

By means of this concept of a canonical derivation the concept of validity of an inference rule can be defined. The intention behind the definition of a valid inference rule is to extend the concept of canonical derivations in such a way that the additional inference rules can be justified by appeal to the introduction rules or to the meaning assigned to complex formulas by the introduction rules for their principal signs. For the elimination rules of the standard intuitionistic connectives Prawitz (1965) formulated an "inversion principle" which describes the intuitive idea that elimination rules are inverses of the corresponding introduction rules (cf. Prawitz, 1965, pp. 32–34). We want to generalize this principle in order to obtain a criterion for the justification of arbitrary inference rules. Thus we read it not as a description of the relation between given introduction and elimation rules but as a criterion to justify an elimination rule relative to a given introduction rule. An alleged elimination rule is justified as an elimination rule if the inversion principle holds for that rule, i.e., if, given a derivation of its major premiss using an introduction rule in the last step and derivations of its minor premisses, a derivation of its conclusion can be found *without* an application of that rule. Obviously this formulation works only for elimination rules of a special form, viz. with one major premiss and (possibly) one or more minor premisses. In order to use it for a definition of validity for *arbitrary* inference rules (including, e.g., introduction rules) we have to reformulate the schema (1) so that it is on the one hand a schema for elimination rules to which the inversion principle can be applied but on the other hand includes as a limit case inference rules which are not to be interpreted as elimination rules in the genuine sense. In other words, we want to have a schema according to which all rules are special cases of elimination rules (in a wider sense) so that an inversion principle can be applied to all rules.

We are led to such a schema by the following consideration: *Genuine* elimination rules are inference rules of the form (1), in which one or more premisses without assumptions and without eigenvariables are distinguished as *major premisses*. A *non-genuine* elimination rule is then simply an inference rule which has no major premisses. So we write the general schema of an inference rule in the form

$$(7)\qquad \frac{\begin{matrix}\vdots\\ {}^*A_1\end{matrix}\ \ldots\ldots\ \begin{matrix}\vdots\\ {}^*A_n\end{matrix}\quad \begin{matrix}\Gamma_1\\ \vdots\,\underline{x}_1\\ B_1\end{matrix}\ \ldots\ldots\ \begin{matrix}\Gamma_m\\ \vdots\,\underline{x}_m\\ B_m\end{matrix}}{A}$$

where the stars indicate the premisses to be counted as major premisses. (We leave open the possibility that a Γ_i and/or $\underline{x}_i$ is empty; we then have assumption-free and/or eigenvariable-free minor premisses, as, e.g., in the case of $\rightarrow$ elimination.) The star is of course not allowed for atomic premisses. The generalisation we thus undertake for elimination rules (in the genuine sense) is that we allow them to have more than one major premiss.

This enables us to formulate an inversion principle as a definition of the validity of a rule of the form (7): A rule of the form (7) is valid iff: given derivations of its premisses, whereby in the case of major premisses introduction rules are used in the last step, a derivation of its conclusion can be found *without* an application of that rule.

This proposal still leaves open what kind of derivations are taken into consideration, i.e., to which formal system they belong. As such derivations we allow derivations in $\mathscr{C}$ possibly depending on assumptions (including assumption rules). We choose $\mathscr{C}$ because the canonical derivations are the basis of our semantic framework. We permit derivations to depend on assumption formulas because the basic notion of "derivation" in natural deduction calculi is that of "derivation from assumption formulas". We furthermore permit assumption rules because we want validity to be preserved in each extension of $\mathscr{C}$; arbitrary assumption rules as possible assumptions guarantee that validity is not affected when we extend $\mathscr{C}$ by rules already shown to be valid.[3, 4] They even guarantee that it is not affected when one adds rules that are not necessarily valid. This is very important: Validity is not considered to be a global property of whole formal systems but a local property of single inference rules. Thus this property should not depend on the special features of the system to which the inference rules in question belong. Besides this there remains the motivation for admitting assumption rules: namely that they allow one to define a satisfactory concept of derivability of rules (see above, Part I). The way assumption rules are related to natural deduction systems is similar to the way assumptions (in the sense of formulas) are related to Hilbert-style systems.

