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Prawitz's completeness conjecture: A reassessment

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Abstract
In 1973, Dag Prawitz conjectured that the calculus of intuitionistic logic is complete with respect to his notion of validity of arguments. On the background of the recent disproof of this conjecture by Piecha, de Campos Sanz and Schroeder-Heister, we discuss possible strategies of saving Prawitz's intentions. We argue that Prawitz's original semantics, which is based on the principal frame of all atomic systems, should be replaced with a general semantics, which also takes into account restricted frames of atomic systems. We discard the option of not considering extensions of atomic systems, but acknowledge the need to incorporate definitional atomic bases in the semantic framework. It turns out that ideas and results by Westerståhl on the Carnap categoricity of intuitionistic logic can be applied to Prawitz semantics. This implies that Prawitz semantics has a status of its own as a genuine, though incomplete, semantics of intuitionstic logic. An interesting side result is the fact that every formula satisfiable in general semantics is satisfiable in an axioms-only frame (a frame whose atomic systems do not contain proper rules). We draw a parallel between this seemingly paradoxical result and Skolem's paradox in first-order model theory.

KEYWORDS
categoricity, completeness, incompleteness, intuitionistic logic, proof theory, proof-theoretic semantics

1 | INTRODUCTION

At the beginning of the 1970s, Dag Prawitz developed a validity-based proof-theoretic semantics (Prawitz, 1971, 1973, 1974) and conjectured that the formal system of intuitionistic logic is sound and complete for it (Prawitz, 1973, p. 246).[1] His completeness conjecture was challenged by Piecha, de Campos Sanz and Schroeder-Heister (Piecha et al., 2015; Piecha & Schroeder-Heister, 2019). The current paper discusses the incompleteness argument and makes a proposal

[1]The completeness conjecture was initially formulated only for minimal logic, but intended, of course, also for full intuitionistic logic, see Prawitz (2014).

for a refinement of this semantics to which the arguments against completeness are no longer applicable, and for which completeness can be established at least by classical reasoning related to Kripke semantics. We confine ourselves to propositional logic, as this is sufficient to make our point.

The argument developed here, namely that incompleteness holds with respect to the distinguished structure of atomic systems considered by Prawitz, but not for a general semantics taking all potential atomic structures into account, is not completely novel. We (Piecha & Schroeder-Heister, 2016) remarked ourselves that Prawitz semantics can be seen as based on a specific Kripke model, and a more general semantics based on all Kripke models might be a possible solution to the completeness problem. Goldfarb (2016), Litland (2012) and Stafford and Nascimento (2023) developed in detail methods to embed Kripke semantics in a general proof-theoretic semantics, allowing them to achieve semantical completeness. We will comment on their methods in Section 7. In this paper, we discuss these issues from a philosophical point of view as near as possible to Prawitz's original point of view, including the problem of referring to extensions of atomic systems in the semantics of hypothetical reasoning, without digging too deep into technical details.[2]

A *caveat* applies to our use of the term 'Prawitz semantics'. Although there is a general framework that can be subsumed under this term, several variants of it have been presented and discussed by Prawitz up to now, some of which differ in crucial respects from one another. Even in the early publications of Prawitz (1971, 1973, 1974), where its fundamentals are laid down, significant differences can be found, which affect the issue of completeness. This paper relies on the version proposed in Schroeder-Heister (2006), which does not in every respect correspond to what Prawitz put forward in the early 1970s. Therefore, in what follows, 'Prawitz semantics' means a certain interpretation of Prawitz's early semantical works, which may deviate in critical aspects from the semantics that Prawitz sees as his conceptual achievement. Section 5 provides further comments on this issue.

2 | PRAWITZ'S PROOF-THEORETIC SEMANTICS: BASIC FEATURES

Prawitz's completeness conjecture refers to a semantics that is primarily a semantics of derivations or proofs and only secondarily a semantics of formulas or sentences. In this semantics, Prawitz defines the validity of arguments with respect to atomic systems S. (For a detailed reconstruction of this semantics, see Schroeder-Heister, 2006; for a comprehensive discussion of Prawitz's validity-related concepts, see Piccolomini d'Aragona, 2023.) In the simplified propositional case considered here, formulas are built up from atomic formulas (in short: atoms), which are sentence letters (denoted by lower case letters $a, a_1, a_2, \ldots, b, c, \ldots$), by means of the logical constants $\wedge$, $\vee$, $\rightarrow$, $\bot$ and $\neg$, where $\neg A$ stands for $A \rightarrow \bot$. Arguments are formula trees, which structurally look like natural deduction derivations, with the crucial difference that their nodes are not necessarily applications of the common introduction and elimination rules for logical connectives, but arbitrary inference steps of the form

$$\frac{\begin{matrix}\Delta_1 \\ \vdots \\ A_1\end{matrix} \quad \ldots \quad \begin{matrix}\Delta_n \\ \vdots \\ A_n\end{matrix}}{B} \tag{1}$$

[2]A preliminary version of the paper by Stafford and Nascimento (2023) came (anonymously) to my attention at the end of May 2022, when the topic and content of my talk at the Rolf-Schock-Prize conference (originally scheduled for 2020) had already been fixed. By then, I had only been aware of Goldfarb's (2016) Kripke-style completeness proof and of Litland's (2012) related proposal to prove completeness.

where the A_i and B are any formulas and the Δ_i any finite sets of formulas, and where the Δ_i tell which formulas can be discharged at this step in the argument tree. As a Δ_i may be empty, it is obvious that (1) is a general schema for inference steps under which the standard introduction and elimination steps fall. Prawitz uses the term 'argument' rather than 'proof' or 'derivation' for such a tree, as one can well talk of a wrong or invalid argument, whereas talking of a 'wrong proof' or a 'wrong derivation' is at least dubious. When we say 'this proof is wrong' we really mean 'this alleged proof is not a proof for what it claims to be'; that is, 'this argument is invalid, i.e., is not a proof'. Other terminologies are 'proof skeleton', 'proof structure', 'proof candidate' or related terms not implying correctness or validity. An atomic system S, also called 'base', is a system of atomic production rules, that is, of rules of the form

$$\frac{a_1 \; \ldots \; a_n}{b}$$

for atomic $a_1, \ldots, a_n, b$, where the premisses can be lacking, in which case the rule is an axiom. A base thus constitutes a formal system for the derivation of atoms. Actually, it is a system for the derivation of atoms from atoms. Conditional atomic derivations do not play a role in the definition of validity, but need to be considered in certain proofs (such as that of Theorem 9) or in the discussion of alternative approaches to semantics (see Definition 11).

A closed argument is an argument, whose end formula does not depend on an assumption. An open argument is an argument, whose end formula depends on at least one (undischarged) assumption. As introduction inferences in the standard sense fall under the general schema (1), we can define a canonical argument to be an argument using an introduction rule in the last step.

The validity of an argument is not only defined with respect to a base S, but also with respect to an argument reduction system (called 'justification'), that is, a set of argument transformations which transform a given argument into another one. We do not discuss here the precise formal structure of such a reduction system, as it is not crucial for the point we are going to make and refer the reader to the discussion in Schroeder-Heister (2006). The reader may imagine reductions and permutations in proofs of normalisation, or, more generally, something analogous to a term rewriting system, where the terms are now arguments.[3]

The definition of S-validity for closed arguments runs roughly[4] as follows.

- Every closed derivation in S is S-valid.
- A closed canonical argument is S-valid, if all its immediate subarguments are S-valid.
- A closed non-canonical argument is S-valid, if it reduces to an S-valid canonical argument or to a closed derivation in S.

These clauses do not say how to handle absurdity $\bot$ and thus negation. Prawitz actually formulates his completeness conjecture for minimal logic, where we have no full negation (Prawitz, 1973, p. 246). The standard procedure to consider $\bot$ to be a constant for which there is no introduction rule and therefore no canonical proof, which gives one the *ex falso quodlibet* principle by vacuous quantification, is not a viable option—see the remark at the end of this section. However, as we are dealing with atomic systems, there is something corresponding to absurdity, namely the fact that an atomic system S may be inconsistent in the sense that any atom is derivable in it. This happens, for example, when S contains all atoms as axioms. If we consider the inconsistency of S, that is, $\vdash_S a$ for all a, to be the content of $\bot$ in S, we may add in the definition of validity as a clause for $\bot$:

[3]As we have not just trees, but also a discharge structure within trees, rewriting of λ-terms may be the closest analogy.

[4]'Roughly' means that the justification on which the reductions are based is not specified here as an explicit parameter in addition to the base S. See Prawitz (1973), Schroeder-Heister (2006) and Piccolomini d'Aragona (2023) (Ch. 3) for details.

