ULF R. SCHMERL, *Diophantine equations in a fragment of number theory.*

We study the following problem: Given a diophantine equation, is it possible to find out whether or not this equation can be proved impossible in the fragment Z_0 of classical first order arithmetic in 0, S, $+$, $\cdot$, and open induction?

Using proof-theoretic methods we prove the following: Let $r(x_1 \cdots x_n) = s(x_1 \cdots x_n)$ be a diophantine equation in the variables $x_1, \ldots, x_n$. Then

$$\forall x_1 \cdots x_n [r(x_1 \cdots x_n) \neq s(x_1 \cdots x_n)] \text{ is provable in } Z_0$$
$$\Leftrightarrow \exists c \in N\, \forall (\hat{x}_1 \cdots \hat{x}_n) \in I^c_{x_1 \cdots x_n} r(\hat{x}_1 \cdots \hat{x}_n) - s(\hat{x}_1 \cdots \hat{x}_n) \mid 1 + \mathbf{N}[x_1 \cdots x_n].$$

Here $I^c_{x_1 \cdots x_n} = \{0, 1, \ldots, c-1, x_1 + c\} \times \cdots \times \{0, 1, \ldots, c-1, x_n + c\}$, $1 + \mathbf{N}[x_1 \cdots x_n]$ is the set of polynomials in $x_1 \cdots x_n$ with coefficients from $\mathbf{N}$ and constant coefficient $\neq 0$, and $r(\hat{x}_1 \cdots \hat{x}_n) - s(\hat{x}_1 \cdots \hat{x}_n) \mid 1 + \mathbf{N}[\cdots]$ means that $r(\hat{x}_1 \cdots \hat{x}_n) - s(\hat{x}_1 \cdots \hat{x}_n)$, understood as a polynomial from $\mathbf{Z}[x_1 \cdots x_n]$, divides some polynomial in $1 + \mathbf{N}[x_1 \cdots x_n]$.

This characterization still holds when the functions P (predecessor), sg, and $\overline{\text{sg}}$ (sign and cosign) are added to Z_0.

PETER H. SCHMITT, *Model- and substructure complete theories of ordered Abelian groups.*

In his pioneering paper [1] Yuri Gurevich associated with every ordered Abelian group G for every $n \geq 2$ a coloured chain (i.e. a linear order with additional unary predicates) $\text{Sp}_n(G)$, called the *n-spine* of G, and proved that $G \equiv H$ if and only if $\text{Sp}_n(G) \equiv \text{Sp}_n(H)$ for all $n \geq 2$.

Thus for every elementary class $\mathscr{M}$ of ordered abelian groups there are theories T_n in the language of n-spines, such that

$$G \in \mathscr{M} \text{ if and only if } \text{Sp}_n(G) \models T_n \text{ for all } n \geq 2.$$

MAIN THEOREM. *If for all $n \geq 2$ T_n is model complete (substructure complete), then $\mathscr{M}$ is model complete (substructure complete) in a certain definitional extension of the language of ordered groups.*

REFERENCE

[1] Y. GUREVICH, *Elementary properties of ordered abelian groups*, ***Algebra i Logika Seminar***, vol. 3 (1964), pp. 5–39; English translation, ***American Mathematical Society Translations***, ser. 2, vol. 46 (1965), pp. 165–192.

P. SCHROEDER-HEISTER, *Natural deduction calculi with rules of higher levels.*

Natural deduction calculi, as introduced by S. Jaśkowski and G. Gentzen, differ from Hilbert-type calculi as well as from sequent calculi in that assumptions may be discharged by the application of inference rules. An inference rule in such calculi can be stated as

$$\frac{\begin{matrix} \Gamma_1 & & \Gamma_l \\ A_1 & \cdots & A_l \end{matrix}}{A}$$

where the Γ's are (possibly empty) sequences of formulas indicating the assumptions which may be discharged. This concept of a calculus can be generalized in the following way. In the first step one allows not only formulas but also rules as assumptions. If a rule R which does not belong to the basic inference rules of the calculus considered is used in a derivation of a formula A, then A is said to *depend* on R. In the second step one defines inference rules which allow one to discharge assumptions which are themselves rules. This leads to the concept of rules of arbitrary (finite) levels: A level-0 rule is a formula, a level-1 rule is a rule not allowing one to discharge any assumption (like rules in Hilbert-type systems), and a level-($m + 2$) rule is a rule allowing one to discharge assumptions which are level-m rules. An example of a level-3 rule is

$$\frac{\begin{matrix} & A \Rightarrow B \\ A \rightarrow B & C \end{matrix}}{C}$$

where $\rightarrow$ is the implication sign and $A \Rightarrow B$ is a linear notation for the level-1 rule $\frac{A}{B}$. This level-3 rule is equivalent to *modus ponens*. With the help of level-m rules for arbitrary (finite) m, a general schema for introduction and elimination rules for n-place sentential connectives and quantifiers is definable, thus yielding a natural deduction system for logical operators in a generalized sense. Derivations in this system are normalizable. Furthermore, the (functional) completeness of the standard intuitionistic operators $\wedge$, $\vee$, $\rightarrow$, $\perp$, $\forall$ and $\exists$ can be proved. The system is not suitable for the interpretation of modal calculi without modifications. So the meaning of level-m rules is somewhat different from the meaning of sequents of higher levels used by K. Došen (***Logical Constants***, Ph.D. thesis, Oxford, 1980) for the interpretation of various logical systems including modal and relevant logics.