So we are led to the following *inversion principle/definition of validity*, where $\vdash^{+}_{\mathscr{C}}$ denotes canonical derivability by means of a derivation using an introduction rule in the last step:

DEFINITION. An inference rule of the form (7) is valid, iff for all systems Δ of assumptions (including assumption rules): If $\Delta \vdash^{+}_{\mathscr{C}} A_i$ for all i $(1 \leqslant i \leqslant n)$ and there is for all j $(1 \leqslant j \leqslant m)$ a subsystem Δ_j of Δ not containing a variable of $\underline{x}_j$ free such that $\Delta_j, \Gamma_j \vdash_{\mathscr{C}} B_j$, then $\Delta \vdash_{\mathscr{C}} A$.

(This generalized implication should be understood constructively as the existence of a procedure which effectively transforms derivations into derivations.)

As a corollary of this definition we formulate:

LEMMA If a valid rule of the form (7) has no major premisses, it is derivable in $\mathscr{C}$.

Proof Since by (4) for each j $(1 \leqslant j \leqslant m)$ we have $\Gamma_j \underset{\underline{x}_j}{\Rightarrow} B_j, \Gamma_j \vdash_{\mathscr{C}} B_j$, and since $\Gamma_j \underset{\underline{x}_j}{\Rightarrow} B_j$ does not contain $\underline{x}_j$ free, we obtain by the definition of validity

$$\Gamma_1 \underset{\underline{x}_1}{\Rightarrow} B_1, \ldots, \Gamma_m \underset{\underline{x}_m}{\Rightarrow} B_m \vdash_{\mathscr{C}} A,$$

i.e., the derivability in $\mathscr{C}$ of the valid inference rule considered.

This result coincides very well with our expectations that valid inference rules which cannot be conceived as genuine elimination rules do not properly extend the introduction rules.

Let $\Delta \Vdash A$ denote the derivability of A from Δ by means of valid rules of form (7). (For the relation of '$\Vdash$' to our notion of validity see below, Part IV.)

III. THE COMPLETENESS OF INTUITIONISTIC FIRST-ORDER LOGIC

Let $\mathscr{J}$ be the calculus of intuitionistic first-order logic, extended by the chosen atomic base $\mathscr{B}$, i.e., $\mathscr{C}$ plus the elimination rules

$$\frac{\overset{\vdots}{{}^{*}A \wedge B}}{A} \qquad \frac{\overset{\vdots}{{}^{*}A \wedge B}}{B} \qquad \frac{\overset{\vdots}{{}^{*}A \vee B} \quad \overset{\overset{A}{\vdots}}{C} \quad \overset{\overset{B}{\vdots}}{C}}{C}$$

$$\frac{\overset{\vdots}{{}^{*}A \rightarrow B} \quad \overset{\vdots}{A}}{B} \qquad \frac{\overset{\vdots}{{}^{*}\bot}}{B}$$

$$\frac{\overset{\vdots}{{}^{*}\forall xA}}{A[x/t]} \qquad \frac{\overset{\vdots}{{}^{*}\exists xA} \quad \overset{\overset{A[x/y]}{\vdots\, y}}{B}}{B} \quad (y \text{ not free in } \exists xA \text{ or in } B)$$

Let $\Gamma \vdash_{\mathscr{J}} A$ denote the derivability of A from the assumptions Γ in $\mathscr{J}$.

THEOREM For each formula A and each system of assumptions Γ: $\Gamma \vdash_{\mathscr{J}} A$ iff $\Gamma \Vdash A$.

Proof 1. From left to right: We show that all inference rules of $\mathscr{J}$ are valid. The assertion, then, follows by induction on the length of derivations in $\mathscr{J}$. For the atomic inference rules and the introduction rules this is trivial since they are part of $\mathscr{C}$. Now we consider the elimination rules. Let Δ be a system of assumptions.