– The one-step derivation $\overline{\bot}$ is S-valid, if S is inconsistent, that is, if for all a: $\vdash_S a$,

and extend the clause for non-canonical arguments to:

– A closed non-canonical argument is S-valid, if it reduces to an S-valid canonical argument or to a closed derivation in S or to an S-valid one-step derivation $\overline{\bot}$.

Adding a clause for open arguments, this gives us the definition of S-validity, which has the form of a generalised inductive definition.

Definition 1. (Validity of arguments)

1. Every closed derivation in S is S-valid.
2. The one-step derivation $\overline{\bot}$ is S-valid, if S is inconsistent, that is, if for all a: $\vdash_S a$.
3. A closed canonical argument is S-valid, if all its immediate subarguments are S-valid.
4. A closed non-canonical argument is S-valid, if it reduces to an S-valid canonical argument or to a closed derivation in S or to an S-valid one-step derivation $\overline{\bot}$.
5. An open argument

$$\begin{array}{c} A_1 \ldots A_n \\ \mathcal{D} \\ B \end{array}$$

where all open assumptions of $\mathcal{D}$ are among $A_1, \ldots, A_n$, is S-valid, if for every extension $S' \supseteq S$ and for every list of closed S'-valid arguments

$$\begin{array}{c} \mathcal{D}_i \\ A_i \end{array}$$

the closed argument

$$\begin{array}{c} \mathcal{D}_1 \quad\;\; \mathcal{D}_n \\ A_1 \ldots A_n \\ \mathcal{D} \\ B \end{array}$$

is S'-valid.

The point of considering extensions of atomic systems in Clause 5 is to guarantee that S-validity is monotone with respect to S; that is, that valid arguments continue to be valid, if our knowledge base S is extended. Furthermore, it excludes certain 'void' validities in open arguments, which are due to the fact that the particular base S under consideration does not allow the derivation of the assumptions. It also ensures that validity of atomic consequences is stable, which means that S-validity of an argument of an atom from atomic assumptions coincides with derivability in S. Finally, and most importantly, it secures that the logical constants characterised in terms of validity are the intuitionistic ones in Heyting's sense (see Section 8). When no base extensions are considered, these properties are lost, and the semantics becomes classical if the metalanguage is classical (see Section 4).

An argument is considered to be logically valid, if it is valid with respect to any base S, which, in view of the monotonicity of S-validity with respect to S, means the same as $\emptyset$-validity, that is, validity with respect to the empty base.

We write $\Gamma \vDash_S A$ (for finite Γ), if there is an S-valid argument of A from Γ (i.e. an S-valid argument for A, whose set of open assumptions is contained in Γ). We write $\Gamma \vDash A$, if there is a logically valid argument of A from Γ. We write $\vdash \Gamma$, $\vDash \Gamma$ and $\vDash_S \Gamma$, when we mean $\vdash A$, $\vDash A$ and $\vDash_S A$, respectively, for every A in Γ.

> *Remark*. Note that 'For every S there is an S-valid argument for A from Γ' and 'there is an argument for A from Γ which is S-valid for every S' are equivalent, as the first formulation implies that there is an $\emptyset$-argument for A from Γ, which, due to the monotonicity with respect to bases, is an S-argument for A from Γ for any S.

It is easy to see that $\vDash_S$ and $\vDash$ are consequence relations satisfying the following conditions, where $\mathscr{S}$ is the set of all bases and $\vdash_S$ means derivability in the base S.

Lemma 2. (Basic properties of $\vDash_S$ and $\vDash$)

(At) $\vDash_S a \Leftrightarrow\ \vdash_S a$.
($\bot$) $\vDash_S \bot \Leftrightarrow$ For all a: $\vDash_S a$.
($\wedge$) $\vDash_S A \wedge B \Leftrightarrow\ \vDash_S A$ and $\vDash_S B$.
($\vee$) $\vDash_S A \vee B \Leftrightarrow\ \vDash_S A$ or $\vDash_S B$.
($\to$) $\vDash_S A \to B \Leftrightarrow A \vDash_S B$.
($\vDash_S^{\text{ext}}$) $\Gamma \vDash_S A \Leftrightarrow$ For all $S' \supseteq S$: $(\vDash_{S'} \Gamma \Rightarrow\ \vDash_{S'} A)$.
($\vDash$) $\Gamma \vDash A \Leftrightarrow$ For all $S \in \mathscr{S}$: $(\vDash_S \Gamma \Rightarrow\ \vDash_S A)$.
($\vDash'$) $\Gamma \vDash A \Leftrightarrow$ For all $S \in \mathscr{S}$: $\Gamma \vDash_S A$.

Note that ($\vDash'$) is immediate from the definition of $\vDash$, and ($\vDash$) is an immediate consequence of $\vDash_S^{\text{ext}}$ and the fact that $\vDash$ is the same as $\vDash_\emptyset$. Note also that in clause ($\bot$) 'For all a: $\vDash_S a$' is equivalent to 'For all A: $\vDash_S A$' (where A can be non-atomic).

For what follows, we just need to deal with these conditions, independent of whether they result from a definition of validity for arguments. We could as well consider the clauses (At),($\bot$),($\wedge$),($\vee$),($\to$),($\vDash_S^{\text{ext}}$) in Lemma 2 to be a definition of S-consequence $\vDash_S$, and define $\Gamma \vDash A$ as $\Gamma \vDash_\emptyset A$, which gives us the clauses ($\vDash$) and ($\vDash'$). This would be a *sentence semantics* rather than a *proof or argument semantics*, as we would *directly define* valid S-consequence between sentences rather than considering it to be established by S-valid proofs from assumptions. Though of less expressive power, sentence semantics is technically easier to handle than proof semantics. Prawitz's completeness conjecture was posed within a framework of proof semantics. See Section 5 for further discussions of this issue.

Now, let IPC be the intuitionistic propositional calculus with the standard constants $\wedge$, $\vee$, $\to$, $\bot$ and $\neg$, where $\neg A$ stands for $A \to \bot$, and let $\vdash_{\text{IPC}}$ stand for derivability in IPC. Then, given the above semantics, *soundness* of IPC means

$$\text{For any } \Gamma \text{ and } A: \text{ if } \Gamma \vdash_{\text{IPC}} A, \text{ then } \Gamma \vDash A;$$

and *completeness* of IPC means:

$$\text{For any } \Gamma \text{ and } A: \text{ if } \Gamma \vDash A, \text{ then } \Gamma \vdash_{\text{IPC}} A.$$

It is easy to see that IPC is sound. Prawitz's completeness conjecture can then be formulated as:

IPC is complete.

Remark on negation. In standard intuitionistic logic, $\neg A$ is understood as $A \rightarrow \bot$, where absurdity $\bot$ is a nullary logical constant which (by definition) never holds. In a natural deduction-based framework this means that $\bot$ is a constant for which there is no introduction rule and therefore *ex falso quodlibet* as elimination rule. If we used this idea in the present framework, we would have to stipulate that $\bot$ holds in no base, that is, $\nvDash_S \bot$ for every S. This would lead to the consequence that $\neg a$ never holds in any base S, for $\neg a$ means $a \rightarrow \bot$, that is, 'a holds in no extension of S' (because $\bot$ holds in no base and thus in no extension of S). However, as a is contained in some extension of S, this statement is false. Thus, $\neg\neg a$ holds in any S. Consequently, as can be easily shown, $\neg\neg A$ holds in any S, provided A does not contain $\bot$ (this corresponds to Counterexample 1 in Goldfarb, 2016). This is due to the specific structure of Prawitz semantics, which, unlike Kripke semantics, is based on a single frame of bases ordered by set inclusion, and in which every atom a not in S becomes eventually valid in an extension $S' \supseteq S$ (just add a as an axiom to S). This again means that we would not be able to semantically force $\neg a$ to be valid (in some S). One way out is to consider $\bot$ an atomic formula rather than a logical constant, for which *ex falso quodlibet* holds in all atomic systems. Then, we can force $\neg a$ to hold in S by having $a \Rightarrow \bot$ as a rule of S. This is the way followed in most presentations of validity-based semantics. Our procedure above corresponds to this. However, as we prefer to have $\bot$ as a logical constant rather than an atom with an associated *ex falso* rule, we define the validity of $\bot$ in S as the derivability of every a in S. Technically this comes to the same as taking $\bot$ to be atomic. Conceptually, it has the advantage that $\bot$ continues to be a logical constant.

