Prepared for the Post-Conference Workshop on "Proof-Theoretical Extensions of Logic Programming" at the Eleventh International Conference on Logic Programming (ICLP'94, Santa Margherita Ligure, Italy, 13-17 June 1994). The topic was taken up in "Restricting initial sequents: the trade-offs between identity, contraction and cut". In: Advances in Proof Theory. Ed. by Reinhard Kahle, Thomas Strahm and Thomas Studer. Basel: Birkhäuser 2016, pp. 339–351. https://doi.org/10.1007/978-3-319-29198-7_10.

Cut Elimination for Logics with Definitional Reflection and Restricted Initial Sequents

Peter Schroeder-Heister
Wilhelm-Schickard-Institut, Universität Tübingen
Sand 13, 72076 Tübingen, Germany
e-mail: psh@logik.informatik.uni-tuebingen.de *

The failure of cut elimination in general has sometimes be considered a deficiency of systems with definitional reflection. If the system is contraction-free or if the definition considered does not contain implication, then the system admits cut elimination (see [5]). Based on considerations unrelated to cut elimination, Kreuger [4] has proposed to restrict initial sequents

$$a \vdash a$$

in the logic of definitional reflection to the case where a is an atomic formula which is not properly defined by the given definition $\mathcal{D}$ in the sense that $a \Leftarrow a$ is the only clause for a in $\mathcal{D}$. It will be shown that in such systems, which contain contraction, cut is eliminable.

Slightly differently, without using clauses like $a \Leftarrow a$, we can describe the situation as follows: Let $\mathcal{D}$ be a definition and $\mathcal{U}$ a distinguished set of atoms which are not defined by $\mathcal{D}$, i.e., which are not head of any clause in $\mathcal{D}$. Elements of $\mathcal{U}$ are called 'uratoms'. Then we consider the system of definitional reflection described in [6] with both thinning and contraction, but replace the rule (I) with

$$(I)_{\mathcal{U}} \ \frac{}{a \vdash a} \quad a \in \mathcal{U}$$

and the rule $(\mathcal{D}\vdash)$ with

$$(\mathcal{D}\vdash)_{\mathcal{U}} \ \frac{\{\Gamma, C \vdash A : C \in \mathcal{D}(a)\}}{\Gamma, a \vdash A} \quad a \notin \mathcal{U}$$

(of course with the usual proviso that guarantees closure under substitution). Furthermore, due to the presence of contraction, we just consider a single conjunction $\wedge$. We call this system $\mathbf{DR}_{\mathcal{U}}(\mathcal{D})$. We show that cut is admissible in this system.

*Draft (Februar 1994) — Comments welcome)

First we transform $\mathbf{DR}_{\mathcal{U}}(\mathcal{D})$ into a system $\mathbf{DR}^*_{\mathcal{U}}(\mathcal{D})$, in which contraction and thinning are no longer explicit rules but built into the other rules. For simplicity, we here only consider the propositional part:

$$(I)\ \frac{}{\Gamma, a \vdash a}\quad a \in \mathcal{U}$$

$$(\top)\ \frac{}{\Gamma \vdash \top} \qquad\qquad (\bot)\ \frac{}{\Gamma, \bot \vdash A}$$

$$(\vdash\wedge)\ \frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \wedge B} \qquad\qquad (\wedge\vdash)\ \frac{\Gamma, A, B \vdash C}{\Gamma, A \wedge B \vdash C}$$

$$(\vdash\vee)\ \frac{\Gamma \vdash A}{\Gamma \vdash A \vee B} \quad \frac{\Gamma \vdash B}{\Gamma \vdash A \vee B} \qquad (\vee\vdash)\ \frac{\Gamma, A \vdash C \quad \Gamma, B \vdash C}{\Gamma, A \vee B \vdash C}$$

$$(\vdash\to)\ \frac{\Gamma, A \vdash B}{\Gamma \vdash A \to B} \qquad\qquad (\to\vdash)\ \frac{\Gamma, A \to B \vdash A \quad \Gamma, B \vdash C}{\Gamma, A \to B \vdash C}$$

$$(\vdash\mathcal{D})\ \frac{\Gamma \vdash C}{\Gamma \vdash a}\ C \in \mathcal{D}(a) \qquad\qquad (\mathcal{D}\vdash)_{\mathcal{U}}\ \frac{\{\Gamma, C \vdash A : C \in \mathcal{D}(a)\}}{\Gamma, a \vdash A}\quad a \notin \mathcal{U}$$

It is obvious that thinning is admissible in $\mathbf{DR}^*_{\mathcal{U}}(\mathcal{D})$. For contraction we argue as follows: We can show that $(\mathcal{D}\vdash)_{\mathcal{U}}$ is invertible in the sense that, if $\Gamma, a \vdash A$ is derivable with n applications of $(\mathcal{D}\vdash)_{\mathcal{U}}$, then $\Gamma, C \vdash A$ is derivable for any $C \in \mathcal{D}(a)$ with $< n$ applications of $(\mathcal{D}\vdash)_{\mathcal{U}}$. Here it is crucial that $\Gamma, a \vdash A$ cannot result from $(I)_{\mathcal{U}}$, since a is no uratom. The admissibility of contraction then follows by induction on the number of applications of $(\mathcal{D}\vdash)_{\mathcal{U}}$ and the length of derivations.