If $\Delta \vdash^{+}_{\mathscr{C}} A \wedge B$, then $\Delta \vdash_{\mathscr{C}} A$ and $\Delta \vdash_{\mathscr{C}} B$, since the derivation of $A \wedge B$ uses $\wedge$ introduction in the last step.

If $\Delta \vdash^{\pm}_{\mathscr{C}} A \wedge B$, then $\Delta \vdash_{\mathscr{C}} A$ or $\Delta \vdash_{\mathscr{C}} B$. Together with $\Delta, A \vdash_{\mathscr{C}} C$ and $\Delta, B \vdash_{\mathscr{C}} C$ we obtain $\Delta \vdash_{\mathscr{C}} C$.

$\Delta \vdash^{\pm}_{\mathscr{C}} \bot$ is always false since $\bot$ has no introduction rule.

If $\Delta \vdash^{\pm}_{\mathscr{C}} \forall x A$, then $\Delta' \vdash_{\mathscr{C}} A[x/y]$ where Δ' is a subsystem of Δ not containing y free. Then $\Delta' \vdash_{\mathscr{C}} A[x/y][y/t]$ for all terms t by condition (2). This equals $\Delta' \vdash_{\mathscr{C}} A[x/t]$ by the condition on y. Therefore $\Delta \vdash_{\mathscr{C}} A[x/t]$.

If $\Delta \vdash^{\pm}_{\mathscr{C}} \exists x A$, then $\Delta \vdash_{\mathscr{C}} A[x/t]$ for a term t. If furthermore $\Delta', A[x/y] \vdash_{\mathscr{C}} B$ for a subsystem Δ' of Δ, where Δ' and B do not contain y free and y is not free in $\exists x A$, we obtain $\Delta', A[x/y][y/t] \vdash_{\mathscr{C}} B$ by condition (2). This equals $\Delta', A[x/t] \vdash_{\mathscr{C}} B$. Therefore $\Delta \vdash_{\mathscr{C}} B$.

2. *From right to left:* We show that each valid inference rule of the form (7) is derivable in $\mathscr{J}$, i.e.,

$$A_1, \ldots, A_n, \Gamma_1 \underset{\underline{x}_1}{\Rightarrow} B_1, \ldots, \Gamma_m \underset{\underline{x}_m}{\Rightarrow} B_m \vdash_{\mathscr{J}} A.$$

The assertation, then, follows by induction on the length of derivations using valid rules: If we have $\Delta \Vdash A$ by means of a derivation using a valid rule of the form (7) in its last step, we have $\Delta \Vdash A_i$ for all i $(1 \leqslant i \leqslant n)$ and $\Delta_j, \Gamma_j \Vdash B_j$ for all j $(1 \leqslant j \leqslant m)$ by means of shorter derivations, where $\{\Delta_j\} \subseteq \{\Delta\}$ and Δ_j does not contain a variable of $\underline{x}_j$ free. By induction hypothesis $\Delta \vdash_{\mathscr{J}} A_i$ for all i $(1 \leqslant i \leqslant n)$ and $\Delta_j, \Gamma_j \vdash_{\mathscr{J}} B_j$ for all j $(1 \leqslant j \leqslant m)$, therefore by conditon (2) $\Delta_j, \Gamma_j[\underline{x}_j/\underline{t}_j] \vdash_{\mathscr{J}} B_j[\underline{x}_j/\underline{t}_j]$ for all $\underline{t}_j$ of the same length as $\underline{x}_j$ and free for $\underline{x}_j$ in Γ_j and B_j. Thus we are able to replace each assumption A_i and each application of $\Gamma_j \underset{\underline{x}_j}{\Rightarrow} B_j$ in the derivation of A from $A_1, \ldots, A_n, \Gamma_1 \underset{\underline{x}_1}{\Rightarrow} B_1, \ldots, \Gamma_m \underset{\underline{x}_m}{\Rightarrow} B_m$ in $\mathscr{J}$ by the given derivations in $\mathscr{J}$. This yields $\Delta \vdash_{\mathscr{J}} A$.