3 | RECAPITULATION OF THE INCOMPLETENESS RESULT

In Piecha et al. (2015) and Piecha and Schroeder-Heister (2019), Piecha, de Campos Sanz and Schroeder-Heister showed that Prawitz's completeness conjecture does not hold. In Piecha et al. (2015), this was shown for a particular form of formal bases, which may contain higher level rules, namely rules allowing one to discharge rules as assumptions (admitting formula-discharging rules would be sufficient). In Piecha and Schroeder-Heister (2019), incompleteness was demonstrated in a more general setting without any presupposition about the form of rules in bases. We recapitulate our findings in a slightly revised form.[5]

By an *abstract semantics*, we understand an arbitrary set $\mathscr{S}$ of entities, for each $S \in \mathscr{S}$ a consequence relation $\vDash_S$ as well as an additional consequence relation $\vDash$ such that the conditions $(\bot)$, $(\wedge)$, $(\vee)$, $(\rightarrow)$, $(\vDash)$, $(\vDash')$ of Lemma 2 are satisfied. As these conditions are part of the lemma, this means that Prawitz semantics as described in the previous section is an abstract semantics. Note that the characteristic conditions chosen for an abstract semantics are those conditions in Lemma 2, which neither refer to the internal structure of bases nor to any ordering relation between bases. In Lemma 2, only the conditions (At) and $(\vDash_S^{\mathrm{ext}})$ refer to the fact that bases are sets of atomic rules generating a derivability relation $\vdash_S$ and that they are ordered by

[5]These revisions concern minor changes in terminology, the explicit consideration of soundness and the explicit consideration of absurdity $\bot$. In Piecha and Schroeder-Heister (2019), the latter could be omitted as we assumed soundness of IPC without further proof.

set inclusion. Because we disregard (At) and ($\vDash_S^{\text{ext}}$) in the definition of an abstract semantics, we must explicitly state ($\vDash$) and ($\vDash'$), which in the presence of (At) and ($\vDash_S^{\text{ext}}$) are deducible from the other clauses of Lemma 2.

Definition 3. An abstract semantics is given by a set of entities $\mathscr{S}$, consequence relations $\vDash_S$ for each $S \in \mathscr{S}$ and a consequence relation $\vDash$, such that the following conditions are satisfied:

($\bot$) $\vDash_S \bot \Leftrightarrow$ For all A: $\vDash_S A$.
($\wedge$) $\vDash_S A \wedge B \Leftrightarrow\ \vDash_S A$ and $\vDash_S B$.
($\vee$) $\vDash_S A \vee B \Leftrightarrow\ \vDash_S A$ or $\vDash_S B$.
($\to$) $\vDash_S A \to B \Leftrightarrow A \vDash_S B$.
($\vDash$) $\Gamma \vDash A \Leftrightarrow$ For all $S \in \mathscr{S}$: ($\vDash_S \Gamma \Rightarrow\ \vDash_S A$).
($\vDash'$) $\Gamma \vDash A \Leftrightarrow$ For all $S \in \mathscr{S}$: $\Gamma \vDash_S A$.

In the rest of this section, when considering $\vDash_S$ and $\vDash$, we always refer to an abstract semantics in the sense of this definition.

The conditions of an abstract semantics establish soundness.

Lemma 4. IPC is sound with respect to any abstract semantics.

For the proof one just has to go through the rules of, for example, a sequent system for IPC.

We consider the generalised disjunction property for an arbitrary consequence relation $\Vdash$ in the language of IPC :

GDP($\Vdash$) If $\Gamma \Vdash A \vee B$, where $\vee$ does not occur in Γ, then $\Gamma \Vdash A$ or $\Gamma \Vdash B$.

This property holds for derivability $\vdash_{\text{IPC}}$ in IPC ; that is, we have GDP($\vdash_{\text{IPC}}$).[6]

We can show the following.

Lemma 5. If the generalised disjunction property holds for any $\vDash_S$, that is, for all S: GDP($\vDash_S$), then for all A, B_1, B_2

$$\neg A \to (B_1 \vee B_2) \vDash (\neg A \to B_1) \vee (\neg A \to B_2),$$

that is, Harrop's rule

$$\frac{\neg A \to (B_1 \vee B_2)}{(\neg A \to B_1) \vee (\neg A \to B_2)}$$

is valid.

The proof essentially relies on the fact that in IPC any negated formula $\neg A$ is equivalent to a disjunction-free formula A'. Using soundness (Lemma 4) and condition ($\vDash'$), we obtain $\neg A_S \dashv\vDash_S A'$. Harrop's rule is the standard example of a rule which, though admissible, is *not derivable in* IPC. Therefore, assuming that the generalised disjunction property holds for any $\vDash_S$, we have refuted Prawitz's completeness conjecture with Harrop's rule as a counterexample.

[6]A stronger version of GDP($\vdash_{\text{IPC}}$), in which it is only assumed that $\vee$ does not occur *positively* in Γ, was proven by Harrop (1960) and, in a natural deduction setting, by Prawitz (1965). Note that 'GDP' for 'generalised disjunction property' has already been used with a somewhat different meaning; see the survey by Chagrov and Zakharyashchev (1991), p. 208.

That GDP($\vDash_S$) holds for any S can be reduced to GDP($\vDash$) by the following principle, which says that the base S of non-logical consequence $\vDash_S$ can be 'exported' as a set of assumptions (S^*) of logical consequence $\vDash$.

(Export) For every base S, there is a set of $\vee$-free formulas S^* such that for all Γ and all A: $\Gamma \vDash_S A \Leftrightarrow \Gamma, S^* \vDash A$.

Lemma 6. (Export) + GDP($\vDash$) $\Rightarrow$ GDP($\vDash_S$) for all S.

Therefore, by Lemma 5 we obtain Harrop's rule as a counterexample to completeness by assuming (Export) and GDP($\vDash$). If we nevertheless assume that IPC is complete, we can infer GDP($\vDash$) from GDP($\vdash_{\text{IPC}}$). This yields the following result:

Lemma 7. (Conditional incompleteness) If IPC is complete, then under the condition of (Export), Harrop's rule is valid; that is, IPC is not complete.

This gives us as a corollary:

Corollary 8. (Export) $\Rightarrow$ IPC is incomplete.

The corollary is based on an indirect argument in the constructive sense. Harrop's rule is not established as a counterexample to completeness outright. By means of Lemma 7 we have reduced the assumption of completeness to its own contradictory and thus to absurdity, which by (constructive) *reductio ad absurdum* yields incompleteness.

This result holds for any abstract semantics in the sense of Definition 3. If we consider the concrete case of Prawitz semantics, in which bases are sets of production rules for atoms, for which the principles (At) and ($\vDash_S^{\text{ext}}$) hold (Lemma 2), we can show that (Export) holds.

Theorem 9. (Export) holds for Prawitz semantics; thus IPC is incomplete with respect to Prawitz semantics. More precisely, (Export) holds for any semantics that has the properties listed in Lemma 2.

The proof is given in Piecha and Schroeder-Heister (2019, Lemma 3.6).

This means that Prawitz's completeness conjecture cannot be upheld as it stands. However, there is a possible way out. Before presenting it, we discuss another option, which at first glance might represent a way out by itself.

4 | NON-EXTENSION SEMANTICS

The proof of Theorem 9 heavily relies on the extension property ($\vDash_S^{\text{ext}}$). Thus, our incompleteness result crucially depends on it. The extension property, which is reminiscent of the interpretation of implication in Kripke semantics, has certainly an intuitive motivation. If a base S represents one's knowledge at a certain stage, extending S means extending this knowledge. The semantical value of an implication should remain stable under such extensions; that is, passing from the antecedent to the conclusion of an implication should remain possible when our knowledge base becomes larger. Technically, this property guarantees monotonicity of consequence statements with respect to bases. It also gives the logical connectives, in particular implication, their intuitionistic meaning, as we see in Section 8.

One might, however, argue that a base is not something that describes factual knowledge, which may increase over time, but something that has a definitional status, that is, something that determines the meaning of atomic expressions. It is therefore not something that may be increased, as extending a definition changes the meaning of the definiendum. This might be the reason why Prawitz after 1971 does not mention the extension clause in validity semantics. In 2016, Prawitz makes this point explicit: 'To consider extensions of the given base in this way is natural when a base is seen as representing a state of knowledge, but is in conflict with the view adopted here that a base is to be understood as giving the meanings of the atomic sentences. For instance, the argument representing reasoning by mathematical induction […] ceases to be valid relative to the arithmetical base […] if we require […] that validity be monotone with respect to the base.' (Prawitz, 2016, p. 18, footnote 12). It is clear that when we consider, for example, a base consisting of principles such as the inductive definitions for plus and times, we would not consider extensions of these definitions (but only extensions of the given base by further definitions).