WILFRIED SIEG, *A note on König's lemma.*

Every finitely branching but infinite tree has an infinite branch. That is König's lemma KL, a most useful tool for mathematical and metamathematical investigations. The Heine/Borel covering theorem and Gödel's completeness theorem, to mention just two examples, can be proved using KL (over a very weak theory; see below). KL can be formulated as an "abstract principle" [2] in the language of second order arithmetic:

$$\text{KL} \qquad (\forall f)[\mathcal{T}(f) \,\&\, (\forall x)(\exists y)(\text{lh}(y) = x \,\&\, f(y) = 0) \rightarrow (\exists g)(\forall x) f(\bar{g}(x)) = 0],$$

where $\mathcal{T}(f)$ abbreviates that $\{x \mid f(x) = 0\}$ forms a finitely branching tree; i.e.

$$(\forall x)(\forall y)(f(x * y) = 0 \rightarrow f(x) = 0) \,\&\, (\forall x)(\exists z)(\forall y)(f(x * \langle y \rangle) = 0 \rightarrow y \leq z).$$

Over BT (the second order version of PRA together with the comprehension principle for quantifier-free formulas) plus Σ^0_1-AC$_0$, KL is equivalent to the full arithmetical choice principle Π^0_∞-AC$_0$ (see [1]). Thus the theory

$$(\text{BT} + \Sigma^0_1\text{-AC}_0 + \text{KL}) \qquad [(\text{BT} + \Sigma^0_1\text{-AC}_0 + \Pi^1_\infty\text{-IA} + \text{KL})]$$

is equivalent to

$$(\Pi^0_\infty\text{-AC}_0) \upharpoonright [(\Pi^0_\infty\text{-AC}_0)]$$

and consequently [NOT] conservative over elementary number theory z.

In the presence of Σ^0_2-AC$_0$, i.e. in effect Π^0_∞-AC$_0$, KL is equivalent over BT to a version in which a bound for the size of the immediate descendants of a node is given by a function:

$$\text{KL}_\text{b} \qquad (\forall f)(\forall g)[\mathcal{T}(f, g) \,\&\, (\forall x)(\exists y)(\text{lh}(y) = x \,\&\, f(y) = 0) \rightarrow (\exists h)(\forall x) f(\bar{h}(x)) = 0]$$

where $\mathcal{T}(f, g)$ now abbreviates

$$(\forall x)(\forall y)(f(x * y) = 0 \rightarrow f(x) = 0) \,\&\, (\forall x)(\forall y)(f(x * \langle y \rangle) = 0 \rightarrow y \leq g(x)).$$

Kl$_\text{b}$ is by itself, however, weaker than KL: if (K) is (BT + Σ^0_1-AC$_0$ + Π^1_∞-IA + KL$_\text{b}$), then (K) is conservative over Z. This is a slight generalization of a result of Kreisel's [2]. For the refined development of analysis and metamathematics (see [4]) other results are more significant.

THEOREM 1. *$(F) := (BT + \Sigma^0_1\text{-}AC_0 + \Sigma^0_1\text{-}IA + KL_b)$ is conservative over PRA for Π^0_2-sentences.*

Friedman's theory WKL$_0$ is essentially (BT + Δ^0_1-CA + Σ^0_1-IA + WKL), where WKL is König's lemma for trees of sequences of zeros and ones, and it is contained in (F). So we have as a corollary a result of Friedman's [4]: WKL$_0$ is conservative over PRA for Π^0_2-sentences. Note that the examples mentioned above can be proved in WKL$_0$; indeed, they are equivalent to WKL (see [4]).

Minc [3] formulated a theory S^+ which is (BT + Π^0_1-CA$^-$ + Π^0_2-IR$^-$); the schemata extending BT are available only for formulas without function parameters. (IR is the induction rule.) WKL for primitive recursive trees can be proved in S^+ and (using it) Gödel's completeness theorem. Minc showed that S^+ is a conservative extension of PRA for Π^0_2-sentences. This fact is an immediate consequence of the following stronger result.

THEOREM 2. *$(M) := (BT + \Sigma^0_2\text{-}AC_0^- + \Pi^0_2\text{-}IR^- + KL_b)$ is conservative over PRA for Π^0_2-sentences.*

The arguments for Theorems 1 and 2 are purely proof theoretic.

REFERENCES

[1] H. FRIEDMAN, ***König's lemma is weak***, mimeographed notes, Stanford University, Stanford, California, c. 1969.