We prove the derivability of valid inference rules in $\mathscr{J}$ by induction on the number of their major premisses (i.e., their starred premisses). Let a valid inference rule R of the form (7) be given. If $n = 0$, then R is derivable in $\mathscr{C}$ by the lemma, thus derivable in $\mathscr{J}$. If $n > 0$, we consider the rightmost major premiss A_n of R.

If A_n has the form $C \wedge D$, then

$$\begin{array}{cccccc} & & & & \Gamma_1 & \Gamma_m \\ \vdots & \vdots & \vdots & \vdots & \vdots \, \underline{x}_1 & \vdots \, \underline{x}_m \\ \hline {}^*A_1 \ldots\ldots & {}^*A_{n-1} & C & D & B_1 \ldots\ldots & B_m \\ \hline & & A & & & \end{array}$$

is a valid rule since from $\Delta \vdash_{\mathscr{G}} C$ and $\Delta \vdash_{\mathscr{G}} D$ we obtain $\Delta \vdash^{+}_{\mathscr{G}} C \wedge D$. Since this rule has a lower number of major premisses than R, we have by induction hypothesis

$$A_1, \ldots, A_{n-1}, C, D, \Gamma_1 \underset{\underline{x}_1}{\Rightarrow} B_1, \ldots, \Gamma_m \underset{\underline{x}_m}{\Rightarrow} B_m \vdash_{\mathscr{S}} A.$$

By $\wedge$ elimination in $\mathscr{S}$ we obtain the derivability of R.

If A_n has the form $C \vee D$, then

$$\begin{array}{cccccc} & & & \Gamma_1 & & \Gamma_m \\ \vdots & & \vdots & \vdots & \vdots\, \underline{x}_1 & & \vdots\, \underline{x}_m \\ {}^*A_1 & \ldots\ldots & {}^*A_{n-1} & C & B_1 \;\ldots\ldots & B_m \\ \hline & & & A & & \end{array}$$

and

$$\begin{array}{cccccc} & & & \Gamma_1 & & \Gamma_m \\ \vdots & & \vdots & \vdots & \vdots\, \underline{x}_1 & & \vdots\, \underline{x}_m \\ {}^*A_1 & \ldots\ldots & {}^*A_{n-1} & D & B_1 \;\ldots\ldots & B_m \\ \hline & & & A & & \end{array}$$

are valid rules with a lower number of major premisses than R. Thus, by induction hypothesis, they are derivable in $\mathscr{S}$. By $\vee$ elimination we obtain the derivability of R in $\mathscr{S}$.

If A_n has the form $C \rightarrow D$, then

$$\begin{array}{cccccc} & & & C & \Gamma_1 & \Gamma_m \\ \vdots & & \vdots & \vdots & \vdots\, \underline{x}_1 & \vdots\, \underline{x}_m \\ {}^*A_1 & \ldots\ldots & {}^*A_{n-1} & D & B_1 \;\ldots\ldots & B_m \\ \hline & & & A & & \end{array}$$

is valid and, by induction hypothesis, derivable in $\mathscr{S}$, i.e.,

$$A_1, \ldots, A_{n-1}, C \Rightarrow D, \Gamma_1 \underset{\underline{x}_1}{\Rightarrow} B_1, \ldots, \Gamma_m \underset{\underline{x}_m}{\Rightarrow} B_m \vdash_{\mathscr{S}} A.$$

Replacing each application

$$\frac{C \Rightarrow D \quad \overset{\vdots}{C}}{D} \text{ of } C \Rightarrow D$$

by an application

$$\frac{C \to D \quad \begin{matrix} \vdots \\ C \end{matrix}}{D} \text{ of } \to \text{ elimination,}$$

we obtain the derivability of R in $\mathscr{I}$.

If A_n is $\perp$, then R is trivially derivable in $\mathscr{I}$, since $\perp \vdash_{\mathscr{I}} A$.