This is certainly a crucial point. However, we cannot do justice to it by just replacing the extension clause ($\vDash_S^{\text{ext}}$) with a corresponding clause not referring to extensions, that is, by replacing Clause 5 in Definition 1 with the following clause

5′. An open argument

$$\begin{array}{c} A_1 \ldots A_n \\ \mathcal{D} \\ B \end{array}$$

where all open assumptions of $\mathcal{D}$ are among $A_1, \ldots, A_n$, is S-valid, if for every list of closed S-valid arguments

$$\begin{array}{c} \mathcal{D}_i \\ A_i \end{array}$$

the closed argument

$$\begin{array}{c} \mathcal{D}_1 \quad\quad \mathcal{D}_n \\ A_1 \ldots A_n \\ \mathcal{D} \\ B \end{array}$$

is S-valid.

This corresponds to replacing clause ($\vDash_S^{\text{ext}}$) in Lemma 2 with

($\vDash_S$) $\Gamma \vDash_S A \Leftrightarrow (\vDash_S \Gamma \Rightarrow \vDash_S A)$.

Such a change in the definition of validity makes the situation even worse, as far as the completeness of IPC is concerned. Whereas for an abstract semantics in the sense of Definition 3 we established incompleteness of IPC only under the condition of (Export), we can now achieve incompleteness unconditionally, albeit using classical logic.

Lemma 10. *(Incompleteness for non-extension semantics)* IPC is incomplete with respect to any abstract semantics satisfying ($\vDash_S$).

Proof. We validate the classical disjunction principle

$$A \to (B_1 \vee B_2) \vDash (A \to B_1) \vee (A \to B_2),$$

that is, the rule

$$\frac{A \to (B_1 \vee B_2)}{(A \to B_1) \vee (A \to B_2)}$$

as follows:

$$\begin{aligned} A \vDash_S \ & B_1 \vee B_2 \\ & \Rightarrow (\vDash_S A \Rightarrow \vDash_S B_1 \vee B_2);\ \text{by } (\vDash_S) \\ & \Rightarrow (\vDash_S A \Rightarrow (\vDash_S B_1 \text{ or } \vDash_S B_2));\ \text{by } (\vee) \\ & \Rightarrow (\vDash_S A \Rightarrow \vDash_S B_1) \text{ or } (\vDash_S A \Rightarrow \vDash_S B_2);\ \text{by classical (meta)logic} \\ & \Rightarrow A \vDash_S B_1 \text{ or } A \vDash_S B_2;\ \text{by } (\vDash_S). \end{aligned}$$

(We do not need the supposition that $\vee$ does not occur in A; so, A, B_1 and B_2 stand for arbitrary formulas.) ■

Thus, we have proved the validity of a rule, which, unlike Harrop's rule, is not even admissible in IPC. This is a classical proof. That is, classically the assumption of completeness of IPC leads to a contradiction. Claiming that completeness can nonetheless be proved by intuitionistic (and thus also by classical) means implies claiming a classical contradiction. Given that these proofs can be coded in first-order arithmetic and that classical arithmetic and Heyting arithmetic are equiconsistent, such a claim cannot be upheld. In simpler terms, inconsistency is a negative result, and on the negative side classical and intuitionistic logics coincide.

There is a further point that relates to atomic stability and that does not use classical logic. In Definition 1 the S-validity of a closed non-canonical argument of an atom a is defined via a closed derivation in S to which it reduces, or in terms of sentence semantics, $\vDash_S a$ is understood as $\vdash_S a$. Thus, it is natural that $a_1, \ldots, a_n \vDash_S a$ is equivalent to $a_1, \ldots, a_n \vdash_S a$. However, the official meaning of $a_1, \ldots, a_n \vDash_S a$, in terms of sentence semantics, is

$$\vDash_S a_1, \ldots, \vDash_S a_n \ \Rightarrow \ \vDash_S a.$$

Definition 11. (Stability) S is called stable, if the following holds:

(At′) $a_1, \ldots, a_n \vDash_S a \Leftrightarrow a_1, \ldots, a_n \vdash_S a.$

In extension semantics stability is easy to prove, as we can add the assumptions $a_1, \ldots, a_n$ to S and thus obtain an extension S' of S with $a_1, \ldots, a_n$ as axioms. (At′) is crucial in establishing (Export) and therefore incompleteness in extension semantics (see Remark 3.5 and Lemma 3.6 in Piecha & Schroeder-Heister, 2019). However, in non-extension semantics (At′) says, in view of (At),

$$(\vdash_S a_1, \ldots, \ \vdash_S a_n \Rightarrow \ \vdash_S a) \Leftrightarrow a_1, \ldots, a_n \vdash_S a.$$

In other words, the rule

$$\frac{a_1 \ \ldots \ a_n}{a}$$

is admissible in S if and only if it is derivable in S. The coincidence of admissibility and derivability is also called 'structural completeness'. This means that in non-extension semantics, S

is stable only if it is structurally complete. Therefore, if we want to have stability, very many bases are disregarded. This is an unwanted, or at least peculiar behaviour of non-extension semantics.

In fact, if we interpret the right arrow on the right side of ($\vDash_S$)

$$\vDash_S \Gamma \Rightarrow \vDash_S A$$

classically as

$$\nvDash_S \Gamma \text{ or } \vDash_S A,$$

then we obtain a semantics which is throughout classical, rather than the intended intuitionistic one. This can be seen as follows. In view of the classical reading of ($\vDash_S$), any implication $A \to B$ with B different from $\bot$ can be read as $\neg A \vee B$. For any A, let $[\![A]\!]$ be the set of bases, in which A holds: $[\![A]\!] = \{S : \vDash_S A\}$. Let $\mathbb{0}$ be the set of inconsistent bases: $\mathbb{0} = \{S : \vDash_S A \text{ for all } A\}$, which means that $\mathbb{0} = [\![\bot]\!]$. Let $\mathbb{1}$ be the set $\mathscr{S}$ of all bases. For a set of bases $\mathscr{M}$, let $\overline{\mathscr{M}} = (\mathscr{S} \setminus \mathscr{M}) \cup \mathbb{0}$. Then, $[\![\neg A]\!] = \overline{[\![A]\!]}$, and the structure $\langle \mathcal{P}(\mathscr{S}), \mathbb{0}, \mathbb{1}, \cup, \cap, \overline{} \rangle$, which is the algebra of semantical values of formulas, is a Boolean algebra.

We have argued that passing from extension semantics with the principle ($\vDash_S^{\text{ext}}$) to non-extension semantics with the principle ($\vDash_S$) is a wrong strategy for intuitionistic semantics. That one needs to incorporate somehow the idea of bases-as-definitions as opposed to bases-as-states-of-knowledge is nonetheless a crucial point. However, by simply removing the extension feature we do justice neither to bases-as-states-of-knowledge nor to bases-as-definitions. The idea of bases-as-definitions introduces an entirely new aspect into validity-based proof-theoretic semantics, which needs further discussion. Piecha and Schroeder-Heister (2016, 2017) have initiated such a discussion by using the idea of definitional reflection, which goes back to Hallnäs (1991). Definitional reasoning does not compete with knowledge-state reasoning, but is an independent additional topic that should be incorporated in a joint framework. In such a framework, atomic bases would consist of a knowledge-representing and a definitional part. Consequence with respect to such a base would be denoted by $\Gamma \vDash_{\mathbb{D},S} A$, where $\mathbb{D}$ would stand for the definitional and S for the knowledge-representing part. With respect to the knowledge-representing part, the semantics would be an extension-semantics, as we definitely would like to have monotonicity for states of knowledge. With respect to the definitional part, the semantics would be a novel non-extension-semantics. It is not clear at all, what such a semantics may look like, but it is definitely not just a simplified or restricted variant of knowledge-state semantics.

We agree completely with Prawitz in that the definitional aspects of atomic bases need to be addressed and that it is a desideratum to develop tools for this. But this issue does not affect the incompleteness of IPC pointed to here, which relates to states of knowledge and not to definitions. The completeness of IPC which can be secured after all according to the proposal made in Section 6, builds on bases which represent states of knowledge.

5 | INTERIM DISCUSSION: PROOF SEMANTICS VERSUS SENTENCE SEMANTICS AND THE ISSUE OF UNIFORM REDUCTION

As indicated in the last paragraph of the introduction (Section 1), we are relying on the semantics presented in Schroeder-Heister (2006). In this section, we sketch in which respects it differs from concepts originally found in Prawitz (1971, 1973, 1974) and why we prefer it in the present context. Nothing in the following sections depends on this discussion; so, a reader who is not so much interested in these conceptual issues may skip it and pass to the presentation of an alternative semantic framework in Section 6.