Remark If we had to formulate a system with implicit contraction, but with the rule (I) unrestricted, we would have to take

$$\frac{\{\Gamma, a, C \vdash A : C \in \mathcal{D}(a)\}}{\Gamma, a \vdash A}$$

with the a repeated above the inference line as a primitive rule. Otherwise, for example, from the definition $p \Leftarrow p \to \bot$ the sequent $p \vdash \bot$ would not be derivable, which is derivable with explicit contraction.

Now in the system $\mathbf{DR}^*_{\mathcal{U}}(\mathcal{D})$ we can easily eliminate cuts: we use induction on the triple $\langle d, c, l \rangle$, where the $\mathcal{D}$-rank d is the maximum number of applications of $\mathcal{D}$-rules in all branches leading to the conclusion of the cut, the cut-degree c is the complexity of the cut formula, and the cut-length l is the number of rule applications above the conclusion of the cut. In the main reductions with $\to$ the $\mathcal{D}$-rank is not increased since in its definition we have taken the maximum and not the sum of applications of $\mathcal{D}$-rules. In the main reduction of $\mathcal{D}$ the $\mathcal{D}$-rank is decreased since we have counted both $(\mathcal{D}\vdash)_{\mathcal{U}}$- and $(\vdash\mathcal{D})$-inferences.

Remarks

1. In the system with unrestricted (I) and implicit contraction within $(\mathcal{D}\vdash)$ we cannot perform main reductions of $\mathcal{D}$, since the $\mathcal{D}$-rank is not necessarily decreased. If we have no contraction at all, then it is possible to work with sums in the computation of the $\mathcal{D}$-rank and just count $(\mathcal{D}\vdash)$-applications (as in [5]).[1]

2. If we use the ω-version of definitional reflection with $(\mathcal{D}\vdash)_\omega$ instead of $(\mathcal{D}\vdash)$, no additional problems arise in principle. The rule $(\mathcal{D}\vdash)_\omega$ is invertible, if we do not require contraction.[2]

3. Jäger and Stärk [3], who work with a multiple succedent calculus with negation as primitive, have proved a result similar to the one given here. The differences between their sequent system and ours are not very important as far as cut elimination is concerned. They translate proofs in the original system into a system with ramified $\mathcal{D}$-rules, for which the cut elimination proof is completely standard, and then retranslate cut-free proofs. Such a translation and retranslation is possible if identity (I) is lacking. This method — which also works if contraction is missing, but full identity is present — can easily be carried over to the situation considered here. Jäger and Stärk arrive at their system without identity from a different point of view, considering the three-valued semantics of logic programs with negation as failure. Kreuger motivated the restrictions on (I) by considerations concerning the operational interpretation of definitional reflection as implemented in GCLA.

4. We do not think that the issue of cut elimination is of any relevance as to whether to restrict (I) (or analogously, whether to reject contraction). The rules of the system have to be justified independently. Unlike Girard [1] we have always taken the view that eliminability of cuts is a feature of the particular definition $\mathcal{D}$ under consideration, and not something that has to be made sure from the beginning. According to Hallnäs [2] a partial inductive definition $\mathcal{D}$ is called total, if the consequence relation generated by $\mathcal{D}$ is transitive (i.e., if we can eliminate cuts). Whether a partial inductive definition is properly partial or whether it is total is something that may (or may not) be proved after stating the definition. This is quite analogous to the definition of a partial recursive function which later on may (or may not) turn out to be total. It seems impossible to single out the definitions which are total by a simple syntactic criterion.

[1] So in that case we either use the maximum und count both $(\vdash\mathcal{D})$- and $(\mathcal{D}\vdash)$-applications, or we use sums and count $(\mathcal{D}\vdash)$. In $\mathbf{DR}^*_{\mathcal{U}}(\mathcal{D})$we have to use the maximum.

[2] This is proved in [6, Lemma 4]. Actually, there the invertibility of $(\mathcal{D}\vdash)_\omega$ was mistakenly claimed also for the system with contraction. The validity of the theorems of that paper is not affected by this fault.

References

[1] Girard, J.-Y. A fixpoint theorem for linear logic. In: P. Lincoln (Ed.), *Linear Logic Mailing List*, linear@cs.stanford.edu, 5 February 1992 (Reply: *ibid.*, 19 February 1992).

[2] Hallnäs, L. Partial inductive definitions. *Theoretical Computer Science*, **87** (1991), 115-142.

[3] Jäger, G. & Stärk, R. F. A proof-theoretic framework for logic programming. In: S. Buss (Ed.), *Handbook of Proof Theory* (to appear).

[4] Kreuger, P. Axioms in definitional calculi. In: R. Dyckhoff (Ed.), *Extensions of Logic Programming. 4th International Workshop (St. Andrews, March 1993). Proceedings.* Springer LNAI (to appear).

[5] Schroeder-Heister, P. Cut-elimination in logics with definitional reflection. In: D. Pearce & H. Wansing (Eds.), *Nonclassical Logics and Information Processing. International Workshop, Berlin 1990, Proceedings.* Springer LNCS, Vol. 619, Berlin 1992, 146–171.

[6] Schroeder-Heister, P. Rules of definitional reflection. In: *8th Annual IEEE Symposium on Logic in Computer Science (Montreal 1993).* IEEE Computer Society Press, Los Alamitos 1993, 222–232.