If A_n has the form $\forall xC$, then

$$\frac{\begin{matrix} \vdots \\ {}^*A_1 \end{matrix} \;\ldots\ldots\; \begin{matrix} \vdots \\ {}^*A_{n-1} \end{matrix} \quad \begin{matrix} \vdots\, y \\ C[x/y] \end{matrix} \quad \begin{matrix} \Gamma_1 \\ \vdots\, \underline{x}_1 \\ B_1 \end{matrix} \;\ldots\ldots\; \begin{matrix} \Gamma_m \\ \vdots\, \underline{x}_m \\ B_m \end{matrix}}{A}$$

is valid. By induction hypothesis we have

$$A_1, \ldots, A_{n-1}, \underset{y}{\Rightarrow} C[x/y], \Gamma_1 \underset{\underline{x}_1}{\Rightarrow} B_1, \ldots, \Gamma_m \underset{\underline{x}_m}{\Rightarrow} B_m \vdash_{\mathscr{I}} A.$$

Since $C[x/y][y/t]$ equals $C[x/t]$ by the condition on y, we may replace each application

$$\frac{\underset{y}{\Rightarrow} C[x/y]}{C[x/y][y/t]} \text{ of } \underset{y}{\Rightarrow} C[x/y]$$

by an application

$$\frac{\forall xC}{C[x/t]} \text{ of } \forall \text{ elimination}$$

and obtain the derivability of R in $\mathscr{I}$.

If A_n has the form $\exists xC$, then

$$\frac{\begin{matrix} \vdots \\ {}^*A_1 \end{matrix} \;\ldots\ldots\; \begin{matrix} \vdots \\ {}^*A_{n-1} \end{matrix} \quad \begin{matrix} \vdots \\ C[x/t] \end{matrix} \quad \begin{matrix} \Gamma_1 \\ \vdots\, \underline{x}_1 \\ B_1 \end{matrix} \;\ldots\ldots\; \begin{matrix} \Gamma_m \\ \vdots\, \underline{x}_m \\ B_m \end{matrix}}{A}$$

is valid for all terms t. By induction hypothesis we have $A_1, \ldots, A_{n-1}$, $C[x/t], \Gamma_1 \underset{\underline{x}_1}{\Rightarrow} B_1, \ldots, \Gamma_m \underset{\underline{x}_m}{\Rightarrow} B_m \vdash_{\mathscr{I}} A$ for all t. If we choose t to be a variable not occurring free in $A_1, \ldots, A_{n-1}, \exists xC, \Gamma_1 \underset{\underline{x}_1}{\Rightarrow} B_1, \ldots, \Gamma_m \underset{\underline{x}_m}{\Rightarrow} B_m, A$, we can apply $\exists$ elimination and obtain the derivability of R in $\mathscr{I}$.

IV. SOME REMARKS

We note some important features of our concept of validity. This may facilitate comparison with other definitions of validity, especially the one given by Prawitz.

(1) Canonical derivations in our sense need not be closed, i.e., they may depend on assumptions. So, for instance, A is a canonical derivation of A depending on itself and $\dfrac{A \Rightarrow B \quad A}{B}$ is a canonical derivation of B depending on A and (the assumption rule) $A \Rightarrow B$.

(2) Our notion of validity, unlike that of Prawitz, is not strongly connected with the notion of normalizability of derivations but only with the invertibility of introduction rules by elimination rules. As a consequence, our proof that each inference rule of $\mathscr{S}$ is valid (i.e., the first part of the proof of the theorem) does not make use of the whole apparatus of normalization procedures but only of proper reductions. So our completeness proof does not give the information obtained by normalization theorems. (However, there seems to be no reason to require that.)