We distinguished between proof semantics and sentence semantics. We started with proof semantics as the Prawitzean framework and then passed over to sentence semantics by identifying certain conditions satisfied by the given proof semantics (Lemma 2). These conditions of sentence semantics we used to demonstrate incompleteness. Moreover, all discussions concerning a possible way out of the incompleteness trap in the following sections are carried out in this framework. Sentence semantics is the framework in which a great deal of proof-theoretic semantics is conducted today, and from which significant results have been obtained. One might even say that sentence semantics is the basis on which the current success and progress of proof-theoretic semantics builds. This does not mean that the clauses of sentence semantics must be exactly of the kind presented here. For example, the widely discussed approach by Sandqvist which he called 'base-extension semantics' (Sandqvist, 2015) is based on different clauses. However, it is still sentence semantics, that is, semantical values are attached to sentences and consequence statements in the form of an inductive definition of validity. In our discussions here we follow this methodology.

The fact that the original proof semantics in Prawitz's sense can be framed as a certain sentence semantics, for which we have incompleteness and possible solutions to it, relies on certain conceptual decisions within proof semantics made in Schroeder-Heister (2006), which are not mandatory. Proof semantics can be formulated in ways that cannot be captured by sentence semantics, and a close inspection of early works of Prawitz (1971, 1973, 1974) shows that he had some such aspects in mind. This can best be illustrated by considering the notion of logical consequence, which in sentence semantics is understood as

$$(\vDash)\ \Gamma \vDash A \Leftrightarrow \text{For all } S \in \mathscr{S} : (\vDash_S \Gamma \Rightarrow \vDash_S A).$$

Logical consequence here means the preservation of validity in every atomic system S, quite analogous to the classical idea of logical consequence as truth-transmission in every model. However, in the corresponding proof semantics in Prawitz (1973, 1974), the transformation of validity is something that is required to happen in a *uniform* way. It is demanded that there is a single reduction procedure which does not depend on specific atomic systems but works for all S in the same way, that is, schematically. More precisely, in Prawitz (1973) validity is defined for pairs $\langle \mathscr{D}, \mathscr{J} \rangle$ consisting of an argument $\mathscr{D}$ and a set of reductions $\mathscr{J}$ ('justifications') such that certain reducibility conditions (corresponding to our Definition 1) are satisfied. Logical validity then means the availability of such a set of reductions $\mathscr{J}$. As this selection of $\mathscr{J}$ is independent of atomic systems, it is uniform with respect to them. In Prawitz (1973, 1974), this uniformity requirement is not explicitly emphasised (this has been done in public first by Piccolomini d'Aragona, 2023, Ch. 7.1 and Piccolomini d'Aragona, 2024 in connection with our incompleteness result), but we clearly think that the definitions given by Prawitz can and should be interpreted that way.

Now, this uniformity of reductions with respect to atomic systems cannot be expressed in sentence semantics. On the right side of $(\vDash)$, we have a quantification over S and an implication between validity statements, which do not say anything about the form of the procedure being used in proof semantics to establish validity. Therefore, insisting on the uniformity of reduction procedures with respect to atomic systems blocks the connection between proof semantics and sentence semantics. This means, in particular, that the incompleteness results as recapitulated here are not applicable to proof semantics.

The incompleteness result might nevertheless hold for uniform proof semantics. To establish it there, one would have to use methods that do justice to the uniformity requirements. One would essentially have to check, whether the export condition (our Theorem 9) can be proven for this semantics. We have not done this so far.

It is not even clear whether uniform proof semantics is stricter in effect than the non-uniform semantics used in this paper. Even though the uniformity requirement intends a stricter reading of logical consequence, no example has been found so far of a formula which is logically valid in non-uniform proof semantics but not logically valid in uniform proof semantics. If we suppose that IPC is complete with respect to uniform proof semantics, such an example would be a direct counterexample to completeness for non-uniform semantics, which we have not found so far. Our proof of incompleteness is indirect, and Harrop's rule was not established as a counterexample to completeness outright (see the paragraph immediately after Corollary 8).[7] So it cannot be excluded that the uniformity requirement, though conceptually ('intensionally') different from the non-uniform semantics on which our investigations are based, actually leads to the same logically valid consequences.

It should also be mentioned that in Prawitz (1973, 1974), the uniformity requirement goes along with a variant of non-extension semantics to which the discussions in Section 4 are not applicable without restrictions. This issue cannot be discussed here and must be left to a thorough discussion and comparison between uniform and non-uniform semantics. Interestingly, in Prawitz (1971) it is still non-uniformly requested for the validity of proofs that for every S, there is a reduction procedure (rather than uniformly: There is a reduction procedure which works for all S); also, the approach there is an extension semantics. This corresponds to our way of proceeding here.

Besides the technical comparison between the uniform and non-uniform versions of proof-based semantics, we should, as another desideratum of validity-based proof-theoretic semantics, answer the question: Are there any philosophical reasons why one of these approaches is preferable over the other one? Perhaps we have two semantical paradigms that have their respective strengths and weaknesses, and whose consequences we need to investigate without philosophical preference.

For the rest of this article, the reader should keep in mind that we are working with a specific variant of proof semantics which can be expressed in sentence semantics. Even though Prawitz would stick to the uniform approach, which cannot be equivalently formulated in terms of sentence semantics, we still use the term 'Prawitz semantics' for what we are discussing here, as it is a framework coined by Prawitz (and present in Prawitz, 1971). Our semantic framework is interesting in any case as it relates a certain variant of proof semantics via a corresponding sentence semantics to novel developments in proof-theoretic semantics. Much of the discussion in what follows, for example on the structure of semantical values, on general versus principal-frame semantics etc., can be understood also from the viewpoint of uniform semantics, even though it would not appear there as motivated by an incompleteness result, which we do not have (yet?) for uniform validity.

6 | GENERAL SEMANTICS VERSUS PRINCIPAL-FRAME SEMANTICS

For an abstract semantics in the sense of Definition 3, instead of bases in the concrete sense of sets of atomic rules, we considered a set of entities $\mathscr{S}$ without specified internal structure. In particular, we did not assume that there was an ordering between elements of $\mathscr{S}$, which in the concrete case would be the subset order. The abstract result that the condition (Export) implies the incompleteness of IPC is independent of the subset order.

However, in the concrete case of Prawitz semantics, where we could verify (Export) and therefore establish incompleteness (Theorem 9), we relied on the particular internal structure of the set of bases $\mathscr{S}$, namely that its elements are atomic systems, that is, sets of atomic rules, which are ordered by the subset relation. Moreover, we assumed

1. that $\mathscr{S}$ is the set of *all* bases, that is, all possible sets of atomic rules and

[7] Actually, as Stafford (2021, Lemma 5.5) has shown, it is not a counterexample.

2. that Prawitz semantics is an extension semantics, that is, that the principle ($\vDash_S^{ext}$) governs hypothetical consequence with respect to A.

Our proposal to reassess Prawitz's completeness conjecture is to revise the first principle and correspondingly change the meaning of ($\vDash_S^{ext}$) in the second principle. As mentioned in Section 1, this corresponds to proposals also made by Goldfarb (2016), Litland (2012) and Stafford and Nascimento (2023).

This revision is not considered just a formal idea, but has a strong philosophical underpinning. When talking about extensions of formal systems to be considered when defining the meaning of hypothetical consequence, the intuition is that a base represents in a way our state of knowledge, and its extension represents a possible extension of our knowledge, based on further 'experience'. However, such an extension ordering should not necessarily be the full subset ordering. We would certainly expect that an extension of our knowledge as given by an atomic base should be a superset of this base, but it is not at all clear that any superset should represent a knowledge extension. We might assume that knowledge extensions are coarser grained than the very fine-grained superset relation. If we take this possibility into account, we might consider the set $\mathscr{S}$ of bases underlying the extension ordering of our semantics to be a proper subset of the set of all bases. Borrowing a terminology from modal logic, we call such a set $\mathscr{S}$ of bases a *frame*. The principle ($\vDash_S^{ext}$) would then be reformulated with a reference to the frame $\mathscr{S}$ as

($\vDash_{\mathscr{S},S}^{ext}$) $\Gamma \vDash_{\mathscr{S},S} A \Leftrightarrow$ For all $S' \in \mathscr{S}$ with $S' \supseteq S$: ($\vDash_{S'}\Gamma \Rightarrow \vDash_{S'} A$).

Consequence in the frame $\mathscr{S}$ is then consequence $\vDash_S$ with respect to all $S \in \mathscr{S}$:

($\vDash_{\mathscr{S}}$) $\Gamma \vDash_{\mathscr{S}} A \Leftrightarrow$ For all $S \in \mathscr{S} : \Gamma \vDash_S A$.

Logical consequence would then be consequence in all frames:

($\vDash$) $\Gamma \vDash A \Leftrightarrow$ For all $\mathscr{S} : \Gamma \vDash_{\mathscr{S}} A$.

From the point of view of this refined semantics, Prawitz semantics is a semantics that considers just a *single frame*, namely the set of all bases (ordered by the subset relation). If we call this the *principal frame* $\mathscr{P}$ and the corresponding semantics the *principal-frame semantics*, then we have shown that the intuitionistic propositional calculus IPC is incomplete with respect to principal-frame semantics. We have shown that it is *not the case* that for all Γ and all A,

$$\Gamma \vDash_{\mathscr{P}} A \Rightarrow \Gamma \vdash_{IPC} A, \tag{2}$$

where $\mathscr{P}$ is the set containing all bases.

However, when we consider a semantics with respect to the set of *all frames*, which we call *general semantics*, this negative result does not continue to hold. For general semantics, completeness means

$$\Gamma \vDash A \Rightarrow \Gamma \vdash_{IPC} A,$$

that is,

$$(\text{For all } \mathscr{S} : \Gamma \vDash_{\mathscr{S}} A) \Rightarrow \Gamma \vdash_{IPC} A. \tag{3}$$

By strengthening the antecedent of the completeness claim from '$\Gamma \vDash_{\mathscr{P}} A$' in (2) to 'For all $\mathscr{S} : \Gamma \vDash_{\mathscr{S}} A$' in (3), we have weakened the completeness claim itself so that (3) may still hold though (2) has been refuted.

It is the case indeed that for general semantics, completeness of IPC in the sense of (3) can be demonstrated. In general semantics, the methods of Kripke semantics can be applied (though the Kripke framework is, of course, non-constructive). Assuming $\Gamma \nvdash_{\mathrm{IPC}} A$, we can construct (in the classical sense of construction) a countermodel, which is given by a frame $\mathscr{S}$ and an element $S \in \mathscr{S}$ such that $\vDash_S \Gamma$ but $\nvDash_S A$. For the details of this construction, see Section 7.

If one inspects the proof of (Export) (our Theorem 9, and Lemma 3.6 in Piecha & Schroeder-Heister, 2019), on which our proof of incompleteness crucially rests, it turns out that it is assumed throughout that arbitrary extensions of a base are available, that is, we are working in the principal frame. And if we inspect the Kripke-style construction of a countermodel, it turns out that, as a countermodel, a base is constructed within a frame, which is not the principal frame (see Section 7); that is, the countermodel is a model in general semantics, but not in principal-frame semantics.

Our conclusion is that with respect to principal-frame semantics, Prawitz's completeness conjecture fails, but that with respect to general semantics, his conjecture holds true, at least if one accepts Kripke's nonconstructive methods. The step from principal-frame semantics to general semantics is not really big: One just would accept that there are various possible ways of extending one's knowledge and that, in order to achieve logical validity, validity should be expected with respect to each of these possibilities.

The situation arrived at is closely related to what we find in second-order logic. There we also have both a completeness and an incompleteness theorem. Second-order logic is incomplete, if we consider only *full structures*, in which predicate variables run over all elements of the corresponding power set. However, it is complete, if we consider *general structures*, in which the domain of variability may be a proper subset of the power set, depending on the structure considered. The semantics based on full structures corresponds to our principal-frame semantics, and the semantics based on general structures corresponds to our general semantics. As in our case, incompleteness and completeness can coexist and depend on which sort of semantics is chosen. See van Dalen (2004), Ch. 4), where also an elementary example of a second-order set of formulas is given, which has a model, but not a principal model, and thus establishes incompleteness for principal models. Completeness for general semantics was proven by Henkin (1950).

A further possible refinement of general semantics consists in defining validity not only with respect to the set of all frames but also with respect to a selected class $\mathfrak{C}$ of frames:

$$(\vDash_{\mathfrak{C}})\quad \Gamma \vDash_{\mathfrak{C}} A \Leftrightarrow \text{For all } \mathscr{S} \in \mathfrak{C} : \Gamma \vDash_{\mathscr{S}} A.$$

This would enable us to investigate whether certain semantic or syntactic features depend on which set of frames is chosen. So far, we have considered the set $\mathfrak{U}$ of all frames and the set $\mathfrak{P}$ that only contains the principal frame. But there are other options. We might consider, for example, the set $\mathfrak{A}$ of all frames, whose bases contain only axioms (see Section 7), or the set $\mathfrak{B}$ of all frames, whose bases contain only axioms but are closed under certain boundary rules, or the set $\mathfrak{V}$ of all frames which are closed under the union of bases, or the set $\mathfrak{E}$ of all frames which are closed under union of bases and extension of bases with axioms, etc. Closer inspection of the proof of (Export) (our Theorem 9 and Lemma 3.6 in Piecha & Schroeder-Heister, 2019), which leads to inconsistency, shows that it presupposes that in the frame $\mathscr{S}$ considered, for two bases $S, S' \in \mathscr{S}$, their union $S \cup S'$ is in $\mathscr{S}$ as well and that, furthermore, every extension of an $S \in \mathscr{S}$ by additional axioms is in $\mathscr{S}$. This means that IPC is incomplete with respect to $\mathfrak{E}$; that is, it does *not* hold that

$$\Gamma \vDash_{\mathfrak{E}} A \Rightarrow \Gamma \vdash_{\mathrm{IPC}} A.$$

A particularly interesting class of frames studied by Nascimento (2024) is the set $\mathfrak{F}$ of focused frames, which have a base S as their bottom node and contain all bases S' extending S, and for which IPC is complete. Conceptually, $\mathfrak{F}$ has the advantage that a frame in $\mathfrak{F}$ is determined by a base S so that, when defining logical validity, we can quantify over bases rather than frames, which is very natural from the model-theoretic point of view. Thus, we have incompleteness with respect to $\mathfrak{P}$ and $\mathfrak{C}$, but completeness with respect to $\mathfrak{U}$ and $\mathfrak{F}$. We also have completeness with respect to $\mathfrak{B}$, if we choose the boundary rules specified by Goldfarb (2016) or Litland (2012) (see the remark at the end of Section 7).

Results of this kind invite one to study in more detail, which classes of frames yield completeness and which classes of frames do not, or even beyond that which logic, possibly deviant from intuitionistic logic, corresponds to which class of frames. Such investigations would be analogous to methods in modal logic, where the study of correspondence between features of frames and syntactic features belongs to the standard repertoire. In general, this means that our results open up a wide field of study beyond Prawitz's completeness conjecture.

7 | THE STRUCTURE OF THE COUNTERMODEL: DO WE NEED PROPER RULES?

A Kripke-style completeness proof proceeds by constructing a countermodel to a non-derivable formula, where this countermodel is build up from syntactic entities. In predicate logic, one speaks of a 'term model'. In fact, this construction uses methods similar to Henkin's completeness proof for first-order logic. Depending on which variant of completeness proof is chosen, the countermodel constructed looks different. Such a countermodel is not normally a proof-theoretic base within a frame in the sense of Section 6. Crucial for our purpose here is that a 'conventional' Kripke countermodel can be isomorphically translated into a frame $\mathscr{S}$ together with a base S in our sense so that the pair $\langle \mathscr{S}, S\rangle$ 'essentially is' a Kripke countermodel.

As we only consider propositional logic, we rely on the presentation in Schütte (1968). Given a formula A which is not derivable in IPC , let $s(A)$ be the set of all subformulas of A. A subset D of $s(A)$ is called distinguished, if the formula $\bigwedge D \to \bigvee (s(A) \setminus D)$ is not derivable in IPC. Let $\mathscr{D}$ be the set of all distinguished subsets of $s(A)$ ordered by the subset relation, together with a valuation which at a point $D \in \mathscr{D}$ assigns truth to all atoms in D and falsity to all atoms not in D. This represents a Kripke-structure. We can show that this Kripke-structure verifies, at a point D, all formulas in D and falsifies all subformulas of A which are not in D. As for a non-derivable A, we can always construct a distinguished $D \subseteq s(A)$ not containing A, $\langle \mathscr{D}, D\rangle$ is a Kripke countermodel to A, as it falsifies A at point D within $\mathscr{D}$.