(3) Besides the concept of canonical derivations ($\mathscr{C}$-derivations) we use in the definition of validity a concept of "canonical derivations applying an introduction rule in the last step"; let us call them "$\mathscr{C}^+$-derivations". Obviously not all subderivations of $\mathscr{C}^+$-derivations are $\mathscr{C}^+$-derivations. For the concept of validity this has the effect that, e.g., $\dfrac{* (A \wedge B) \wedge C}{A}$ is not a valid inference rule: A valid inference rule transforms $\mathscr{C}^+$-derivations of its major premisses and $\mathscr{C}$-derivations of the other (minor) premisses into a $\mathscr{C}$-derivation of its conclusion. From a $\mathscr{C}^+$-derivation of $(A \wedge B) \wedge C$ we can obtain a $\mathscr{C}$-derivation of $A \wedge B$ and a $\mathscr{C}$-derivation of C; but in general we cannot obtain a $\mathscr{C}^+$-derivation of $A \wedge B$ (and thus a $\mathscr{C}$-derivation of A).
$\dfrac{A \wedge B \quad C}{(A \wedge B) \wedge C}$ is, for example, a $\mathscr{C}^+$-derivation of $(A \wedge B) \wedge C$.

(4) Since

$$\frac{* (A \wedge B) \wedge C}{A}$$

is derivable in $\mathscr{S}$, the non-validity of that inference rule shows that there are inference rules derivable in $\mathscr{S}$ which are not valid. So we have not proved in our completeness result that all inference rules derivable in $\mathscr{J}$ are valid; we have proved only that all inference rules derivable in $\mathscr{J}$ are derivable by use of valid inference rules of form (7). This can be stated more generally in the following way: Let $\|$ be the relation between sets $\{{}^*A_1, \ldots, {}^*A_n, \Gamma_1 \underset{\underline{x}_1}{\Rightarrow} B_1, \ldots, \Gamma_n \underset{\underline{x}_n}{\Rightarrow} B_m\}$ and formulas A which holds if (7) is valid. Then for $\|$ we can define a kind of "transitivity" in an obvious way with the result that $\|\!\vdash$ is exactly the transitive closure of $\|$. (We roughly speak of the transitive closure of validity.) Since the inference rules

$$\frac{{}^*(A \wedge B) \wedge C}{A \wedge B} \quad \text{and} \quad \frac{{}^*A \wedge B}{A}$$

are valid our example shows that $\|$ is not transitive, i.e., that $\|\!\vdash$ is different from $\|$ (whereas derivability in $\mathscr{J}$ and its transitive closure are identical). We can now reformulate our completeness theorem. It says that the transitive closures of derivability in $\mathscr{J}$ and of validity coincide; but it does not say that derivability in $\mathscr{J}$ are validity coincide.

(5) The validity of a rule of form (7) depends on whether premisses are starred or not. ${}^*A \wedge B/A$ is valid whereas $A \wedge B/B$ is not. Our derivability concept was so defined that it does not count such rules as different. If we assume, however, that all premisses without assumptions and without eigenvariables of a rule of form (7) are starred, we can state the problem of finding a subsystem $\mathscr{J}'$ of $\mathscr{J}$ so that a rule of this kind is derivable in $\mathscr{J}'$ iff it is valid.[5] Such a system is obtained if we define an $\mathscr{J}'$-derivation to be an $\mathscr{J}$-derivation in which major premisses of elimination rules occur only as top-formulas (i.e. assumptions).

(6) That our concept of validity is not transitive in the sense explained above (contrary, e.g., to the validity concept of Prawitz) seems philosophically (i.e., from the standpoint of the theory of meaning) to be no defect. We require that a valid inference rule can be eliminated from a derivation if its major premisses are used according to their meanings, that is, if they are derived by an application of an introduction rule. That they should be eliminable also in other cases is, as it seems, not a reasonable demand. If we nevertheless want to have a transitive validity concept, we can simply take the transitive closure of our validity concept as the 'real' validity concept, i.e., define an inference rule of form (7) to be valid iff

$$A_1, \ldots, A_n, \Gamma_1 \underset{\underline{x}_1}{\Rightarrow} B_1, \ldots, \Gamma_m \underset{\underline{x}_m}{\Rightarrow} B_m \|\!\vdash A.$$

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NOTES

[1] Our use of the word "valid" differs of course from the use of this word within model-theoretic semantics where it is connected with a certain concept of "interpretation" of signs. One could introduce another word to distinguish validity in our sense from validity in the model-theoretic sense (e.g., "operationally valid", as proposed by the anonymous referee of *JPL*). Following Prawitz, however, we prefer to speak simply of "validity" since this word denotes also in our context the central semantic (or meaning-theoretic) concept which is used to justify a certain derivability notion based on syntactic rules.