This Kripke-structure is no frame in the sense of our semantics, as its reference points D are sets of subformulas of D rather than bases (i.e. sets of atomic rules). Such a D can, however, isomorphically be turned into a base, and the Kripke structure into a frame. Take sufficiently many sentence letters $c_1, c_2, \ldots$ not occurring in A, and use them to code all compound subformulas of A, that is, all subformulas of A except the sentence letters occurring in A. Let D' result from D by replacing compound subformulas of A by their codes. Then D' is just a set of sentence letters. For any $D_1, D_2 \in \mathscr{D}$, it holds that $D_1 \subseteq D_2$ if and only if $D'_1 \subseteq D'_2$, which means that we have an order isomorphism. Let $\mathscr{D}'$ be the set of all D' such that $D \in \mathscr{D}$. Then, $\mathscr{D}'$ represents a frame in our sense which is isomorphic to the Kripke structure. If A is false in a reference point of the Kripke structure, it is invalid in the corresponding reference point in the frame.

This means that Kripke-(Henkin-)style reasoning gives us a frame and a base in this frame, which invalidates a formula underivable in IPC. To mimic the structure of the Kripke frame, we used some coding of formulas via additional propositional letters, but we did not bother about these propositional letters, as they are not contained in the formula A under consideration.

From the proof-theoretic viewpoint, it is interesting that we do not need proper rules in the countermodel constructed, but just axioms, namely the propositional letters representing true atomic propositions. This means that we have completeness not only with respect to all frames, but already with respect to the set of frames, whose bases contain only axioms and no proper rules. Let us call such a frame an *axioms-only frame*. IPC is complete with respect to the class $\mathfrak{A}$ of axioms-only frames. If by satisfiability of a formula A we understand the availability of a frame $\mathscr{S}$ and an element $S \in \mathscr{S}$ such that A is valid in S with respect to $\mathscr{S}$, we obtain the following:

Theorem 12. If a formula A is satisfiable at all, it is satisfiable in an axioms-only frame.

This sounds paradoxical at first glance, given that it is one of the crucial issues of proof-theoretic validity that its bases are genuine production systems and are not just sets of axioms representing a valuation of atoms. Satisfiability in axioms-only frames thus appears to go against the spirit of proof-theoretic semantics. However, this situation is not more paradoxical than Skolem's paradox. Even though every satisfiable first-order formula is satifiable in a countable structure, this is not normally considered to speak against the uncountable in itself, as uncountable structures have significant features of their own. Similarly, bases with proper rules have significant features of their own that deserve to be (and are being) studied, even though satisfiability per se does not depend on the availability of proper rules. Both Skolem's paradox and the above theorem result from the particular feature of the counterexample constructed in the completeness proof—in the case of Skolem's paradox, it is the countability of the constructed term model; in our case, it is the fact that the model construction only needs axioms.

Remark. Goldfarb (2016), Litland (2012) and Stafford and Nascimento (2023) prove completeness in a more general setting. Whereas above a specific form of the Kripke-style completeness proof was considered, and the constructed countermodel was coded in the proof-theoretic framework, Goldfarb, Litland and Stafford & Nascimento present methods to code arbitrary finite-tree Kripke models and thus can adopt any kind of completeness proof based on such models. Goldfarb works in a framework in which bases only consist of axioms, but where a frame is related to a global set of boundary rules. These boundary rules are used to code the accessibility structure of a given Kripke model: Reference points are coded by atoms and the accessibility of a from b by the atomic rule $a \Rightarrow b$. Litland works in a similar framework, which is technically related to Golfarb's approach, but with the crucial difference that he formally distinguishes verifiers of atoms from the atoms themselves and lets the boundary rules operate on these verifiers.[8] Stafford and Nascimento work (essentially) in the same framework as defined in this paper. Like Goldfarb, they code reference points by atoms. Given a base S_w corresponding to a reference point w and coded by c, this base contains the atoms evaluated as true in w and in addition the vacuous rule $c' \Rightarrow c'$, where c' is the code of the immediate predecessor (accessibility-wise) of S_w. To all authors, the remarks made above on the axioms-only paradox apply: Satisfiability implies axioms-only satisfiability. Even though they are using proper inference rules to code the order structure of their respective frame, this usage is not 'genuine', as these rules do not contribute to the validity of a formula in a base (in Stafford and Nascimento's version the rules are deductively void anyway).

[8]As to the publishing dates, though only published in 2016, Goldfarb's manuscript dates from 1999 (and was made available by Michael Dummett at the first Tübingen conference on proof-theoretic semantics in that year—the manuscript of Dummett's reply to Goldfarb's theses is lost). Thus, Litland's doctoral dissertation of 2012 was able to build on Goldfarb's ideas and to develop them further.

To Goldfarb and Litland, the axioms-only paradox may perhaps not sound paradoxical, as they work in an axioms-only framework with added boundary rules.

8 | CARNAP CATEGORICITY OF PRAWITZ SEMANTICS AND THE INTENDED MEANING OF THE LOGICAL CONSTANTS

If we consider just the calculus of intuitionistic propositional logic IPC, the meaning of the logical connectives is not fully determined. For example, both classical and intuitionistic connectives obey the laws of IPC. However, given a specific semantics for which IPC is sound, we may ask whether this semantics delivers a single interpretation of the IPC connectives, or whether there are different interpretations compatible with it. This question, which for classical logic was posed by Carnap (1943), and which has occasionally been referred to in inferentialism and proof-theoretic semantics (e.g. Murzi & Hjortland, 2009), was given a precise model-theoretic rendering by Westerståhl et al. (Bonnay & Westerståhl, 2016 for classical logic, and Tong & Westerståhl, 2023 for intuitionistic logic). Tong and Westerståhl show, for various intuitionistic interpretations of IPC, that they are unique. In Kripke semantics, such an interpretation, and thus the notion of uniqueness, is relative to a given frame. As this uniqueness property holds for all frames, Tong and Westerståhl speak of 'Carnap categoricity' of IPC with respect to Kripke semantics.

As Carnap categoricity is based on a uniqueness concept for single frames, it can be applied to any single-frame semantics, in particular to Prawitz's principal-frame semantics. Here, we would speak of Carnap categoricity if uniqueness of interpretation holds for the single frame under consideration. It turns out that in this sense IPC is indeed Carnap categorical under Prawitz's principal-frame semantics. This can be seen as follows. We rely on the terminology and the results of Tong and Westerståhl.

For any A, let again $[\![A]\!]$ be the set of bases, in which A holds: $[\![A]\!] = \{S : \vDash_S A\}$. Let $\mathbb{0}$ be the set of inconsistent bases: $\mathbb{0} = \{S : \vDash_S A \text{ for all } A\}$, which means that $\mathbb{0} = [\![\bot]\!]$. Let $\mathbb{1}$ be the set $\mathscr{S}$ of all bases. In order to interpret intuitionistic implication, a set $\mathscr{M}$ of bases is called upward closed if for bases S, S':

$$S \in \mathscr{M} \text{ and } S' \supseteq S \Rightarrow S' \in \mathscr{M}.$$

Due to the extension condition ($\vDash_S^{\text{ext}}$), $[\![A]\!]$ (for any A) and $\mathbb{0}$ are upward closed; $\mathbb{1}$ is trivially upward closed. Let $S\uparrow$ be the set of all bases $S' \supseteq S$ (i.e. the smallest upward-closed set containing S). Let $\mathbf{U}$ be the set of all upward-closed sets except the empty set.[9] Then Prawitz semantics can be viewed as assigning to every A an upward-closed set $[\![A]\!] \in \mathbf{U}$ as its semantical value in the following way.

Let v be a valuation function, which assigns an upward-closed set to every atom a, that is, $v(a) \in \mathbf{U}$. Then, we assign semantical values to formulas according to the following definition.

Definition 13. Assignment of semantical values to formulas.

$$[\![a]\!]_v = v(a) \quad [\![\bot]\!]_v = \mathbb{0} \quad [\![\top]\!]_v = \mathbb{1}$$
$$[\![A \wedge B]\!]_v = [\![A]\!]_v \cap [\![B]\!]_v \quad [\![A \vee B]\!]_v = [\![A]\!]_v \cup [\![B]\!]_v$$
$$[\![A \rightarrow B]\!]_v = \{S : S\uparrow \cap [\![A]\!]_v \subseteq [\![B]\!]_v\}.$$

[9]That we exclude the empty set of bases as a possible semantic value of formulas is due to our non-standard way of treating absurdity and negation, which is itself due to the specific single-frame structure of Prawitz semantics. The semantical value of absurdity, that is, $\mathbb{0}$, is the set of all inconsistent bases rather than the empty set of bases. See the remark on negation at the end of Section 2.

If we deal not just with principal-frame semantics but with semantics based on more than a single frame, we would have to relativise $[\![A]\!]_v$ to the frame considered, for example, by writing $[\![A]\!]_v^{\mathcal{S}}$. The valuation $v_0(a) := \{S : \vdash_S a\}$ is distinguished. $[\![A]\!]_{v_0}$ gives us the intended semantical value of the formula A in Prawitz semantics, so we may define $[\![A]\!] := [\![A]\!]_{v_0}$. However, it should be emphasised that Prawitz semantics makes perfect sense with respect to arbitrary valuations v as presented in Definition 13 and that IPC is sound for interpretations using such valuations. Carnap categoricity crucially depends on this feature.

If we denote the set-theoretical operation on upward-closed sets that in Definition 13 corresponds to $\rightarrow$ by $\sqsubseteq$:

$$\mathcal{M} \sqsubseteq \mathcal{N} := \{S : S\uparrow \cap \mathcal{M} \subseteq \mathcal{N}\},$$

then it can easily be seen that

$$\langle \mathbf{U}, \mathbb{0}, \mathbb{1}, \cap, \cup, \sqsubseteq \rangle$$

is a Heyting algebra. Tong and Westerståhl demonstrated that this is the only possibility to interpret the logical connectives in $\mathbf{U}$, with the sole precondition that conjunction is interpreted by intersection. If the logical constants are interpreted by $\mathbb{0}', \mathbb{1}', \cap, \cup', \sqsubseteq'$ rather than $\mathbb{0}, \mathbb{1}, \cap, \cup, \sqsubseteq$, and IPC is sound with respect to this interpretation, then the resulting algebra

$$\langle \mathbf{U}, \mathbb{0}', \mathbb{1}', \cap, \cup', \sqsubseteq' \rangle$$

is a Heyting algebra identical to

$$\langle \mathbf{U}, \mathbb{0}, \mathbb{1}, \cap, \cup, \sqsubseteq \rangle.$$

In this sense, Prawitz semantics is a (Carnap) categorical semantics of intuitionistic logic: intuitionistic, as the interpretation of the connectives yields a Heyting algebra, which is the intended interpretation; and categorical, as it has no unintended ('non-standard') interpretation. Prawitz's principal-frame semantics describes exactly the intuitionistic connectives. The arguments above can easily be carried over to general semantics and to semantics with respect to certain classes of frames. However, the point we wanted to make here is that *Prawitz semantics as it stands* has the fundamental property of categoricity.

Even though IPC is incomplete for Prawitz semantics, that is, does not generate all laws that are semantically valid, it nevertheless gives the connectives a unique meaning in the light of this semantics. One might even argue that categoricity is more important than completeness. When passing to advanced systems, we are loosing completeness anyway, so it is perhaps not wise to insist on it for propositional logic. What is more important from the viewpoint of proof-theoretic semantics is its unique determination of meaning. We distinguish between the meaning of the connectives and the laws that hold in virtue of this meaning. The first might be called the intensional and the second the extensional context of semantics. Intensionally, Prawitz semantics does for IPC what it supposes to do: giving the connectives their intuitionistic meaning in a proof-theoretic manner. Only extensionally, it lacks a certain feature, namely completeness, which calls for an investigation into which superintuitionistic logic calculi might be complete for Prawitz semantics. However, in such a logic (see Stafford, 2021), the meaning of the connectives would still be intuitionistic and would be described by the above Heyting algebra.[10]In any

[10]Incidentally, there is an analogous situation with the proof-theoretic concept of uniqueness. In intuitionistic logic, the introduction and elimination rules for a connective c uniquely determine its meaning in the sense that two copies of the introduction and elimination rules for c and c^*, respectively, yield the interderivability of c and c^*. This interderivability persists if the logic is extended with additional superintuitionistic principles.

case, Carnap categoricity, which is a concept at the level of the widely discussed notions of harmony and uniqueness, deserves to be investigated in much more depth in proof-theoretic semantics.

9 | CONCLUDING DISCUSSION

Prawitz's validity-based semantics renders intuitionistic propositional logic IPC incomplete, because Harrop's rule, which is only admissible, but not derivable in IPC , can be validated. Therefore his completeness conjecture cannot be upheld in its original form. Two basic strategies are possible as way out.

1. We might change the semantics and look for a plausible modified version of it, for which IPC is complete.
2. We might change the logic and look for a plausible modified version of it, which is complete for Prawitz semantics.

The second option leads into the area of logic intermediate between intuitionistic and classical logic. In view of the counterexample to completeness, of particular interest is logic stronger than IPC ('superintuitionistic logics'), in which the disjunction property still holds (see Chagrov & Zakharyashchev, 1991; Kleene, 1962). Stafford (2021) has made a big step in this direction by showing that general inquisitive semantics is equivalent to Prawitz semantics when in atomic systems rules that discharge assumptions are admitted.

Using the first option, we would look for modifications of Prawitz semantics, which are conceptually plausible and render IPC complete. In this paper, we have discussed two possibilities, a favourable one and the one that we discarded. The one we favoured is to give up Prawitz's focus on a single structure of atomic systems, according to which formulas are evaluated in the frame of all bases, which we called 'principal-frame semantics'. Formulas should instead be evaluated in the multiplicity of all subsets of the set of atomic systems, called 'general semantics'. Through general semantics, we gain access to Kripke-style completeness proofs and can re-establish completeness, at least if we are prepared to accept the classical methods on which they rest. We have completeness with respect to general semantics and incompleteness with respect to principal-frame semantics. This coexistence of incompleteness and completeness is analogous to the situation in second-order logic, where we have incompleteness with respect to full models, but completeness with respect to general models; so, it is not an absolutely novel phenomenon.

What we discarded was a specific modification of Prawitz semantics supposed to take definitional reasoning into account: namely the idea to remove the consideration of extensions of bases for the validation of hypothetical reasoning. Simply disregarding extensions leads to even bigger problems with completeness; it actually leads to a classical rather than intuitionistic semantics. We nevertheless totally agree with Prawitz that definitional reasoning must be incorporated in proof-theoretic semantics. However, this should be done in a different way, being an issue on top of extension semantics and nothing that replaces extension semantics. A potential candidate for definitional clauses are boundary rules that work for all bases likewise and that are not extended when bases representing one's knowledge state are extended.

We discussed alternatives to Prawitz's validity semantics as first laid out by him in Prawitz (1971, 1973), still keeping intact its fundamental orientation: the view of introduction rules as primary meaning-giving inferences. In particular, a valid canonical proof of a disjunction $A \vee B$ assumes that either a valid proof of A or a valid proof of B is given, to which the step of disjunction introduction is applied. There are many other approaches to validity in proof-theoretic semantics; some come to similar, some to different conclusions as far as completeness is concerned. For an overview of such approaches, see the survey by Piecha (2016) and the discussion by Stafford (2024). In particular, we have not dealt here with validity concepts based on elimination rules, which would interpret disjunction by the ('indirect') rule

$$\frac{A \vee B \quad \begin{matrix} A \\ \vdots \\ C \end{matrix} \quad \begin{matrix} B \\ \vdots \\ C \end{matrix}}{C}.$$

In particular, the fact that in a semantic clause for disjunction based on this rule, the minor premiss and conclusion C can be assumed to be atomic—as proposed by various authors including Prawitz[11]—leads to interesting semantic frameworks beyond our scope here, which include completeness proofs for IPC : in Litland (2012) by using Kripke semantics, in Sandqvist (2015) by using atomic higher-level rules and in Oliveira (2021) by adapting a method of complementation of arguments originally suggested by Dummett (1991, Ch.13).

An interesting general phenomenon is the axioms-only paradox, namely the fact that if a formula is satisfiable at all, it is satisfiable in a frame whose bases have only axioms and no proper rules. This requires a further discussion of the role of rules at the atomic level. To describe one's state of knowledge, proper rules are perhaps not needed. On the other hand, one would need them to represent definitional reasoning, as inductive definitions are essentially systems of rules (see Aczel, 1977). This again points to the desideratum to develop a semantic framework that includes both states of knowledge and definitions.

It could be shown that Tong and Westerståhl's (2023) results on Carnap categoricity of IPC are applicable to Prawitz semantics in a straightforward manner. This semantics thus exhibits a fundamental feature, which in proof-theoretic semantics has not received the attention it deserves. The property of categoricity qualifies Prawitz's principal-frame semantics as a genuine intuitionistic semantics for IPC , even though completeness is lacking.

I would like to finish with a personal remark of gratitude to Dag Prawitz. My work on the proof-theoretic foundations of logic, and indeed most of my career, would not have been possible without Dag's guidance both through his publications and through numerous discussions we had on issues of proof-theoretic semantics. I am extremely grateful to Dag for this, and I am delighted that his outstanding achievements in logic and the philosophy of logic have been recognised and honoured by the Rolf Schock prize. I am proud to be a contributor to this special issue of *Theoria* commemorating the event.

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[11]See Ferreira (2006), Prawitz (2007), Litland (2012), Schroeder-Heister (2015), Sandqvist (2015).

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