[2] That is for the present purpose. For other purposes it is possible and makes sense to introduce calculi with a discharge operation for assumption rules, cf. Schroeder-Heister (1981).

[3] In the last two points the inversion principle we are formulating according to Prawitz (1965) differs entirely from Lorenzen's "Inversionsprinzip" (cf. Lorenzen 1955, p. 30). Lorenzen's Inversionsprinzip is formulated by use of the notion of "Zulässigkeit" of an inference rule, and this notion is a useful tool only for assumption-free derivations whose subderivations are also assumption-free. (For derivations depending on assumptions it coincides with derivability.) However, it is a central feature of natural deduction calculi that assumption-free derivations may contain subderivations depending on assumptions (e.g. a derivation of $A \to B$, using $\to$ introduction in the last step).

[4] Note that assumption rules of the form $\Gamma \underset{\underline{x}}{\Rightarrow} A$ can represent a justified inference rule of the form (1), since in $\mathscr{C}\to$ introduction and $\forall$ introduction are available. The application, e.g., of an inference rule

$$\frac{\begin{array}{c} A, B \\ \vdots\ \underline{x} \\ C \end{array}}{D}$$

can be replaced by an application of the assumption rule $\forall x(B \to (A \to C)) \Rightarrow D$ in the following way:

$$\dfrac{\forall x\,(B\to(A\to C))\Rightarrow D \qquad \dfrac{\dfrac{\dfrac{\begin{matrix}\overset{1}{(A)} & \overset{2}{(B)}\\ \vdots & \\ C & \end{matrix}}{A\to C}\,1}{B\to(A\to C)}\,2}{\forall x\,(B\to(A\to C))}}{D}\,.$$

[5] Neil Tennant drew my attention to this.

REFERENCES

Dummett, M., 1975: 'The Philosophical Basis of Intuitionistic Logic', in: H.E. Rose and J.E. Shepherdson (eds.), *Logic Colloquium '73*, Proceedings of the Logic Colloquium, Bristol, July 1973, Amsterdam, pp. 5-40.

Dummett, M., (with the assistance of R. Minio), 1977: *Elements of Intuitionism*, Oxford.

Gentzen, G., 1935: 'Untersuchungen über das logische Schliessen', *Mathematische Zeitschrift* 39, 176-210, 405-431, English translation in: *The Collected Papers of Gerhard Gentzen*, ed. M.E. Szabo, Amsterdam, 1969, pp. 68-131.

Lorenzen, P., 1955: *Einführung in die operative Logik und Mathematik*, Berlin (2nd edn. 1969).

Prawitz, D., 1965: *Natural Deduction, A Proof-Theoretical Study*, Stockholm.

Prawitz, D., 1971: 'Ideas and Results in Proof Theory', in: J.E. Fenstad (ed.), *Proceedings of the 2nd Scandinavian Logic Symposium* (Oslo 1970), Amsterdam, pp. 235-308.

Prawitz, D., 1973: 'Towards a Foundation of a General Proof Theory', in P. Suppes *et al.* (eds.), *Logic, Methodology and Philosophy of Science* vol. IV, Amsterdam, pp. 225-250.

Prawitz, D., 1977: 'Meaning and Proofs: On the Conflict between Classical and Intuitionistic Logic', *Theoria* 43, 2-40.

Prawitz, D., 1979: 'Proofs and the Meaning and Completeness of the Logical Constants', in: J. Hintikka *et al.* (eds.), Essays on Mathematical and Philosophical Logic, Dordrecht, pp. 25-40.

Schroeder-Heister, P., 1981: 'Untersuchungen zur regellogischen Deutung von Aussagenverknüpfungen', Dissertation, Bonn.

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